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Hopf bifurcation in a delayed reaction-diffusion-advection Nicholson's blowfly model

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A B S T R A C T

In this paper, we investigate a delayed reaction-diffusion-advection Nicholson's blowfly model under the influence of predation. By analyzing the principal eigenvalue of the elliptic operator, we establish the stability of the positive steady state solution and the existence of a Hopf bifurcation. Furthermore, employing normal form theory and center manifold theory, we examine the stability and direction of the periodic solutions arising from the Hopf bifurcation. In addition, numerical simulations are conducted to validate our theoretical results. The numerical findings indicate that population density decreases with increasing advection rate, saturation predation rate, and mortality rate. Moreover, Hopf bifurcation induced by time delay is more likely to occur when these parameters are relatively low. Conversely, population density increases with the half-saturation constant, and Hopf bifurcation is more likely to arise when the half-saturation constant is small.

1. Introduction

Mathematical biology is an interdisciplinary field combining biology and mathematics. Using mathematical tools, it explores and reveals the underlying principles of biological systems, holding significant theoretical and practical value. In recent years, mathematical biology has become one of the most active research directions in applied mathematics. The research in this field typically involves two aspects: on the one hand, mathematical models are established and analyzed to understand and predict the intrinsic mechanisms of biological processes; on the other hand, population models not only help discover new mathematical problems but also promote the development of related fields. For example, in epidemic modeling, mathematical analysis can be used to optimize control strategies; in resource competition and population dynamics research, mathematical models can reveal novel dynamic behaviors.

Early population models were primarily based on ordinary differential equations to describe changes in individual numbers over time. In studying Nicholson's blowfly model, Nicholson [1] suggested that the primary cause of oscillations was the time delay between density-dependent responses and their effects. In 1976, May [2] used a delayed logistic model to simulate Nicholson's experiments and inferred that the development time from egg to adult was 9 days. However, this differed significantly from Nicholson's observed value of approximately 15 days. To address this discrepancy, in 1990, Gurney et al. [3] proposed the following delay differential equation to describe the dynamics of Nicholson's blowfly population:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\delta u(t) + pu(t-\tau)e^{-au(t-\tau)},$$

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where p is the maximum per capita egg-laying rate per day, $\frac{1}{a}$ represents the population size at which the Nicholson's blowfly reproduces at its maximum rate, δ is the average adult mortality rate per day, and τ denotes the maturation time.

Although this model effectively describes the population fluctuations of Nicholson's blowflies, its fundamental assumption is that the population is spatially homogeneous, meaning that interactions among individuals are global and do not account for spatial diffusion or migration. However, in real ecological systems, biological individuals are often not uniformly distributed, and spatial structure may lead to encountering frequencies differing from the homogeneous assumption.

To address this issue, in 1998, So and Yang [4], building upon the model of Gurney, introduced a diffusion term and formulated the following delayed reaction-diffusion model for Nicholson's blowfly population:

$$\frac{\partial u}{\partial t} = d\Delta u(x,t) - \delta u(x,t) + pu(x,t-\tau)e^{-u(x,t-\tau)}.$$

This model is defined on a finite domain with homogeneous Neumann boundary conditions. It analyzed the effect of the non-monotonicity of the delay term on system stability under Dirichlet boundary conditions and established a novel mathematical approach to study the global attractivity of equilibrium states.

In 2000, So et al. [5] conducted numerical simulations of this model, exploring the Hopf bifurcation phenomenon. In 2016, Guo and Ma [6] applied the Lyapunov-Schmidt reduction method to investigate the existence of spatially nonhomogeneous steady-state solutions. By analyzing the distribution of eigenvalues, they derived conditions for the existence of Hopf bifurcation at these steady states. Using normal form theory and center manifold reduction, they further examined the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions.

Although the above model incorporates diffusion effects, it still assumes that individual movement is directionless, meaning that diffusion is random. However, in real ecological systems, individual movement is often influenced by environmental gradients, such as water currents, wind direction, or heterogeneous resource distribution. Therefore, in recent years, researchers have further introduced advection terms into population diffusion models to more accurately describe migration patterns under environmental gradients or external disturbances.

Reaction-diffusion-advection models effectively describe population movement patterns and biological processes, making them a powerful tool for studying spatial population dynamics. A significant problem in spatial ecology is understanding the impact of spatially heterogeneous environments on species invasion. A heterogeneous environment refers to the spatially varying distribution of environmental conditions. For example, phytoplankton in oceans or lakes require light, whose intensity varies with depth in the vertical direction. In such heterogeneous environments, species movement involves not only random diffusion but also advection.

To address the above, in 2022, Zhang and Wei [7] studied the following delayed reaction-diffusion-advection population model:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u - \alpha \nabla \cdot (u\nabla m) + u(x,t-\tau) f(u(x,t-\tau)) - \delta u, & x \in \Omega, t > 0, \\ d\partial_n u - \alpha u \partial_n m = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where τ represents the maturation time, δ denotes the mortality rate, and the advection term $\alpha\nabla \cdot (u\nabla m)$ describes population movement biased along resource gradients or advection due to water flow with velocity $\alpha m(x)$. The parameters d, α, τ, δ are positive constants, and $m(x) \in C^2(\bar{\Omega})$. The study results indicate that delay-induced Hopf bifurcation is more likely to occur at lower advection rates. The impact of advection on the spatial distribution of general competitive populations has been investigated in previous studies [8–11].

In this paper, we further consider additional factors that inhibit population growth, such as low mating rates, artificial harvesting, and predation by potential predators. Compared to previous models, our study incorporates the effects of time delay, spatial diffusion, advection, and predation, providing a more realistic representation of population dynamics in ecosystems.

By analyzing the steady state solutions, stability, and delay-induced bifurcations of the model, we explore the effects of predation, time delay, and environmental gradients on population dynamics, offering theoretical support for ecosystem management and conservation. In this paper, we study the following reaction-diffusion-advection model of the Nicholson's blowfly population under predation effects:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) - \nabla \cdot \left[\alpha_1 u \nabla m(x)\right] - \frac{bu^2(x,t)}{c^2+u^2(x,t)} - \delta u(x,t) + pu(x,t-\tau)e^{-au(x,t-\tau)}, \quad x \in \Omega, t > 0, \\ d\partial_n u - \alpha u \partial_n m = 0, \quad x \in \partial\Omega, t > 0. \end{cases} \tag{1}$$

where $u(x,t)$ represents the population density at location x and time t, d_1 is the random diffusion rate, b is the saturation predation rate, representing the maximum predation capacity at high prey density, and c is the half-saturation constant, representing the prey population density at which the predation rate reaches half of its maximum value. The function $m(x)$ describes the resource distribution, and α_1 is the advection rate describing population movement along the gradient of $m(x)$.

The domain Ω is a bounded region with a smooth boundary $\partial\Omega$, and n is the outward unit normal vector on $\partial\Omega$. The no-flux boundary condition implies that no individuals pass through the boundary.

Let $\hat{u}(x,\hat{t}) = e^{(-\alpha_1/d_1)m(x)}u(x,t)$, $\hat{t} = d_1 t$, and $\hat{\tau} = d_1\tau$. Defining $\lambda = \frac{1}{d_1}$ and $\alpha = \frac{\alpha_1}{d_1}$, we omit the hat notation for simplicity. Consequently, system Eq. (1) is transformed into the following form:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = e^{-\alpha m(x)}\nabla \cdot \left[e^{\alpha m(x)}\nabla u(x,t)\right] - \frac{\lambda b e^{\alpha m(x)} u^2(x,t)}{c^2+e^{2\alpha m(x)}u^2(x,t)} - \lambda\delta u(x,t) + \lambda p u(x,t-\tau)e^{-ae^{\alpha m(x)}u(x,t-\tau)}, x \in \Omega, t > 0, \\ \partial_n u = 0, \quad x \in \partial\Omega, t > 0. \end{cases} \tag{2}$$

In this paper, we assume that $m(x)$ satisfies the following condition:

$(\mathbf{H}_1)$ $m(x) \in C^2(\bar{\Omega})$ and $m(x) \geq 0, m(x) \not\equiv 0$.

The rest of this paper is organized as follows: Section 2 studies the existence of positive steady state solutions. Section 3 investigates the corresponding eigenvalue problem and the existence of Hopf bifurcations. Section 4 analyzes the direction of Hopf bifurcations and the stability of bifurcating periodic solutions using normal form theory and center manifold reduction. Section 5 provides numerical simulations and discusses the impact of parameters on Hopf bifurcations.

According to the literature [12,13], we define the following spaces as

$$X = \left\{u \in H^2(\Omega) : \partial_n u = 0,\ x \in \partial\Omega\right\}, Y = L^2(\Omega), C = C([-\tau, 0], Y),$$

and $C = C([-1, 0], Y)$. The complexification of the linear space Z is defined as

$$Z_{\mathbb{C}} := Z + \mathrm{i}Z = \{a + \mathrm{i}b : a, b \in Z\}.$$

For a linear operator T, we define its domain as $\mathcal{D}(T)$, its kernel as $\mathcal{N}(T)$, and its range as $\mathcal{R}(T)$. Furthermore, for the Hilbert space $Y_{\mathbb{C}}$, the standard inner product is

$$\langle u, v\rangle = \int_\Omega \bar{u}(x)v(x)\,dx.$$

2. Existence of positive steady state solution

This section focuses on the existence of positive steady state solutions for model (2), which satisfies the following elliptic equation:

$$\begin{cases} \nabla\cdot\left[e^{\alpha m(x)}\nabla u\right] + \lambda u e^{\alpha m(x)}\left(-\frac{be^{\alpha m(x)}u}{c^2+e^{2\alpha m(x)}u^2} - \delta + pe^{-ae^{\alpha m(x)}u}\right) = 0, & x \in \Omega, \\ \partial_n u = 0, \quad x \in \partial\Omega. \end{cases} \tag{3}$$

Define the operator

$$P_0 := \nabla\cdot\left[e^{\alpha m(x)}\nabla\right],$$

and assume that it satisfies the homogeneous Neumann boundary condition. According to Belgacem and Cosner [14], Lou and Zhou [15] , the principal eigenvalue of the operator $-P_0$ under the homogeneous Neumann boundary condition is $\lambda_1 = 0$, and the corresponding eigenfunction ϕ can be chosen as a constant. For simplicity, we take $\phi = 1$.

It is easy to verify that P_0 is a self-adjoint Fredholm operator from space X to Y. Therefore, the spaces X and Y can be decomposed as follows:

$$X = \mathcal{N}\left(P_0\right) \oplus X_1, Y = \mathcal{N}\left(P_0\right) \oplus Y_1,$$

where

$$\mathcal{N}\left(P_0\right) = \mathrm{span}\{\phi\} = \mathrm{span}\{1\}, X_1 = \left\{y \in X : \int_\Omega y(x)\,dx = 0\right\},$$
$$Y_1 = \mathcal{R}\left(P_0\right) = \left\{y \in Y : \int_\Omega y(x)\,dx = 0\right\}.$$

By an argument similar to Theorem 2.1 in Cantrell et al. [16], we obtain the following existence result for positive steady-state solutions.

Theorem 2.1. *Assume that condition $(\mathbf{H}_1)$ holds and the following conditions are satisfied: (H_2) $p > \delta$, $0 < \beta e^{\alpha m(x)} \leq c$. Then, there exists $\lambda^* > 0$ and a continuously differentiable mapping $\lambda \mapsto u_\lambda$ from the interval $[0, \lambda^*]$ into X, such that for $\lambda \in (0, \lambda^*]$, u_λ is a positive steady state solution of Eq. (3). Moreover,*

$$\lim_{\lambda\to 0} u_\lambda = \beta_0,$$

where β_0 is the unique solution of the following equation:

$$\int_\Omega pe^{\alpha m(x)}e^{-ae^{\alpha m(x)}\beta}\,dx - \int_\Omega \frac{be^{2\alpha m(x)}\beta}{c^2 + e^{2\alpha m(x)}\beta^2}\,dx - \delta\int_\Omega e^{\alpha m(x)}\,dx = 0. \tag{4}$$

Proof. The positive steady-state solutions of model (2) satisfy Eq. (3). Now, we define the mapping $H : \mathbb{R}\times X_1 \times \mathbb{R} \mapsto Y$:

$$H(\beta, \eta, \lambda) = P_0\eta + \lambda p(\beta+\eta)e^{\alpha m(x)}e^{-ae^{\alpha m(x)(\beta+\eta)}} - \lambda\frac{be^{2\alpha m(x)}(\beta+\eta)^2}{c^2 + e^{2\alpha m(x)}(\beta+\eta)^2} - \lambda\delta e^{\alpha m(x)}(\beta+\eta).$$

Let $u = \beta + \eta$, where $\beta \in \mathbb{R}$, $\eta \in X_1$. Substituting it into Eq. (3), we see that Eq. (3) has a solution (u, λ) with $u \in X$ and $\lambda > 0$ if and only if there exist some $\beta \in \mathbb{R}$ and $\eta \in X_1$, along with $\lambda > 0$, such that the equation $H(\beta, \eta, \lambda) = 0$ is solvable.

