The Journal of Geometric Analysis (2025) 35:356
https://doi.org/10.1007/s12220-025-02192-0

Wavelet Characterization of Inhomogeneous Lipschitz Spaces on Spaces of Homogeneous Type and Its Applications

Fan Wang[1]

Received: 6 March 2025 / Accepted: 5 September 2025

Abstract
In this article, the author establishes a wavelet characterization of inhomogeneous Lipschitz space $\mathrm{lip}_\theta(\mathcal{X})$ via Carleson sequence, where $\mathcal{X}$ is a space of homogeneous type introduced by R. R. Coifman and G. Weiss. As applications, characterizations of several geometric conditions on $\mathcal{X}$, involving the upper bound, the lower bound, and the Ahlfors regular condition, are obtained.

Keywords Space of homogeneous type · Inhomogeneous Lipschitz space · Wavelet · Ahlfors regular condition

Mathematics Subject Classification Primary 46E36 · Secondary 46E35 · 46E39 · 42B25 · 30L99

1 Introduction

As very fundamental function spaces, (inhomogeneous) Lipschitz spaces permeate both pure and applied disciplines. Their significance extends ubiquitously across diverse mathematical domains, such as ordinary and partial differential equations, measure-theoretic analysis, and nonlinear functional analysis, as well as geometric-topological contexts including metric geometry, fractal theory, and topological dynamics. Beyond theoretical mathematics, these functions demonstrate remarkable versatility in computational science, finding essential applications in image processing

This project is partially supported by the National Natural Science Foundation of China (Grant No. 12401115), the Open Project Program of Key Laboratory of Mathematics and Complex System of Beijing Normal University (Grant No. K202304), the Hebei Natural Science Foundation (Grant No. A2024201004), and the Excellent Youth Research Innovation Team of Hebei University (QNTD202414).

✉ Fan Wang
fanwang@hbu.edu.cn

1 College of Mathematics and Information Science, Hebei University, Baoding 071002, The People's Republic of China

algorithms, search engine optimization architectures, and stability analysis of machine learning models.

During the 1970s, Coifman and Weiss [13, 14] introduced the groundbreaking concept of spaces $(\mathcal{X}, d, \mu)$ of homogeneous type (see Definition 2.1 below), a framework extending classical Euclidean analysis to general metric measure spaces. This innovation catalyzed profound investigations into Lipschitz spaces over such structures. In [38], Macías and Segovia made contributions by elucidating the geometric structure of $\mathcal{X}$ and unifying several definitions of Lipschitz functions on these spaces. In 2020, Zheng et al. [46] established the Littlewood–Paley characterization of Lipschitz spaces on Ahlfors regular spaces. Later, Li and Zheng [31] obtained the boundedness of Caldrón–Zygmund operators on Lipschitz spaces. Motivated by the advent of wavelets system constructed in [2], Liu et al. [36] developed an wavelets characterization of homogeneous Lipschitz spaces. On the other hand, He et al. [22, Definition 2.7] introduced a new kind of *approximations of the identity with exponential decay*, a pivotal tool to establish (in)homogeneous continuous/discrete Calderón reproducing formulae on $\mathcal{X}$. Building on this foundation, He et al. [21] obtained a complete real-variable theory of atomic Hardy spaces on $\mathcal{X}$. Recently, based on the concept of inhomogeneous approximation of the identity with exponential decay, He et al. [25] established several characterizations of local Hardy space $h^p(\mathcal{X})$ and showed that the dual of $h^p(\mathcal{X})$ is the inhomogeneous Lipschitz space $\mathrm{lip}_{1/p-1}(\mathcal{X})$. We refer the reader to [10, 11, 32, 42, 43] for more recent progress on the topic of (local) Hardy spaces and their duals on spaces of homogeneous type.

Theoretically, an important significance of Lipschitz spaces lies in their role as the dual of Hardy-type spaces. Thus, the products of functions in Hardy spaces and Lipschitz spaces have also garnered significant research interest. Inspired by the progress on geometric function theory (see, for instance, [1]) and the nonlinear elasticity (see, for instance, [3, 39]), Bonami et al. [8] pioneered the investigation into bilinear decompositions involving products of functions in Hardy spaces and Lipschitz spaces. Subsequent developments by Bonami and Feuto [5, 15] established the linear decomposition of product of functions in $H^p(\mathbb{R}^n)$ and its dual space. Concurrently, Li and Peng [33] obtained a linear decomposition of product of functions in $H^1_L(\mathbb{R}^n)$ and its dual space $\mathrm{BMO}_L(\mathbb{R}^n)$, where $L := -\Delta + V$ is a Schrödinger operator; see also Ky [29] for a bilinear version. In the context of local Hardy space $h^p(\mathbb{R}^n)$, Cao et al. [12] established a bilinear decomposition of products for functions in $h^p(\mathbb{R}^n)$ and its dual spaces with $p \in (0, 1]$, which was further refined by Yang et al. [44]. In [44], Yang et al. obtained alternative bilinear decomposition of products for functions in $h^p(\mathbb{R}^n)$ and its dual spaces with $p \in (0, 1)$, which was shown to be sharp in the dual sense. Moreover, using this bilinear decomposition, Yang et al. [44] obtained some div-curl estimates. These results of bilinear decomposition also play key roles in the estimates of weak Jacobians (see, for instance, [6, 7]) and commutators (see, for instance, [28, 34]). These works further inspire many new ideas in the research of nonlinear partial differential equations; see, for instance, [8, 27, 30] and their references therein for more details. Recent advances in (bi)linear decomposition theory for (local) Hardy space products and their duals continue to emerge, as documented in [4, 9, 17, 35, 37], highlighting the enduring vitality of this research direction.

Notice that wavelet characterization for (inhomogeneous) Lipschitz spaces plays a key role when analyzing products of functions in (local) Hardy spaces and their duals. This naturally raises an question: Can inhomogeneous Lipschitz spaces on general spaces of homogeneous type admit analogous wavelet characterizations? The main target of this paper is to give an affirmative answer. Precisely, we develop a wavelet characterization of the inhomogeneous Lipschitz space $\mathrm{lip}_\theta(\mathcal{X})$ via Carleson sequence, where ω is the *upper dimension* in 2.3, η is the smooth index of wavelets in Lemma 3.1, and $L^2_{\mathcal{B}}(\mathcal{X})$ is the collection of all measurable functions f on $\mathcal{X}$ such that $f\mathbf{1}_B \in L^2(\mathcal{X})$ for any ball $B \subset \mathcal{X}$.

Theorem 1.1 *Let ω be as in (2.3), $\eta \in (0, 1]$ be as in Lemma 3.1, and $\theta \in (0, \eta/\omega)$. Then, for any $f \in L^2_{\mathcal{B}}(\mathcal{X})$, the following statements are equivalent:*

(i) $f \in \mathrm{lip}_\theta(\mathcal{X})$;
(ii)

$$f = \sum_{\alpha\in\mathcal{A}_0} \left\langle f, \phi^0_\alpha \right\rangle \phi^0_\alpha + \sum_{k=0}^{\infty} \sum_{\beta\in\mathcal{G}_k} \left\langle f, \psi^{k+1}_\beta \right\rangle \psi^{k+1}_\beta$$

in $L^2_{\mathcal{B}}(\mathcal{X})$ and

$$\|f\|_* = \sup_{Q\in\mathcal{D}_0} \left\{ \frac{1}{[\mu(Q)]^{1+2\theta}} \left[\sum_{\{\alpha\in\mathcal{A}_0:Q^0_\alpha\subset Q\}} \left|\left\langle f, \phi^0_\alpha \right\rangle\right|^2 \right.\right.$$
$$\left.\left. + \sum_{k=0}^{\infty} \sum_{\{\beta\in\mathcal{G}_k:Q^{k+1}_\beta\subset Q\}} \left|\left\langle f, \psi^{k+1}_\beta \right\rangle\right|^2 \right] \right\}^{\frac{1}{2}} < \infty.$$

Moreover, there exists a constant $C \in [1, \infty)$, such that

$$C^{-1}\|f\|_* \le \|f\|_{\mathrm{lip}_\theta(\mathcal{X})} \le C\|f\|_*.$$

This result crucially eliminates dependencies on the reverse doubling condition of the measure and the metric condition of the quasi-metric under consideration. Moreover, using this characterization, we discuss several geometric conditions on $\mathcal{X}$, involving the upper bound, the lower bound, and the Ahlfors regular condition, and obtain some equivalence characterizations.

The organization of the remainder of this article is as follows.

In Section 2, we first recall some basic preliminaries on spaces of homogeneous type, inhomogeneous Lipschitz spaces, dyadic cube system established in [26], spaces of test functions, and spaces of distributions. We show that all test functions are pointwise multipliers on the inhomogeneous Lipschitz spaces; see Proposition 2.8 below.

Section 3 is devoted to proving Theorem 1.1. To this end, we first recall the wavelets system obtained in [2]. Using theses wavelets, we establish an equivalence characterization of imhomogenous Lipschitz spaces via Carleson sequences.

Differently from the proof of [12, Theorem 2.8] on Euclidean spaces $\mathbb{R}^n$, the relation $\mathrm{lip}_\theta(\mathcal{X}) = F^{\omega\theta}_{\infty,\infty}(\mathcal{X}) = C^{\omega\theta}(\mathcal{X})$ may not holds true in the general setting of spaces of homogeneous type; see Corollary 4.3 below. To overcome this, we fully use the exponential decay of the wavelets to obtain that the integrals of the products of functions in $\mathrm{lip}_\theta(\mathcal{X})$ and wavelets also have enough decay; see Proposition 3.5 below.

In Section 4, we give some applications. As corollaries of Theorem 1.1, we develop three equivalent characterizations of geometric conditions on $\mathcal{X}$, involving the upper bound, the lower bound, and the Ahlfors regular condition. Corollary 4.3 extends results in [42, Theorem 3.2] to the inhomogeneous version.

Finally, let us make some conventions on notation. Throughout this article, A_0 is used to denote the positive constant appearing in (2.1), ω is used to to denote the *upper dimension* in (2.3), and η is used to denote the smoothness index of wavelets in Lemma 3.1. Moreover, δ is a small positive number coming from the construction of the dyadic cubes on $\mathcal{X}$ (see Lemma 2.6 below). We use C to denote a positive constant which is independent of the main parameters involved, but may vary from line to line. The symbol $C_{(\alpha,\beta,\dots)}$ denotes a positive constant depending on the indicated parameters α, β, ... The symbol $A \lesssim B$ means that $A \le CB$ for some positive constant C, while $A \sim B$ means $A \lesssim B \lesssim A$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g = h$ or $f \lesssim g \le h$. For any set $E \subset \mathcal{X}$, $\mathbf{1}_E$ means the characteristic function of E. For any set F, $\#F$ denotes its *cardinality*.

2 Inhomogeneous Lipschitz Spaces on Spaces of Homogeneous Type

In this section, we mainly investigate the inhomogeneous Lipschitz spaces $\mathrm{lip}_\theta(\mathcal{X})$ on spaces of homogeneous type, including the relation between $\mathrm{lip}_\theta(\mathcal{X})$ and distributions spaces. Moreover, we establish an equivalent characterization of $\mathrm{lip}_\theta(\mathcal{X})$ via Carleson sequences. Let us first recall the concept of space of homogeneous type in the sense of Coifman and Weiss [13, 14].

Definition 2.1 Let $\mathcal{X}$ be a non-empty set, d a non-negative function defined on $\mathcal{X} \times \mathcal{X}$, and μ a measure on $\mathcal{X}$. $(\mathcal{X}, d, \mu)$ is called a *space of homogeneous type* provided that d and μ satisfy the following conditions:

(I) for any $x, y, z \in \mathcal{X}$,

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) there exists a constant $A_0 \in [1, \infty)$, independent of x, y, and z, such that

$$d(x, z) \le A_0[d(x, y) + d(y, z)]; \tag{2.1}$$

(II) there exists a constant $C \in [1, \infty)$ such that, for any ball $B \subset \mathcal{X}$,

$$\mu(2B) \le C\mu(B), \tag{2.2}$$

where, the *ball* B, centered at $x_B \in \mathcal{X}$ with radius $r_B \in (0, \infty)$, of $\mathcal{X}$ is defined by setting

$$B := B(x_B, r_B) := \{x \in \mathcal{X} : \ d(x_B, x) < r_B\}$$

and, for any $\tau \in (0, \infty)$, $\tau B := B(x_B, \tau r_B)$.

Observe that, for $\mu(\mathcal{X}) < \infty$, He et al. [25, Proposition 6.5] showed that $H^p(\mathcal{X}) = h^p(\mathcal{X})$ with equivalent norms. By the duality theory of $H^p(\mathcal{X})$ and $h^p(\mathcal{X})$, we know that homogeneous Lipschitz spaces coincide with inhomogeneous Lipschitz spaces if $\mu(\mathcal{X}) < \infty$. In [36, Theorems 3.6, 3.7, and 3.8], Liu et al. established the wavelet characterization of homogeneous Lipschitz spaces. Thus, concentrating on inhomogeneous Lipschitz spaces, in what follows, we always assume that $\mu(\mathcal{X}) = \infty$. Note that diam$\mathcal{X} = \infty$ implies $\mu(\mathcal{X}) = \infty$ (see, for instance, [40, Lemma 5.1] and [2, Lemma 8.1]). Therefore, under the assumptions of this article, $\mu(\mathcal{X}) = \infty$ if and only if diam$\mathcal{X} = \infty$. For any $x \in \mathcal{X}$, we also assume that the balls $\{B(x, r)\}_{r\in(0,\infty)}$ form a basis of open neighborhoods of x. Moreover, we also assume that μ is Borel regular, that is, all open sets are measurable and every set $A \subset \mathcal{X}$ is contained in a Borel set E satisfying that $\mu(A) = \mu(E)$. For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, we suppose that $\mu(B(x, r)) \in (0, \infty)$ and $\mu(\{x\}) = 0$. Let

$$C_{(\mu)} := \sup_{\text{ball } B\subset\mathcal{X}} \mu(2B)/\mu(B).$$

Then it is easy to prove that $C_{(\mu)}$ is the smallest positive constant satisfying (2.2). Moreover, (2.2) further implies that, for any ball B and any $\lambda \in [1, \infty)$,

$$\mu(\lambda B) \le C_{(\mu)}\lambda^{\omega}\mu(B), \tag{2.3}$$

where $\omega := \log_2 C_{(\mu)}$ is called the *upper dimension* of $\mathcal{X}$.

The following lemma includes some useful estimates related to the measure of balls; see, for instance, [20, Lemma 2.1] for more details (see also [22, Lemma 2.4]). For any $r \in (0, \infty)$ and $x, y \in \mathcal{X}$ with $x \ne y$, let

$$V(x, y) := \mu(B(x, d(x, y))) \text{ and } V_r(x) := \mu(B(x, r)).$$

Lemma 2.2 (i) *Let $x, y \in \mathcal{X}$ with $x \ne y$ and $r \in (0, \infty)$. Then $V(x, y) \sim V(y, x)$ and*

$$V_r(x) + V_r(y) + V(x, y) \sim V_r(x) + V(x, y) \sim V_r(y) + V(x, y) \sim \mu(B(x, r + d(x, y))).$$

Moreover, if $d(x, y) \le r$, then $V_r(x) \sim V_r(y)$. Here all the above positive equivalence constants are independent of x, y, and r.

(ii) *Let $\gamma \in (0, \infty)$. There exists a positive constant C such that, for any $x_1 \in \mathcal{X}$ and $r \in (0, \infty)$,*

$$\int_{\mathcal{X}} \frac{1}{V_r(x_1) + V(x_1, y)} \left[\frac{r}{r + d(x_1, y)}\right]^{\gamma} d\mu(y) \le C.$$

(iii) *There exists a positive constant* C *such that, for any* $x \in \mathcal{X}$ *and* $R \in (0, \infty)$,

$$\int_{\{z\in\mathcal{X}:\, d(x,z)\geq R\}} \frac{1}{V(x,y)} \left[\frac{R}{d(x,y)}\right]^{\beta} d\mu(y) \leq C.$$

On the ratio of measures of two balls, we have the following lemma.

Lemma 2.3 *Let* $x, y \in \mathcal{X}$ *and* $r_1, r_2 \in (0, \infty)$. *If* $r_1 + d(x, y) \geq r_2$, *then*

$$\frac{\mu(B(x,r_1))}{\mu(B(y,r_2))} \leq A_0^{\omega} \left[\frac{r_1 + d(x,y)}{r_2}\right]^{\omega}.$$

Proof Let $x, y \in \mathcal{X}$ and $r_1, r_2 \in (0, \infty)$. By (2.1), we find that, for any $z \in B(x, r_1)$,

$$d(z, y) \leq A_0[d(z, x) + d(x, y)] < A_0[r_1 + d(x, y)],$$

which further implies that

$$B(x, r_1) \subset B(y, A_0[r_1 + d(x, y)]) = B\left(y, A_0\frac{r_1 + d(x,y)}{r_2} r_2\right).$$

This, together with (2.3), further implies that

$$\mu(B(x, r_1)) \leq A_0^{\omega} \left[\frac{r_1 + d(x,y)}{r_2}\right]^{\omega} \mu(B(y, r_2)),$$

which completes the proof of Lemma 2.3. □

Now, we recall the concept of inhomogeneous Lipschitz spaces on spaces of homogeneous type. For any $q \in [1, \infty]$, the set $L^q_{\mathcal{B}}(\mathcal{X})$ denotes the collection of all measurable functions f on $\mathcal{X}$ such that $f\mathbf{1}_B \in L^q(\mathcal{X})$ for any ball $B \subset \mathcal{X}$. For any $\{f_n\}_{n\in\mathbb{N}} \subset L^q_{\mathcal{B}}(\mathcal{X})$ and $f \in L^q_{\mathcal{B}}(\mathcal{X})$, if, for any $B \subset \mathcal{X}$,

$$\lim_{n\to\infty} \|f_n - f\|_{L^q(B)} = 0,$$

then we say that $\{f_n\}_{n\in\mathbb{N}}$ converges to f in $L^q_{\mathcal{B}}(\mathcal{X})$.

Definition 2.4 For any $B := B(x_B, r_B) \subset \mathcal{X}$, $\theta \in (0, 1)$, and $f \in L^{\infty}_{\mathcal{B}}(\mathcal{X})$, let

$$\mathcal{M}^B_{\theta}(f) := \begin{cases} \sup\limits_{x,y\in B} \dfrac{|f(x) - f(y)|}{[\mu(B)]^{\theta}} & \text{if } r_B \in (0, 1], \\[2ex] \dfrac{\|f\|_{L^{\infty}(B)}}{[\mu(B)]^{\theta}} & \text{if } r_B \in (1, \infty). \end{cases}$$

The *inhomogeneous Lipschitz spaces* $\mathrm{lip}_\theta(\mathcal{X})$ is defined by setting

$$\mathrm{lip}_\theta(\mathcal{X}) := \left\{ f \in L^q_{\mathcal{B}}(\mathcal{X}) : \|f\|_{\mathrm{lip}_\theta(\mathcal{X})} := \sup_{\text{ball } B\subset\mathcal{X}} \mathcal{M}^B_\theta(f) < \infty \right\}.$$

By Definition 2.4, we find that, for any $f \in \mathrm{lip}_\theta(\mathcal{X})$, f is continuous. Moreover, we have the following properties of $\mathrm{lip}_\theta(\mathcal{X})$.