Clearly, for any $\beta \in \mathbb{R}$, we have $H(\beta, 0, 0) = 0$. We compute the Fréchet derivative of $H(\beta, \eta, \lambda)$ with respect to (η, λ):

$$D_{(\eta,\lambda)} H(\beta, \eta, \lambda)[\vartheta, \sigma] = P_0 \vartheta + p(\beta + \eta) e^{\alpha m(x)} e^{-a e^{\alpha m(x)}(\beta+\eta)} \sigma - \frac{b e^{2\alpha m(x)} (\beta + \eta)^2}{c^2 + e^{2\alpha m(x)} (\beta + \eta)^2} \sigma - \delta e^{\alpha m(x)} (\beta + \eta) \sigma$$
$$- \lambda \frac{2bc^2 e^{2\alpha m(x)} (\beta + \eta)}{\left(c^2 + e^{2\alpha m(x)} (\beta + \eta)^2\right)^2} \vartheta - \lambda \delta e^{\alpha m(x)} \vartheta + \lambda p e^{\alpha m(x)} (\beta + \eta) e^{-a e^{\alpha m(x)}(\beta+\eta)} \vartheta - \lambda a p (\beta + \eta)^2 e^{2\alpha m(x)} e^{-a e^{\alpha m(x)}(\beta+\eta)} \vartheta.$$

Then

$$D_{(\eta,\lambda)} H(\beta, 0, 0)[\vartheta, \sigma] = P_0 \vartheta + \beta e^{\alpha m(x)} \sigma \left(- \frac{b e^{\alpha m(x)} \beta}{c^2 + e^{2\alpha m(x)} \beta^2} - \delta + p e^{-a e^{\alpha m(x)} \beta} \right).$$

Noting that

$$-\beta_0 e^{\alpha m(x)} \sigma \left(- \frac{b e^{\alpha m(x)} \beta_0}{c^2 + e^{2\alpha m(x)} \beta_0^2} - \delta + p e^{-a e^{\alpha m(x)} \beta_0} \right) \in Y_1 = \mathcal{R}(T),$$

there exists a unique $\vartheta_* \in X_1$ satisfying

$$P_0 \vartheta * = -\beta_0 e^{\alpha m(x)} \left(- \frac{b e^{\alpha m(x)} \beta_0}{c^2 + e^{2\alpha m(x)} \beta_0^2} - \delta + p e^{-a e^{\alpha m(x)} \beta_0} \right),$$

thus

$$\mathcal{N}\left(D_{(\eta,\lambda)} H\left(\beta_0, 0, 0\right)\right) = \left\{ \left(s \vartheta_*, s\right) : s \in \mathbb{R} \right\}.$$

Further computation gives

$$D_\beta D_{(\eta,\lambda)} H\left(\beta_0, 0, 0\right)\left[\vartheta_*, 1\right] = e^{\alpha m(x)} \left(- \frac{b e^{\alpha m(x)} \beta}{c^2 + e^{2\alpha m(x)} \beta^2} - \delta + p e^{-a e^{\alpha m(x)} \beta} \right) - \frac{b e^{2\alpha m(x)} (c^2 - \beta^2 e^{2\alpha m(x)})}{(c^2 + \beta^2 e^{2\alpha m(x)})^2} - a p e^{2\alpha m(x)} e^{-a e^{\alpha m(x)} \beta},$$

where $D_\beta D_{(\eta,\lambda)} H(\beta_0, 0, 0)$ is the Fréchet derivative of $D_{(\eta,\lambda)} H(\beta, \eta, \lambda)$ with respect to β at $(\beta_0, 0, 0)$. We claim that

$$D_\beta D_{(\eta,\lambda)} H\left(\beta_0, 0, 0\right)\left[\eta_*, 1\right] \notin \mathcal{R}\left(D_{(\eta,\lambda)} H\left(\beta_0, 0, 0\right)\right).$$

Assume that the statement is false. Then, there exists $\left(\widetilde{\vartheta}, \widetilde{\sigma}\right)$ such that

$$D_{(\eta,\lambda)} H(\beta_0, 0, 0)[\widetilde{\vartheta}, \widetilde{\sigma}] = P_0 \widetilde{\vartheta} - \frac{b e^{2\alpha m(x)} \beta^2}{c^2 + e^{2\alpha m(x)} \beta^2} \widetilde{\sigma} - \delta \beta e^{\alpha m(x)} \widetilde{\sigma} + p \beta e^{\alpha m(x)} e^{-a e^{\alpha m(x)} \beta} \widetilde{\sigma}$$
$$= e^{\alpha m(x)} \left(- \frac{b e^{\alpha m(x)} \beta}{c^2 + e^{2\alpha m(x)} \beta^2} - \delta + p e^{-a e^{\alpha m(x)} \beta} \right) + e^{\alpha m(x)} \left(- \frac{b e^{\alpha m(x)} (c^2 - \beta^2 e^{2\alpha m(x)})}{(c^2 + \beta^2 e^{2\alpha m(x)})^2} - a p e^{\alpha m(x)} e^{-a e^{\alpha m(x)} \beta} \right),$$

which implies that

$$A(x) = e^{\alpha m(x)} \left(- \frac{b e^{\alpha m(x)} \beta}{c^2 + e^{2\alpha m(x)} \beta^2} - \delta + p e^{-a e^{\alpha m(x)} \beta} \right) + e^{\alpha m(x)} \left(- \frac{b e^{\alpha m(x)} (c^2 - \beta^2 e^{2\alpha m(x)})}{(c^2 + \beta^2 e^{2\alpha m(x)})^2} - a p e^{\alpha m(x)} e^{-a e^{\alpha m(x)} \beta} \right) \in \mathcal{R}(T).$$

By direct computation, we obtain

$$\int_\Omega A(x) dx = -\beta_0 \int_\Omega \frac{b e^{2\alpha m(x)} (c^2 - e^{2\alpha m(x)} \beta_0^2)}{(c^2 + e^{2\alpha m(x)} \beta_0^2)^2} dx - a p \beta_0 \int_\Omega e^{2\alpha m(x)} e^{-a e^{\alpha m(x)} c_0} dx$$
$$\neq 0.$$

This leads to a contradiction, and hence the above statement holds.

Using the Crandall-Rabinowitz [17] bifurcation theorem, the solutions of $H(\beta, \eta, \lambda) = 0$ near $(\beta_0, 0, 0)$ form the curve $\{(\beta, 0, 0) : \beta \in \mathbb{R}\}$ and

$$\{(\beta(s), \eta(s), \lambda(s)) : s \in (-\epsilon, \epsilon)\},$$

here, $\beta(s)$, $\eta(s)$, and $\lambda(s)$ are continuously differentiable, satisfying $\beta(0) = \beta_0$, $\eta(0) = 0$, $\lambda(0) = 0$, $\eta'(0) = \eta_*$, and $\lambda'(0) = 1$. Thus, $\lambda(s)$ has an inverse $s(\lambda)$ near zero. Since $\beta_0 > 0$, there exists $\lambda^* > 0$ such that Eq. (3) has a positive solution $u_\lambda = \beta(s(\lambda)) + \eta(s(\lambda))$ for $\lambda \in (0, \lambda^*)$. Moreover,

$$u_0 = \beta(s(0)) + \eta(s(0)) = \beta(0) + \eta(0) = \beta_0.$$

This completes the proof. □

3. Hopf bifurcation analysis

This section considers the eigenvalue problem associated with the positive steady-state solution of (2). Linearizing system (2) at u_λ, we obtain

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = e^{-\alpha m(x)}\nabla\cdot\left[e^{\alpha m(x)}\nabla u(x,t)\right] - \dfrac{2\lambda bc^2 e^{\alpha m(x)}u_\lambda u(x,t)}{\left(c^2+e^{2\alpha m(x)}u_\lambda^2\right)^2} - \lambda\delta u(x,t) + \lambda B_1(\lambda,x)u(x,t-\tau), & x\in\Omega, t>0, \\ \partial_n u = 0, \quad x\in\partial\Omega, t>0, \end{cases} \tag{5}$$

where

$$B_1(\lambda,x) = p\left(1-au_\lambda e^{\alpha m(x)}\right)e^{-ae^{\alpha m(x)}u_\lambda}, B_1(x) = p\left(1-a\beta_0 e^{\alpha m(x)}\right)e^{-ae^{\alpha m(x)}\beta_0}.$$

According to Wu [18], the solution semigroup of Eq. (5) has an infinitesimal generator $\mathcal{A}_{\tau,\lambda}$ satisfying

$$\mathcal{A}_{\tau,\lambda}\psi = \dot{\psi},$$

for $\psi\in\mathcal{D}(\mathcal{A}_{\tau,\lambda})$, where

$$\mathcal{D}(\mathcal{A}_{\tau,\lambda}) = \{\psi\in\mathcal{C} \mid \psi(0)\in X, \dot{\psi}(0) = \mathcal{L}_{\tau,\lambda}\psi\},$$

and

$$\mathcal{L}_{\tau,\lambda}\psi = \left(e^{-\alpha m(x)}P_0 - \frac{2\lambda bc^2 e^{\alpha m(x)}u_\lambda}{\left(c^2+e^{2\alpha m(x)}u_\lambda^2\right)^2} - \lambda\delta\right)\psi(0) + \lambda B_1(\lambda,x)\psi(-\tau),$$

where $C^1_{\mathbb{C}} = C^1\left([-\tau,0], Y_{\mathbb{C}}\right)$. The complex number μ is an eigenvalue of $\mathcal{A}_{\tau,\lambda}$ if and only if

$$\mu\in\sigma(\mathcal{A}_{\tau,\lambda}) = \{\mu\in\mathbb{C} \mid \Delta(\lambda,\mu,\tau)\psi = 0, \psi\in X_{\mathbb{C}}\backslash\{0\}\},$$

where

$$\Delta(\lambda,\mu,\tau)\psi := \left(e^{-\alpha m(x)}P_0 - \frac{2\lambda bc^2 e^{\alpha m(x)}u_\lambda}{\left(c^2+e^{2\alpha m(x)}u_\lambda^2\right)^2} - \lambda\delta\right)\psi + \lambda B_1(\lambda,x)e^{-\mu\tau}\psi - \mu\psi.$$

Lemma 3.1. *Suppose that $\left(\mu_\lambda,\tau_\lambda,\psi_\lambda\right)$ is a solution of $\Delta(\lambda,\mu,\tau)\psi = 0$, where $\mathrm{Re}\,\mu_\lambda\ge 0$, $\tau_\lambda\ge 0$, and $\psi_\lambda(\ne 0)\in X_{\mathbb{C}}$. Then, $\left|\frac{\mu_\lambda}{\lambda}\right|$ is bounded for $\lambda\in(0,\lambda^*]$.*

Proof. Multiplying both sides of $\Delta\left(\lambda,\mu_\lambda,\tau_\lambda\right)\psi_\lambda = 0$ by $e^{\alpha m(x)}\bar{\psi}_\lambda$ and integrating over Ω, we obtain

$$\langle\psi_\lambda, P_0\psi_\lambda\rangle + \lambda e^{-\mu_\lambda\tau_\lambda}\int_\Omega e^{\alpha m(x)}B_1(\lambda,x)\left|\psi_\lambda\right|^2 dx - \mu_\lambda\int_\Omega e^{\alpha m(x)}\left|\psi_\lambda\right|^2 dx - 2\lambda bc^2\int_\Omega \frac{e^{2\alpha m(x)}u_\lambda}{\left(c^2+e^{2\alpha m(x)}u_\lambda^2\right)^2}dx - \lambda\delta\int_\Omega e^{\alpha m(x)}\left|\psi_\lambda\right|^2 dx = 0.$$

Noting that

$$\langle\psi_\lambda, P_0\psi_\lambda\rangle = -\int_\Omega e^{\alpha m(x)}\left|\nabla\psi_\lambda\right|^2 dx \le 0,$$

and since $\mathrm{Re}\,\mu_\lambda\ge 0$, $\tau_\lambda\ge 0$, we have

$$\begin{aligned}\mathrm{Re}\left(\frac{\mu_\lambda}{\lambda}\right) &\le \frac{\mathrm{Re}\left(e^{-\mu_\lambda\tau_\lambda}\int_\Omega e^{\alpha m(x)}B_1(\lambda,x)\left|\psi_\lambda\right|^2 dx\right)}{\int_\Omega e^{\alpha m(x)}\left|\psi_\lambda\right|^2 dx} \\ &\le \left\|B_1(\lambda,x)\right\|_\infty,\end{aligned}$$

and

$$\begin{aligned}\left|\mathrm{Im}\left(\frac{\mu_\lambda}{\lambda}\right)\right| &\le \frac{\mathrm{Im}\left(e^{-\mu_\lambda\tau_\lambda}\int_\Omega e^{\alpha m(x)}B_1(\lambda,x)\left|\psi_\lambda\right|^2 dx\right)}{\int_\Omega e^{\alpha m(x)}\left|\psi_\lambda\right|^2 dx} \\ &\le \left\|B_1(\lambda,x)\right\|_\infty.\end{aligned}$$

Since the mapping $\lambda\mapsto\|u_\lambda\|_\infty$ is continuous, we conclude that $\left|\frac{\mu_\lambda}{\lambda}\right|$ is bounded. This completes the proof. □

Lemma 3.2. *Suppose $z\in\left(X_1\right)_{\mathbb{C}}$. Then*

$$\left|\langle P_0 z, z\rangle\right| \ge \lambda_2\|z\|^2_{Y_{\mathbb{C}}},$$

where λ_2 is the second eigenvalue of the operator $-P_0$ under homogeneous Neumann boundary conditions.

Proof. It is well known that the operator $-P_0$ on the domain Ω with zero Neumann boundary conditions has a sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ satisfying

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots, \lim_{n\to\infty} \lambda_n = \infty,$$

and the corresponding eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ form an orthogonal basis of Y_c, Moreover, $\phi_1 = 1$. In particular, for each $v \in X_c$ satisfying $\langle v, 1\rangle = 0$, there exists a sequence of real numbers $\{c_n\}_{n=2}^{\infty}$ such that

$$v = \sum_{n=2}^{\infty} c_n \phi_n.$$

Thus

$$P_0 v = \sum_{n=2}^{\infty} c_n P_0 \phi_n = \sum_{n=2}^{\infty} c_n \lambda_n \phi_n.$$

From the above equation, we obtain

$$|\langle P_0 v, v\rangle| = \sum_{n=2}^{\infty} c_n^2 \lambda_n \|\phi_n\|_{L^2}^2 \ge \lambda_2 \sum_{n=2}^{\infty} c_n^2 \|\phi_n\|_{Y_{\mathbb{C}}}^2 = \lambda_2 \|v\|_{Y_{\mathbb{C}}}^2.$$

This completes the proof. □

Theorem 3.3. *If $L_0 < 0$, then there exists λ^* such that*

$$\sigma\big(A_{\tau,\lambda}\big) \subset \{x + \mathrm{i}y : x, y \in \mathbb{R}, x < 0\}$$

for $\lambda \in (0, \lambda^*]$ and $\tau \ge 0$, where

$$L_0 = \int_\Omega \left[pa\beta_0 e^{2\alpha m(x)} e^{-ae^{\alpha m(x)}\beta_0} - 2\left(\frac{2bc^2 e^{2\alpha m(x)}\beta_0}{c^2 + e^{2\alpha m(x)}\beta_0^2} + \delta e^{\alpha m(x)} \right) \right] dx.$$

Proof. If the contrary, then there exists a positive sequence $\{(\lambda_n, \mu_n, \tau_n, \psi_n)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \lambda_n = 0$, and for $n \ge 1$, $\lambda_n > 0$, satisfying

$$\Delta(\lambda_n, \mu_n, \tau_n)\psi_n = 0,$$

with $\mathrm{Re}\,(\mu_n) \ge 0$, $\tau_n \ge 0$, and $\psi_n (\neq 0) \in X_{\mathbb{C}}$. Ignoring a scalar factor, suppose $\mu_n = \lambda_n h_n$ and express ψ_n as

$$\begin{aligned} &\psi_n = r_n \beta_0 + \lambda_n z_n,\ z_n \in (X_1)_{\mathbb{C}}, r_n \ge 0, \\ &\|\psi_n\|_{Y_{\mathbb{C}}}^2 = r_n^2 \beta_0^2 |\Omega| + \lambda_n^2 \|z_n\|_{Y_{\mathbb{C}}}^2 = \beta_0^2 |\Omega|. \end{aligned} \tag{6}$$

Substituting $\mu_n = \lambda_n h_n$ and (6) into $\Delta(\lambda_n, \mu_n, \tau_n)\psi_n = 0$, we obtain

$$\begin{aligned} H_1(z_n, r_n, h_n, \tau_n, \lambda_n) &= P_0 z_n + e^{-\lambda_n h_n \tau_n} B_1(\lambda, x) e^{\alpha m(x)} (r_n \beta_0 + \lambda_n z_n) - \frac{2bc^2 e^{2\alpha m(x)} u_\lambda}{(c^2 + e^{2\alpha m(x)} u_\lambda^2)^2} (r_n \beta_0 + \lambda_n z_n) \\ &\quad - \big(\delta e^{\alpha m(x)} + h_n e^{\alpha m(x)}\big)(r_n \beta_0 + \lambda_n z_n) = 0, \\ H_2(z_n, r_n, \lambda_n) &= (r_n^2 - 1)\beta_0^2 |\Omega| + \lambda_n^2 \|z_n\|_{Y_{\mathbb{C}}}^2 = 0. \end{aligned}$$

By Lemma 3.1, we obtain that for $\lambda \in (0, \lambda^*]$, $|h_n|$ is bounded, and $|r_n| \le 1$. By Lemma 3.2, there exist constants $M_1, M_2 > 0$ such that

$$\lambda_2 \|z_n\|_{Y_{\mathbb{C}}}^2 \le |\langle P_0 z_n, z_n\rangle| \le M_1 \|z_n\|_{Y_{\mathbb{C}}} + M_2 \lambda_n \|z_n\|_{Y_{\mathbb{C}}}^2.$$

Thus, for $\lambda \in (0, \lambda^*]$, $\{z_n\}_{n=1}^{\infty}$ is bounded in $Y_{\mathbb{C}}$. Since the operator

$$P_0 : (X_1)_{\mathbb{C}} \to (Y_1)_{\mathbb{C}}$$

has a bounded inverse P_0^{-1}, it follows that $P_0^{-1} H_1(z_n, r_n, h_n, \tau_n, \lambda_n) = 0$, implying that $\{z_n\}_{n=1}^{\infty}$ is bounded in $(X_1)_{\mathbb{C}}$. Hence, the sequence

$$\left\{ \left(z_n, r_n, h_n, e^{-\mathrm{Re}\,(\lambda_n \tau_n h_n)}, e^{-\mathrm{i}\,\mathrm{Im}\,(\lambda_n \tau_n h_n)} \right) \right\}_{n=1}^{\infty}$$

is bounded. This sequence is precompact in $Y_{\mathbb{C}} \times \mathbb{R}^3 \times \mathbb{C}$, so there exists a convergent subsequence

$$\left\{ \left(z_{n_k}, r_{n_k}, h_{n_k}, e^{-\mathrm{Re}\,(\lambda_{n_k} \tau_{n_k} h_{n_k})}, e^{-\mathrm{i}\,\mathrm{Im}\,(\lambda_{n_k} \tau_{n_k} h_{n_k})} \right) \right\}_{k=1}^{\infty}$$

with limit $(z_*, r_*, h_*, \sigma_*, e^{-\mathrm{i}\theta_*})$, where

$$r_* = 1, z_* \in Y_{\mathbb{C}}, h_* \in \mathbb{C}\,(\mathrm{Re}\, h_* \ge 0), \theta_* \in [0, 2\pi), \sigma_* \in [0, 1].$$

Noting that

$$\lim_{k\to\infty} P_0^{-1} H_1\left(z_{n_k}, r_{n_k}, h_{n_k}, \tau_{n_k}, \lambda_{n_k}\right) = 0,$$

we obtain $z_* \in \left(X_1\right)_{\mathbb{C}}$, and $\left(z_*, r_*, h_*, \sigma_*, \theta_*\right)$ satisfies

$$P_0 z_* - \left(\frac{2bc^2 e^{2\alpha m(x)}\beta_0}{(c^2 + e^{2\alpha m(x)}\beta_0^2)^2} + \delta e^{\alpha m(x)} + h_* e^{\alpha m(x)}\right)\beta_0 + \sigma_* e^{-i\theta_*} B(\lambda, x) e^{\alpha m(x)}\beta_0 = 0.$$

Thus, we obtain

$$\begin{cases} \sigma_* \cos\theta_* \int_\Omega B_1(\lambda, x) e^{\alpha m(x)} dx = \operatorname{Re} h_* \int_\Omega e^{\alpha m(x)} dx + \int_\Omega \delta e^{\alpha m(x)} dx, + \int_\Omega \frac{2bc^2 e^{2\alpha m(x)} c_0}{(c^2 + e^{2\alpha m(x)} c_0^2)^2} dx \\ -\sigma_* \sin\theta_* \int_\Omega B_1(\lambda, x) e^{\alpha m(x)} dx = \operatorname{Im} h_* \int_\Omega e^{\alpha m(x)} dx. \end{cases}$$

Since $\operatorname{Re} h_* \geq 0$, we have

$$\sigma_*^2 \left(\int_\Omega B_1(x) e^{\alpha m(x)}\, dx\right)^2 \geq \left(\int_\Omega \frac{2bc^2 e^{2\alpha m(x)}\beta_0}{(c^2 + e^{2\alpha m(x)}\beta_0^2)^2} dx + \int_\Omega \delta e^{\alpha m(x)} dx\right)^2,$$

which implies that

$$\sigma_*^2 L_0 \int_\Omega p a \beta_0 e^{-\alpha_1 e^{\alpha m(x)}\beta_0} e^{2\alpha m(x)} dx \geq \left(\int_\Omega \frac{2bc^2 e^{2\alpha m(x)}\beta_0}{(c^2 + e^{2\alpha m(x)}\beta_0^2)^2} dx + \int_\Omega \delta e^{\alpha m(x)} dx\right)^2. \tag{7}$$

Noting that $L_0 < 0$ and $\sigma_* \in [0, 1]$, it follows from (7) that

$$0 \leq \sigma_*^2 L_0 \int_\Omega p a \beta_0 e^{-\alpha_1 e^{\alpha m(x)}\beta_0} e^{2\alpha m(x)} dx < 0.$$

This contradicts the previous derivation, thus proving the theorem. □

According to Theorem 3.3, when $L_0 < 0$, we can see that all eigenvalues of $\mathcal{A}_{\tau,\lambda}$ have negative real parts for $\lambda \in (0, \lambda^*]$. The following discussion focuses on the case $L_0 > 0$. Next, we analyze the scenario when $\mathcal{A}_{\tau,\lambda}$ has a pair of purely imaginary eigenvalues $\mu = \pm \mathrm{i}\omega$ with $\omega > 0$.

From the previous arguments, we conclude that if $\mu = \mathrm{i}\omega \in \sigma(A_{\tau,\lambda})$ for some $\tau > 0$, then the following holds if and only if

$$\Delta(\lambda, \mathrm{i}\omega, \tau)\psi := \left(e^{-am(x)} P_0 - \frac{2\lambda bc^2 e^{\alpha m(x)} u_\lambda}{\left(c^2 + e^{2\alpha m(x)} u_\lambda^2\right)^2} - \lambda\delta\right)\psi + \lambda e^{-\mathrm{i}\theta} B_1(\lambda, x)\psi - \mathrm{i}\omega\psi, \tag{8}$$

is solvable for some $\omega > 0$, $\theta \in [0, 2\pi)$, and $\psi \in X_{\mathbb{C}} (\neq 0)$, where $\theta := \omega\tau$.

If (ω, θ, ψ) satisfies (8), then ψ can be decomposed and normalized as

$$\begin{aligned} &\psi = r\beta_0 + \lambda z, z \in \left(X_1\right)_{\mathbb{C}}, r \geq 0, \\ &\|\psi\|_{Y_{\mathbb{C}}}^2 = r^2 \beta_0^2 |\Omega| + \lambda^2 \|z\|_{Y_{\mathbb{C}}}^2 = \beta_0^2 |\Omega|. \end{aligned} \tag{9}$$

Substituting Eq. (9) and $\omega = \lambda h$ into Eq. (8), we obtain the following equivalent system:

$$\begin{aligned} &g_1(z, r, h, \theta, \lambda) := P_0 z + \left[e^{-\mathrm{i}\theta} B_1(\lambda, x) e^{\alpha m(x)}\right]\left(r\beta_0 + \lambda z\right) - \left[\frac{2bc^2 e^{2\alpha m(x)} u_\lambda}{\left(c^2 + e^{2\alpha m(x)} u_\lambda^2\right)^2} + \delta e^{\alpha m(x)} + i h e^{\alpha m(x)}\right]\left(r\beta_0 + \lambda z\right) = 0, \\ &g_2(z, r, \lambda) := \left(r^2 - 1\right)\beta_0^2 |\Omega| + \lambda^2 \|z\|_{Y_{\mathbb{C}}}^2 = 0. \end{aligned} \tag{10}$$

Define the mapping $G : \left(X_1\right)_{\mathbb{C}} \times \mathbb{R}^4 \mapsto Y_{\mathbb{C}} \times \mathbb{R}$ as

$$G(z, r, h, \theta, \lambda) := \left(g_1, g_2\right).$$

We use the implicit function theorem to analyze the solvability of the equation $G(z, r, h, \theta, \lambda) = 0$ when $\lambda = 0$.

Lemma 3.4. *Suppose $L_0 > 0$. Then the following equation*

$$\begin{cases} G(z, r, h, \theta, 0) = 0, \\ z \in \left(X_1\right)_{\mathbb{C}}, h > 0, r \geq 0, \theta \in [0, 2\pi], \end{cases}$$

has a unique solution $\left(z_0, r_0, h_0, \theta_0\right)$, *where*

$$\begin{aligned} &\sin\theta_0=\frac{-h_0\int_\Omega e^{\alpha m(x)}dx}{\int_\Omega B_1(x)e^{\alpha m(x)}dx},\cos\theta_0=\frac{\int_\Omega \frac{2bc^2e^{2\alpha m(x)}\beta_0}{\left(c^2+e^{2\alpha m(x)}\beta_0^2\right)^2}dx+\delta\int_\Omega e^{\alpha m(x)}dx}{\int_\Omega B_1(x)e^{\alpha m(x)}dx},\\ &h_0=\frac{\left[\left(\int_\Omega B_1(x)e^{\alpha m(x)}\,dx\right)^2-\left(\int_\Omega \frac{2bc^2e^{2\alpha m(x)}\beta_0}{\left(c^2+e^{2\alpha m(x)}\beta_0^2\right)^2}dx+\delta\int_\Omega e^{\alpha m(x)}\,dx\right)^2\right]^{\frac12}}{\int_\Omega e^{\alpha m(x)}\,dx}>0, \end{aligned} \tag{11}$$

and $z_0 \in \left(X_1\right)_{\mathbb{C}}$ *is the unique solution to the following equation:*

$$P_0 z=-e^{-\mathrm{i}\theta}B_1(x)e^{\alpha m(x)}\beta_0+\left[\frac{2bc^2e^{2\alpha m(x)}\beta_0}{\left(c^2+e^{2\alpha m(x)}\beta_0^2\right)^2}+\delta e^{\alpha m(x)}+ihe^{\alpha m(x)}\right]\beta_0.$$