Lemma 2.5 *Let $\theta \in (0, 1)$. Then there exists a positive constant C such that, for any $x \in \mathcal{X}$, $r \in (1, \infty)$, and $f \in \mathrm{lip}_\theta(\mathcal{X})$,*

$$|f(x)| \le C\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}[V_r(x)]^\theta, \tag{2.4}$$

and, for any $x, y \in \mathcal{X}$ with $x \neq y$,

$$|f(x) - f(y)| \le C\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}[V(x, y)]^\theta. \tag{2.5}$$

To prove Lemma 2.5, we need the following dyadic cube system established by Hytönen and Kairema in [26, Theorem 2.2].

Lemma 2.6 *Let $c_0, C_0 \in (0, \infty)$ and $\delta \in (0, 1)$ be such that $c_0 < C_0$ and $12A_0^3C_0\delta \le c_0$. Assume that $\mathcal{A}_k$, a set of indices for any $k \in \mathbb{Z}$, and a set of points, $\{x^k_\alpha : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\} \subset \mathcal{X}$, have the following properties: for any $k \in \mathbb{Z}$,*

$$d(x^k_\alpha, x^k_\beta) \ge c_0\delta^k \text{ if } \alpha \neq \beta, \text{ and } \min_{\alpha\in\mathcal{A}_k} d(x, x^k_\alpha) < C_0\delta^k \text{ for any } x \in \mathcal{X}. \tag{2.6}$$

Then there exists a family of sets, $\{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\}$, such that

- (i) *for any $k \in \mathbb{Z}$, $\{Q^k_\alpha : \alpha \in \mathcal{A}_k\}$ is disjoint and $\bigcup_{\alpha\in\mathcal{A}_k} Q^k_\alpha = \mathcal{X}$;*
- (ii) *if $l, k \in \mathbb{Z}$ and $l \le k$, then, for any $\alpha \in \mathcal{A}_l$ and $\beta \in \mathcal{A}_k$, either $Q^k_\beta \subset Q^l_\alpha$ or $Q^k_\beta \cap Q^l_\alpha = \emptyset$;*
- (iii) *for any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_k$, $B(x^k_\alpha, c_\#\delta^k) \subset Q^k_\alpha \subset B(x^k_\alpha, C_\#\delta^k)$, where $c_\# := (3A_0^2)^{-1}c_0$ and $C_\# := \max\{2A_0C_0, 1\}$.*

Points in $\{x^k_\alpha : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\} \subset \mathcal{X}$ are called *dyadic points*. For any $k \in \mathbb{Z}$, let

$$\mathcal{X}^k := \left\{x^k_\alpha : \alpha \in \mathcal{A}_k\right\}.$$

By the construction of dyadic points in [26, 2.21], we may assume that $\mathcal{X}^k$ is countable and $\mathcal{X}^k \subset \mathcal{X}^{k+1}$ for any $k \in \mathbb{Z}$. For any $k \in \mathbb{Z}$, define

$$\mathcal{Y}^k := \mathcal{X}^{k+1} \setminus \mathcal{X}^k$$

and

$$\mathcal{G}_k := \left\{\alpha \in \mathcal{A}_{k+1} : y^k_\alpha := x^{k+1}_\alpha \in \mathcal{Y}^k\right\}.$$

For any $x \in \mathcal{X}$, let

$$d(x, \mathcal{Y}^k) := \inf_{y \in \mathcal{Y}^k} d(x, y).$$

Next, we prove Lemma 2.5.

Proof of Lemma 2.5 Notice that f is continuous. To show (2.4), it suffices to prove that (2.4) holds true for almost every $x \in \mathcal{X}$. For this purpose, assume that

$$E := \left\{x \in \mathcal{X} : |f(x)| > C\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}[V_r(x)]^\theta\right\} \text{ and } \mu(E) > 0, \tag{2.7}$$

where C is determined later. Find $k_0 \in \mathbb{Z}$ such that $C_\# \delta^{k_0} < r$ where $C_\#$ is the same as in Lemma 2.6(iii). For any $\alpha \in \mathcal{A}_{k_0}$, let $E_\alpha := E \cap B(x_\alpha^{k_0}, r)$. Then, by (i) and (ii) of Lemma 2.6, we have $E = \bigcup_{\alpha \in \mathcal{A}_{k_0}} E_\alpha$. From this, the fact that $\mathcal{A}_{k_0}$ is countable, and (2.7), we deduce that there exists $\alpha_0 \in \mathcal{A}_{k_0}$ such that

$$\mu\left(E_{\alpha_0}\right) > 0. \tag{2.8}$$

By (2.1), we conclude that, for any $y \in B(x_{\alpha_0}^{k_0}, r)$ and $x \in E_{\alpha_0}$,

$$d(y, x) \le A_0\left[d\left(y, x_{\alpha_0}^{k_0}\right) + d\left(x_{\alpha_0}^{k_0}, x\right)\right] < 2A_0 r,$$

which further implies that $B(x_{\alpha_0}^{k_0}, r) \subset B(x, 2A_0 r)$. Combining this and (2.2), we find that, for any $x \in E_{\alpha_0}$,

$$\mu\left(B\left(x_{\alpha_0}^{k_0}, r\right)\right) \le C_{(\mu)}(2A_0)^\omega \mu(B(x, r)). \tag{2.9}$$

This, together with (2.7), further implies that, for any $x \in E_{\alpha_0}$,

$$|f(x)| > CC_{(\mu)}^{-\theta}(2A_0)^{-\theta\omega}\left[\mu\left(B\left(x_{\alpha_0}^{k_0}, r\right)\right)\right]^\theta \|f\|_{\mathrm{lip}_\theta(\mathcal{X})}.$$

Using this and (2.8), we infer that

$$\|f\|_{L^\infty(B(x_{\alpha_0}^{k_0}, r))} > CC_{(\mu)}^{-\theta}(2A_0)^{-\theta\omega}\left[\mu\left(B\left(x_{\alpha_0}^{k_0}, r\right)\right)\right]^\theta \|f\|_{\mathrm{lip}_\theta(\mathcal{X})}. \tag{2.10}$$

Choose $C := 2C_{(\mu)}^\theta(2A_0)^{\theta\omega}$. Then (2.10) implies that

$$\|f\|_{\mathrm{lip}_\theta(\mathcal{X})} \ge \frac{\|f\|_{L^\infty(B(x_{\alpha_0}^{k_0}, r))}}{[\mu(B(x_{\alpha_0}^{k_0}, r))]^\theta} > 2\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}.$$

This is a contradiction and hence $\mu(E) = 0$. This show that (2.4) holds true for almost every $x \in \mathcal{X}$ and finishes the prove of (2.4).

To prove (2.5), we consider the following two cases on $d(x, y)$.

Case 1. $d(x, y) < 1$. In this case, since $d(x, y) < 1$, if follows that there exists $N \in \mathbb{N}$ such that

$$\frac{N+1}{N} d(x, y) < 1. \tag{2.11}$$

Let $B_N := B(x, \frac{N+1}{N} d(x, y))$. Then $x, y \in B_N$. By (2.11), Definition 2.4, (2.3), and $(N+1)/N \le 2$, we obtain

$$|f(x) - f(y)| \le \|f\|_{\mathrm{lip}_\theta(\mathcal{X})} [\mu(B_N)]^\theta \lesssim \|f\|_{\mathrm{lip}_\theta(\mathcal{X})} [V(x, y)]^\theta.$$

This is the desired estimate.

Case 2. $d(x, y) \ge 1$. In this case, applying (2.4), Lemma 2.2, and (2.3), we deduce that

$$\begin{aligned} |f(x) - f(y)| &\le |f(x)| + |f(y)| \\ &\lesssim \|f\|_{\mathrm{lip}_\theta(\mathcal{X})} [V_2(x)]^\theta + \|f\|_{\mathrm{lip}_\theta(\mathcal{X})} [V_2(y)]^\theta \\ &\lesssim \|f\|_{\mathrm{lip}_\theta(\mathcal{X})} [V_2(x) + V_2(y) + V(x, y)]^\theta \\ &\lesssim \|f\|_{\mathrm{lip}_\theta(\mathcal{X})} [V(x, y)]^\theta. \end{aligned}$$

This is the desired estimate. Combining the above two cases, we finish the proof of (2.5) and hence Lemma 2.5. □

Now, we recall the concepts of test functions and distributions on $\mathcal{X}$; see, for instance, [19, 20]. For any $\gamma \in (0, \infty)$, the function P_γ with *Polynomial decay* is defined by setting, for any $x, y \in \mathcal{X}$ and $r \in (0, \infty)$,

$$P_\gamma(x, y; r) := \frac{1}{V_r(x) + V(x, y)} \left[\frac{r}{r + d(x, y)}\right]^\gamma. \tag{2.12}$$

Definition 2.7 (test functions) Let $x_0 \in \mathcal{X}$, $\beta \in (0, 1]$, and $r, \gamma \in (0, \infty)$. If a measurable function f on $\mathcal{X}$ satisfies that there exists a positive constant C such that

(i) for any $x \in \mathcal{X}$,

$$|f(x)| \le C P_\gamma(x_0, x; r); \tag{2.13}$$

(ii) for any $x, y \in \mathcal{X}$ satisfying $d(x, y) \le (2A_0)^{-1}[r + d(x_0, x)]$,

$$|f(x) - f(y)| \le C \left[\frac{d(x, y)}{r + d(x_0, x)}\right]^\beta P_\gamma(x_0, x; r), \tag{2.14}$$

where, for any $x, y \in \mathcal{X}$ and $r \in (0, \infty)$, P_γ is the same as in (2.12), then f is called a *test function of type* (x_0, r, β, γ).

The symbol $\mathcal{G}(x_0, r, \beta, \gamma)$ denotes the collection of all test functions of type (x_0, r, β, γ). For any $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, its *norm* $\|f\|_{\mathcal{G}(x_0, r, \beta, \gamma)}$ in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by setting

$$\|f\|_{\mathcal{G}(x_0, r, \beta, \gamma)} := \inf\{C \in (0, \infty) : \text{ (2.13) and (2.14) hold}\}.$$

Observe that, for any $x_1, x_2 \in \mathcal{X}$ and $r_1, r_2 \in (0, \infty)$,

$$\mathcal{G}(x_1, r_1, \beta, \gamma) = \mathcal{G}(x_2, r_2, \beta, \gamma)$$

with equivalent norms while the positive equivalence constants may depend on x_1, x_2, r_1, and r_2. For a fixed point $x_0 \in \mathcal{X}$, the space $\mathcal{G}(x_0, 1, \beta, \gamma)$ is simplified by $\mathcal{G}(\beta, \gamma)$. Usually, $\mathcal{G}(\beta, \gamma)$ is called the *spaces of test functions* on $\mathcal{X}$.

Fix $\epsilon \in (0, 1]$ and $\beta, \gamma \in (0, \epsilon]$. The symbol $\mathcal{G}_0^\epsilon(\beta, \gamma)$ denotes the completion of the set $\mathcal{G}(\epsilon, \epsilon)$ in $\mathcal{G}(\beta, \gamma)$ with the norm of $\mathcal{G}_0^\epsilon(\beta, \gamma)$ defined by setting $\|\cdot\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} := \|\cdot\|_{\mathcal{G}(\beta,\gamma)}$. The dual space $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ is defined to be the collection of all continuous linear functionals from $\mathcal{G}_0^\epsilon(\beta, \gamma)$ to $\mathbb{C}$, equipped with the weak-$*$ topology. Usually, $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ is called the *spaces of distributions* on $\mathcal{X}$.

The following proposition indicates that all test functions are *pointwise multipliers* on the inhomogeneous Lipschitz space.

Proposition 2.8 *Let ω be as in (2.3), $\theta \in (0, 1/\omega]$, $\beta \in [\theta\omega, 1]$, and $\gamma \in (0, \infty)$. Then there exists a positive constant C such that, for any $\psi \in \mathcal{G}(\beta, \gamma)$ and $f \in \mathrm{lip}_\theta(\mathcal{X})$,*

$$\|\psi f\|_{\mathrm{lip}_\theta(\mathcal{X})} \le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}.$$

Proof Without loss of generality, we assume that $\|f\|_{\mathrm{lip}_\theta(\mathcal{X})} = 1$. Let $B := B(x_B, r_B) \subset \mathcal{X}$. We consider two cases on r_B.

Case 1. $r_B \in (1, \infty)$. In this case, by $\mathcal{G}(\beta, \gamma) \subset L^\infty(\mathcal{X})$, we have, for any $x \in \mathcal{X}$,

$$|\psi(x)f(x)| \le \|\psi\|_{L^\infty(\mathcal{X})}|f(x)| \le \|\psi\|_{\mathcal{G}(\beta,\gamma)}|f(x)|,$$

which further implies that

$$\frac{\|\psi f\|_{L^\infty(B)}}{[\mu(B)]^\theta} \le \|\psi\|_{L^\infty(\mathcal{X})}\frac{\|f\|_{L^\infty(B)}}{[\mu(B)]^\theta} \le \|\psi\|_{\mathcal{G}(\beta,\gamma)}\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}. \tag{2.15}$$

Case 2. $r_B \in (0, 1]$. In this case, for any $x, y \in B$ with $x \ne y$,

$$d(x, y) \le A_0[d(x, x_B) + d(x_B, y)] < 2A_0 r_B$$

and hence $V(x, y) \lesssim \mu(B)$. Moreover,

$$\begin{aligned}|\psi(x)f(x) - \psi(y)f(y)| &= |\psi(x)f(x) - \psi(y)f(x) + \psi(y)f(x) - \psi(y)f(y)|\\ &\le |\psi(x) - \psi(y)||f(x)| + |\psi(y)||f(x) - f(y)|\\ &=: I_1 + I_2.\end{aligned}$$

For I_2, by (2.5), we find that

$$I_2 \lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)}[V(x, y)]^\theta \lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)}[\mu(B)]^\theta. \tag{2.16}$$

To estimate I_1, fix $x_0 \in \mathcal{X}$. Then, from Lemma 2.5, we deduce that

$$\begin{aligned} I_1 &\le |\psi(x) - \psi(y)|[|f(x) - f(x_0)| + |f(x_0)|] \\ &\lesssim |\psi(x) - \psi(y)| \left\{[V(x, x_0)]^\theta + 1\right\}. \end{aligned}$$

In what follows, we consider two subcases on $d(x, y)$.

Case 2.1. $d(x, y) > (2A_0)^{-1}[d(x, x_0) + 1]$. In this case, by Lemma 2.2(i), we conclude that

$$\begin{aligned} [V(x, x_0)]^\theta + 1 &\sim [V(x, x_0) + 1]^\theta \sim [\mu(B(x, d(x, x_0) + 1))]^\theta \\ &\lesssim [\mu(B(x, d(x, y)))]^\theta \sim [\mu(B)]^\theta. \end{aligned}$$

Thus,

$$I_1 \lesssim [|\psi(x)| + |\psi(y)|][\mu(B)]^\theta \lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)}[\mu(B)]^\theta. \tag{2.17}$$

Case 2.2. $d(x, y) \le (2A_0)^{-1}[d(x, x_0)+1]$. In this case, by Lemmas 2.2(i) and 2.3, we obtain

$$\begin{aligned} [V(x, x_0)]^\theta + 1 &\sim [\mu(B(x, d(x, x_0) + 1))]^\theta \\ &\lesssim \left[\frac{d(x, x_0) + 1}{d(x, y)}\right]^{\theta\omega} [V(x, y)]^\theta \lesssim \left[\frac{d(x, x_0) + 1}{d(x, y)}\right]^{\theta\omega} [\mu(B)]^\theta. \end{aligned}$$

Using this, (2.14), and $\beta \in [\theta\omega, 1]$, we infer that

$$\begin{aligned} I_1 &\lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)} \left[\frac{d(x, y)}{d(x, x_0) + 1}\right]^{\beta-\theta\omega} [\mu(B)]^\theta \\ &\lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)}[\mu(B)]^\theta. \end{aligned}$$

Combining this, (2.16), and (2.17), we conclude that, for any $x, y \in B$,

$$\frac{|\psi(x)f(x) - \psi(y)f(y)|}{[\mu(B)]^\theta} \lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)},$$

which, together with (2.15), further implies that

$$\|\psi f\|_{\mathrm{lip}_\theta(\mathcal{X})} \le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}.$$

This finishes the proof of Proposition 2.8. □

Proposition 2.9 *Let ω be as in (2.3), $\theta \in (0, \infty)$, $\beta \in (0, 1]$, and $\gamma \in (\theta\omega, \infty)$. Then there exists a positive constant C such that, for any $\psi \in \mathcal{G}(\beta, \gamma)$ and $f \in \mathrm{lip}_\theta(\mathcal{X})$,*

$$|\langle f, \psi\rangle| \le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}. \tag{2.18}$$

Proof Without loss of generality, we assume that $\|f\|_{\mathrm{lip}_\theta(\mathcal{X})} = 1$. To show this proposition, fix $x_0 \in \mathcal{X}$, and, for any $f \in \mathrm{lip}_\theta(\mathcal{X})$ and $\psi \in \mathcal{G}(\beta, \gamma)$, write

$$\begin{aligned}&\left|\int_{\mathcal{X}} f(x)\psi(x)\,d\mu(x)\right|\\&\quad\le \int_{\mathcal{X}} |f(x) - f(x_0)||\psi(x)|\,d\mu(x) + |f(x_0)|\int_{\mathcal{X}} |\psi(x)|\,d\mu(x)\\&\quad=: \mathrm{I} + \mathrm{II}.\end{aligned}$$

For I, from (2.5) and (2.13), we deduce that

$$\mathrm{I} \lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)} \int_{\mathcal{X}} [V(x, x_0)]^\theta P_\gamma(x, x_0; 1)\,d\mu(x),$$

where P_γ is as in (2.12). Notice that, by Lemma 2.2(i) and (2.3),

$$V(x, x_0) \sim \mu(B(x_0, d(x, x_0))) \le \mu(B(x_0, d(x, x_0) + 1)) \lesssim [d(x, x_0) + 1]^\omega,$$

which, combined with Lemma 2.2(ii) and $\gamma \in (\theta\omega, \infty)$, further implies that

$$\mathrm{I} \lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)} \int_{\mathcal{X}} P_{\gamma-\theta\omega}(x, x_0; 1)\,d\mu(x) \lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)}.$$

For II, using (2.4), (2.13), and Lemma 2.2(ii), we infer that

$$\mathrm{II} \lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)} \int_{\mathcal{X}} P_\gamma(x, x_0; 1)\,d\mu(x) \lesssim \|\psi\|_{\mathcal{G}(\beta,\gamma)}.$$

This, together with the estimate of I, finishes the proof of Proposition 2.9. □

As a direct corollary of Proposition 2.9, we have the following conclusion.

Corollary 2.10 *Let ω be the same as in (2.3), $\theta \in (0, 1/\omega)$, $\eta \in (\theta\omega, 1]$, $\beta \in (0, \eta]$, and $\gamma \in (\theta\omega, \eta)$. Then $\mathrm{lip}_\theta(\mathcal{X}) \subset (\mathcal{G}_0^\eta(\beta, \gamma))'$ continuously.*

3 Proof of Theorem 1.1

Now, we establish an equivalence characterization of $\mathrm{lip}_\theta(\mathcal{X})$ via Carleson sequences. Let us recall the wavelet systems in [2, Theorems 6.1 and 7.1 and Corollary 10.4]. For any $k \in \mathbb{Z}$, denote by $V_k \subset L^2(\mathcal{X})$ the closed linear span of spline functions in [2]. Let $s \in (0, 1]$ and $\nu \in (0, \infty)$. For any $k \in \mathbb{Z}$, the function E_k with *exponential decay* is defined by setting, for any $x, y \in \mathcal{X}$ and $r \in (0, \infty)$,

$$E_k(x, y; r) := \exp\left(-\frac{\nu}{r}\left[\frac{d(x, y)}{\delta^k}\right]^s\right). \tag{3.1}$$

Lemma 3.1 *There exist constants* $s \in (0, 1]$, $\eta \in (0, 1)$, $C, \nu \in (0, \infty)$, *and wavelet functions* $\{\phi_\alpha^k : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\}$ *satisfying, for any* $k \in \mathbb{Z}$ *and* $\alpha \in \mathcal{A}_k$,

(i) *for any* $x \in \mathcal{X}$,

$$\left|\phi_\alpha^k(x)\right| \le C \left[V_{\delta^k}(x_\alpha^k)\right]^{-1/2} E_k\left(x, x_\alpha^k; 1\right);$$

(ii) *for any* $x, x' \in \mathcal{X}$ *with* $d(x, x') \le \delta^k$,

$$\left|\phi_\alpha^k(x) - \phi_\alpha^k(x')\right| \le C \left[V_{\delta^k}(x_\alpha^k)\right]^{-1/2} \left[\frac{d(x, x')}{\delta^k}\right]^\eta E_k\left(x, x_\alpha^k; 1\right),$$

where E_k *is the same as in* (3.1). *Moreover, for any* $k \in \mathbb{Z}$, *the functions* $\{\phi_\alpha^k\}_k$ *form an orthonormal base of* V_k.

Lemma 3.2 *There exist constants* $s \in (0, 1]$, $\eta \in (0, 1)$, $C, \nu \in (0, \infty)$, *and wavelet functions* $\{\psi_\beta^{k+1} : k \in \mathbb{Z}, \beta \in \mathcal{G}_k\}$ *satisfying, for any* $k \in \mathbb{Z}$ *and* $\beta \in \mathcal{G}_k$,

(i) *for any* $x \in \mathcal{X}$,

$$\left|\psi_\beta^{k+1}(x)\right| \le C \left[V_{\delta^k}(x_\beta^{k+1})\right]^{-1/2} E_k\left(x, x_\beta^{k+1}; 1\right);$$

(ii) *for any* $x, x' \in \mathcal{X}$ *with* $d(x, x') \le \delta^k$,

$$\left|\psi_\beta^{k+1}(x) - \psi_\beta^{k+1}(x')\right| \le C \left[V_{\delta^k}(x_\beta^{k+1})\right]^{-1/2} \left[\frac{d(x, x')}{\delta^k}\right]^\eta E_k\left(x, x_\beta^{k+1}; 1\right);$$

(iii)

$$\int_{\mathcal{X}} \psi_\beta^{k+1}(x)\, d\mu(x) = 0,$$

where E_k *is the same as in* (3.1). *Moreover, the functions* $\{\psi_\alpha^k\}_{k,\alpha}$ *form an orthonormal base of* $L^2(\mathcal{X})$ *and an unconditional base of* $L^p(\mathcal{X})$ *for any given* $p \in (1, \infty)$.

Remark 3.3 (i) The constant η in Lemmas 3.1 and 3.2 comes from the construction of random dyadic cubes in [2], which is very important because it characterizes the smoothness of the wavelets. Moreover, from the construction of $\{\psi_\alpha^k\}_{k,\beta}$ and $\{\phi_\alpha^k\}_{k,\alpha}$ in [2], we deduce that, for any $k_0 \in \mathbb{Z}$, $\{\phi_\alpha^{k_0}\}_{\alpha \in \mathcal{A}_{k_0}} \cup \{\psi_\beta^k : k \in \mathbb{Z},\ k \ge k_0,\ \text{and}\ \beta \in \mathcal{G}_k\}$ form an orthonormal base of $L^2(\mathcal{X})$. Moreover, for any $k, l \in \mathbb{Z}$, $\alpha \in \mathcal{A}_k$, and $\beta \in \mathcal{G}_l$,

$$\int_{\mathcal{X}} \psi_\beta^{l+1}(x)\phi_\alpha^k(x)\, d\mu(x) = 0.$$

(ii) Using the wavelet systems in Lemmas 3.1 and 3.2, He et al. [22] introduced a kind of approximations of the identity with exponential decay (for short, exp-ATI) and obtained new Calderón reproducing formulae on $\mathcal{X}$, which proves necessary to establish various real-variable characterizations of Hardy spaces. Motivated by this, He et al. [21] developed a complete real-variable theory of Hardy spaces on $\mathcal{X}$ including various real-variable equivalent characterizations, which solves an *open problem* on the radial characterization of the Hardy space on $\mathcal{X}$ raised in [14], and the boundedness of sublinear operators. We refer the readers to [16, 24, 45] for more applications of exp-ATIs.

(iii) The constants s and ν in Lemmas 3.1 and 3.2 are the same; see [2, Theorems 6.1 and 7.1 and Corollary 10.4] for more details. Thus, in what follows, we always use s and ν to denote the same constant in Lemmas 3.1 and 3.2.

Now, we establish an equivalent characterization of imhomogenous Lipschitz spaces via Carleson sequences. To this end, let

$$\mathcal{D}_0 := \bigcup_{k=0}^{\infty} \left\{ Q_\alpha^k : \alpha \in \mathcal{A}_k \right\}.$$

Proposition 3.4 *Let ω be as in (2.3), $\eta \in (0, 1]$ be as in Lemma 3.1, and $\theta \in (0, \eta/\omega)$. Then there exists a positive constant C such that, for any $f \in \mathrm{lip}_\theta(\mathcal{X})$,*

$$\sup_{Q\in\mathcal{D}_0} \left\{ \frac{1}{[\mu(Q)]^{1+2\theta}} \left[\sum_{\{\alpha\in\mathcal{A}_0: Q_\alpha^0\subset Q\}} \left|\left\langle f, \phi_\alpha^0 \right\rangle\right|^2 \right.\right. \\ \left.\left. + \sum_{k=0}^{\infty} \sum_{\{\beta\in\mathcal{G}_k: Q_\beta^{k+1}\subset Q\}} \left|\left\langle f, \psi_\beta^{k+1} \right\rangle\right|^2 \right] \right\}^{\frac{1}{2}} \le C\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}.$$

To prove Proposition 3.4, we need the following lemma which contains some useful estimates on the pair $\langle f, \phi_\alpha^k\rangle$ and $\langle f, \psi_\beta^{k+1}\rangle$. For any $B \subset \mathcal{X}$ and $f \in L_{\mathcal{B}}^\infty(\mathcal{X})$, let

$$f_B := \frac{1}{\mu(B)} \int_B f(x)\, d\mu(x).$$

Lemma 3.5 *Let $\theta \in (0, \infty)$ and $B := B(x_B, r_B) \subset \mathcal{X}$. Then there exists a positive constant C such that, for any $k \in \mathbb{Z}$, $\alpha \in \mathcal{A}_k$, and $f \in \mathrm{lip}_\theta(\mathcal{X})$,*

(i) *in general,*

$$\int_{\mathcal{X}} |f(x) - f_B| \left|\phi_\alpha^k(x)\right| d\mu(x) \\ \le C\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}[\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \left[1 + \frac{\delta^k + d(x_B, x_\alpha^k)}{r_B}\right]^{\theta\omega};$$

(ii) *if* $r_B \in (1, \infty)$*, then*

$$\int_{\mathcal{X}} |f(x)| \left|\phi_\alpha^k(x)\right| d\mu(x) \le C\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}[\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \left[1 + \frac{\delta^k + d(x_B, x_\alpha^k)}{r_B}\right]^{\theta\omega};$$

(iii) *if* $\delta^k \le r_B$ *and* $d(x_B, x_\alpha^k) < 2\tau r_B$ *for some* $\tau \in [1, \infty)$*, then*

$$\begin{aligned}&\int_{\{x\in\mathcal{X}:d(x_B,x)>4\tau A_0 r_B\}} |f(x) - f_B| \left|\phi_\alpha^k(x)\right| d\mu(x)\\&\quad\le C\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}[\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \exp\left[-\frac{\nu}{3}\left(\frac{\tau r_B}{\delta^k}\right)^s\right];\end{aligned}$$

(iv) *if* $\delta^k \le r_B$*,* $r_B \in (1, \infty)$*, and* $d(x_B, x_\alpha^k) < 2\tau r_B$ *for some* $\tau \in [1, \infty)$*, then*

$$\begin{aligned}&\int_{\{x\in\mathcal{X}:d(x_B,x)>4\tau A_0 r_B\}} |f(x)| \left|\phi_\alpha^k(x)\right| d\mu(x)\\&\quad\le C\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}[\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \exp\left[-\frac{\nu}{3}\left(\frac{\tau r_B}{\delta^k}\right)^s\right];\end{aligned}$$

(v) *items through (i) to (iv) still hold true if* ϕ_α^k *and* x_α^k *are replaced, respectively, by* ψ_β^{k+1} *and* x_β^{k+1}.

Proof Without loss of generality, we assume that $\|f\|_{\mathrm{lip}_\theta(\mathcal{X})} = 1$. We first prove (i). By (2.5), we have, for any $x \in \mathcal{X}$,

$$\begin{aligned}|f(x) - f_B| &\le \frac{1}{\mu(B)} \int_B |f(x) - f(y)|\, d\mu(y)\\&\lesssim \frac{1}{\mu(B)} \int_B [V(x, y)]^\theta\, d\mu(y). \qquad (3.2)\end{aligned}$$

From (2.1), we deduce that, for any $y \in B$ and $z \in B(x, d(x, y))$,

$$\begin{aligned}d(z, x_B) &\le A_0[d(z, x) + d(x, x_B)] < A_0 d(x, y) + A_0 d(x, x_B)\\&\le A_0^2[d(x, x_B) + d(x_B, y)] + A_0 d(x, x_B)\\&< \left(A_0^2 + A_0\right) d(x, x_B) + A_0^2 r_B,\end{aligned}$$

which further implies that $B(x, d(x, y)) \subset B(x_B, (A_0^2 + A_0)d(x, x_B) + A_0^2 r_B)$. By this, we conclude that, for $x \in \mathcal{X}$ and $y \in B$,

$$V(x, y) \lesssim \left[\frac{r_B + d(x, x_B)}{r_B}\right]^\omega \mu(B). \qquad (3.3)$$

By this, (3.2), and Lemma 3.1(i), we obtain

$$\begin{aligned}&\int_{\mathcal{X}} |f(x)-f_B| \left|\phi_\alpha^k(x)\right| \, d\mu(x)\\&\lesssim [\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \int_{\mathcal{X}} \left[\frac{r_B+d(x,x_B)}{r_B}\right]^{\theta\omega} \frac{1}{V_{\delta^k}(x_\alpha^k)} E_k\left(x, x_\alpha^k; 1\right) \, d\mu(x),\end{aligned}$$

where E_k is as in (3.1). To estimate the above integral, write

$$\begin{aligned}&\int_{\mathcal{X}} \left[\frac{r_B+d(x,x_B)}{r_B}\right]^{\theta\omega} \frac{1}{V_{\delta^k}(x_\alpha^k)} E_k\left(x, x_\alpha^k; 1\right) \, d\mu(x)\\&= \int_{B(x_\alpha^k, r_B+d(x_\alpha^k, x_B))} \left[\frac{r_B+d(x,x_B)}{r_B}\right]^{\theta\omega} \frac{1}{V_{\delta^k}(x_\alpha^k)} E_k\left(x, x_\alpha^k; 1\right) \, d\mu(x)\\&\quad + \int_{\mathcal{X}\setminus B(x_\alpha^k, r_B+d(x_\alpha^k, x_B))} \cdots\\&=: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

We first estimate I_1. Using (2.1), we infer that, for any $x \in B(x_\alpha^k, r_B + d(x_\alpha^k, x_B))$,

$$d(x,x_B) \lesssim d(x,x_\alpha^k) + d(x_\alpha^k, x_B) \lesssim r_B + d(x_\alpha^k, x_B). \tag{3.4}$$

From Lemmas 2.2(i) and 2.3, we deduce that, for any $x \in \mathcal{X}$,

$$\begin{aligned}\frac{1}{V_{\delta^k}(x_\alpha^k)} &\sim \frac{1}{V_{\delta^k}(x_\alpha^k) + V(x_\alpha^k, x)} \frac{V(x_\alpha^k, \delta^k + d(x_\alpha^k, x))}{V_{\delta^k}(x_\alpha^k)}\\&\lesssim \frac{1}{V_{\delta^k}(x_\alpha^k) + V(x_\alpha^k, x)} \left[\frac{\delta^k + d(x_\alpha^k, x)}{\delta^k}\right]^\omega .\end{aligned} \tag{3.5}$$

Moreover, notice that, for any $x \in \mathcal{X}$ and $\Gamma \in (0, \infty)$,

$$E_k\left(x, x_\alpha^k; 1\right) \lesssim \left[\frac{\delta^k}{\delta^k + d(x_\alpha^k, x)}\right]^\Gamma . \tag{3.6}$$

Combining this with $\Gamma := \omega + 1$, (3.4), (3.5), and Lemma 2.2(ii), we find that

$$\begin{aligned}\mathrm{I}_1 &\lesssim \left[\frac{r_B + d(x_\alpha^k, x_B)}{r_B}\right]^{\theta\omega} \int_{\mathcal{X}} P_1\left(x, x_\alpha^k; \delta^k\right) \, d\mu(x)\\&\lesssim \left[\frac{r_B + d(x_\alpha^k, x_B)}{r_B}\right]^{\theta\omega},\end{aligned} \tag{3.7}$$

where P_1 is as in (2.12) with $\gamma = 1$. To estimate I_2, by (2.1), we conclude that, for any $x \in \mathcal{X} \setminus B(x_\alpha^k, r_B + d(x_\alpha^k, x_B))$,

$$r_B + d(x, x_B) \lesssim r_B + d(x, x_\alpha^k) + d(x_\alpha^k, x_B) \lesssim d(x, x_\alpha^k).$$

This, together with (3.5), (3.6) with $\Gamma := \theta\omega + \omega + 1$, and Lemma 2.2(ii), further implies that

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \left(\frac{\delta^k}{r_B}\right)^{\theta\omega} \int_{\mathcal{X}} \left[\frac{d(x, x_\alpha^k)}{\delta^k}\right]^{\theta\omega} \frac{1}{V_{\delta^k}(x_\alpha^k) + V(x_\alpha^k, x)} \\
&\quad \times \left[\frac{\delta^k + d(x_\alpha^k, x)}{\delta^k}\right]^{\omega} \left[\frac{\delta^k}{\delta^k + d(x_\alpha^k, x)}\right]^{\theta\omega+\omega+1} d\mu(x) \\
&\lesssim \left(\frac{\delta^k}{r_B}\right)^{\theta\omega} \int_{\mathcal{X}} P_1\left(x, x_\alpha^k; \delta^k\right) d\mu(x) \lesssim \left(\frac{\delta^k}{r_B}\right)^{\theta\omega}.
\end{aligned}$$

Combining the estimates of I_1 and I_2, we obtain

$$\int_{\mathcal{X}} |f(x) - f_B| \left|\phi_\alpha^k(x)\right| d\mu(x) \lesssim [\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \left[1 + \frac{\delta^k + d(x_B, x_\alpha^k)}{r_B}\right]^{\theta\omega},$$

which completes the proof of (i).

Next, we prove (ii). Since $r_B \in (1, \infty)$, by (2.4) and (2.3), it follows that, for any $x \in \mathcal{X}$,

$$|f(x)| \lesssim [\mu(B(x, r_B))]^\theta \lesssim [\mu(B)]^\theta \left[\frac{r_B + d(x, x_B)}{r_B}\right]^{\theta\omega}. \tag{3.8}$$

Using this, Lemma 3.1(i), and the estimates of I_1 and I_2, we infer that

$$\begin{aligned}
&\int_{\mathcal{X}} |f(x)| \left|\phi_\alpha^k(x)\right| d\mu(x) \\
&\quad \lesssim [\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \\
&\qquad \times \int_{\mathcal{X}} \left[\frac{r_B + d(x, x_B)}{r_B}\right]^{\theta\omega} \frac{1}{V_{\delta^k}(x_\alpha^k)} E_k\left(x, x_\alpha^k; 1\right) d\mu(x) \\
&\quad \lesssim [\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \left[1 + \frac{\delta^k + d(x_B, x_\alpha^k)}{r_B}\right]^{\theta\omega}.
\end{aligned}$$

This finishes the proof of (ii).

Now, we show (iii). By (3.2), (3.3), and Lemma 3.1(i), we find that

$$\begin{aligned}
&\int_{\{x \in \mathcal{X}: d(x_B, x) > 4\tau A_0 r_B\}} |f(x) - f_B| \left|\phi_\alpha^k(x)\right| d\mu(x) \\
&\quad \lesssim [\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \int_{\{x \in \mathcal{X}: d(x_B, x) > 4\tau A_0 r_B\}} \left[\frac{r_B + d(x, x_B)}{r_B}\right]^{\theta\omega} \\
&\qquad \times \frac{1}{V_{\delta^k}(x_\alpha^k)} E_k\left(x, x_\alpha^k; 1\right) d\mu(x).
\end{aligned} \tag{3.9}$$

Notice that, if $\delta^k \le r_B$ and $d(x_B, x_\alpha^k) < 2\tau r_B$ for some $\tau \in [1, \infty)$, then, by (2.1), we have, for any $x \in \{x \in \mathcal{X} : d(x_B, x) > 4\tau A_0 r_B\}$,

$$\begin{aligned} 4\tau A_0 r_B < d(x_B, x) &\le A_0 \left[d(x_B, x_\alpha^k) + d(x_\alpha^k, x) \right] \\ &< A_0 \left[2\tau r_B + d(x_\alpha^k, x) \right], \end{aligned} \tag{3.10}$$

which further implies that $2\tau r_B < d(x_\alpha^k, x)$ and hence

$$\begin{aligned} \left[\frac{d(x, x_\alpha^k)}{\delta^k} \right]^s &> \frac{1}{3} \left\{ \left[\frac{\tau r_B}{\delta^k} \right]^s + \left[\frac{d(x, x_\alpha^k)}{4\delta^k} \right]^s + \left[\frac{d(x, x_\alpha^k)}{4\delta^k} \right]^s \right\} \\ &> \frac{1}{3} \left\{ \left[\frac{\tau r_B}{\delta^k} \right]^s + \left[\frac{d(x, x_\alpha^k)}{4\delta^k} \right]^s + \left(\frac{\tau}{2} \right)^s \right\}. \end{aligned} \tag{3.11}$$

Moreover, from $\delta^k \le r_B$ and (3.10), we deduce that

$$\frac{r_B + d(x, x_B)}{r_B} \lesssim \tau \frac{r_B + d(x, x_\alpha^k)}{r_B} \lesssim \tau \left[1 + \frac{d(x, x_\alpha^k)}{\delta^k} \right].$$

Applying this, (3.5), and (3.11), we infer that, for any $x \in \{x \in \mathcal{X} : d(x_B, x) > 4\tau A_0 r_B\}$,

$$\begin{aligned} &\left[\frac{r_B + d(x, x_B)}{r_B} \right]^{\theta\omega} \frac{1}{V_{\delta^k}(x_\alpha^k)} E_k \left(x, x_\alpha^k; 1 \right) \\ &\quad \lesssim \tau^{\theta\omega} \left[1 + \frac{d(x, x_\alpha^k)}{\delta^k} \right]^{\theta\omega} \frac{1}{V_{\delta^k}(x_\alpha^k) + V(x_\alpha^k, x)} \left[\frac{\delta^k + d(x_\alpha^k, x)}{\delta^k} \right]^{\omega} \\ &\qquad \times \exp \left[-\frac{\nu}{3} \left(\frac{\tau r_B}{\delta^k} \right)^s \right] E_k \left(x, x_\alpha^k; 3 \cdot 4^s \right) \exp \left(-\frac{\nu \tau^s}{3 \cdot 2^s} \right) \\ &\quad \lesssim \exp \left[-\frac{\nu}{3} \left(\frac{\tau r_B}{\delta^k} \right)^s \right] P_1 \left(x, x_\alpha^k; \delta^k \right). \end{aligned}$$

This, together with Lemma 2.3, further implies that

$$\begin{aligned} &\int_{\{x \in \mathcal{X} : d(x_B, x) > 4\tau A_0 r_B\}} \left[\frac{r_B + d(x, x_B)}{r_B} \right]^{\theta\omega} \frac{1}{V_{\delta^k}(x_\alpha^k)} E_k \left(x, x_\alpha^k; 1 \right) d\mu(x) \\ &\quad \lesssim \exp \left[-\frac{\nu}{3} \left(\frac{\tau r_B}{\delta^k} \right)^s \right] \int_{\mathcal{X}} P_1 \left(x, x_\alpha^k; \delta^k \right) d\mu(x) \lesssim \exp \left[-\frac{\nu}{3} \left(\frac{\tau r_B}{\delta^k} \right)^s \right]. \end{aligned} \tag{3.12}$$

Combining this and (3.9), we finish the proof of (iii).