Proof. Clearly, when $\lambda=0$, we have $r=r_0=1$, and

$$g_1\left(z,r_0,h,\theta,0\right)=P_0z+pe^{-\mathrm{i}\theta}e^{-ae^{\alpha m(x)\beta_0}}(1-a\beta_0e^{\alpha m(x)})e^{\alpha m(x)}\beta_0-\delta e^{\alpha m(x)}\beta_0-\frac{2bc^2e^{2\alpha m(x)}\beta_0}{\left(1+e^{2\alpha m(x)}\beta_0^2\right)^2}-\mathrm{i}he^{\alpha m(x)}\beta_0,$$

thus

$$\begin{cases} g_1\left(z,r_0,h,\theta,0\right)=0,\\ z\in\left(X_1\right)_{\mathbb{C}}, h\geq 0, r\geq 0, \theta\in[0,2\pi], \end{cases}$$

is solvable if and only if

$$\begin{cases} \cos\theta\int_\Omega B_1(x)e^{\alpha m(x)}dx=\int_\Omega \frac{2bc^2e^{2\alpha m(x)}\beta_0}{\left(c^2+e^{2\alpha m(x)}\beta_0^2\right)^2}dx+\delta\int_\Omega e^{\alpha m(x)}dx,\\ -\sin\theta\int_\Omega B_1(x)e^{\alpha m(x)}dx=h\int_\Omega e^{\alpha m(x)}dx, \end{cases}$$

admits a solution (θ, h), where $h\geq 0$ and $\theta\in[0,2\pi]$. According to (4), from $L_0>0$ we obtain

$$h_0^2=\frac{\left(\int_\Omega B_1(x)e^{\alpha m(x)}\,dx\right)^2-\left(\int_\Omega \frac{2bc^2e^{2\alpha m(x)}\beta_0}{\left(c^2+e^{2\alpha m(x)}\beta_0^2\right)^2}dx+\delta\int_\Omega e^{\alpha m(x)}\,dx\right)^2}{\left(\int_\Omega e^{\alpha m(x)}\,dx\right)^2}>0.$$

Thus, $\sin\theta_0$ and $\cos\theta_0$ satisfy Eq. (11). This completes the proof. □

Lemma 3.5. *Assume* $L_0>0$. *Then there exists* $\lambda\in(0,\tilde\lambda^*]$ *and a continuously differentiable mapping*

$$\lambda\mapsto\left(z_\lambda,r_\lambda,h_\lambda,\theta_\lambda\right)$$

from $[0,\tilde\lambda^*]$ *to* $\left(X_1\right)_{\mathbb{C}}\times\mathbb{R}^3$, *such that* $\left(z_\lambda,r_\lambda,h_\lambda,\theta_\lambda\right)$ *is the unique solution to the equation*

$$G\left(z_\lambda,r_\lambda,h_\lambda,\theta_\lambda,\lambda\right)=0,$$

where G is the mapping defined in Lemma 3.4:

$$\begin{cases} G(z,r,h,\theta,\lambda)=0,\\ z\in\left(X_1\right)_{\mathbb{C}}, h>0, r\geq 0, \theta\in[0,2\pi). \end{cases}$$

Proof. Let $T=\left(T_1,T_2\right):\left(X_1\right)_{\mathbb{C}}\times\mathbb{R}^3\to Y_{\mathbb{C}}\times\mathbb{R}$ be the Fréchet derivative of the mapping G with respect to (z,r,h,θ) at $\left(z_0,r_0,h_0,\theta_0\right)$. Then, for $(\chi,\kappa,\epsilon,\vartheta)\in\left(X_1\right)_{\mathbb{C}}\times\mathbb{R}^3$, we compute

$$\begin{aligned} T_1[\chi,\kappa,\epsilon,\vartheta]=&P_0\chi-\mathrm{i}\beta_0e^{\alpha m(x)}\epsilon-\mathrm{i}e^{-\mathrm{i}\theta}\beta_0B_1(x)e^{\alpha m(x)}\vartheta-\mathrm{i}h_0\beta_0e^{\alpha m(x)}\kappa+\left[\beta_0e^{-\mathrm{i}\theta_0}B_1(x)e^{\alpha m(x)}\frac{-2bc^2e^{2\alpha m(x)}\beta_0^2}{\left(c^2+e^{2\alpha m(x)}\beta_0^2\right)^2}-\delta e^{\alpha m(x)}\beta_0\right]\kappa,\\ T_2[\chi,\kappa,\epsilon,\vartheta]=&2\beta_0^2|\Omega|\kappa. \end{aligned}$$

We now prove that $T:\left(X_1\right)_{\mathbb{C}}\times\mathbb{R}^3\to Y_{\mathbb{C}}\times\mathbb{R}$ is bijective. In fact, T is linear with respect to $\chi,\kappa,\epsilon,\vartheta$, then for any $(y,s)\in Y_{\mathbb{C}}\times\mathbb{R}$, there exists $v=(\chi,\kappa,\epsilon,\vartheta)$ such that $T(v)=(y,s)$, this implies that T is surjective, in the following it suffices to show that T is injective. If $T_2(\chi,\kappa,\epsilon,\vartheta)=0$, then $\kappa=0$. Substituting $\kappa=0$ into T_1, we obtain $T_1(\chi,\kappa,\epsilon,\vartheta)=0$, which implies $\chi=\epsilon=\vartheta=0$. Thus, T is injective. By the implicit function theorem, there exists $\lambda>0$ and a continuously differentiable mapping $\lambda\mapsto\left(z_\lambda,r_\lambda,h_\lambda,\theta_\lambda\right)$ defined on $[0,\tilde\lambda^*]$ with values in $\left(X_1\right)_{\mathbb{C}}\times\mathbb{R}^3$, satisfying $G\left(z_\lambda,r_\lambda,h_\lambda,\theta_\lambda,\lambda\right)=0$. We now prove uniqueness. It suffices to show that if $G\left(z^\lambda,r^\lambda,h^\lambda,\theta^\lambda\right)=0$, where $z^\lambda\in\left(X_1\right)_{\mathbb{C}}, h^\lambda>0, r^\lambda\geq 0$, and $\theta^\lambda\in[0,2\pi)$, then

$$\left(z^\lambda,r^\lambda,h^\lambda,\theta^\lambda\right)\to\left(z_0,r_0,h_0,\theta_0\right)=\left(z_0,1,h_0,\theta_0\right),$$

as $\lambda \to 0$, in the norm of $(X_1)_{\mathbb{C}} \times \mathbb{R}^3$, By Lemma 3.1, the sequences $\{h^\lambda\}$, $\{r^\lambda\}$, and $\{\theta^\lambda\}$ are bounded for $\lambda \in [0, \tilde{\lambda}^*]$. Multiplying the first equation in Eq. (10) by $\bar{z}_\lambda$ and integrating over Ω, we obtain the existence of constants $\widetilde{M}_1, \widetilde{M}_2 > 0$ such that

$$\lambda_2 \left\| z^\lambda \right\|_{Y_{\mathbb{C}}}^2 \le \left| \left\langle L_0 z^\lambda, z^\lambda \right\rangle \right| \le \widetilde{M}_1 \left\| z^\lambda \right\|_{Y_{\mathbb{C}}} + \widetilde{M}_2 \lambda \left\| z^\lambda \right\|_{Y_{\mathbb{C}}}^2,$$

which implies that $\{z^\lambda\}$ is bounded in $Y_{\mathbb{C}}$ for sufficiently small λ. Since P_0^{-1} exists and is a bounded mapping from $(Y_1)_{\mathbb{C}}$ to $(X_1)_{\mathbb{C}}$, it follows that $\{z^\lambda\}$ is bounded in $X_{\mathbb{C}}$. Therefore, $\{(z^\lambda, r^\lambda, h^\lambda, \theta^\lambda) : \lambda \in (0, \tilde{\lambda}]\}$ is precompact in $Y_{\mathbb{C}} \times \mathbb{R}^3$.

Consequently, there exists a subsequence $\{(z^{\lambda_n}, r^{\lambda_n}, h^{\lambda_n}, \theta^{\lambda_n})\}$ such that

$$(z^{\lambda_n}, r^{\lambda_n}, h^{\lambda_n}, \theta^{\lambda_n}) \to (z^0, r^0, h^0, \theta^0) \text{ in } Y_{\mathbb{C}} \times \mathbb{R}^3, \lambda_n \to 0 \text{ as } n \to \infty.$$

Taking the limit of the equation $P_0^{-1} g_1(z^{\lambda_n}, r^{\lambda_n}, h^{\lambda_n}, \theta^{\lambda_n}) = 0$, as $n \to \infty$, we obtain

$$(z^{\lambda_n}, r^{\lambda_n}, h^{\lambda_n}, \theta^{\lambda_n}) \to (z^0, r^0, h^0, \theta^0) \text{ in } X_{\mathbb{C}} \times \mathbb{R}^3, \text{ as } n \to \infty,$$

and $G(z^0, r^0, h^0, \theta^0, 0) = 0$. By Lemma 3.4, we conclude that

$$(z^0, r^0, h^0, \theta^0) = (z_0, r_0, h_0, \theta_0).$$

This completes the proof. □

The following theorem follows directly from Lemma 3.5.

Theorem 3.6. *Assume that $L_0 > 0$, for $\lambda \in (0, \tilde{\lambda}^*]$, the following eigenvalue problem*

$$\Delta(\lambda, \mathrm{i}\omega, \tau)\psi = 0, \omega > 0, \tau \ge 0, \psi (\neq 0) \in X_{\mathbb{C}}$$

has a solution (ω, τ, ψ) if and only if the following conditions are satisfied:

$$\omega = \omega_\lambda = \lambda h_\lambda, \psi = c\psi_\lambda, \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{\omega_\lambda}, n = 0, 1, 2, \dots,$$

where $\psi = k\psi_\lambda = k(r_\lambda \beta_0 + \lambda z_\lambda)$, k is a nonzero constant, and $z_\lambda, r_\lambda, h_\lambda, \theta_\lambda$ are defined according to Lemma 3.5.

To prove that $\mathrm{i}\omega$ is a simple eigenvalue and satisfies the transversality condition, we provide the following estimates.

Lemma 3.7. *Assume that $L_0 > 0$ and define*

$$S_n(\lambda) := \int_\Omega \psi_\lambda^2 \, dx + \lambda \tau_n e^{-\mathrm{i}\theta_\lambda} \int_\Omega B(\lambda, x) \psi_\lambda^2 dx,$$

where ψ_λ, τ_n, and θ_λ are given as in Theorem 3.6. Then, for $n = 0, 1, 2, \ldots$, we have

$$\lim_{r \to 0} S_n(\lambda) \neq 0.$$

Proof. According to Theorem 3.6, we have

$$\begin{aligned} \lim_{\lambda \to 0} \operatorname{Re} S_n(\lambda) &= \lim_{\lambda \to 0} \int_\Omega \psi_\lambda^2 dx + \lambda \tau_n \cos \theta_\lambda \int_\Omega B_1(\lambda, x) \psi_\lambda^2 dx \\ &= \beta_0^2 |\Omega| + \frac{\theta_0 + 2n\pi}{h_0} \beta_0^2 \left(\int_\Omega \frac{2bc^2 e^{2\alpha m(x)} \beta_0}{(c^2 + e^{2\alpha m(x)} \beta_0^2)^2} dx + \delta |\Omega| \right) \neq 0, \end{aligned}$$

$$\begin{aligned} \lim_{\lambda \to 0} \operatorname{Im} S_n(\lambda) &= \lim_{\lambda \to 0} -\lambda \tau_n \sin \theta_\lambda \int_\Omega B_1(x) \psi_\lambda^2 dx \\ &= \lim_{\lambda \to 0} \lambda \frac{\theta_\lambda + 2n\pi}{\lambda h_\lambda} h |\Omega| \\ &= \beta_0^2 (\theta_0 + 2n\pi) |\Omega| \neq 0, \end{aligned}$$

thus, we have

$$\lim_{r \to 0} S_n(\lambda) \neq 0.$$

This completes the proof. □

Next, we prove that $\mathrm{i}\omega_\lambda$ is a simple eigenvalue.

Theorem 3.8. *Assume that $L_0 > 0$. Then $\mu = \mathrm{i}\omega_\lambda$ is a simple eigenvalue of $\mathcal{A}_{\tau_n, \lambda}$ for $\lambda \in (0, \tilde{\lambda}^*]$ and $n = 0, 1, 2, \ldots$.*

Proof. We know that

$$\mathcal{N}\left[\mathcal{A}_{\tau_n, \lambda} - \mathrm{i}\omega_\lambda\right] = \operatorname{span}\{e^{\mathrm{i}\omega_\lambda s} \psi_\lambda\},$$

where $s \in [-\tau_n, 0]$. Suppose that $\phi_1 \in \mathcal{N}\left[\mathcal{A}_{\tau_n,\lambda} - \mathrm{i}\omega_\lambda\right]^2$, then

$$\left[\mathcal{A}_{\tau_n,\lambda} - \mathrm{i}\omega_\lambda\right]\phi_1 \in \mathcal{N}\left[\mathcal{A}_{\tau_n,\lambda} - \mathrm{i}\omega_\lambda\right] = \mathrm{span}\left\{e^{\mathrm{i}\omega_\lambda s}\psi_\lambda\right\}.$$

Thus, there exists a constant a such that

$$\left[\mathcal{A}_{\tau_n,\lambda} - \mathrm{i}\omega_\lambda\right]\phi_1 = a e^{\mathrm{i}\omega_\lambda s}\psi_\lambda.$$

Therefore,

$$\begin{aligned} &\dot{\phi}_1(s) = \mathrm{i}\omega_\lambda\phi_1(s) + a e^{\mathrm{i}\omega_\lambda s}\psi_\lambda, \quad s \in [-\tau_n, 0], \\ &\dot{\phi}_1(0) = \left(e^{-am(x)}P_0 - \frac{2\lambda b c^2 e^{\alpha m(x)} u_\lambda}{\left(c^2 + e^{2\alpha m(x)} u_\lambda^2\right)^2} - \lambda\delta\right)\phi(0) + \lambda B(\lambda, x)\phi(-\tau_n). \end{aligned} \tag{12}$$

From the first equation in Eq. (12), we obtain

$$\begin{aligned} &\phi_1(s) = \phi_1(0)e^{\mathrm{i}\omega_\lambda s} + a s e^{\mathrm{i}\omega_\lambda s}\psi_\lambda, \\ &\dot{\phi}_1(0) = \mathrm{i}\omega_\lambda\phi_1(0) + a\psi_\lambda. \end{aligned} \tag{13}$$

From the second equations in Eq. (12) and Eq. (13), we get

$$\begin{aligned} \Delta(\lambda, \mathrm{i}\omega_\lambda, \tau_n)\phi_1(0) =& \left(e^{-\alpha m(x)}P_0 - \frac{2\lambda b c^2 e^{\alpha m(x)} u_\lambda}{\left(c^2 + e^{2\alpha m(x)} u_\lambda^2\right)^2} - \lambda\delta\right)\phi_1(0) + \lambda e^{-\mathrm{i}\omega_\lambda\tau_n}B_1(\lambda, x)\phi_1(0) - \mathrm{i}\omega_\lambda e^{\alpha m(x)}\phi_1(0) \\ =& a e^{\alpha m(x)}\left[\psi_\lambda + \lambda\tau_n e^{-\mathrm{i}\omega_\lambda\tau_n}B_1(\lambda, x)\psi_\lambda\right]. \end{aligned}$$

Noting that $\Delta(\lambda, -\mathrm{i}\omega_\lambda, \tau_n)\bar{\psi}_\lambda = 0$, it follows that

$$\begin{aligned} a S_n(\lambda) &= a\left(\int_\Omega e^{\alpha m(x)}\psi_\lambda^2\, dx + \lambda\tau_n e^{-\mathrm{i}\theta_\lambda}\int_\Omega B_1(\lambda, x)e^{\alpha m(x)}\psi_\lambda^2\, dx\right) \\ &= \left\langle e^{\alpha m(x)}\Delta(\lambda, -\mathrm{i}\omega_\lambda, \tau_n)\bar{\psi}_\lambda, \phi(0)\right\rangle = \left\langle \bar{\psi}_\lambda, e^{\alpha m(x)}\Delta(\lambda, \mathrm{i}\omega_\lambda, \tau_n)\phi(0)\right\rangle = 0. \end{aligned}$$

By Lemma 3.7, for $\lambda \in (0, \tilde{\lambda}^*]$, it follows that $a = 0$. Therefore, $\phi_1 \in \mathcal{N}\left[\mathcal{A}_{\tau_n,\lambda} - \mathrm{i}\omega_\lambda\right]$. By induction, we obtain

$$\mathcal{N}\left[\mathcal{A}_{\tau_n,\lambda} - \mathrm{i}\omega_\lambda\right]^j = \mathcal{N}\left[\mathcal{A}_{\tau_n,\lambda} - \mathrm{i}\omega_\lambda\right], j = 2, 3, 4, \dots, n = 0, 1, 2, \dots.$$

This completes the proof. □

Since $\mu = \mathrm{i}\omega_\lambda$ is a simple eigenvalue of $\mathcal{A}_{\tau_n,\lambda}$, the implicit function theorem implies that there exists a neighborhood $O_n \times D_n \times H_n \subset \mathbb{R} \times \mathbb{C} \times X_{\mathbb{C}}$, containing the point $(\tau_n, \mathrm{i}\omega_\lambda, \psi_\lambda)$, and a continuously differentiable mapping $(\mu(\tau), \psi(\tau)) : O_n \mapsto D_n \times H_n$, such that for each $\tau \in O_n$, $\mu(\tau)$ is the unique eigenvalue of $\mathcal{A}_{\tau,\lambda}$ in D_n, satisfying

$$\Delta(\lambda, \mu(\tau), \tau)\psi(\tau) = \left(e^{-\alpha m(x)}P_0 - \frac{2\lambda b c^2 e^{\alpha m(x)} u_\lambda}{(c^2 + e^{2\alpha m(x)} u_\lambda^2)^2} - \lambda\delta\right)\psi(\tau) + \lambda e^{-\mu(\tau)\tau}B_1(\lambda, x)\psi(\tau) - \mu(\tau)\psi(\tau) = 0. \tag{14}$$

where $\psi(\tau_n) = \psi_\lambda$. Now, we verify the transversality condition for the Hopf bifurcation.

Theorem 3.9. *Suppose $L_0 > 0$. Then, for $\lambda \in (0, \tilde{\lambda}^*]$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,\mathrm{Re}\left[\mu(\tau_n)\right] > 0, n = 0, 1, 2, \dots.$$

Proof. Differentiating Eq. (14) with respect to τ at $\tau = \tau_n$, we obtain

$$\frac{\mathrm{d}\mu(\tau_n)}{\mathrm{d}\tau}\left[\lambda\tau_n e^{-i\theta_\lambda}B_1(\lambda, x)\psi_\lambda + \psi_\lambda\right] = \Delta(\lambda, i\omega_\lambda, \tau_n)\frac{\mathrm{d}\psi(\tau_n)}{\mathrm{d}\tau} - i\omega_\lambda\lambda e^{-i\theta_\lambda}B_1(\lambda, x)\psi_\lambda. \tag{15}$$

Noting that

$$\left\langle \bar{\psi}_\lambda, \Delta(\lambda, i\omega_\lambda, \tau_n)\frac{\mathrm{d}\psi(\tau_n)}{\mathrm{d}\tau}\right\rangle = \left\langle \Delta(\lambda, -i\omega_\lambda, \tau_n)\bar{\psi}_\lambda, \frac{\mathrm{d}\psi(\tau_n)}{\mathrm{d}\tau}\right\rangle = 0,$$

multiplying both sides of Eq. (15) by ψ_λ and integrating over Ω, we obtain

$$\begin{aligned}\frac{\mathrm{d}\mu(\tau_n)}{\mathrm{d}\tau} &= \frac{-i\omega\lambda e^{-i\theta_\lambda}\int_\Omega B_1(\lambda,x)\psi_\lambda^2\,dx}{\int_\Omega \psi_\lambda^2\,dx + r\tau_n e^{-i\theta_\lambda}\int_\Omega B_1(\lambda,x)\psi_\lambda^2\,dx}\\ &= \frac{-i\omega\lambda e^{-i\theta_\lambda}\int_\Omega B_1(\lambda,x)\psi_\lambda^2\,dx\left[\int_\Omega \psi_\lambda^2\,dx + r\tau_n e^{-i\theta_\lambda}\int_\Omega B_1(\lambda,x)\psi_\lambda^2\,dx\right]}{\left|S_n(\lambda)\right|^2}\\ &= -\frac{1}{\left|S_n(\lambda)\right|^2}\left[i\omega\lambda e^{-i\theta_\lambda}\int_\Omega \psi_\lambda^2\,dx\int_\Omega B_1(\lambda,x)\psi_\lambda^2\,dx\right]\\ &\quad -\frac{1}{\left|S_n(\lambda)\right|^2}\left[i\omega\tau_n\lambda^2 e^{-i\theta_\lambda}\left(\int_\Omega B_1(\lambda,x)\psi_\lambda^2\,dx\right)^2\right].\end{aligned}$$

Since $e^{-i\theta_\lambda} = \cos\theta_\lambda - i\sin\theta_\lambda$, we obtain

$$\mathrm{Re}\,\frac{\mathrm{d}\mu(\tau_n)}{\mathrm{d}\tau} = -\frac{1}{\left|S_n(\lambda)\right|^2}\left(\sin\theta_\lambda\omega\lambda\int_\Omega \psi_\lambda^2\,dx\int_\Omega B_1(\lambda,x)\psi_\lambda^2\,dx\right).$$

Thus, we get

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Re}\left[\mu(\tau_n)\right] = \frac{1}{\lim_{\lambda\to 0}\left|S_n(\lambda)\right|^2}h_0^2 > 0.$$

This completes the proof. □

Theorem 3.10. *Suppose $\beta_0 > 0$. Then model Eq. (2) has a unique positive steady state solution u_λ. Moreover, for any $\lambda \in (0, \tilde{\lambda}^*]$, where $0 < \tilde{\lambda}^* \ll 1$, the following conclusions hold:*

(i) If $L_0 < 0$, then u_λ is locally asymptotically stable for $\tau \in [0, +\infty)$;
(ii) If $L_0 > 0$, then there exists a sequence $\{\tau_n\}_{n=0}^{\infty}$ (given by Theorem 3.6), such that: when $\tau < \tau_0$, u_λ is locally asymptotically stable; when $\tau > \tau_0$, u_λ is unstable; when $\tau = \tau_n$ (i.e., $n = 0, 1, \dots$), model (2) undergoes a Hopf bifurcation.

4. The direction of the Hopf bifurcation

This section adopts the method from to study the direction of the Hopf bifurcation [19,20] for model Eq. (2). Let $U(t) = u(\cdot,t) - u_\lambda$, where $t = \tau\tilde{t}$ and $\tau = \tau_n + \varrho$. To simplify notation, we omit the tilde symbol, thus transforming system Eq. (2) into the following form:

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \tau_n e^{-\alpha m(x)}P_0 U(t) + \tau_n P_1(U_t) + J(U_t,\varrho), \tag{16}$$

where $U_t = U(t+s) \in C = C([-1,0],Y)$, and

$$\begin{aligned}P_1(U_t) &= -\lambda\frac{2bc^2e^{\alpha m(x)}u_\lambda}{\left(c^2+e^{2\alpha m(x)}u_\lambda^2\right)^2}U(t) - \lambda\delta(x)U(t) + \lambda B_1(\lambda,x)U(t-1),\\ J(U_t,\varrho) &= \varrho e^{-\alpha m(x)}P_0U(t) + \varrho P_1(U_t)\\ &\quad + \lambda(\varrho+\tau_n)\times\left[\frac{B_2(\lambda,x)}{2}U^2(t-1)\right]\\ &\quad + \lambda(\varrho+\tau_n)\times\left[\frac{B_3(\lambda,x)}{6}U^3(t)U^3(t-1) + O(U^4(t-1))\right]\\ &\quad - \lambda(\varrho+\tau_n)\times\left[\frac{C_2(\lambda,x)}{2}U^2(t) + \frac{C_3(\lambda,x)}{6}U^3(t) + O(U^4(t))\right],\end{aligned}$$

where

$$\begin{aligned}B_2(\lambda,x) &= p\left(2ae^{\alpha m(x)} + a^2u_\lambda e^{2\alpha m(x)}\right)e^{-ae^{\alpha m(x)}u_\lambda},\\ B_2(x) &= p\left(2ae^{\alpha m(x)} + a^2\beta_0 e^{2\alpha m(x)}\right)e^{-ae^{\alpha m(x)}\beta_0},\\ B_3(\lambda,x) &= p\left(3a^2e^{2\alpha m(x)} - a^3u_\lambda e^{3\alpha m(x)}\right)e^{-ae^{\alpha m(x)}u_\lambda},\\ B_3(x) &= p\left(3a^2e^{2\alpha m(x)} - a^3\beta_0 e^{3\alpha m(x)}\right)e^{-ae^{\alpha m(x)}\beta_0},\\ C_2(\lambda,x) &= \frac{2bc^2(c^2-3e^{2\alpha m(x)}u_\lambda^2)}{(c^2+e^{2\alpha m(x)}u_\lambda^2)^3}, C_3(\lambda,x) = \frac{24bc^2e^{5\alpha m(x)}u_\lambda^3}{(c^2+e^{2\alpha m(x)}u_\lambda^2)^4},\\ C_2(x) &= \frac{2bc^2(c^2-3e^{2\alpha m(x)}\beta_0^2)}{(c^2+e^{2\alpha m(x)}\beta_0^2)^3}, C_3(x) = \frac{24bc^2e^{5\alpha m(x)}\beta_0^3}{(c^2+e^{2\alpha m(x)}\beta_0^2)^4}.\end{aligned}$$

Clearly, when $\varrho = 0$, model Eq. (18) undergoes a Hopf bifurcation at the zero equilibrium. For $\varrho = 0$, the linearized equation of (18) at $U_t = 0$ is given by

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \tau_n e^{-\alpha m(x)} P_0 U(t) + \tau_n P_1 U_t. \tag{17}$$

Let $\mathcal{A}_{\tau_n}$ be the infinitesimal generator of the solution semigroup of Eq. (17). Crandall and Rabinowitz [17] proved that for all $\Psi \in \mathcal{D}(\mathcal{A}_{\tau_n})$, we have

$$\mathcal{A}_{\tau_n}\Psi = \dot{\Psi},$$

$$\mathcal{D}(\mathcal{A}_{\tau_n}) = \left\{ \Psi \in C_{\mathbb{C}} \cap C^1_{\mathbb{C}} : \Psi(0) \in X_{\mathbb{C}}, \dot{\Psi}(0) = \tau_n e^{-\alpha m(x)} P_0 \Psi(0) - \lambda\tau_n(\delta + \frac{2bc^2 e^{\alpha m(x)} u_\lambda}{\left(c^2 + e^{2\alpha m(x)} u_\lambda^2\right)^2})\Psi(0) + \lambda\tau_n B(\lambda, x)\Psi(-1) \right\},$$

where $C^1_{\mathbb{C}} = C^1([-1,0], Y_{\mathbb{C}})$. Then Eq. (18) can be rewritten in an abstract form as

$$\frac{\mathrm{d}U_t}{\mathrm{d}t} = \mathcal{A}_{\tau_n} U_t + X_0 J(U_t, \varrho), \tag{18}$$

where

$$X_0(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ I, & \theta = 0. \end{cases}$$

From the previous discussion, $\mathcal{A}_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\omega_\lambda \tau_n$, which are simple. The corresponding eigenfunctions are $\varrho(s) = \psi_\lambda e^{i\omega_\lambda \tau_n s}$ and $\bar{\varrho}(s) = \bar{\psi}_\lambda e^{-i\omega_\lambda \tau_n s}$ for $s \in [-1, 0]$, where ψ_λ is defined as in Theorem 3.6.