We next prove (iv). By $r_B \in (1, \infty)$, (3.8), (3.2), (3.3), Lemma 3.1(i), and (3.12), we have

$$\begin{aligned}
&\int_{\{x\in\mathcal{X}:d(x_B,x)>4\tau A_0 r_B\}} |f(x)| \left|\phi_\alpha^k(x)\right| d\mu(x) \\
&\lesssim [\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \int_{\{x\in\mathcal{X}:d(x_B,x)>4\tau A_0 r_B\}} \left[\frac{r_B + d(x, x_B)}{r_B}\right]^{\theta\omega} \\
&\quad \times \frac{1}{V_{\delta^k}(x_\alpha^k)} E_k\left(x, x_\alpha^k; 1\right) d\mu(x) \\
&\lesssim [\mu(B)]^\theta \sqrt{V_{\delta^k}(x_\alpha^k)} \exp\left[-\frac{\nu}{3}\left(\frac{\tau r_B}{\delta^k}\right)^s\right],
\end{aligned}$$

which completes the proof of (iv).

Finally, in the proofs of (i) through (iv), we only use the size condition of ϕ_α^k, which ψ_β^{k+1} also satisfies. Thus, repeating the arguments in the proofs of (i) through (iv), we show that items through (i) to (iv) still hold true if ϕ_α^k and x_α^k are replaced, respectively, by ψ_β^{k+1} and x_β^{k+1}. This finishes the proof of (v) and hence of Lemma 3.5. □

Now, we show Proposition 3.4.

Proof of Proposition 3.4 Without loss of generality, we assume that $\|f\|_{\mathrm{lip}_\theta(\mathcal{X})} = 1$. To prove this proposition, fix $Q \in \mathcal{D}_0$. Notice that, for any $Q \in \mathcal{D}_0$, there exist $k_0 \in \mathbb{Z}_+$ and $\alpha_0 \in \mathcal{A}_{k_0}$ such that $Q = Q_{\alpha_0}^{k_0}$.

We first show that

$$\left\{\frac{1}{[\mu(Q_{\alpha_0}^{k_0})]^{1+2\theta}}\left[\sum_{\{\alpha\in\mathcal{A}_0:Q_\alpha^0\subset Q_{\alpha_0}^{k_0}\}} \left|\left\langle f, \phi_\alpha^0\right\rangle\right|^2\right]\right\}^{\frac{1}{2}} \lesssim 1. \tag{3.13}$$

If $\{\alpha \in \mathcal{A}_0 : Q_\alpha^0 \subset Q_{\alpha_0}^{k_0}\} = \emptyset$, then (3.13) holds true. If $\{\alpha \in \mathcal{A}_0 : Q_\alpha^0 \subset Q_{\alpha_0}^{k_0}\} \neq \emptyset$, then, by Lemma 2.6(ii) and $k_0 \in \mathbb{Z}_+$, we find that, for any $\tilde{\alpha} \in \{\alpha \in \mathcal{A}_0 : Q_\alpha^0 \subset Q_{\alpha_0}^{k_0}\}$, either $Q_{\alpha_0}^{k_0} \subset Q_{\tilde{\alpha}}^0$ or $Q_{\tilde{\alpha}}^0 \cap Q_{\alpha_0}^{k_0} = \emptyset$. Thus,

$$Q_{\alpha_0}^{k_0} = Q_{\tilde{\alpha}}^0 \quad \text{and} \quad x_{\alpha_0}^{k_0} = x_{\tilde{\alpha}}^0. \tag{3.14}$$

From Lemma 2.6(ii) agagin, we deduce that $\{\alpha \in \mathcal{A}_0 : Q_\alpha^0 \subset Q_{\alpha_0}^{k_0}\}$ has only one element $\tilde{\alpha}$. To estimate $|\langle f, \phi_{\tilde{\alpha}}^0\rangle|$, let $B := B(x_{\tilde{\alpha}}^0, 2)$ and write

$$\begin{aligned}
\left|\left\langle f, \phi_{\tilde{\alpha}}^0\right\rangle\right| &= \left|\int_{\mathcal{X}} f(x)\phi_{\tilde{\alpha}}^0(x)\, d\mu(x)\right| \\
&\le \int_{\mathcal{X}} |f(x) - f_B| \left|\phi_{\tilde{\alpha}}^0(x)\right| d\mu(x) + |f_B| \int_{\mathcal{X}} \left|\phi_{\tilde{\alpha}}^0(x)\right| d\mu(x) \\
&=: \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned}$$

From Lemmas 3.5(i) and 2.6(iii) and (3.14), we deduce that

$$\mathrm{J}_1 \lesssim [\mu(B)]^{\theta}\sqrt{V_1(x^0_{\tilde{\alpha}})}\left[1+\frac{1+d(x^0_{\tilde{\alpha}},x^0_{\tilde{\alpha}})}{1}\right]^{\theta\omega} \lesssim \left[\mu\left(Q^{k_0}_{\alpha_0}\right)\right]^{\theta+\frac{1}{2}}.$$

For J_2, notice that, $|f_B| \le \|f\|_{L^\infty(B)}$. On the other hand, using Lemmas 3.1(i), 2.2, 2.6(iii), and 2.3, we infer that

$$\begin{aligned}
\int_{\mathcal{X}}\left|\phi^0_{\tilde{\alpha}}(x)\right| d\mu(x) &\lesssim \int_{\mathcal{X}}\frac{1}{\sqrt{V_1(x^0_{\tilde{\alpha}})}}E_0\left(x,x^0_{\tilde{\alpha}};1\right)d\mu(x)\\
&\lesssim \left[V_1(x^0_{\tilde{\alpha}})\right]^{\frac{1}{2}}\int_{\mathcal{X}}\frac{V_{d(x,x^0_{\tilde{\alpha}})+1}(x^0_{\tilde{\alpha}})}{V_1(x^0_{\tilde{\alpha}})}\\
&\quad\times\frac{1}{V_1(x^0_{\tilde{\alpha}})+V(x,x^0_{\tilde{\alpha}})}E_0\left(x,x^0_{\tilde{\alpha}};1\right)d\mu(x)\\
&\lesssim \left[V_1(x^0_{\tilde{\alpha}})\right]^{\frac{1}{2}}\int_{\mathcal{X}}P_1\left(x,x^0_{\tilde{\alpha}};1\right)d\mu(x) \lesssim \left[V_1(x^0_{\tilde{\alpha}})\right]^{\frac{1}{2}},
\end{aligned}$$

where E_0 is as in (3.1) with $k=0$ and P_1 is as in (2.12) with $\gamma=1$. This, together with Lemma 2.6(iii), Definition 2.4, (2.3), and (3.14), further implies that

$$\mathrm{J}_2 \lesssim \|f\|_{L^\infty(B)}\left[V_1(x^0_{\tilde{\alpha}})\right]^{\frac{1}{2}} \lesssim \left[\mu\left(Q^0_{\tilde{\alpha}}\right)\right]^{\theta+\frac{1}{2}}.$$

Combining the estimates of J_1 and J_2, we conclude that

$$\begin{aligned}
&\left\{\frac{1}{[\mu(Q^{k_0}_{\alpha_0})]^{1+2\theta}}\left[\sum_{\{\alpha\in\mathcal{A}_0:\,Q^0_\alpha\subset Q^{k_0}_{\alpha_0}\}}\left|\left\langle f,\phi^0_\alpha\right\rangle\right|^2\right]\right\}^{\frac{1}{2}}\\
&=\left\{\frac{1}{[\mu(Q^{k_0}_{\alpha_0})]^{1+2\theta}}\left|\left\langle f,\phi^0_{\tilde{\alpha}}\right\rangle\right|^2\right\}^{\frac{1}{2}} \lesssim 1.
\end{aligned}$$

This finishes the proof of (3.13).

Next, we show that

$$\left\{\frac{1}{[\mu(Q^{k_0}_{\alpha_0})]^{1+2\theta}}\left[\sum_{k=0}^{\infty}\sum_{\{\beta\in\mathcal{G}_k:\,Q^{k+1}_\beta\subset Q^{k_0}_{\alpha_0}\}}\left|\left\langle f,\psi^{k+1}_\beta\right\rangle\right|^2\right]\right\}^{\frac{1}{2}} \lesssim 1. \tag{3.15}$$

For this purpose, write

$$
\begin{aligned}
&\left\{\frac{1}{[\mu(Q_{\alpha_0}^{k_0})]^{1+2\theta}}\left[\sum_{k=0}^{\infty}\sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset Q_{\alpha_0}^{k_0}\}}\left|\left\langle f,\psi_\beta^{k+1}\right\rangle\right|^2\right]\right\}^{\frac{1}{2}}\\
&\lesssim\left\{\frac{1}{[\mu(Q_{\alpha_0}^{k_0})]^{1+2\theta}}\left[\sum_{k=0}^{k_0-1}\sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset Q_{\alpha_0}^{k_0}\}}\left|\left\langle f,\psi_\beta^{k+1}\right\rangle\right|^2\right]\right\}^{\frac{1}{2}}\\
&\quad+\left\{\frac{1}{[\mu(Q_{\alpha_0}^{k_0})]^{1+2\theta}}\left[\sum_{k=k_0}^{\infty}\sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset Q_{\alpha_0}^{k_0}\}}\left|\left\langle f,\psi_\beta^{k+1}\right\rangle\right|^2\right]\right\}^{\frac{1}{2}}\\
&=:\mathrm{J}_3+\mathrm{J}_4.
\end{aligned}
$$

We now estimate J_3. Let, if $k_0>0$,

$$
E:=\left\{(k,\beta):\ k\in\{0,\cdots,k_0-1\},\beta\in\mathcal{G}_k,\,Q_\beta^{k+1}\subset Q_{\alpha_0}^{k_0}\right\}
$$

and, if $k_0=0$, $E:=\emptyset$. If $E=\emptyset$, then $\mathrm{J}_3=0$. If $E\neq\emptyset$, we claim that E has only one element. Indeed, assume $(k_1,\beta_1),(k_2,\beta_2)\in E$. From Lemma 2.6(ii) and the definition of E, we deduce that

$$
Q_{\beta_1}^{k_1+1}=Q_{\alpha_0}^{k_0}=Q_{\beta_2}^{k_2+1}\quad\text{and}\quad x_{\beta_1}^{k_1+1}=x_{\alpha_0}^{k_0}=x_{\beta_2}^{k_2+1}. \tag{3.16}
$$

Notice that $x_{\beta_1}^{k_1+1}\in\mathcal{Y}^{k_1}$ and $x_{\beta_2}^{k_2+1}\in\mathcal{Y}^{k_2}$. By the definition of $\mathcal{Y}^k$, we find that, if $k_1\neq k_2$, then $\mathcal{Y}^{k_1}\cap\mathcal{Y}^{k_2}=\emptyset$ and hence $x_{\beta_1}^{k_1+1}\neq x_{\beta_2}^{k_2+1}$. Therefore, $k_1=k_2$ and $\beta_1=\beta_2$, which completes the proof of the above claim. Denote the only element in E by $(\tilde{k},\tilde{\beta})$. Then, by (3.16), we have

$$
\mathrm{J}_3=\frac{1}{[\mu(Q_{\alpha_0}^{k_0})]^{\frac{1}{2}+\theta}}\left|\left\langle f,\psi_{\tilde{\beta}}^{\tilde{k}+1}\right\rangle\right|=\frac{1}{[\mu(Q_{\tilde{\beta}}^{\tilde{k}+1})]^{\frac{1}{2}+\theta}}\left|\left\langle f,\psi_{\tilde{\beta}}^{\tilde{k}+1}\right\rangle\right|.
$$

Let $B_1:=(x_{\tilde{\beta}}^{\tilde{k}+1},\delta^{\tilde{k}+1})$. Then, using Lemma 3.2(iii), (i) and (v) of Lemma 3.5, and (3.16), we infer that

$$
\begin{aligned}
\mathrm{J}_3&=\frac{1}{[\mu(Q_{\tilde{\beta}}^{\tilde{k}+1})]^{\frac{1}{2}+\theta}}\left|\left\langle f-f_{B_1},\psi_{\tilde{\beta}}^{\tilde{k}+1}\right\rangle\right|\\
&\lesssim\frac{1}{[\mu(Q_{\tilde{\beta}}^{\tilde{k}+1})]^{\frac{1}{2}+\theta}}[\mu(B_1)]^{\theta}\sqrt{V_{\delta^{\tilde{k}}}(x_{\tilde{\beta}}^{\tilde{k}+1})}\lesssim 1.
\end{aligned}
$$

To estimate J_4, let $B_2 := B(x_{\alpha_0}^{k_0}, 2C_\# \delta^{k_0})$ with $C_\#$ as the same in Lemma 2.6(iii). By Lemma 3.2(iii), we have

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset Q_{\alpha_0}^{k_0}\}} \left|\left\langle f, \psi_\beta^{k+1}\right\rangle\right|^2 \\
&\lesssim \sum_{k=k_0}^{\infty} \sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset Q_{\alpha_0}^{k_0}\}} \left|\left\langle [f - f_{B_2}]\mathbf{1}_{4A_0B_2}, \psi_\beta^{k+1}\right\rangle\right|^2 \\
&\quad + \sum_{k=k_0}^{\infty} \sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset Q_{\alpha_0}^{k_0}\}} \left|\left\langle [f - f_{B_2}]\mathbf{1}_{\mathcal{X}\setminus(4A_0B_2)}, \psi_\beta^{k+1}\right\rangle\right|^2 \\
&=: \mathrm{J}_{4,1} + \mathrm{J}_{4,2}.
\end{aligned}$$

For $\mathrm{J}_{4,1}$, notice that, by Definition 2.4, we have $[f - f_{B_2}]\mathbf{1}_{4A_0B_2} \in L^2(\mathcal{X})$. From this, Lemma 3.2, and (2.5), we deduce that

$$\begin{aligned}
\mathrm{J}_{4,1} &\leq \left\|[f - f_{B_2}]\mathbf{1}_{4A_0B_2}\right\|_{L^2(\mathcal{X})}^2 \\
&\leq \int_{4A_0B_2} \left[\frac{1}{\mu(B_2)} \int_{B_2} |f(x) - f(y)|\, d\mu(y)\right]^2 d\mu(x) \\
&\lesssim \int_{4A_0B_2} \left[\frac{1}{\mu(B_2)} \int_{B_2} [V(x, y)]^\theta\, d\mu(y)\right]^2 d\mu(x) \\
&\lesssim [\mu(B_2)]^{1+2\theta}.
\end{aligned}$$

For $\mathrm{J}_{4,2}$, since $k \geq k_0$ and $Q_\beta^{k+1} \subset Q_{\alpha_0}^{k_0}$, it follows that

$$\delta^{k+1} < C_\# \delta^{k_0} \quad \text{and} \quad d(x_\beta^{k+1}, x_{\alpha_0}^{k_0}) \leq C_\# \delta^{k_0}.$$

By this, (iii) and (v) of Lemma 3.5, and Lemma 2.6(ii), we obtain

$$\begin{aligned}
\mathrm{J}_{4,2} &\lesssim \sum_{k=k_0}^{\infty} \sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset Q_{\alpha_0}^{k_0}\}} [\mu(B_2)]^{2\theta} V_{\delta^k}(x_\beta^{k+1}) \exp\left[-\frac{2\nu}{3}\left(\frac{C_\#\delta^{k_0}}{\delta^k}\right)^s\right] \\
&\sim [\mu(B_2)]^{2\theta} \sum_{k=k_0}^{\infty} \exp\left[-\frac{2\nu}{3}\left(\frac{C_\#\delta^{k_0}}{\delta^k}\right)^s\right] \sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset Q_{\alpha_0}^{k_0}\}} \mu(Q_\beta^{k+1}) \\
&\lesssim [\mu(B_2)]^{2\theta+1},
\end{aligned}$$

which, together with the estimate of $\mathrm{J}_{4,1}$, further implies that

$$\mathrm{J}_4 \lesssim 1.$$

Combining the estimates of J_3 and J_4, we find that (3.15) holds true, which completes the proof of Proposition 3.4. □

Proposition 3.6 *Let* $\eta \in (0, 1]$ *as the same in Lemma 3.1,* $\theta \in (0, \frac{\eta}{\omega})$. *If a sequence*

$$c := \left\{c_\alpha^0\right\}_{\alpha\in\mathcal{A}_0} \cup \left\{c_\beta^{k+1}\right\}_{k\in\mathbb{Z}_+,\beta\in\mathcal{G}_k} \subset \mathbb{C}$$

satisfies that

$$\|c\|_* := \sup_{Q\in\mathcal{D}_0} \left\{\frac{1}{[\mu(Q)]^{1+2\theta}} \left[\sum_{\{\alpha\in\mathcal{A}_0:Q_\alpha^0\subset Q\}} \left|c_\alpha^0\right|^2 + \sum_{k=0}^{\infty} \sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset Q\}} \left|c_\beta^{k+1}\right|^2\right]\right\}^{\frac{1}{2}} < \infty, \tag{3.17}$$

then

$$\sum_{\alpha\in\mathcal{A}_0} c_\alpha^0 \phi_\alpha^0 + \sum_{k=0}^{\infty} \sum_{\beta\in\mathcal{G}_k} c_\beta^{k+1} \psi_\beta^{k+1}$$

converges in $L^2_{\mathcal{B}}(\mathcal{X})$. *Denote the limit by* f. *Then there exists a positive constant* C, *independent of* c, *such that*

$$\|f\|_{\mathrm{lip}_\theta(\mathcal{X})} \le C\|c\|_*.$$

To prove Proposition 3.6, we need several lemmas. We first recall the following very useful inequality.