Since advection is present, the standard inner product on $Y_{\mathbb{C}}$ is unsuitate for computing the normal form. Following Chen et al. [21] , we introduce the following weighted inner product on $Y_{\mathbb{C}}$:

$$\langle u, v\rangle_1 = \int_\Omega e^{\alpha m(x)} \bar{u}(x) v(x)\, dx, \text{for } u, v \in Y_{\mathbb{C}}.$$

Since $m(x)$ is bounded in Ω and $e^{\alpha m(x)}$ is positive, we verify that $Y_{\mathbb{C}}$ remains a Hilbert space under this inner product.

Lemma 4.1. *The formal adjoint operator $\mathcal{A}^*_{\tau_n}$ of $\mathcal{A}_{\tau_n}$ is defined as*

$$\mathcal{A}^*_{\tau_n} \tilde{\Psi}(s) = -\dot{\tilde{\Psi}}(s)$$

with the domain

$$\mathcal{D}(\mathcal{A}^*_{\tau_n}) = \left\{ \tilde{\Psi} \in C^*_{\mathbb{C}} \cap \left(C^*_{\mathbb{C}}\right)^1 : \tilde{\Psi}(0) \in X_{\mathbb{C}}, -\dot{\tilde{\Psi}}(0) = \tau_n e^{-\alpha m(x)} P_0 \tilde{\Psi}(0) - \lambda\tau_n \left(\delta + \frac{2bc^2 e^{\alpha m(x)} u_\lambda}{\left(c^2 + e^{2\alpha m(x)} u_\lambda^2\right)^2}\right)\tilde{\Psi}(0) + \lambda\tau_n B_1(\lambda, x)\tilde{\Psi}(1) \right\},$$

where $\left(C^*_{\mathbb{C}}\right)^1 = C^1\left([0,1], Y_{\mathbb{C}}\right)$. Then, $\mathcal{A}_{\tau_n}$ and $\mathcal{A}^*_{\tau_n}$ satisfy

$$\left\langle\left\langle \mathcal{A}^*_{\tau_n}\tilde{\Psi}, \Psi \right\rangle\right\rangle = \left\langle\left\langle \tilde{\Psi}, \mathcal{A}_{\tau_n}\Psi \right\rangle\right\rangle.$$

Proof. For $\Psi \in \mathcal{D}(\mathcal{A}_{\tau_n})$ and $\tilde{\Psi} \in \mathcal{D}(\mathcal{A}^*_{\tau_n})$, we have

$$\begin{aligned}
&\left\langle\left\langle \tilde{\Psi}, \mathcal{A}_{\tau_n}\Psi \right\rangle\right\rangle \\
&= \left\langle \tilde{\Psi}(0), \left(\mathcal{A}_{\tau_n}\Psi\right)(0) \right\rangle_1 + \lambda\tau_n \int_{-1}^0 \left\langle \tilde{\Psi}(s+1), B_1(\lambda, x)\dot{\Psi}(s)(y) \right\rangle_1 ds \\
&= \left\langle \tilde{\Psi}(0), \tau_n e^{-\alpha m(x)} P_0 \Psi(0) \right\rangle_1 + \lambda\tau_n \left\langle \tilde{\Psi}(0), B_1(\lambda, x)\Psi(-1) \right\rangle_1 \\
&\quad - \lambda\tau_n \left\langle \tilde{\Psi}(0), \left(\frac{2bc^2 e^{\alpha m(x)} u_\lambda}{\left(c^2 + e^{2\alpha m(x)} u_\lambda^2\right)^2} + \delta \right)\Psi(0) \right\rangle_1 \\
&\quad + r\tau_n \left[\left\langle \tilde{\Psi}(s+1), B_1(\lambda, x)\Psi(s) \right\rangle_1 \right]_{-1}^0 \\
&\quad - \lambda\tau_n \int_{-1}^0 \left\langle \dot{\tilde{\Psi}}(s+1), B_1(\lambda, x)\Psi(s) \right\rangle_1 ds \\
&= \left\langle \left(\mathcal{A}^*_{\tau_n}\tilde{\Psi}\right)(0), \Psi(0) \right\rangle_1 + \lambda\tau_n \int_{-1}^0 \left\langle -\dot{\tilde{\Psi}}(s+1), B_1(\lambda, x)\Psi(s) \right\rangle_1 ds \\
&= \left\langle\left\langle \mathcal{A}^*_{\tau_n}\tilde{\Psi}, \Psi \right\rangle\right\rangle.
\end{aligned}$$

This completes the proof. $\square$

Similarly, we know that $\mathcal{A}^*_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\omega_\lambda\tau_n$, and they are simple. The eigenfunctions associated with $i\omega_\lambda\tau_n$ ($-i\omega_\lambda\tau_n$) are

$$q(\tilde{s}) = \psi_\lambda e^{-i\omega_\lambda\tau_n\tilde{s}} \quad \left(\overline{q}(\tilde{s}) = \bar{\psi}_\lambda e^{i\omega_\lambda\tau_n\tilde{s}}\right)$$

for $\tilde{s}\in[0,1]$, where ψ_λ is defined as in Theorem 3.6. The center subspace of Eq. (18) is $P = \mathrm{span}\{p(s), \bar{p}(s)\}$. Moreover, the basis of the eigenfunction space of the adjoint operator $\mathcal{A}^*_{\tau_n}$ associated with the eigenvalues $\pm i\omega_\lambda\tau_n$ is

$$P^* = \mathrm{span}\{q(\tilde{s}), \bar{q}(\tilde{s})\}.$$

Furthermore, the formal adjoint subspace of P is P^*. As usual, $C_{\mathbb{C}}$ can be decomposed as

$$C_{\mathbb{C}} = P\oplus Q,\ \text{where } Q = \left\{\psi\in C_{\mathbb{C}} \mid \langle\langle\widehat{\psi},\psi\rangle\rangle = 0,\ \forall\widehat{\psi}\in P^*\right\}.$$

Define

$$\Phi_p = (p(s), \bar{p}(s)),\ \text{for } s\in[-1,0],$$
$$\Psi_p = \left(\frac{q(\tilde{s})}{\bar{S}_n(\lambda)}, \frac{\bar{q}(\tilde{s})}{S_n(\lambda)}\right),\ \text{for } \tilde{s}\in[0,1].$$

It can be easily verified that

$$\left\langle\left\langle\Phi_p,\Psi_p\right\rangle\right\rangle = I,$$

where I is the identity matrix in $\mathbb{R}^{2\times 2}$. Since the bifurcation direction and stability formula we will study next are only related to $\varrho = 0$, we set $\varrho = 0$ in the system Eq. (18) and define

$$z(t) = \frac{1}{S_n(\lambda)}\langle\langle q, U_t\rangle\rangle.$$

Let

$$W(z,\bar{z})(s) = W_{20}(s)\frac{z^2}{2} + W_{11}(s)z\bar{z} + W_{02}(s)\frac{\bar{z}^2}{2} + \cdots,$$

be the center manifold with range in Q. Then the flow on the center manifold for Eq. (18) can be written as

$$U_t = \Phi_p\cdot\begin{pmatrix} z(t)\\ \bar{z}(t)\end{pmatrix} + W(z(t),\bar{z}(t)).$$

Since $\varrho = 0$, we obtain

$$\dot{z}(t) = \frac{1}{S_n(\lambda)}\frac{d\langle\langle q, U_t\rangle\rangle}{dt} = i\omega_\lambda\tau_n z(t) + g(z(t),\bar{z}(t)), \tag{19}$$

where

$$\begin{aligned} g(z(t),\bar{z}(t)) &= \frac{1}{S_n(\lambda)}\left\langle q(0), F\left(U_t, 0\right)\right\rangle_1\\ &= \frac{1}{S_n(\lambda)}\left\langle q(0), F\left(\Phi_p\cdot\begin{pmatrix} z(t)\\ \bar{z}(t)\end{pmatrix} + W(z(t),\bar{z}(t)), 0\right)\right\rangle_1\\ &= g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \end{aligned}$$

Clearly, simple calculations yield

$$\begin{aligned} g_{20} =& -\frac{\lambda\tau_n}{S_n(\lambda)}\int_\Omega C_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^3 dx + \frac{\lambda\tau_n}{S_n(\lambda)}e^{-2\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^3 dx,\\ g_{11} =& -\frac{\lambda\tau_n}{S_n(\lambda)}\int_\Omega C_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda|\psi_\lambda|^2 dx + \frac{\lambda\tau_n}{S_n(\lambda)}\int_\Omega B_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda|\psi_\lambda|^2 dx,\\ g_{02} =& -\frac{\lambda\tau_n}{S_n(\lambda)}\int_\Omega C_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda\bar{\psi}_\lambda^2 dx + \frac{\lambda\tau_n}{S_n(\lambda)}e^{2\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda\bar{\psi}_\lambda^2 dx,\\ g_{21} =& -\frac{2\lambda\tau_n}{S_n(\lambda)}\int_\Omega C_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2 w_{11}(0)dx + \frac{2\lambda\tau_n}{S_n(\lambda)}e^{-\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2 w_{11}(-1)dx - \frac{\lambda\tau_n}{S_n(\lambda)}\int_\Omega C_2(\lambda,x)e^{\alpha m(x)}|\psi_\lambda|^2 w_{20}(0)dx\\ &+ \frac{\lambda\tau_n}{S_n(\lambda)}e^{\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_2(\lambda,x)e^{\alpha m(x)}|\psi_\lambda|^2 w_{20}(-1)dx - \frac{\lambda\tau_n}{3S_n(\lambda)}\int_\Omega C_3(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2|\psi_\lambda|^2 dx\\ &+ \frac{\lambda\tau_n}{3S_n(\lambda)}e^{\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_3(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2|\psi_\lambda|^2 dx - \frac{\lambda\tau_n}{S_n(\lambda)}\int_\Omega C_3(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2|\psi_\lambda|^2 dx\\ &+ \frac{\lambda\tau_n}{S_n(\lambda)}e^{-\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_3(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2|\psi_\lambda|^2 dx. \end{aligned}$$

Now, only $W_{20}(s)$ and $W_{11}(s)$ remain to be computed in g_{21}.

$$\dot{W} = \begin{cases} \mathcal{A}_{\tau_n}W - gp(s) - \overline{gp}(s), & s\in[-1,0),\\ \mathcal{A}_{\tau_n}W - gp(0) - \overline{gp}(0) + J(2\,\mathrm{Re}\{z(t)p\} + W(z(t),\bar{z}(t)), 0), & s = 0.\end{cases} \tag{20}$$

On the other hand, according to the definition of W, when the center manifold C_0 approaches the origin,

$$\begin{aligned}\dot{W} =& W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}\\ =& \left[W_{20}(s)z + W_{11}(s)\bar{z}\right]\dot{z} + \left[W_{11}(s)z + W_{02}(s)\bar{z}\right]\dot{\bar{z}} + \cdots\\ =& \left[W_{20}(s)z + W_{11}(s)\bar{z}\right]\left(\mathrm{i}\theta_{n\lambda}z + g(z,\bar{z})\right)\\ &+ \left[W_{11}(s)z + W_{02}(s)\bar{z}\right]\left(-\mathrm{i}\theta_{n\lambda}\bar{z} + \bar{g}(z,\bar{z})\right) + \cdots .\end{aligned}$$

Combining the above equation with Eq. (20), we obtain

$$\left(2i\theta_{n\lambda}I - \mathcal{A}_{\tau_n}\right)W_{20}(s) = \begin{cases}-g_{20}p(s) - \bar{g}_{02}\bar{p}(s), & s \in [-1,0),\\ -g_{20}p(0) - \bar{g}_{02}\bar{p}(0) - \lambda\tau_n C_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2 & \\ +\lambda\tau_n e^{-2\mathrm{i}\omega_\lambda\tau_n}B_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2, & s = 0.\end{cases} \tag{21}$$

Moreover,

$$-\mathcal{A}_{\tau_n}W_{11}(s) = \begin{cases}-g_{11}p(s) - \bar{g}_{11}\bar{p}(s), & s \in [-1,0),\\ -g_{11}p(0) - \bar{g}_{11}\bar{p}(0) - \lambda\tau_n C_2(\lambda,x)e^{\alpha m(x)}\left|\psi_\lambda\right|^2 & \\ +\lambda\tau_n B_2(\lambda,x)e^{\alpha m(x)}\left|\psi_\lambda\right|^2, & s = 0.\end{cases}$$

To compute W_{20}, from Eq. (21), we obtain

$$W_{20}'(s) = 2\mathrm{i}\theta_{n\lambda}W_{20}(s) + g_{20}p(s) + \bar{g}_{02}\bar{p}(s), s \in [-1,0).$$

Noting that $p(s) = \psi_\lambda e^{i\omega_\lambda\tau_n s}$, we derive the following relation:

$$W_{20}(s) = \frac{\mathrm{i}g_{20}}{\omega_\lambda\tau_n}p(s) + \frac{\mathrm{i}\bar{g}_{02}}{3\omega_\lambda\tau_n}\bar{p}(s) + Ee^{2\mathrm{i}\omega_\lambda\tau_n s}. \tag{22}$$

In particular, Eqs. (21) and Eq. (22) indicate:

$$\left(2i\omega_\lambda\tau_n I - \mathcal{A}_{\tau_n}\right)Ee^{2\mathrm{i}\omega_\lambda\tau_n s}\Big|_{s=0} = -\lambda\tau_n C_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2 + \lambda\tau_n e^{-2\mathrm{i}\omega_\lambda\tau_n}B_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2,$$

which is equivalent to

$$\Delta\left(\lambda, 2\mathrm{i}\omega_\lambda, \tau_n\right)E = \lambda\tau_n C_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2 - \lambda\tau_n e^{-2\mathrm{i}\omega_\lambda\tau_n}B_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2. \tag{23}$$