Lemma 3.7 *For any* $\theta \in (0, 1]$ *and* $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$, *it holds true that*

$$\left(\sum_{j=1}^{\infty} |a_j|\right)^\theta \le \sum_{j=1}^{\infty} |a_j|^\theta.$$

The following lemma comes from [36, Lemma 2.21], whose proof is still valid if d only satisfies (2.1). We omit details here.

Lemma 3.8 *There exists a positive constant* C *such that, for any* $b, c \in (0, \infty)$, $k \in \mathbb{Z}$, *and* $x \in \mathcal{X}$,

$$\sum_{\alpha\in\mathcal{A}_k} \exp\left(-b\left[\frac{d(x_\alpha^k, x)}{\delta^k}\right]^c\right) \le C.$$

Now, we show Proposition 3.6.

Proof of Proposition 3.6 Let $B := (x_B, r_B) \subset \mathcal{X}$, we consider two cases on r_B.

Case 1. $r_B \in (1, \infty)$. In this case, write

$$
\begin{aligned}
&\sum_{\alpha\in\mathcal{A}_0} c_\alpha^0\phi_\alpha^0+\sum_{k=0}^\infty\sum_{\beta\in\mathcal{G}_k} c_\beta^{k+1}\psi_\beta^{k+1}\\
&\quad=\sum_{\{\alpha\in\mathcal{A}_0:d(x_\alpha^0,x_B)<2A_0r_B\}} c_\alpha^0\phi_\alpha^0+\sum_{\{\alpha\in\mathcal{A}_0:d(x_\alpha^0,x_B)\ge 2A_0r_B\}} c_\alpha^0\phi_\alpha^0\\
&\qquad+\sum_{k=0}^\infty\sum_{\{\beta\in\mathcal{G}_k:d(x_\beta^{k+1},x_B)<2A_0r_B\}} c_\beta^{k+1}\psi_\beta^{k+1}\\
&\qquad+\sum_{k=0}^\infty\sum_{\{\beta\in\mathcal{G}_k:d(x_\beta^{k+1},x_B)\ge 2A_0r_B\}} c_\beta^{k+1}\psi_\beta^{k+1}\\
&\quad=: F_{1,1}+F_{1,2}+F_{1,3}+F_{1,4}.
\end{aligned}
$$

For $F_{1,1}$, by (2.1) and Lemma 2.6(iii), we find that, for any $\alpha\in\mathcal{A}_0$ such that $d(x_\alpha^0,x_B)<2A_0r_B$ and any $y\in Q_\alpha^0$,

$$
d(y,x_B)\le A_0\left[d(y,x_\alpha^0)+d(x_\alpha^0,x_B)\right]<A_0(C_\#+2A_0)r_B,
$$

which further implies that $Q_\alpha^0\subset B(x_B,A_0(C_\#+2A_0)r_B)$ and hence

$$
\left\{\alpha\in\mathcal{A}_0: d(x_\alpha^0,x_B)<2A_0r_B\right\}\subset\left\{\alpha\in\mathcal{A}_0: Q_\alpha^0\subset B(x_B,A_0(C_\#+2A_0)r_B)\right\}.
$$

Using this, Lemma 3.1, (3.17), and Lemmas 3.7 and 2.6(ii), we infer that

$$
\begin{aligned}
\|F_{1,1}\|_{L^2(\mathcal{X})}&\le\left[\sum_{\{\alpha\in\mathcal{A}_0:Q_\alpha^0\subset B(x_B,A_0(C_\#+2A_0)r_B)\}}\left|c_\alpha^0\right|^2\right]^{\frac12}\\
&\le\|c\|_*\left(\sum_{\{\alpha\in\mathcal{A}_0:Q_\alpha^0\subset B(x_B,A_0(C_\#+2A_0)r_B)\}}\left[\mu(Q_\alpha^0)\right]^{2\theta+1}\right)^{\frac12}\\
&\le\|c\|_*\left[\sum_{\{\alpha\in\mathcal{A}_0:Q_\alpha^0\subset B(x_B,A_0(C_\#+2A_0)r_B)\}}\mu(Q_\alpha^0)\right]^{\theta+\frac12}\\
&\lesssim\|c\|_*[\mu(B)]^{\theta+\frac12}.
\end{aligned}
$$

For $F_{1,2}$, by (3.17) and Lemma 3.1(i), we have, for any $x\in B$,

$$
|F_{1,2}(x)|\le\sum_{\{\alpha\in\mathcal{A}_0:d(x_\alpha^0,x_B)\ge 2A_0r_B\}}\left|c_\alpha^0\right|\left|\phi_\alpha^0(x)\right|
$$

$$\lesssim \|c\|_* \sum_{\{\alpha\in\mathcal{A}_0:d(x_\alpha^0,x_B)\geq 2A_0r_B\}} \left[\mu(Q_\alpha^0)\right]^\theta E_0\left(x_\alpha^0, x; 1\right),$$

where E_0 is as in (3.1) with $k=0$. Observe that, since $d(x_\alpha^0, x_B)\geq 2A_0r_B$, it follows that, for any $x\in B$,

$$2A_0r_B \leq d(x_\alpha^0, x_B) \leq A_0\left[d(x_\alpha^0, x)+d(x, x_B)\right] < A_0d(x_\alpha^0, x)+A_0r_B. \quad (3.18)$$

Thus, $d(x, x_B) < r_B < d(x_\alpha^0, x)$, which, together with (3.18), further implies that

$$d(x_\alpha^0, x_B) < 2A_0d(x_\alpha^0, x).$$

From this and Lemmas 2.3 and 3.8, we deduce that, for any $x\in B$,

$$\begin{aligned}
|F_{1,2}(x)| &\lesssim \|c\|_* \sum_{\{\alpha\in\mathcal{A}_0:d(x_\alpha^0,x_B)\geq 2A_0r_B\}} \left[\mu(Q_\alpha^0)\right]^\theta E_0\left(x_\alpha^0, x_B; 2^sA_0^s\right)\\
&\lesssim \|c\|_*[\mu(B)]^\theta \sum_{\{\alpha\in\mathcal{A}_0:d(x_\alpha^0,x_B)\geq 2A_0r_B\}} \left[1+d(x_\alpha^0, x_B)\right]^{\theta\omega} E_0\left(x_\alpha^0, x_B; 2^sA_0^s\right)\\
&\lesssim \|c\|_*[\mu(B)]^\theta \sum_{\{\alpha\in\mathcal{A}_0:d(x_\alpha^0,x_B)\geq 2A_0r_B\}} E_0\left(x_\alpha^0, x_B; 2^{s+1}A_0^s\right)\\
&\lesssim \|c\|_*[\mu(B)]^\theta
\end{aligned}$$

and hence

$$\|F_{1,2}\|_{L^2(B)} \lesssim \|c\|_*[\mu(B)]^{\theta+\frac{1}{2}}.$$

Next, we estimate $F_{1,3}$. Applying (2.1) and Lemma 2.6(iii), we infer that, for any $k\in\mathbb{Z}_+$, $\beta\in\mathcal{G}_k$ such that $d(x_\beta^{k+1}, x_B) < 2A_0r_B$ and any $y\in Q_\beta^{k+1}$,

$$d(y, x_B)\leq A_0\left[d(y, x_\beta^{k+1})+d(x_\beta^{k+1}, x_B)\right] < A_0(C_\# + 2A_0)r_B,$$

which further implies that $Q_\beta^{k+1}\subset B(x_B, A_0(C_\# + 2A_0)r_B)$ and hence

$$\left\{\beta\in\mathcal{G}_k : d(x_\beta^{k+1}, x_B) < 2A_0r_B\right\} \subset \left\{\beta\in\mathcal{G}_k : Q_\beta^{k+1}\subset B(x_B, A_0(C_\# + 2A_0)r_B)\right\}.$$

Using this, Lemma 3.2, (3.17), and Lemmas 3.7 and 2.6(ii), we infer that

$$\|F_{1,3}\|_{L^2(\mathcal{X})} \leq \left[\sum_{k=0}^\infty \sum_{\{\beta\in\mathcal{G}_k:d(x_\beta^{k+1},x_B)<2A_0r_B\}} \left|c_\beta^{k+1}\right|^2\right]^{\frac{1}{2}}$$

$$
\begin{aligned}
&\le \left[\sum_{k=0}^{\infty}\sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset B(x_B,A_0(C_\#+2A_0)r_B)\}}\left|c_\beta^{k+1}\right|^2\right]^{\frac{1}{2}}\\
&\le \left[\sum_{\{\alpha\in\mathcal{A}_0:Q_\alpha^0\cap B(x_B,A_0(C_\#+2A_0)r_B)\neq\emptyset\}}\sum_{k=0}^{\infty}\sum_{\{\beta\in\mathcal{G}_k:Q_\beta^{k+1}\subset Q_\alpha^0\}}\left|c_\beta^{k+1}\right|^2\right]^{\frac{1}{2}}\\
&\le \|c\|_*\left[\sum_{\{\alpha\in\mathcal{A}_0:Q_\alpha^0\cap B(x_B,A_0(C_\#+2A_0)r_B)\neq\emptyset\}}\left[\mu(Q_\alpha^0)\right]^{2\theta+1}\right]^{\frac{1}{2}}\\
&\le \|c\|_*\left[\sum_{\{\alpha\in\mathcal{A}_0:Q_\alpha^0\cap B(x_B,A_0(C_\#+2A_0)r_B)\neq\emptyset\}}\mu(Q_\alpha^0)\right]^{\frac{1}{2}+\theta}\\
&\lesssim \|c\|_*[\mu(B)]^{\theta+\frac{1}{2}}.
\end{aligned}
$$

Now, we estimate $F_{1,4}$. By (3.17) and Lemma 3.1(i), we have, for any $x\in B$,

$$
\begin{aligned}
|F_{1,4}(x)| &\le \sum_{k=0}^{\infty}\sum_{\{\beta\in\mathcal{G}_k:d(x_\beta^{k+1},x_B)\ge 2A_0r_B\}}\left|c_\beta^{k+1}\right|\left|\psi_\beta^{k+1}(x)\right|\\
&\lesssim \|c\|_*\sum_{k=0}^{\infty}\sum_{\{\beta\in\mathcal{G}_k:d(x_\beta^{k+1},x_B)\ge 2A_0r_B\}}\left[\mu(Q_\beta^{k+1})\right]^{\theta}E_k\left(x_\beta^{k+1},x;1\right).
\end{aligned}
$$

From $d(x_\beta^{k+1},x_B)\ge 2A_0r_B$, we deduce that, for any $x\in B$,

$$
2A_0r_B\le d(x_\beta^{k+1},x_B)\le A_0\left[d(x_\beta^{k+1},x)+d(x,x_B)\right]<A_0d(x_\beta^{k+1},x)+A_0r_B,
$$

which further implies that

$$
d(x,x_B)<r_B<d(x_\beta^{k+1},x) \tag{3.19}
$$

and hence

$$
d(x_\beta^{k+1},x_B)<2A_0d(x_\beta^{k+1},x).
$$

By this and (3.19), we find that, for any $x\in B$,

$$
d(x_\beta^{k+1},x)>\frac{d(x_\beta^{k+1},x_B)}{4A_0}+\frac{r_B}{2}.
$$

Using this, Lemmas 2.3 and 3.8, we deduce that, for any $x \in B$,

$$
\begin{aligned}
|F_{1,4}(x)| &\lesssim \|c\|_* \sum_{k=0}^{\infty} \sum_{\{\beta\in\mathcal{G}_k : d(x_\beta^{k+1}, x_B)\ge 2A_0 r_B\}} \left[\mu(Q_\beta^{k+1})\right]^\theta \exp\left(-\nu\left[\frac{d(x_\beta^{k+1}, x_B)}{4A_0\delta^k} + \frac{r_B}{2\delta^k}\right]^s\right)\\
&\lesssim \|c\|_*[\mu(B)]^\theta \sum_{k=0}^{\infty} \sum_{\{\beta\in\mathcal{G}_k : d(x_\beta^{k+1}, x_B)\ge 2A_0 r_B\}} \left[1 + \frac{d(x_\beta^{k+1}, x_B)}{r_B}\right]^{\theta\omega}\\
&\quad \times E_k\left(x_\beta^{k+1}, x_B; 2^{2s+1}A_0^s\right) \exp\left[-\frac{\nu}{2^{a+1}}\left(\frac{r_B}{\delta^k}\right)^s\right]\\
&\lesssim \|c\|_*[\mu(B)]^\theta \sum_{k=0}^{\infty} \exp\left[-\frac{\nu}{2^{a+1}}\left(\frac{1}{\delta^k}\right)^s\right]\\
&\quad \times \sum_{\{\beta\in\mathcal{G}_k : d(x_\beta^{k+1}, x_B)\ge 2A_0 r_B\}} E_k\left(x_\beta^{k+1}, x_B; 4^{s+1}A_0^s\right)\\
&\lesssim \|c\|_*[\mu(B)]^\theta
\end{aligned}
$$

and hence

$$
\|F_{1,4}\|_{L^2(B)} \lesssim \|c\|_*[\mu(B)]^{\theta+\frac{1}{2}}.
$$

To summarize the estimates of $F_{1,1}$, $F_{1,2}$, $F_{1,3}$, and $F_{1,4}$, we conclude that, for almost everywhere $x \in B$,

$$
\sum_{\alpha\in\mathcal{A}_0} c_\alpha^0 \phi_\alpha^0(x) + \sum_{k=0}^{\infty} \sum_{\beta\in\mathcal{G}_k} c_\beta^{k+1} \psi_\beta^{k+1}(x)
$$

converges. Letting

$$
f := \sum_{\alpha\in\mathcal{A}_0} c_\alpha^0 \phi_\alpha^0 + \sum_{k=0}^{\infty} \sum_{\beta\in\mathcal{G}_k} c_\beta^{k+1} \psi_\beta^{k+1}, \tag{3.20}
$$

then we find that (3.20) converges in $L^2_{\mathcal{B}}(\mathcal{X})$ and, for any $B \subset \mathcal{X}$ with $r_B \in (1, \infty)$,

$$
\|f\|_{L^2(B)} \lesssim \|c\|_*[\mu(B)]^{\theta+\frac{1}{2}}. \tag{3.21}
$$

Case 2. $r_B \in (0, 1]$. In this case, write, for any $x \in B$,

$$
\begin{aligned}
f(x) = &\sum_{\alpha\in\mathcal{A}_0} c_\alpha^0 \left[\phi_\alpha^0(x) - \phi_\alpha^0(x_B)\right] + \sum_{\alpha\in\mathcal{A}_0} c_\alpha^0 \phi_\alpha^0(x_B)\\
&+ \sum_{\{k\in\mathbb{Z}_+ : \delta^k \le r_B\}} \sum_{\{\beta\in\mathcal{G}_k : d(x_\beta^{k+1}, x_B) < 2A_0 r_B\}} c_\beta^{k+1} \psi_\beta^{k+1}(x)\\
&+ \sum_{\{k\in\mathbb{Z}_+ : \delta^k \le r_B\}} \sum_{\{\beta\in\mathcal{G}_k : d(x_\beta^{k+1}, x_B) \ge 2A_0 r_B\}} c_\beta^{k+1} \psi_\beta^{k+1}(x)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\{k\in\mathbb{Z}_+:\delta^k>r_B\}}\sum_{\beta\in\mathcal{G}_k} c_\beta^{k+1}\left[\psi_\beta^{k+1}(x)-\psi_\beta^{k+1}(x_B)\right]\\
&+\sum_{\{k\in\mathbb{Z}_+:\delta^k>r_B\}}\sum_{\beta\in\mathcal{G}_k} c_\beta^{k+1}\psi_\beta^{k+1}(x_B)\\
&=: F_{2,1}+F_{2,2}+F_{2,3}+F_{2,4}+F_{2,5}+F_{2,6}.
\end{aligned}
$$

We first estimate $F_{2,1}$. Applying Lemma 3.1(ii), (3.17), Lemma 2.3, $r_B\in(0,1]$, $\theta\in(0,\frac{\eta}{\omega})$, and Lemma 3.8, we infer that, for any $x\in B$,

$$
\begin{aligned}
|F_{2,1}(x)| &\lesssim \|c\|_*\sum_{\alpha\in\mathcal{A}_0}\left[\mu(Q_\alpha^0)\right]^\theta [d(x,x_B)]^\eta E_0\left(x_\alpha^0,x_B;1\right)\\
&\lesssim \|c\|_*[\mu(B)]^\theta\sum_{\alpha\in\mathcal{A}_0}\left[\frac{1+d(x_\alpha^0,x_B)}{r_B}\right]^{\theta\omega} r_B^\eta E_0\left(x_\alpha^0,x_B;1\right)\\
&\lesssim \|c\|_*[\mu(B)]^\theta\sum_{\alpha\in\mathcal{A}_0}E_0\left(x_\alpha^0,x_B;2\right)\lesssim \|c\|_*[\mu(B)]^\theta,
\end{aligned}
$$

which further implies that

$$
\|F_{2,1}\|_{L^2(B)}\lesssim \|c\|_*[\mu(B)]^{\theta+\frac{1}{2}}.
$$

Now, we estimate $F_{2,2}$. By Lemma 3.1(i), (3.17), and Lemma 2.3, we conclude that,

$$
\begin{aligned}
|F_{2,2}| &\lesssim \sum_{\alpha\in\mathcal{A}_0}\left[\mu(Q_\alpha^0)\right]^\theta E_0\left(x_\alpha^0,x_B;1\right)\\
&\lesssim \sum_{\alpha\in\mathcal{A}_0}\left[\frac{1+d(x_\alpha^0,x_B)}{r_B}\right]^{\theta\omega} E_0\left(x_\alpha^0,x_B;1\right)\\
&\lesssim \sum_{\alpha\in\mathcal{A}_0}E_0\left(x_\alpha^0,x_B;2\right)\lesssim 1,
\end{aligned}
$$

where the implicit positive constants may depend on B. Using an argument similar to that used in the estimate of $F_{1,3}$ and $F_{1,4}$, we obtain

$$
\|F_{2,3}+F_{2,4}\|_{L^2(B)}\lesssim \|c\|_*[\mu(B)]^{\theta+\frac{1}{2}}.
$$

For $F_{2,5}$, by an argument similar to that used in the estimate of $F_{2,1}$, we find that, for any $x\in B$,

$$
\sum_{\beta\in\mathcal{G}_k}\left|c_\beta^{k+1}\right|\left|\psi_\beta^{k+1}(x)-\psi_\beta^{k+1}(x_B)\right|\lesssim \|c\|_*[\mu(B)]^\theta\left(\frac{r_B}{\delta^k}\right)^{\eta-\theta\omega},
$$

which, together with $\theta \in (0, \frac{\eta}{\omega})$, further implies that,

$$|F_{2,5}(x)| \lesssim \|c\|_*[\mu(B)]^\theta \sum_{\{k\in\mathbb{Z}_+:\delta^k>r_B\}} \left(\frac{r_B}{\delta^k}\right)^{\eta-\theta\omega} \lesssim \|c\|_*[\mu(B)]^\theta$$

and hence

$$\|F_{2,5}\|_{L^2(B)} \lesssim \|c\|_*[\mu(B)]^{\theta+\frac{1}{2}}.$$

To estimate $F_{2,6}$, using an argument similar to that used in the estimate of $F_{2,2}$, we have

$$\sum_{\beta\in\mathcal{G}_k} \left|c_\beta^{k+1}\right| \left|\psi_\beta^{k+1}(x_B)\right| \lesssim 1,$$

which, combining the fact that $\#\{k \in \mathbb{Z}_+ : \delta^k > r_B\}$ is a finite number only depending on r_B, further implies that

$$|F_{2,6}| \lesssim 1,$$

where the implicit positive constants may depend on B. Let $c_B := F_{2,2} + F_{2,6}$. From the estimates of $F_{2,1}$ through $F_{2,6}$, we deduce that

$$\|f - c_B\|_{L^2(B)} \lesssim \|c\|_*[\mu(B)]^{\theta+\frac{1}{2}},$$

which, together with the Hölder inequality, further implies that

$$\|f - f_B\|_{L^2(B)} \le \|f - c_B\|_{L^2(B)} + \|c_B - f_B\|_{L^2(B)} \lesssim \|c\|_*[\mu(B)]^{\theta+\frac{1}{2}}.$$

Applying this, (3.21), and [25, Corollary 7.5], we infer that $f \in \mathrm{lip}_\theta(\mathcal{X})$ and

$$\|f\|_{\mathrm{lip}_\theta(\mathcal{X})} \le C\|c\|^*.$$

This finishes the proof of Proposition 3.6. □

Using Propositions 3.4 and 3.6, we have the following wavelet reproducing formula of functions in $\mathrm{lip}_\theta(\mathcal{X})$.

Proposition 3.9 *Let $\eta \in (0, 1]$ as the same in Lemma 3.1, $\theta \in (0, \frac{\eta}{\omega})$. Then, for any $f \in \mathrm{lip}_\theta(\mathcal{X})$,*

$$f = \sum_{\alpha\in\mathcal{A}_0} \left\langle f, \phi_\alpha^0\right\rangle \phi_\alpha^0 + \sum_{k=0}^{\infty} \sum_{\beta\in\mathcal{G}_k} \left\langle f, \psi_\beta^{k+1}\right\rangle \psi_\beta^{k+1}$$

in $L^2_{\mathcal{B}}(\mathcal{X})$.

To prove Proposition 3.9, we need the following lemma.

Lemma 3.10 *Let $\eta \in (0, 1]$ be as the same in Lemma 3.1, $\theta \in (0, \frac{\eta}{\omega})$, and $f \in \mathrm{lip}_\theta(\mathcal{X})$. If, for any $\alpha \in \mathcal{A}_0$, $\langle f, \phi_\alpha^0\rangle = 0$, and, for any $k \in \mathbb{Z}_+$ and $\beta \in \mathcal{G}_k$, $\langle f, \psi_\beta^{k+1}\rangle = 0$, then, for any $x \in \mathcal{X}$, $f(x) = 0$.*

Proof Without loss of generality, we assume that $\|f\|_{\mathrm{lip}_\theta(\mathcal{X})} = 1$. To prove Lemma 3.10, we only need to show that, for any $\varepsilon \in (0, \infty)$ and $B := B(x_B, r_B) \subset \mathcal{X}$,

$$\|f\|_{L^\infty(B)} < \varepsilon. \tag{3.22}$$

Let $\widetilde{B} := A_0 t^2 B$, where t is a large number such that $A_0 t^2 r_B > 1$ and $B \subset 4\widetilde{B}$. By the definition of $\mathrm{lip}_\theta(\mathcal{X})$, we find that $\mathbf{1}_{4\widetilde{B}} f \in L^2(\mathcal{X})$. Applying Lemmas 3.1 and 3.2, write

$$\begin{aligned}
\mathbf{1}_B f &= \mathbf{1}_B \mathbf{1}_{4\widetilde{B}} f \\
&= \mathbf{1}_B \sum_{\alpha \in \mathcal{A}_0} \left\langle \mathbf{1}_{4\widetilde{B}} f, \phi_\alpha^0 \right\rangle \phi_\alpha^0 + \mathbf{1}_B \sum_{k=0}^{\infty} \sum_{\beta \in \mathcal{G}_k} \left\langle \mathbf{1}_{4\widetilde{B}} f, \psi_\beta^{k+1} \right\rangle \psi_\beta^{k+1} \\
&= \mathbf{1}_B \sum_{\{\alpha \in \mathcal{A}_0 : d(x_\alpha^0, x_B) < 2A_0 t^2 r_B\}} \left\langle \mathbf{1}_{4\widetilde{B}} f, \phi_\alpha^0 \right\rangle \phi_\alpha^0 \\
&\quad + \mathbf{1}_B \sum_{\{\alpha \in \mathcal{A}_0 : d(x_\alpha^0, x_B) \geq 2A_0 t^2 r_B\}} \left\langle \mathbf{1}_{4\widetilde{B}} f, \phi_\alpha^0 \right\rangle \phi_\alpha^0 \\
&\quad + \mathbf{1}_B \sum_{k=0}^{\infty} \sum_{\{\beta \in \mathcal{G}_k : d(x_\beta^{k+1}, x_B) < 2A_0 t^2 r_B\}} \left\langle \mathbf{1}_{4\widetilde{B}} f, \psi_\beta^{k+1} \right\rangle \psi_\beta^{k+1} \\
&\quad + \mathbf{1}_B \sum_{k=0}^{\infty} \sum_{\{\beta \in \mathcal{G}_k : d(x_\beta^{k+1}, x_B) \geq 2A_0 t^2 r_B\}} \left\langle \mathbf{1}_{4\widetilde{B}} f, \psi_\beta^{k+1} \right\rangle \psi_\beta^{k+1} \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

We first consider J_1. By $\langle f, \phi_\alpha^0 \rangle = 0$ and Lemma 3.5(iv), we have, for any $\alpha \in \mathcal{A}_0$ with $d(x_\alpha^0, x_B) < 2A_0 t^2 r_B$,

$$\begin{aligned}
\left| \left\langle \mathbf{1}_{4\widetilde{B}} f, \phi_\alpha^0 \right\rangle \right| &= \left| \left\langle \mathbf{1}_{\mathcal{X} \setminus (4\widetilde{B})} f, \phi_\alpha^0 \right\rangle \right| \\
&\leq \int_{\{x \in \mathcal{X} : d(x_B, x) > 4A_0 t^2 r_B\}} |f(x)| \left| \phi_\alpha^0(x) \right| \, d\mu(x) \\
&\lesssim [\mu(\widetilde{B})]^\theta \sqrt{V_1(x_\alpha^0)} \exp\left[-\frac{\nu}{3} \left(A_0 t^2 r_B \right)^s \right].
\end{aligned} \tag{3.23}$$

From this and Lemmas 3.1(i) and 3.8, we deduce that, for any $x \in B$,

$$\begin{aligned}
|J_1(x)| &\leq \sum_{\{\alpha \in \mathcal{A}_0 : d(x_\alpha^0, x_B) < 2A_0 t^2 r_B\}} \left| \left\langle \mathbf{1}_{4\widetilde{B}} f, \phi_\alpha^0 \right\rangle \right| \left| \phi_\alpha^0(x) \right| \\
&\lesssim \left[\mu(\widetilde{B}) \right]^\theta \exp\left[-\frac{\nu}{3} \left(A_0 t^2 r_B \right)^s \right] \\
&\quad \times \sum_{\{\alpha \in \mathcal{A}_0 : d(x_\alpha^0, x_B) < 2A_0 t^2 r_B\}} \exp\left(-\nu \left[d(x_\alpha^0, x) \right]^s \right)
\end{aligned}$$

$$\lesssim \left[\mu(\widetilde{B})\right]^{\theta} \exp\left[-\frac{\nu}{3}\left(A_0 t^2 r_B\right)^s\right].$$

To estimate J_2, observe that, for any $\alpha \in \mathcal{A}_0$ with $d(x_\alpha^0, x_B) \geq 2A_0 t^2 r_B$ and any $x \in B$,

$$2A_0 t^2 r_B \leq d(x_\alpha^0, x_B) \leq A_0\left[d(x_\alpha^0, x) + d(x, x_B)\right] < A_0\left[d(x_\alpha^0, x) + r_B\right]$$

and hence

$$t^2 r_B < d(x_\alpha^0, x). \tag{3.24}$$

Moreover,

$$\begin{aligned} 1 + d(x_B, x_\alpha^0) &\leq 1 + A_0\left[d(x_B, x) + d(x, x_\alpha^0)\right] \\ &< 1 + A_0 r_B + A_0 d(x, x_\alpha^0) \\ &< 2A_0 t^2 r_B + A_0 d(x, x_\alpha^0). \end{aligned}$$

Using this, (3.24), and Lemmas 3.1(i) and 3.5(ii), we infer that, for any $\alpha \in \mathcal{A}_0$ with $d(x_\alpha^0, x_B) \geq 2A_0 t^2 r_B$ and for any $x \in B$,

$$\begin{aligned} &\left|\left\langle \mathbf{1}_{4\widetilde{B}} f, \phi_\alpha^0 \right\rangle\right| \left|\phi_\alpha^0(x)\right| \\ &\quad\lesssim \left[\mu(\widetilde{B})\right]^{\theta}\left[1 + \frac{1 + d(x_B, x_\alpha^0)}{A_0 t^2 r_B}\right]^{\theta\omega} E_0\left(x, x_\alpha^0; 1\right) \\ &\quad\lesssim \left[\mu(\widetilde{B})\right]^{\theta}\left[1 + d(x, x_\alpha^0)\right]^{\theta\omega} E_0\left(x, x_\alpha^0; 1\right) \\ &\quad\lesssim \left[\mu(\widetilde{B})\right]^{\theta} \exp\left[-\frac{\nu}{4}\left(t^2 r_B\right)^s\right] E_0\left(x, x_\alpha^0; 4\right), \end{aligned} \tag{3.25}$$

where E_0 is as in (3.1) with $k = 0$. This, together with Lemma 3.8, further implies that, for any $x \in B$,

$$\begin{aligned} |J_2(x)| &\leq \sum_{\{\alpha \in \mathcal{A}_0 : d(x_\alpha^0, x_B) \geq 2A_0 t^2 r_B\}} \left|\left\langle \mathbf{1}_{4\widetilde{B}} f, \phi_\alpha^0 \right\rangle\right| \left|\phi_\alpha^0(x)\right| \\ &\lesssim \left[\mu(\widetilde{B})\right]^{\theta} \exp\left[-\frac{\nu}{4}\left(t^2 r_B\right)^s\right] \\ &\quad\times \sum_{\{\alpha \in \mathcal{A}_0 : d(x_\alpha^0, x_B) \geq 2A_0 t^2 r_B\}} E_0\left(x, x_\alpha^0; 4\right) \\ &\lesssim \left[\mu(\widetilde{B})\right]^{\theta} \exp\left[-\frac{\nu}{4}\left(t^2 r_B\right)^s\right]. \end{aligned}$$

Next, we estimate J_3. Using an argument similar to that used in (3.23), we have, for any $\beta \in \mathcal{G}_k$ with $d(x_\beta^{k+1}, x_B) < 2A_0t^2r_B$,

$$\left|\left\langle \mathbf{1}_{4\widetilde{B}} f, \psi_\beta^{k+1}\right\rangle\right| \lesssim \left[\mu(\widetilde{B})\right]^\theta \sqrt{V_{\delta^k}(x_\beta^{k+1})} \exp\left[-\frac{\nu}{3}\left(\frac{A_0t^2r_B}{\delta^k}\right)^s\right].$$

By this and Lemmas 3.2(i) and 3.8, we conclude that, for any $x \in B$,

$$\begin{aligned}|J_3(x)| &\lesssim \sum_{k=0}^\infty \sum_{\{\beta\in\mathcal{G}_k: d(x_\beta^{k+1}, x_B)<2A_0t^2r_B\}} \left|\left\langle \mathbf{1}_{4\widetilde{B}} f, \psi_\beta^{k+1}\right\rangle\right| \left|\psi_\beta^{k+1}(x)\right| \\ &\lesssim \left[\mu(\widetilde{B})\right]^\theta \sum_{k=0}^\infty \exp\left[-\frac{\nu}{3}\left(\frac{A_0t^2r_B}{\delta^k}\right)^s\right] \\ &\quad \times \sum_{\{\beta\in\mathcal{G}_k: d(x_\beta^{k+1}, x_B)<2A_0t^2r_B\}} E_k\left(x, x_\beta^{k+1}; 1\right) \\ &\lesssim \left[\mu(\widetilde{B})\right]^\theta \exp\left[-\frac{\nu}{6}\left(A_0t^2r_B\right)^s\right].\end{aligned}$$

Finally, we estimate J_4. Applying an argument similar to that used in (3.25), we infer that, for any $\beta \in \mathcal{G}_k$ with $d(x_\beta^{k+1}, x_B) \geq 2A_0t^2r_B$,

$$\begin{aligned}&\left|\left\langle \mathbf{1}_{4\widetilde{B}} f, \psi_\beta^{k+1}\right\rangle\right| \left|\psi_\beta^{k+1}(x)\right| \\ &\quad \lesssim \left[\mu(\widetilde{B})\right]^\theta \exp\left[-\frac{\nu}{4}\left(\frac{t^2r_B}{\delta^k}\right)^s\right] E_k\left(x, x_\beta^{k+1}; 4\right),\end{aligned}$$

which, combined with Lemma 3.8, further implies that, for any $x \in B$,

$$\begin{aligned}|J_4(x)| &\lesssim \sum_{k=0}^\infty \sum_{\{\beta\in\mathcal{G}_k: d(x_\beta^{k+1}, x_B)\geq 2A_0t^2r_B\}} \left|\left\langle \mathbf{1}_{4\widetilde{B}} f, \psi_\beta^{k+1}\right\rangle\right| \left|\psi_\beta^{k+1}(x)\right| \\ &\lesssim \left[\mu(\widetilde{B})\right]^\theta \sum_{k=0}^\infty \exp\left[-\frac{\nu}{4}\left(\frac{t^2r_B}{\delta^k}\right)^s\right] \\ &\quad \times \sum_{\{\beta\in\mathcal{G}_k: d(x_\beta^{k+1}, x_B)\geq 2A_0t^2r_B\}} E_k\left(x, x_\beta^{k+1}; 4\right) \\ &\lesssim \left[\mu(\widetilde{B})\right]^\theta \exp\left[-\frac{\nu}{8}\left(t^2r_B\right)^s\right].\end{aligned}$$

To summarize the estimates of J_1, J_2, J_3, and J_4, we find that

$$\|\mathbf{1}_B f\|_{L^\infty(B)} \leq \|J_1\|_{L^\infty(B)} + \|J_2\|_{L^\infty(B)} + \|J_3\|_{L^\infty(B)} + \|J_4\|_{L^\infty(B)}$$

$$\lesssim \left[\mu(\widetilde{B})\right]^{\theta} \exp\left[-\frac{\nu}{8}\left(t^2 r_B\right)^s\right] \to 0$$

as $t \to \infty$. This finishes the proof of (3.22) and hence Lemma 3.10. □

Now, we show Proposition 3.9.

Proof of Proposition 3.9 Without loss of generality, we assume that $\|f\|_{\mathrm{lip}_\theta(\mathcal{X})} = 1$. By Propositions 3.4 and 3.6, it follows that

$$\sum_{\alpha\in\mathcal{A}_0} \left\langle f, \phi_\alpha^0 \right\rangle \phi_\alpha^0 + \sum_{k=0}^{\infty} \sum_{\beta\in\mathcal{G}_k} \left\langle f, \psi_\beta^{k+1} \right\rangle \psi_\beta^{k+1}$$

converges in $L^2_{\mathcal{B}}(\mathcal{X})$. Denote this limit by $\widetilde{f}$. To show Proposition 3.9, we only need to prove that $f = \widetilde{f}$ pointwise. Using Lemma 3.10, we further reduce to show that, for any $\alpha_0 \in \mathcal{A}_0$,

$$\left\langle f - \widetilde{f}, \phi_{\alpha_0}^0 \right\rangle = 0, \tag{3.26}$$

and, for any $k_0 \in \mathbb{Z}_+$ and $\beta_0 \in \mathcal{G}_{k_0}$,

$$\left\langle f - \widetilde{f}, \psi_{\beta_0}^{k_0+1} \right\rangle = 0. \tag{3.27}$$

To this end, let, for any $\alpha \in \mathcal{A}_0$, $c_\alpha^0 := \langle f, \phi_\alpha^0 \rangle$ and, for any $k \in \mathbb{Z}_+$ and $\beta \in \mathcal{G}_k$, $c_\beta^{k+1} := \langle f, \psi_\beta^{k+1} \rangle$.