Noting that $2\mathrm{i}\omega_\lambda$ is not an eigenvalue of $A_{\tau_n,\lambda}$, for $\lambda \in (\lambda_*, \tilde{\lambda}^*]$, we have:

$$E = \lambda\Delta\left(\lambda, 2\mathrm{i}\omega_\lambda, \tau_n\right)^{-1}C_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2 - \lambda e^{-2\mathrm{i}\omega_\lambda\tau_n}\Delta\left(\lambda, 2\mathrm{i}\omega_\lambda, \tau_n\right)^{-1}B_2(\lambda,x)e^{\alpha m(x)}\psi_\lambda^2.$$

Similarly, from Eq. (21), we deduce that for $s \in [-1,0)$,

$$W_{11}(s) = -\frac{\mathrm{i}g_{11}}{\omega_\lambda\tau_n}p(s) + \frac{\mathrm{i}\bar{g}_{11}}{\omega_\lambda\tau_n}\bar{p}(s) + F.$$

At $s = 0$, F satisfies the following relation:

$$-\mathcal{A}_\tau F = \lambda\tau_n\left(B_2(\lambda,x)e^{\alpha m(x)} - C_2(\lambda,x)e^{\alpha m(x)}\right)\left|\psi_\lambda\right|^2.$$

Thus, we obtain

$$F = -\lambda\Lambda\left(\lambda, 0, \tau_{n\lambda}\right)^{-1}\left[\left(B_2(\lambda,x)e^{\alpha m(x)} - C_2(\lambda,x)e^{\alpha m(x)}\right)\left|\psi_\lambda\right|^2\right]. \tag{24}$$

Lemma 4.2. *Assume that $\lambda \in (0, \tilde{\lambda}^*]$, and let E_λ and F_λ be defined by Eq. (23) and Eq. (24), respectively. Then, we have*

$$E_\lambda = \rho_\lambda\beta_0 + \eta_\lambda, F_\lambda = k_\lambda\beta_0 + \tilde{\eta}_\lambda, \tag{25}$$

where η_λ and $\tilde{\eta}_\lambda$ satisfy

$$\eta_\lambda, \tilde{\eta}_\lambda \in X_1, \lim_{\lambda\to 0}\|\eta_\lambda\|_{X_\mathbb{C}} = 0, \lim_{\lambda\to 0}\|\tilde{\eta}_\lambda\|_{X_\mathbb{C}} = 0,$$

and the constants ρ_λ and k_λ satisfy

$$\lim_{\lambda\to 0}\rho_\lambda = \frac{e^{-2i\theta_0}\beta_0\int_\Omega B_2(x)e^{2\alpha m(x)}dx - \beta_0\int_\Omega C_2(x)e^{2\alpha m(x)}dx}{D(x) + 2ih_0\int_\Omega e^{\alpha m(x)}\,dx - e^{-2i\theta_0}\int_\Omega B_1(x)e^{\alpha m(x)}\,dx},$$

$$\lim_{\lambda\to 0}k_\lambda = \frac{\beta_0\int_\Omega B_2(x)e^{2\alpha m(x)}dx - \beta_0\int_\Omega C_2(x)e^{2\alpha m(x)}dx}{D(x) - \int_\Omega B_1(x)e^{\alpha m(x)}\,dx},$$

where θ_0,h_0 are defined in Eq. (11), $D(x) = \int_\Omega \frac{2bc^2e^{2\alpha m(x)}\beta_0}{(c^2+e^{2\alpha m(x)}\beta_0^2)}dx + \delta\int_\Omega e^{\alpha m(x)}dx$.

Proof. We only prove the estimate for E_λ, as the estimate for F_λ can be obtained similarly. Substituting the expression of E_λ from Eq. (25) into Eq. (23), we obtain

$$\begin{aligned}&P_0\eta_\lambda-\left(\frac{2\lambda bc^2e^{2\alpha m(x)}u_\lambda}{(c^2+e^{2\alpha m(x)}u_\lambda^2)^2}+\lambda\delta e^{\alpha m(x)}+2\mathrm{i}\omega_\lambda\tau_n\right)(\rho_\lambda\beta_0+\eta_\lambda)+\lambda e^{-2\mathrm{i}\omega_\lambda\tau_n}B_1(\lambda,x)e^{\alpha m(x)}(\rho_\lambda\beta_0+\eta_\lambda)\\&=\lambda C_2(\lambda,x)e^{2\alpha m(x)}\psi_\lambda^2-\lambda e^{-2\mathrm{i}\omega_\lambda\tau_n}B_2(\lambda,x)e^{2\alpha m(x)}\psi_\lambda^2.\end{aligned}\tag{26}$$

Integrating Eq. (26) over Ω, we get

$$\begin{aligned}&\rho_\lambda\left(-\lambda\beta_0\int_\Omega\frac{2bc^2e^{2\alpha m(x)}u_\lambda}{(c^2+e^{2\alpha m(x)}u_\lambda^2)^2}\,dx-\lambda\beta_0\delta\int_\Omega e^{\alpha m(x)}\,dx\right)+\rho_\lambda\left(-2\mathrm{i}\omega_\lambda\beta_0\int_\Omega e^{\alpha m(x)}\,dx+\lambda\beta_0e^{-2\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_1(\lambda,x)e^{\alpha m(x)}\,dx\right)\\&=\lambda\int_\Omega\frac{2bc^2e^{2\alpha m(x)}u_\lambda\eta_\lambda}{(c^2+e^{2\alpha m(x)}u_\lambda^2)^2}\,dx+\lambda\delta\int_\Omega e^{\alpha m(x)}\eta_\lambda dx+2\mathrm{i}\omega_\lambda\int_\Omega e^{\alpha m(x)}\eta_\lambda dx\\&\quad+\lambda\int_\Omega C_2(\lambda,x)e^{2\alpha m(x)}\psi_\lambda^2\,dx-\lambda e^{-2\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_2(\lambda,x)e^{2\alpha m(x)}\psi_\lambda^2\,dx.\end{aligned}\tag{27}$$

Since $|\omega_\lambda|, \left|\frac{\omega_\lambda}{\lambda}\right|, \|u_\lambda\|_\infty, \|\psi_\lambda\|_\infty$, and $\|B_2(\lambda,x)\|_\infty, \|C_2(\lambda,x)\|_\infty$ are bounded for $\lambda\in(0,\tilde\lambda^*]$, there exist constants $M_0>0$ and $M_1>0$ such that

$$|\rho_\lambda|\le M_0\|\eta_\lambda\|_{Y_\mathbb{C}}+M_1,\ \lambda\in(0,\tilde\lambda^*].$$

Multiplying Eq. (26) by $\bar\eta_\lambda$ and integrating over Ω, we obtain

$$\begin{aligned}&\rho_\lambda\left(-\lambda\beta_0\int_\Omega\frac{2bc^2e^{2\alpha m(x)}u_\lambda}{(c^2+e^{2\alpha m(x)}u_\lambda^2)^2}\bar\eta_\lambda dx-\lambda\beta_0\delta\int_\Omega e^{\alpha m(x)}\bar\eta_\lambda dx-2\mathrm{i}\omega_\lambda\beta_0\int_\Omega e^{\alpha m(x)}\bar\eta_\lambda dx\right)\\&\quad+\langle\eta_\lambda,P_0\eta_\lambda\rangle+\rho_\lambda\lambda\beta_0e^{-2\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_1(\lambda,x)e^{\alpha m(x)}\bar\eta_\lambda dx\\&=\lambda\int_\Omega\frac{2bc^2e^{2\alpha m(x)}u_\lambda}{(c^2+e^{2\alpha m(x)}u_\lambda^2)^2}|\eta_\lambda|^2dx+\lambda\delta\int_\Omega e^{\alpha m(x)}|\eta_\lambda|^2dx+2\mathrm{i}\omega_\lambda\int_\Omega e^{\alpha m(x)}|\eta_\lambda|^2dx\\&\quad+\lambda\int_\Omega C_2(\lambda,x)e^{2\alpha m(x)}\psi_\lambda^2\bar\eta_\lambda dx-\lambda e^{-2\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_2(\lambda,x)e^{2\alpha m(x)}\psi_\lambda^2\bar\eta_\lambda dx.\end{aligned}\tag{28}$$

By Lemma 3.2 and Eq. (28), there exist constants $M_2>0$ and $M_3>0$ such that

$$|\lambda_2|\|\eta_\lambda\|_{Y_\mathbb{C}}^2\le\lambda M_2\|\eta_\lambda\|_{Y_\mathbb{C}}^2+\lambda M_3\|\eta_\lambda\|_{Y_\mathbb{C}},\ \lambda\in(0,\tilde\lambda^*].$$

Thus, $\lim_{\lambda\to0}\|\eta_\lambda\|_{Y_\mathbb{C}}=0$. Multiplying Eq. (27) by $\frac{1}{\lambda}$ and taking the limit as $\lambda\to0$, we obtain that ρ_λ satisfies

$$\lim_{\lambda\to0}\rho_\lambda=\frac{e^{-2i\theta_0}\beta_0\int_\Omega B_2(x)e^{2\alpha m(x)}dx-\beta_0\int_\Omega C_2(x)e^{2\alpha m(x)}dx}{D(x)+2ih_0\int_\Omega e^{\alpha m(x)}\,dx-e^{-2i\theta_0}\int_\Omega B_1(x)e^{\alpha m(x)}\,dx}.$$

This completes the proof. □

Thus, by computing the coefficients g_{20}, g_{11}, g_{02}, and g_{21}, we obtain the normal form Eq. (19) restricted to the center manifold $\mathbf{C}_0$. Define

$$C_1(0)=\frac{i}{2\theta_{n\lambda}}\left[g_{11}g_{20}-2|g_{11}|^2-\frac{|g_{02}|^2}{3}\right]+\frac{g_{21}}{2}.$$

Then, we have

$$\begin{aligned}\mu_2&=-\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\mu'(\tau_n))},\\\beta_2&=2\,\mathrm{Re}(C_1(0)),\\T_2&=-\frac{\mathrm{Im}(C_1(0))+\mu_2\,\mathrm{Im}(\mu'(\tau_n))}{\tau_n}.\end{aligned}$$

These quantities determine the properties of the bifurcating periodic solutions at the critical value τ_n, namely:

(i) μ_2 determines the direction of the Hopf bifurcation: if $\mu_2>0$ (<0), then the Hopf bifurcation is forward (backward), and the bifurcating periodic solutions exist for $\tau>\tau_n$ ($\tau<\tau_n$).
(ii) β_2 determines the stability of the bifurcating periodic solutions: if $\beta_2<0$ (>0), then the periodic solutions are orbitally asymptotically stable (unstable) on the center manifold.
(iii) T_2 determines the period of the bifurcating periodic solutions: if $T_2>0$ (<0), then the period increases (decreases).

Theorem 4.3. *Assume that $L_0 > 0$ and $\lambda \in (0, \tilde{\lambda}^*]$, where $0 < \tilde{\lambda}^* \ll 1$. Let $\{\tau_n\}_{n=0}^{\infty}$ be the sequence of Hopf bifurcation values. Then, for each $n = 0, 1, 2, \dots$, the Hopf bifurcation at τ_n is forward, and when $\mathrm{Re}\,[C_1(0)] < 0$, the periodic solution bifurcating from $\tau = \tau_0$ is orbitally asymptotically stable; when $\mathrm{Re}\,[C_1(0)] > 0$, the Hopf bifurcation is backward, and the periodic solution bifurcating from $\tau = \tau_0$ is unstable.*

According to the study by Zhang and Wei [7], we know that when convection terms are present and $\alpha > 0$, the sign of $\mathrm{Re}\,[C_1(0)]$ is unknown. Similarly, we provide an alternative method of calculation here.

Theorem 4.4. *It is shown by calculation that*

$$
\begin{aligned}
\lim_{\lambda\to 0} \mathrm{Re}\, C_1(0) &= \lim_{\lambda\to 0} \mathrm{Re}\left\{ \frac{\mathrm{i}}{2\omega_\lambda \tau_n}\left[g_{11} g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right] + \frac{g_{21}}{2} \right\} \\
&= \beta_0^3 \left(e^{-\mathrm{i}\omega_\lambda \tau_n} \int_\Omega B_2(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \mathrm{Re}\, \frac{\lambda \tau_n}{2 S_n(\lambda)} \left(\frac{2F_\lambda}{\beta_0} + \frac{E_\lambda}{\beta_0} \right) \\
&\quad - \beta_0^3 \left(\int_\Omega C_2(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \mathrm{Re}\, \frac{\lambda \tau_n}{2 S_n(\lambda)} \left(\frac{2F_\lambda}{\beta_0} + \frac{E_\lambda}{\beta_0} \right) \\
&\quad + \beta_0^4 \left(e^{\mathrm{i}\omega_\lambda \tau_n} \int_\Omega B_3(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \mathrm{Re}\, \frac{\lambda \tau_n}{6 S_n(\lambda)} \\
&\quad - \beta_0^4 \left(\int_\Omega C_3(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \mathrm{Re}\, \frac{\lambda \tau_n}{6 S_n(\lambda)} \\
&\quad + \beta_0^4 \left(e^{-\mathrm{i}\omega_\lambda \tau_n} \int_\Omega B_3(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \mathrm{Re}\, \frac{\lambda \tau_n}{2 S_n(\lambda)} \\
&\quad - \beta_0^4 \left(\int_\Omega C_3(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \mathrm{Re}\, \frac{\lambda \tau_n}{2 S_n(\lambda)}.
\end{aligned}
$$

where β_0 is given in Theorem 2.1, and F_λ and E_λ are given in Lemma 4.2.