We first show (3.26). By Lemma 2.9, we find that, for any $\alpha_0 \in \mathcal{A}_0$,

$$\begin{aligned}
\left\langle \widetilde{f}, \phi_{\alpha_0}^0 \right\rangle &= \int_{\mathcal{X}} \widetilde{f}(x)\phi_{\alpha_0}^0(x)\,d\mu(x) \\
&= \sum_{\tilde{\alpha}\in\mathcal{A}_0} \int_{\mathcal{X}} \widetilde{f}(x)\mathbf{1}_{Q_{\tilde{\alpha}}^0}(x)\phi_{\alpha_0}^0(x)\widetilde{f}(x)\mathbf{1}_{Q_{\tilde{\alpha}}^0}(x)\,d\mu(x) \\
&= \sum_{\tilde{\alpha}\in\mathcal{A}_0} \sum_{\alpha\in\mathcal{A}_0} \left\langle c_\alpha^0 \phi_\alpha^0 \mathbf{1}_{Q_{\tilde{\alpha}}^0}, \phi_{\alpha_0}^0 \mathbf{1}_{Q_{\tilde{\alpha}}^0} \right\rangle \\
&\quad + \sum_{\tilde{\alpha}\in\mathcal{A}_0} \sum_{\alpha\in\mathcal{A}_0} \left\langle \sum_{\{(k,\beta):k\in\mathbb{Z}_+,\beta\in\mathcal{G}_k,Q_\beta^{k+1}\subset Q_\alpha^0\}} c_\beta^{k+1}\psi_\beta^{k+1}\mathbf{1}_{Q_{\tilde{\alpha}}^0}, \phi_{\alpha_0}^0\mathbf{1}_{Q_{\tilde{\alpha}}^0} \right\rangle \\
&=: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

To estimate I_1, by Lemma 2.2(ii), we have, for any $\alpha \in \mathcal{A}_0$,

$$\left\| \phi_\alpha^0 \mathbf{1}_{Q_{\tilde{\alpha}}^0} \right\|_{L^2(\mathcal{X})} \lesssim \left[\int_{Q_{\tilde{\alpha}}^0} \frac{1}{V_1(x_\alpha^0)} E_0\left(x_\alpha^0, x; \frac{1}{2}\right) d\mu(x) \right]^{\frac{1}{2}},$$

where E_0 is as in (3.1) with $k = 0$. If $d(x_\alpha^0, x_{\tilde{\alpha}}^0) \le 2A_0C_\#$, then, for any $x \in Q_{\tilde{\alpha}}^0$,

$$d(x_\alpha^0, x) \le A_0\left[d(x_\alpha^0, x_{\tilde{\alpha}}^0) + d(x_{\tilde{\alpha}}^0, x)\right] \lesssim 1.$$

From this and Lemma 2.2, we deduce that

$$\left\|\phi_\alpha^0 \mathbf{1}_{Q_{\tilde{\alpha}}^0}\right\|_{L^2(\mathcal{X})} \lesssim 1 \lesssim E_0\left(x_\alpha^0, x_{\tilde{\alpha}}^0; 2\right).$$

If $d(x_\alpha^0, x_{\tilde{\alpha}}^0) > 2A_0C_\#$, then, for any $x \in Q_{\tilde{\alpha}}^0$,

$$d(x_\alpha^0, x_{\tilde{\alpha}}^0) \le A_0\left[d(x_\alpha^0, x) + d(x, x_{\tilde{\alpha}}^0)\right] < A_0 d(x_\alpha^0, x) + \frac{1}{2} d(x_\alpha^0, x_{\tilde{\alpha}}^0)$$

and hence $\frac{1}{2A_0} d(x_\alpha^0, x_{\tilde{\alpha}}^0) < d(x_\alpha^0, x)$. Using this and Lemmas 2.3 and 2.2, we infer that

$$\begin{aligned}
\left\|\phi_\alpha^0 \mathbf{1}_{Q_{\tilde{\alpha}}^0}\right\|_{L^2(\mathcal{X})} &\lesssim \left[\int_{Q_{\tilde{\alpha}}^0} \frac{1}{V_1(x_\alpha^0)} E_0\left(x_\alpha^0, x; \frac{1}{2}\right) d\mu(x)\right]^{\frac{1}{2}} \\
&\lesssim E_0\left(x_\alpha^0, x_{\tilde{\alpha}}^0; 2A_0\right)\left[\int_{Q_{\tilde{\alpha}}^0} \frac{1}{V_1(x_\alpha^0)} E_0\left(x_\alpha^0, x; 1\right) d\mu(x)\right]^{\frac{1}{2}} \\
&\lesssim E_0\left(x_\alpha^0, x_{\tilde{\alpha}}^0; 2A_0\right).
\end{aligned}$$

By the Hölder inequality, Proposition 3.4, and Lemmas 2.3 and 3.8, we conclude that

$$\begin{aligned}
&\sum_{\tilde{\alpha}\in\mathcal{A}_0}\sum_{\alpha\in\mathcal{A}_0}\left|\left\langle c_\alpha^0 \phi_\alpha^0 \mathbf{1}_{Q_{\tilde{\alpha}}^0}, \phi_{\alpha_0}^0 \mathbf{1}_{Q_{\tilde{\alpha}}^0}\right\rangle\right| \\
&\quad\lesssim \sum_{\tilde{\alpha}\in\mathcal{A}_0}\sum_{\alpha\in\mathcal{A}_0}\left[\mu(Q_\alpha^0)\right]^{\theta+\frac{1}{2}}\left\|\phi_\alpha^0 \mathbf{1}_{Q_{\tilde{\alpha}}^0}\right\|_{L^2(\mathcal{X})}\left\|\phi_{\alpha_0}^0 \mathbf{1}_{Q_{\tilde{\alpha}}^0}\right\|_{L^2(\mathcal{X})} \\
&\quad\lesssim \left[\mu(Q_{\alpha_0}^0)\right]^{\theta+\frac{1}{2}}\sum_{\tilde{\alpha}\in\mathcal{A}_0}\sum_{\alpha\in\mathcal{A}_0}\left[\frac{\mu(Q_\alpha^0)}{\mu(Q_{\alpha_0}^0)}\right]^{\theta+\frac{1}{2}} \\
&\qquad\times E_0\left(x_\alpha^0, x_{\tilde{\alpha}}^0; 2A_0\right) E_0\left(x_{\alpha_0}^0, x_{\tilde{\alpha}}^0; 2A_0\right) \\
&\quad\lesssim \left[\mu(Q_{\alpha_0}^0)\right]^{\theta+\frac{1}{2}}.
\end{aligned}$$

This implies that I_1 converges absolutely.

Next, we estimate I_2. To this end, we consider two cases on $d(x_\alpha^0, x_{\tilde{\alpha}}^0)$.

Case 1. $d(x_\alpha^0, x_{\tilde{\alpha}}^0) \le 4A_0^2 C_\#$. In this case, by Lemmas 3.1 and 3.2, Proposition 3.4, and Lemma 2.2(i), we obtain

$$\begin{aligned}
&\left\| \sum_{\{(k,\beta):k\in\mathbb{Z}_+,\beta\in\mathcal{G}_k, Q_\beta^{k+1}\subset Q_\alpha^0\}} c_\beta^{k+1}\psi_\beta^{k+1}\mathbf{1}_{Q_{\tilde{\alpha}}^0}\right\|_{L^2(\mathcal{X})}\\
&\quad\lesssim \left(\sum_{\{(k,\beta):k\in\mathbb{Z}_+,\beta\in\mathcal{G}_k, Q_\beta^{k+1}\subset Q_\alpha^0\}} \left|c_\beta^{k+1}\right|^2\right)^{\frac{1}{2}}\\
&\quad\lesssim \left[\mu(Q_\alpha^0)\right]^{\theta+\frac{1}{2}} \lesssim \left[\mu(Q_{\tilde{\alpha}}^0)\right]^{\theta+\frac{1}{2}} E_0\left(x_\alpha^0, x_{\tilde{\alpha}}^0; 1\right).
\end{aligned}$$

Case 2. $d(x_\alpha^0, x_{\tilde{\alpha}}^0) > 4A_0^2 C_\#$. In this case, for any (k,β) such that $Q_\beta^{k+1} \subset Q_\alpha^0$ and any $x \in Q_{\tilde{\alpha}}^0$,

$$d(x_\alpha^0, x_{\tilde{\alpha}}^0) < A_0 d(x_\alpha^0, x) + A_0 C_\# < A_0^2 C_\# + A_0^2 d(x_\beta^{k+1}, x) + A_0 C_\#,$$

which further implies that

$$\frac{1 + d(x_\alpha^0, x_{\tilde{\alpha}}^0)}{4A_0^2} < d(x_\beta^{k+1}, x). \tag{3.28}$$

Applying this, Lemma 3.2(i), Proposition 3.4, and Lemmas 2.3 and 3.8, we deduce that, for any $x \in Q_{\tilde{\alpha}}^0$,

$$\begin{aligned}
&\left| \sum_{\{(k,\beta):k\in\mathbb{Z}_+,\beta\in\mathcal{G}_k, Q_\beta^{k+1}\subset Q_\alpha^0\}} c_\beta^{k+1}\psi_\beta^{k+1}(x)\right|\\
&\quad\lesssim \sum_{k=0}^\infty \sum_{\{\beta\in\mathcal{G}_k: Q_\beta^{k+1}\subset Q_\alpha^0\}} \left[\mu(Q_\beta^{k+1})\right]^\theta E_k\left(x_\beta^{k+1}, x; 2\right) \exp\left\{-\frac{\nu}{2}\left[\frac{1 + d(x_\alpha^0, x_{\tilde{\alpha}}^0)}{4A_0^2\delta^k}\right]^s\right\}\\
&\quad\lesssim \left[\mu(Q_{\tilde{\alpha}}^0)\right]^\theta \sum_{k=0}^\infty \left[1 + d(x_\alpha^0, x_{\tilde{\alpha}}^0)\right]^{\theta\omega} \exp\left\{-\frac{\nu}{2}\left[\frac{1 + d(x_\alpha^0, x_{\tilde{\alpha}}^0)}{4A_0^2\delta^k}\right]^s\right\}\\
&\qquad\times \sum_{\{\beta\in\mathcal{G}_k: Q_\beta^{k+1}\subset Q_\alpha^0\}} E_k\left(x_\beta^{k+1}, x; 2\right)\\
&\quad\lesssim \left[\mu(Q_{\tilde{\alpha}}^0)\right]^\theta \exp\left\{-\frac{\nu}{4}\left[\frac{1 + d(x_\alpha^0, x_{\tilde{\alpha}}^0)}{4A_0^2}\right]^s\right\} \sum_{k=0}^\infty \delta^{k\theta\omega}\\
&\quad\lesssim \left[\mu(Q_{\tilde{\alpha}}^0)\right]^\theta E_0\left(x_\alpha^0, x_{\tilde{\alpha}}^0; 4^{s+1}A_0^{2s}\right)
\end{aligned}$$

and hence

$$\left\| \sum_{\{(k,\beta):k\in\mathbb{Z}_+,\beta\in\mathcal{G}_k,Q_\beta^{k+1}\subset Q_\alpha^0\}} c_\beta^{k+1}\psi_\beta^{k+1}\mathbf{1}_{Q_{\tilde{\alpha}}^0} \right\|_{L^2(\mathcal{X})} \lesssim \left[\mu(Q_{\tilde{\alpha}}^0)\right]^{\theta+\frac{1}{2}} E_0\left(x_\alpha^0, x_{\tilde{\alpha}}^0; 4^{s+1}A_0^{2s}\right).$$

Combining the estimates of Cases 1 and 2, the Hölder inequality, Proposition 3.4, and Lemma 3.8, we conclude that

$$\begin{aligned}
&\sum_{\tilde{\alpha}\in\mathcal{A}_0}\sum_{\alpha\in\mathcal{A}_0} \left| \left\langle \sum_{\{(k,\beta):k\in\mathbb{Z}_+,\beta\in\mathcal{G}_k,Q_\beta^{k+1}\subset Q_\alpha^0\}} c_\beta^{k+1}\psi_\beta^{k+1}\mathbf{1}_{Q_{\tilde{\alpha}}^0}, \phi_{\alpha_0}^0\mathbf{1}_{Q_{\tilde{\alpha}}^0} \right\rangle \right| \\
&\lesssim \sum_{\tilde{\alpha}\in\mathcal{A}_0}\sum_{\alpha\in\mathcal{A}_0} \left\| \sum_{\{(k,\beta):k\in\mathbb{Z}_+,\beta\in\mathcal{G}_k,Q_\beta^{k+1}\subset Q_\alpha^0\}} c_\beta^{k+1}\psi_\beta^{k+1}\mathbf{1}_{Q_{\tilde{\alpha}}^0} \right\|_{L^2(\mathcal{X})} \left\| \phi_{\alpha_0}^0\mathbf{1}_{Q_{\tilde{\alpha}}^0} \right\|_{L^2(\mathcal{X})} \\
&\lesssim \sum_{\tilde{\alpha}\in\mathcal{A}_0}\sum_{\alpha\in\mathcal{A}_0} \left[\mu(Q_{\tilde{\alpha}}^0)\right]^{\theta+\frac{1}{2}} E_0\left(x_\alpha^0, x_{\tilde{\alpha}}^0; 4^{s+1}A_0^{2s}\right) E_0\left(x_\alpha^0, x_{\tilde{\alpha}}^0; 2A_0\right) \\
&\lesssim \left[\mu(Q_{\alpha_0}^0)\right]^{\theta+\frac{1}{2}},
\end{aligned}$$

which further implies that I_2 converges absolutely. Therefore, by the Fubini theorem and the orthogonality of wavelets, we have

$$\begin{aligned}
\left\langle \widetilde{f}, \phi_{\alpha_0}^0 \right\rangle &= \sum_{\alpha\in\mathcal{A}_0}\sum_{\tilde{\alpha}\in\mathcal{A}_0} \left\langle c_\alpha^0\phi_\alpha^0\mathbf{1}_{Q_{\tilde{\alpha}}^0}, \phi_{\alpha_0}^0\mathbf{1}_{Q_{\tilde{\alpha}}^0} \right\rangle \\
&\quad + \sum_{\alpha\in\mathcal{A}_0}\sum_{\tilde{\alpha}\in\mathcal{A}_0} \left\langle \sum_{\{(k,\beta):k\in\mathbb{Z}_+,\beta\in\mathcal{G}_k,Q_\beta^{k+1}\subset Q_\alpha^0\}} c_\beta^{k+1}\psi_\beta^{k+1}\mathbf{1}_{Q_{\tilde{\alpha}}^0}, \phi_{\alpha_0}^0\mathbf{1}_{Q_{\tilde{\alpha}}^0} \right\rangle \\
&= \sum_{\alpha\in\mathcal{A}_0} c_\alpha^0 \left\langle \phi_\alpha^0, \phi_{\alpha_0}^0 \right\rangle + \sum_{\alpha\in\mathcal{A}_0} \left\langle \sum_{\{(k,\beta):k\in\mathbb{Z}_+,\beta\in\mathcal{G}_k,Q_\beta^{k+1}\subset Q_\alpha^0\}} c_\beta^{k+1}\psi_\beta^{k+1}, \phi_{\alpha_0}^0 \right\rangle \\
&= c_{\alpha_0}^0 = \left\langle f, \phi_{\alpha_0}^0 \right\rangle.
\end{aligned}$$

This proves (3.26).

Now, we show (3.27). Using an argument similar to that used in the proof of (3.26), we infer that, for any $k_0 \in \mathbb{Z}_+$ and $\beta_0 \in \mathcal{G}_{k_0}$,

$$\left\langle \sum_{\alpha\in\mathcal{A}_0} c_\alpha^0\phi_\alpha^0 + \sum_{k=k_0}^{\infty}\sum_{\beta\in\mathcal{G}_k} c_\beta^{k+1}\psi_\beta^{k+1}, \psi_{\beta_0}^{k_0+1} \right\rangle = c_{\beta_0}^{k_0+1} = \left\langle f, \psi_{\beta_0}^{k_0+1} \right\rangle.$$

Thus, to show (3.27), it suffices to show that

$$\left\langle \sum_{k=0}^{k_0-1} \sum_{\beta\in\mathcal{G}_k} c_\beta^{k+1} \psi_\beta^{k+1}, \psi_{\beta_0}^{k_0+1} \right\rangle = 0. \tag{3.29}$$

If $k_0 = 0$, then (3.29) holds true. If $k_0 > 0$, by Lemma 3.2(i) and (2.1), we obtain, for any $k \in \{0, \cdots, k_0 - 1\}$, $\beta \in \mathcal{G}_k$, and $x \in \mathcal{X}$,

$$\begin{aligned}
&\left|\psi_\beta^{k+1}(x)\psi_{\beta_0}^{k_0+1}(x)\right| \\
&\quad\lesssim \frac{1}{\sqrt{V_{\delta^k}(x_\beta^{k+1})}} E_k\left(x, x_\beta^{k+1}; 1\right) E_{k_0}\left(x, x_{\beta_0}^{k_0+1}; 1\right) \\
&\quad\lesssim \frac{1}{\sqrt{V_{\delta^k}(x_\beta^{k+1})}} E_k\left(x, x_\beta^{k+1}; 2\right) \exp\left(-\frac{\nu}{2^{s+1}}\left[d(x, x_{\beta_0}^{k_0+1}) + d(x, x_\beta^{k+1})\right]^s\right) \\
&\quad\lesssim \frac{1}{\sqrt{V_{\delta^k}(x_\beta^{k+1})}} E_k\left(x, x_\beta^{k+1}; 2\right) E_k\left(x_{\beta_0}^{k_0+1}, x_\beta^{k+1}; \frac{2^{s+1}A_0}{\delta^{k_0}}\right),
\end{aligned}$$

where the implicit positive constants may depend on k_0 and β_0. From this, Proposition 3.6, and Lemmas 2.3, 2.2(ii), and 3.8, we deduce that

$$\begin{aligned}
&\sum_{k=0}^{k_0-1} \sum_{\beta\in\mathcal{G}_k} \left|c_\beta^{k+1}\right| \int_{\mathcal{X}} \left|\psi_\beta^{k+1}(x)\psi_{\beta_0}^{k_0+1}(x)\right| d\mu(x) \\
&\quad\lesssim \sum_{k=0}^{k_0-1} \sum_{\beta\in\mathcal{G}_k} \left[\mu(Q_\beta^{k+1})\right]^{\theta+\frac{1}{2}} E_k\left(x_{\beta_0}^{k_0+1}, x_\beta^{k+1}; \frac{2^{s+1}A_0}{\delta^{k_0}}\right) \\
&\qquad\times \int_{\mathcal{X}} \frac{1}{V_{\delta^k}(x_\beta^{k+1})} E_k\left(x, x_\beta^{k+1}; 2\right) d\mu(x) \\
&\quad\lesssim \sum_{k=0}^{k_0-1} \sum_{\beta\in\mathcal{G}_k} E_k\left(x_{\beta_0}^{k_0+1}, x_\beta^{k+1}; \frac{2^{s+2}A_0}{\delta^{k_0}}\right) \lesssim 1,
\end{aligned}$$

where the implicit positive constants depend on k_0 and β_0. Applying this, the Fubini theorem, and the orthogonality of wavelets, we infer that

$$\left\langle \sum_{k=0}^{k_0-1} \sum_{\beta\in\mathcal{G}_k} c_\beta^{k+1} \psi_\beta^{k+1}, \psi_{\beta_0}^{k_0+1} \right\rangle = \sum_{k=0}^{k_0-1} \sum_{\beta\in\mathcal{G}_k} c_\beta^{k+1} \left\langle \psi_\beta^{k+1}, \psi_{\beta_0}^{k_0+1} \right\rangle = 0.$$

This finishes the proof of (3.29) and hence that of Proposition 3.9. □

Theorem 1.1 follows from Propositions 3.4, 3.6, and 3.9 derictly; we omit details here. Moreover, using Propositions 3.4 and 3.6 and the proof of Proposition 3.9, we also have the following conclusion; we omit details here.

Corollary 3.11 *Let $\eta \in (0, 1]$ as the same in Lemma 3.1, $\theta \in (0, \frac{\eta}{\omega})$. For any $n \in \mathbb{N}$ and $f \in \mathrm{lip}_\theta(\mathcal{X})$, let*

$$f_n := \sum_{\alpha \in \mathcal{A}_0} \left\langle f, \phi_\alpha^0 \right\rangle \phi_\alpha^0 + \sum_{k=0}^{n} \sum_{\beta \in \mathcal{G}_k} \left\langle f, \psi_\beta^{k+1} \right\rangle \psi_\beta^{k+1},$$

Then $f = \lim_{n\to\infty} f_n$ converges in $L^2_{\mathcal{B}}(\mathcal{X})$ and there exists a positive constant C, independent of f, such that, for any $n \in \mathbb{N}$,

$$\|f_n\|_{\mathrm{lip}_\theta(\mathcal{X})} \le C\|f\|_{\mathrm{lip}_\theta(\mathcal{X})}.$$

4 Applications

In this section, we establish several equivalence characterizations of geometric conditions on $\mathcal{X}$. The first one is as following related to the lower bound.

Corollary 4.1 *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type and $\theta \in (0, 1)$. Then $1 \in \mathrm{lip}_\theta(\mathcal{X})$ if and only if there exists a positive constant C such that, for any $x \in \mathcal{X}$, $\mu(x, 1) \ge C$.*

Proof We first show the necessity. Assume that $1 \in \mathrm{lip}_\theta(\mathcal{X})$. Then, by the definition of $\mathrm{lip}_\theta(\mathcal{X})$, we have, for any $x \in \mathcal{X}$,

$$\frac{\|1\|_{L^\infty(B(x,2))}}{[\mu(B(x,2))]^\theta} \lesssim 1$$

and hence $1 \lesssim [\mu(B(x,2))]^\theta$, which further implies that $1 \lesssim \mu(B(x,1))$.

Conversely, suppose $1 \lesssim \mu(B(x,1))$. Let $B := (x_B, r_B)$. If $r_B \in (0, 1]$, then

$$\sup_{x,y\in B, x\neq y} \frac{|1-1|}{[\mu(B)]^\theta} = 0;$$

while, if $r_B \in (1, \infty)$, then

$$\frac{\|1\|_{L^\infty(B)}}{[\mu(B)]^\theta} \lesssim \frac{1}{[\mu(B(x_B, 1))]^\theta} \lesssim 1.$$

This implies that $1 \in \mathrm{lip}_\theta(\mathcal{X})$ and finishes the proof of Corollary 4.1. □

Using Theorem 1.1, we have one equivalence characterization of the upper bound.

Corollary 4.2 *Let* $(\mathcal{X}, d, \mu)$ *be a space of homogeneous type with upper dimension* ω, $\eta \in (0, 1]$ *as the same in Lemma 3.1, and* $\theta \in (0, \frac{\eta}{\omega})$*. Then the following statements are equivalent.*

(i) $\mathrm{lip}_\theta(\mathcal{X}) \subset L^\infty(\mathcal{X})$;
(ii) *there exists a positive constant* C *such that, for any* $x \in \mathcal{X}$, $\mu(B(x, 1)) \le C$.

Proof (ii) $\Rightarrow$ (i) follows from (2.4) directly. Thus, to show Corollary 4.2, it suffices to prove (i) $\Rightarrow$ (ii). By Theorem 1.1, we find that, for any $\alpha \in \mathcal{A}_0$,

$$\left\|\phi_\alpha^0\right\|_{\mathrm{lip}_\theta(\mathcal{X})} \sim \left[\mu(Q_\alpha^0)\right]^{-\frac{1}{2}-\theta}. \tag{4.1}$$

On the other hand, by the proof of [2, Theorem 6.1], we obtain, for any $\alpha \in \mathcal{A}_0$,

$$\left\|\phi_\alpha^0\right\|_{L^\infty(\mathcal{X})} \sim \left|\phi_\alpha^0(x_\alpha^0)\right| \sim \left[\mu(Q_\alpha^0)\right]^{-\frac{1}{2}},$$

which, together with (i) and (4.1), further implies that

$$\mu(Q_\alpha^0) \lesssim 1.$$

For any $x \in \mathcal{X}$, by (2.6), we find that there exists $\alpha_0 \in \mathcal{A}_0$ such that $d(x, x_{\alpha_0}^0) < C_0$. This, combined Lemma 2.2(i), further implies that

$$\mu(B(x, 1)) \sim \mu(B(x_{\alpha_0}^0, 1)) \lesssim 1$$

and hence finishes the proof of Corollary 4.2. □

At the end of this section, we establish a equivalence characterization of the so-called *Ahlfors regular space* via Theorem 1.1.

Corollary 4.3 *Let* $(\mathcal{X}, d, \mu)$ *be a space of homogeneous type with upper dimension* ω, $\eta \in (0, 1]$ *as the same in Lemma 3.1, and* $\theta \in (0, \frac{\eta}{\omega})$*. Then the following statements are equivalent.*

(i) $\mathcal{X}$ *is an Ahlfors regular space: there exists a constant* $C \in [1, \infty)$ *such that, for any* $x \in \mathcal{X}$ *and* $r \in (0, \infty)$,

$$C^{-1} r^\omega \le \mu(B(x, r)) \le C r^\omega. \tag{4.2}$$

(ii) $\mathrm{lip}_\theta(\mathcal{X}) = C^{\theta\omega}(\mathcal{X})$ *with equivalence quasi-norms, where*

$$C^{\theta\omega}(\mathcal{X}) := \left\{f \in C(\mathcal{X}) : \|f\|_{C^{\theta\omega}(\mathcal{X})} < \infty\right\}$$

with, for any $f \in C(\mathcal{X})$,

$$\|f\|_{C^{\theta\omega}(\mathcal{X})} := \sup_{x,\, y \in \mathcal{X},\, x \ne y} \frac{|f(x) - f(y)|}{[d(x, y)]^{\theta\omega}} + \|f\|_{L^\infty(\mathcal{X})}.$$

Proof We first show (i) $\Rightarrow$ (ii). To this end, suppose that $f \in \mathrm{lip}_\theta(\mathcal{X})$. By Lemma 2.5 and (4.2), we find that, for any $x \in \mathcal{X}$,

$$|f(x)| \lesssim \|f\|_{\mathrm{lip}_\theta(\mathcal{X})},$$

and, for any $x, y \in \mathcal{X}$ with $x \neq y$,

$$|f(x) - f(y)| \lesssim \|f\|_{\mathrm{lip}_\theta(\mathcal{X})}[V(x, y)]^\theta \lesssim \|f\|_{\mathrm{lip}_\theta(\mathcal{X})}[d(x, y)]^{\theta\omega}.$$

This further implies that $f \in C^{\theta\omega}(\mathcal{X})$ and $\|f\|_{C^{\theta\omega}(\mathcal{X})} \lesssim \|f\|_{\mathrm{lip}_\theta(\mathcal{X})}$. On the contrary, assume that $f \in C^{\theta\omega}(\mathcal{X})$ and $B := (x_B, r_B)$ for some $x_B \in \mathcal{X}$ and $r_B \in (0, \infty)$. If $r_B \in (1, \infty)$, then, by Definition 2.4 and (4.2), we have

$$\mathcal{M}_\theta^B(f) = \frac{\|f\|_{L^\infty(B)}}{[\mu(B)]^\theta} \lesssim \|f\|_{L^\infty(\mathcal{X})} \leq \|f\|_{C^{\theta\omega}(\mathcal{X})}. \tag{4.3}$$

If $r_B \in (0, 1]$, then, for any $x, y \in B$ with $x \neq y$, $d(x, y) \leq 2A_0 r_B$. From this and (4.2), we deduce that, for any $x, y \in B$ with $x \neq y$,

$$|f(x) - f(y)| \leq \|f\|_{C^{\theta\omega}(\mathcal{X})}[d(x, y)]^{\theta\omega} \lesssim \|f\|_{C^{\theta\omega}(\mathcal{X})} r^{\theta\omega} \sim \|f\|_{C^{\theta\omega}(\mathcal{X})}[\mu(B)]^{\theta\omega}$$

and hence

$$\mathcal{M}_\theta^B(f) \lesssim \|f\|_{C^{\theta\omega}(\mathcal{X})},$$

which, combined with (4.3), further implies that $f \in \mathrm{lip}_\theta(\mathcal{X})$ and $\|f\|_{\mathrm{lip}_\theta(\mathcal{X})} \lesssim \|f\|_{C^{\theta\omega}(\mathcal{X})}$. This finishes the proof of (i) $\Rightarrow$ (ii).

Next, we show (ii) $\Rightarrow$ (i). By [41, Theorem 6.15] and [23, Theorem 7.4(i)], we conclude that, for any $k \in \mathbb{Z}_+$ and $\beta \in \mathcal{G}_k$,

$$\left\|\psi_\beta^{k+1}\right\|_{\mathrm{lip}_\theta(\mathcal{X})} \sim \left\|\psi_\beta^{k+1}\right\|_{\dot{C}^{\theta\omega}(\mathcal{X})} \sim \delta^{-k\theta\omega}\left[\mu(Q_\beta^{k+1})\right]^{-\frac{1}{2}}. \tag{4.4}$$

On the other hand, from Theorem 1.1, we infer that, for any $k \in \mathbb{Z}_+$ and $\beta \in \mathcal{G}_k$,

$$\left\|\psi_\beta^{k+1}\right\|_{\mathrm{lip}_\theta(\mathcal{X})} \sim \left[\mu(Q_\beta^{k+1})\right]^{-\frac{1}{2}-\theta}.$$

This, together with (4.4), further implies that, for any $k \in \mathbb{Z}_+$ and $\beta \in \mathcal{G}_k$,

$$\mu(Q_\beta^{k+1}) \sim \delta^{k\omega}. \tag{4.5}$$

Using this and [18, Corollary 3.4], we deduce that, for any $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \gtrsim r^\omega. \tag{4.6}$$

Moreover, applying an argument similar to that used in the proof of [42, Lemma 2.9] and (4.5), we have, for any $k \in \mathbb{Z}_+$ and $\alpha \in \mathcal{A}_k$,

$$\mu(Q_\alpha^k) \lesssim \delta^{k\omega}$$

and hence, for any $\alpha \in \mathcal{A}_0$,

$$\mu(B(x_\alpha^0, 1)) \sim \mu(Q_\alpha^k) \lesssim 1.$$

By this, Lemma 2.6(iii), and $\mathcal{X}^k \subset \mathcal{X}^0$ for any $k \in \mathbb{Z} \setminus \mathbb{Z}_+$, we obtain, for any $k \in \mathbb{Z} \setminus \mathbb{Z}_+$ and $\alpha \in \mathcal{A}_k$,

$$\mu(Q_\alpha^k) \le \mu(B(x_\alpha^k, C_\# \delta^k)) \lesssim \delta^k \mu(B(x_\alpha^k, 1)) \lesssim \delta^k,$$

which, combined with [42, Lemma 2.9], further implies that for any $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \lesssim r^\omega.$$

This, together with (4.6), finishes the proof of (ii) $\Rightarrow$ (i) and hence that of Corollary 4.3. □

Acknowledgements The author would like to thank Prof.s D. Yang and W. Yuan for their valuable suggestions on this article. The author is also very grateful to both referees for their very carefully reading and several useful and enlightening comments which indeed improve the quality and presentation of this article.

References

1. Astala, K., Iwaniec, T., Koskela, P., Martin, G.: Mappings of BMO-bounded distortion. Math. Ann. **317**, 703–726 (2000)
2. Auscher, P., Hytönen, T.: Orthonormal bases of regular wavelets in spaces of homogeneous type. Appl. Comput. Harmon. Anal. **34**, 266–296 (2013)
3. Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ration. Mech. Anal. **63**, 337–403 (1976/1977)
4. Bonami, A., Cao, J., Ky, L.D., Liu, L., Yang, D., Yuan, W.: Multiplication between Hardy spaces and their dual spaces. J. Math. Pures Appl. (9) **131**, 130–170 (2019)
5. Bonami, A., Feuto, J.: Products of functions in Hardy and Lipschitz or BMO spaces. In: Recent Developments in Real and Harmonic Analysis, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 57–71 (2010)
6. Bonami, A., Feuto, J., Grellier, S.: Endpoint for the DIV-CURL lemma in Hardy spaces. Publ. Mat. **54**, 341–358 (2010)
7. Bonami, A., Grellier, S., Ky, L.D.: Paraproducts and products of functions in BMO($\mathbb{R}^n$) and $H^1(\mathbb{R}^n)$ through wavelets. J. Math. Pures Appl. (9) **97**, 230–241 (2012)
8. Bonami, A., Iwaniec, T., Jones, P., Zinsmeister, M.: On the product of functions in BMO and H^1. Ann. Inst. Fourier (Grenoble) **57**, 1405–1439 (2007)
9. Bonami, A., Jiao, Y., Xie, G., Yang, D., Zhou, D.: Products and commutators of martingales in H^1 and BMO. J. Math. Pures Appl. **180**, 188–229 (2023)
10. Bui, T.A., Duong, X.-T., Ly, F.K.: Maximal function characterizations for new local Hardy-type spaces on spaces of homogeneous type. Trans. Amer. Math. Soc. **370**, 7229–7292 (2018)
11. Bui, T.A., Duong, X.-T., Ly, F.K.: Maximal function characterizations for Hardy spaces on spaces of homogeneous type with finite measure and applications. J. Funct. Anal. **278**, 108423 (2020)
12. Cao, J., Ky, L.D., Yang, D.: Bilinear decompositions of products of local Hardy and Lipschitz or BMO spaces through wavelets. Comm. Contemp. Math. **20**, 1750025 (2018)

13. Coifman, R.R., Weiss, G.: Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes, (French) Étude de Certaines Intégrales Singulières, Lecture Notes in Mathematics, vol. 242. Springer-Verlag, Berlin-New York (1971)
14. Coifman, R.R., Weiss, G.: Extensions of Hardy spaces and their use in analysis. Bull. Amer. Math. Soc. **83**, 569–645 (1977)
15. Feuto, J.: Products of functions in BMO and H^1 spaces on spaces of homogeneous type. J. Math. Anal. Appl. **359**, 610–620 (2009)
16. Fu, X., Ma, T., Yang, D.: Real-variable characterizations of Musielak-Orlicz Hardy spaces on spaces of homogeneous type. Ann. Acad. Sci. Fenn. Math. **45**, 343–410 (2020)
17. Fu, X., Yang, D., Liang, Y.: Products of functions in BMO($\mathcal{X}$) and $H^1_{\mathrm{at}}(\mathcal{X})$ via wavelets over spaces of homogeneous type. J. Fourier Anal. Appl. **23**, 919–990 (2017)
18. Han, Y., Han, Y., He, Z., Li, J., Pereyra, C.: Geometric characterizations of embedding theorems: For Sobolev, Besov, and Triebel-Lizorkin spaces on spaces of homogeneous type – via orthonormal wavelets. J. Geom. Anal. **31**, 8947–8978 (2021)
19. Han, Y., Müller, D., Yang, D.: Littlewood-Paley characterizations for Hardy spaces on spaces of homogeneous type. Math. Nachr. **279**, 1505–1537 (2006)
20. Han, Y., Müller, D., Yang, D.: A theory of Besov and Triebel–Lizorkin spaces on metric measure spaces modeled on Carnot–Carathéodory spaces. Abstr. Appl. Anal., Art. ID 893409, 1–250 (2008)
21. He, Z., Han, Y., Li, J., Liu, L., Yang, D., Yuan, W.: A complete real-variable theory of Hardy spaces on spaces of homogeneous type. J. Fourier Anal. Appl. **25**, 2197–2267 (2019)
22. He, Z., Liu, L., Yang, D., Yuan, W.: New Calderón reproducing formulae with exponential decay on spaces of homogeneous type. Sci China Math **62**, 283–350 (2019)
23. He, Z., Wang, F., Yang, D., Yuan, W.: Wavelet characterizations of Besov and Triebel-Lizorkin spaces on spaces of homogeneous type and its applications. Appl. Comput. Harmon. Anal. **54**, 176–226 (2021)
24. He, Z., Yan, X., Yang, D.: Calderón reproducing formulae on product spaces of homogeneous type and their applications. Math. Nachr. **298**, 1839–1921 (2025)
25. He, Z., Yang, D., Yuan, W.: Real-variable characterizations of local Hardy spaces on spaces of homogeneous type. Math. Nachr. **249**, 900–955 (2021)
26. Hytönen, T., Kairema, A.: Systems of dyadic cubes in a doubling metric space. Colloq. Math. **126**, 1–33 (2012)
27. Iwaniec, T., Sbordone, C.: Quasiharmonic fields. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **18**, 519–572 (2001)
28. Ky, L.D.: Bilinear decompositions and commutators of singular integral operators. Trans. Amer. Math. Soc. **365**, 2931–2958 (2013)
29. Ky, L.D.: Bilinear decompositions for the product space $H^1_L \times \mathrm{BMO_L}$. Math. Nachr. **287**, 1288–1297 (2014)
30. Lemarié-Rieusset, P.G.: Recent Developments in the Navier-Stokes Problem, Chapman & Hall/CRC Research Notes in Mathematics, vol. 431. Chapman & Hall/CRC, Boca Raton, FL (2002)
31. Li, H., Zheng, T.: Calderón-Zygmund operators on Lipschitz spaces over RD spaces. Quaest. Math. **44**(4), 473–494 (2021)
32. Li, J.: Atomic decomposition of weighted Triebel-Lizorkin spaces on spaces of homogeneous type. J. Aust. Math. Soc. **89**, 255–275 (2010)
33. Li, P., Peng, L.: The decomposition of product space $H^1_L \times \mathrm{BMO}_L$. J. Math. Anal. Appl. **349**, 484–492 (2009)
34. Liu, L., Chang, D.-C., Fu, X., Yang, D.: Endpoint boundedness of commutators on spaces of homogeneous type. Appl. Anal. **96**, 2408–2433 (2017)
35. Liu, L., Chang, D.-C., Fu, X., Yang, D.: Endpoint estimates of linear commutators on Hardy spaces over spaces of homogeneous type. Math. Methods Appl. Sci. **41**, 5951–5984 (2018)
36. Liu, L., Yang, D., Yuan, W.: Bilinear decompositions for products of Hardy and Lipschitz spaces on spaces of homogeneous type. Dissertationes Math. **533**, 1–93 (2018)
37. Liu, J., Yang, D., Zhang, M.: Sharp bilinear decomposition for products of both anisotropic Hardy spaces and their dual spaces with its applications to endpoint boundedness of commutators. Sci China Math **67**, 2091–2152 (2024)
38. Macías, R.A., Segovia, C.: Lipschitz functions on spaces of homogeneous type. Adv. Math. **33**, 257–270 (1979)
39. Müller, S.: Weak continuity of determinants and nonlinear elasticity. C. R. Acad. Sci., Sér. 1 Math. **307**, 501–506 (1988)

40. Nakai, E., Yabuta, K.: Pointwise multipliers for functions of weighted bounded mean oscillation on spaces of homogeneous type. Math. Japon. **46**, 15–28 (1997)
41. Wang, F., Han, Y., He, Z., Yang, D.: Besov and Triebel-Lizorkin spaces on spaces of homogeneous type with applications to boundedness of Calderón-Zygmund operators. Dissertationes Math. **565**, 1–113 (2021)
42. Wang, F., Yang, D., Yuan, W.: Geometric characterization of Ahlfors regular spaces in terms of dyadic cubes related to wavelets with its applications to equivalences of Lipschitz spaces. Expo. Math. **42**, 125574 (2024)
43. Yan, X., He, Z., Yang, D., Yuan, W.: Hardy spaces associated with ball quasi-Banach function spaces on spaces of homogeneous type: Characterizations of maximal functions, decompositions, and dual spaces. Math. Nachr. **296**, 3056–3116 (2023)
44. Yang, D., Yuan, W., Zhang, Y.: Bilinear decomposition and divergence-curl estimates on products related to local Hardy spaces and their dual spaces. J. Funct. Anal. **280**, 108796 (2021)
45. Zhou, X., He, Z., Yang, D.: Real-variable characterizations of Hardy-Lorentz spaces on spaces of homogeneous type with applications to real interpolation and boundedness of Calderón-Zygmund operators. Anal. Geom. Metr. Spaces **8**, 182–260 (2020)
46. Zheng, T., Li, H., Tao, X.: The boundedness of Calderón-Zygmund operators on Lipschitz spaces over spaces of homogeneous type. Bull. Braz. Math. Soc. New Series **51**, 653–669 (2020)