Proof. By Lemmas 3.4, 4.2, and Theorem 3.6, we have:

$$
\lim_{\lambda\to 0} u_\lambda = \beta_0, \lim_{\lambda\to 0} \psi_\lambda = \beta_0, \lim_{\lambda\to 0} \bar{\psi}_\lambda = \beta_0, \lim_{\lambda\to 0} \omega_\lambda \tau_n = \theta_0 + 2n\pi,
$$

and

$$
\begin{aligned}
\lim_{\lambda\to 0} g_{20} &= \beta_0^3 \left(e^{-2\mathrm{i}\theta_0} \int_\Omega B_2(x) e^{\alpha m(x)} dx - \int_\Omega C_2(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \frac{\lambda \tau_n}{S_n(\lambda)}, \\
\lim_{\lambda\to 0} g_{11} &= \beta_0^3 \left(\int_\Omega B_2(x) e^{\alpha m(x)} dx - \int_\Omega C_2(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \frac{\lambda \tau_n}{S_n(\lambda)}, \\
\lim_{\lambda\to 0} g_{02} &= \beta_0^3 \left(e^{2\mathrm{i}\theta_0} \int_\Omega B_2(x) e^{\alpha m(x)} dx - \int_\Omega C_2(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \frac{\lambda \tau_n}{S_n(\lambda)}, \\
\lim_{\lambda\to 0} g_{21} &= -\beta_0^3 \int_\Omega C_2(x) e^{\alpha m(x)} dx \lim_{\lambda\to 0} \frac{2\lambda \tau_n}{S_n(\lambda)} \left(-\frac{\mathrm{i} g_{11}}{\omega_\lambda \tau_n} + \frac{\mathrm{i}\bar{g}_{11}}{\omega_\lambda \tau_n} + \frac{F_\lambda}{\beta_0} \right) \\
&\quad + \beta_0^3 e^{-\mathrm{i}\theta_0} \int_\Omega B_2(x) e^{\alpha m(x)} dx \lim_{\lambda\to 0} \frac{2\lambda \tau_n}{S_n(\lambda)} \left(-\frac{\mathrm{i} g_{11}}{\omega_\lambda \tau_n} e^{-\mathrm{i}\omega_\lambda \tau_n} + \frac{\mathrm{i}\bar{g}_{11}}{\omega_\lambda \tau_n} e^{\mathrm{i}\omega_\lambda \tau_n} \right) \\
&\quad + \beta_0^3 e^{-\mathrm{i}\theta_0} \int_\Omega B_2(x) e^{\alpha m(x)} dx \lim_{\lambda\to 0} \frac{2\lambda \tau_n}{S_n(\lambda)} \left(\frac{F_\lambda}{\beta_0} \right) \\
&\quad - \beta_0^3 \int_\Omega C_2(x) e^{\alpha m(x)} dx \lim_{\lambda\to 0} \frac{\lambda \tau_n}{S_n(\lambda)} \left(\frac{\mathrm{i} g_{20}}{\omega_\lambda \tau_n} + \frac{\mathrm{i}\bar{g}_{02}}{3\omega_\lambda \tau_n} + \frac{E_\lambda}{\beta_0} \right) \\
&\quad + \beta_0^3 e^{\mathrm{i}\theta_0} \int_\Omega B_2(x) e^{\alpha m(x)} dx \lim_{\lambda\to 0} \frac{\lambda \tau_n}{S_n(\lambda)} \left(\frac{\mathrm{i} g_{20}}{\omega_\lambda \tau_n} e^{-\mathrm{i}\omega_\lambda \tau_n} + \frac{\mathrm{i}\bar{g}_{02}}{3\omega_\lambda \tau_n} e^{\mathrm{i}\omega_\lambda \tau_n} \right) \\
&\quad + \beta_0^3 e^{\mathrm{i}\theta_0} \int_\Omega B_2(x) e^{\alpha m(x)} dx \lim_{\lambda\to 0} \frac{\lambda \tau_n}{S_n(\lambda)} \left(\frac{E_\lambda}{\beta_0} e^{-2\mathrm{i}\omega_\lambda \tau_n} \right) \\
&\quad + \beta_0^4 \left(e^{\mathrm{i}\theta_0} \int_\Omega B_3(x) e^{\alpha m(x)} dx - \int_\Omega C_3(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \frac{\lambda \tau_n}{3 S_n(\lambda)} \\
&\quad + \beta_0^4 \left(e^{-\mathrm{i}\theta_0} \int_\Omega B_3(x) e^{\alpha m(x)} dx - \int_\Omega C_3(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \frac{\lambda \tau_n}{S_n(\lambda)} \\
&= \lim_{\lambda\to 0} \left[\frac{-2\mathrm{i} g_{11} g_{20}}{\omega_\lambda \tau_n} + \frac{2\mathrm{i}|g_{11}|^2}{\omega_\lambda \tau_n} + \frac{\mathrm{i} g_{11} g_{20}}{\omega_\lambda \tau_n} + \frac{\mathrm{i}|g_{02}|^2}{3\omega_\lambda \tau_n} \right] \\
&\quad + \beta_0^3 \left(e^{-\mathrm{i}\omega_\lambda \tau_n} \int_\Omega B_2(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \frac{\lambda \tau_n}{S_n(\lambda)} \left(\frac{2F_\lambda}{\beta_0} + \frac{E_\lambda}{\beta_0} \right) \\
&\quad - \beta_0^3 \left(\int_\Omega C_2(x) e^{\alpha m(x)} dx \right) \lim_{\lambda\to 0} \frac{\lambda \tau_n}{S_n(\lambda)} \left(\frac{2F_\lambda}{\beta_0} + \frac{E_\lambda}{\beta_0} \right)
\end{aligned}
$$

$$+\beta_0^4\left(e^{\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_3(x)e^{\alpha m(x)}dx-\int_\Omega C_3(x)e^{\alpha m(x)}dx\right)\lim_{\lambda\to 0}\frac{\lambda\tau_n}{3S_n(\lambda)}$$
$$+\beta_0^4\left(e^{-\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_3(x)e^{\alpha m(x)}dx-\int_\Omega C_3(x)e^{\alpha m(x)}dx\right)\lim_{\lambda\to 0}\frac{\lambda\tau_n}{S_n(\lambda)}.$$

Substituting the above results into the expression for $C_1(0)$, we obtain

$$\begin{aligned}\lim_{\lambda\to 0}\mathrm{Re}\,C_1(0)&=\lim_{\lambda\to 0}\mathrm{Re}\left\{\frac{\mathrm{i}}{2\omega_\lambda\tau_n}\left[g_{11}g_{20}-2|g_{11}|^2-\frac{|g_{02}|^2}{3}\right]+\frac{g_{21}}{2}\right\}\\
&=\beta_0^3\left(e^{-\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_2(x)e^{\alpha m(x)}dx\right)\lim_{\lambda\to 0}\mathrm{Re}\frac{\lambda\tau_n}{2S_n(\lambda)}\left(\frac{2F_\lambda}{\beta_0}+\frac{E_\lambda}{\beta_0}\right)\\
&\quad-\beta_0^3\left(\int_\Omega C_2(x)e^{\alpha m(x)}dx\right)\lim_{\lambda\to 0}\mathrm{Re}\frac{\lambda\tau_n}{2S_n(\lambda)}\left(\frac{2F_\lambda}{\beta_0}+\frac{E_\lambda}{\beta_0}\right)\\
&\quad+\beta_0^4\left(e^{\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_3(x)e^{\alpha m(x)}dx\right)\lim_{\lambda\to 0}\mathrm{Re}\frac{\lambda\tau_n}{6S_n(\lambda)}\\
&\quad-\beta_0^4\left(\int_\Omega C_3(x)e^{\alpha m(x)}dx\right)\lim_{\lambda\to 0}\mathrm{Re}\frac{\lambda\tau_n}{6S_n(\lambda)}\\
&\quad+\beta_0^4\left(e^{-\mathrm{i}\omega_\lambda\tau_n}\int_\Omega B_3(x)e^{\alpha m(x)}dx\right)\lim_{\lambda\to 0}\mathrm{Re}\frac{\lambda\tau_n}{2S_n(\lambda)}\\
&\quad-\beta_0^4\left(\int_\Omega C_3(x)e^{\alpha m(x)}dx\right)\lim_{\lambda\to 0}\mathrm{Re}\frac{\lambda\tau_n}{2S_n(\lambda)}.\end{aligned}$$

This completes the proof. □

5. Numerical simulations and conclusions

To verify the correctness of the theoretical results in this paper and further explore the impact of key parameters on Nicholson's blowfly population dynamics, we conducted a numerical simulation to analyze the proposed predator-affected Nicholson's blowfly population delay reaction-diffusion-advection model in depth. First, we verified the impact of the delay parameter τ on Nicholson's blowfly model population pattern formation through numerical simulations. Additionally, from a biological perspective, we focused on the effects of the advection rate α, saturation predation rate b, half-saturation constant c, and mortality rate δ on the stability of Nicholson's blowfly population. During the numerical simulation, the parameters were selected based on specific observational data from previous studies, ensuring the mathematical model's validity.

5.1. Effect of time delay

This section will demonstrate the impact of the delay parameter τ. According to Theorem 3.10, when $L_0>0$, the positive steady state solution of Eq. (3) is locally asymptotically stable for $\tau\in\left[0,\frac{\tau_0}{d_1}\right)$. However, when $\tau\in\left(\frac{\tau_0}{d_1},\infty\right)$, a Hopf bifurcation occurs, and the positive steady state solution u_λ loses stability.

The following parameter sets are chosen:

$$(P_1)d_1=2,\alpha_1=2,p=30,a=2,b=0.5,c=1.5,\delta=0.5,m(x)=\sin x,x\in(0,\pi),$$
$$(P_2)d_1=2,\alpha_1=2,p=10,a=2,b=0.5,c=1.5,\delta=0.5,m(x)=\sin x,x\in(0,\pi).$$

Initial conditions are:

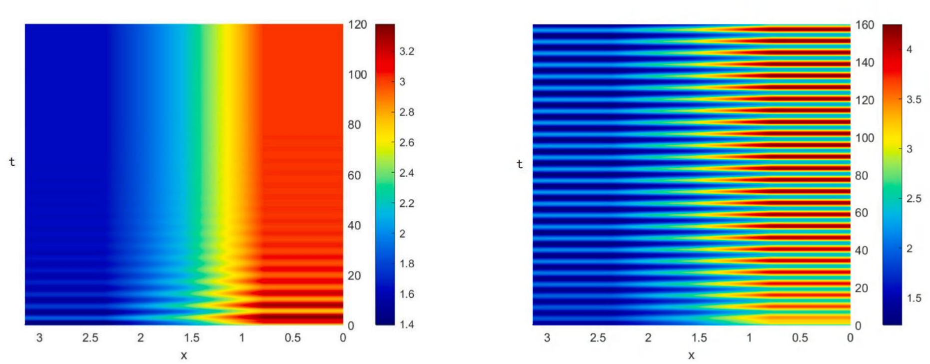
$$(IC)\quad u(x,t)=0.8+0.2\sin x,\ x\in\Omega,\ t\in[-\tau,0].$$

Clearly, the parameter set (P_1) ensures $L_0>0$, with $\beta_0\approx 2.146$ and $L_0\approx 0.374$. The parameter set (P_2) ensures $L_0<0$, with $\beta_0\approx 1.436$ and $L_0\approx -3.957$. Under the condition of (P_1), the first critical value for Eq. (3) is $\tau_0\approx 1.423$.

To illustrate the impact of the delay parameter τ, numerical simulation results for $\tau=1$ and $\tau=2$ are given. Fig. 1 shows that when $L_0>0$: for $\tau=1$, the solution of Eq. (3) converges to a spatially non-uniform positive steady-state solution, while for $\tau=2$, the solution converges to a time-periodic solution. Fig. 2 shows that when $L_0<0$, the solutions for both $\tau=1$ and $\tau=2$ converge to a spatially non-uniform positive steady-state solution. These numerical simulation results are consistent with the conclusions of Theorem 3.10.

5.2. Effect of advection rate

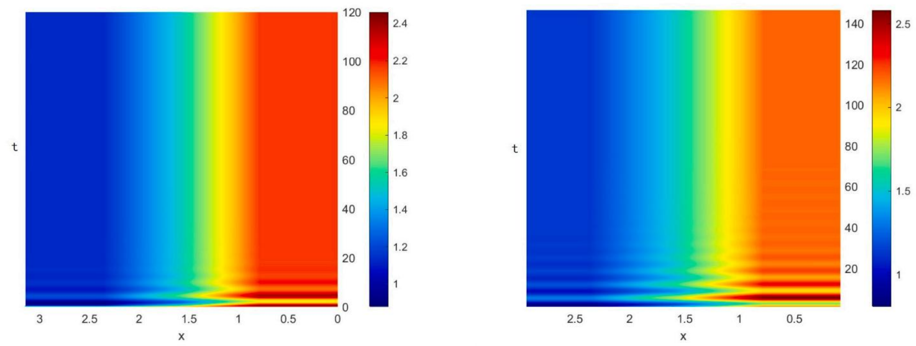
In this section, we will analyze the impact of the convection rate α on the Nicholson fruit fly population. According to the previous discussion, the positive steady state solution β_0 is affected by the convection rate α. Furthermore, based on Theorems 3.3 and 3.10, the existence of a Hopf bifurcation is related to the sign of L_0. Below, we conduct a numerical analysis of the relationship between α, β_0, and L_0.



(a) $\tau = 1$, u_λ is locally asymptotically stable

(b) $\tau = 2$, oscillation periodic solution appears

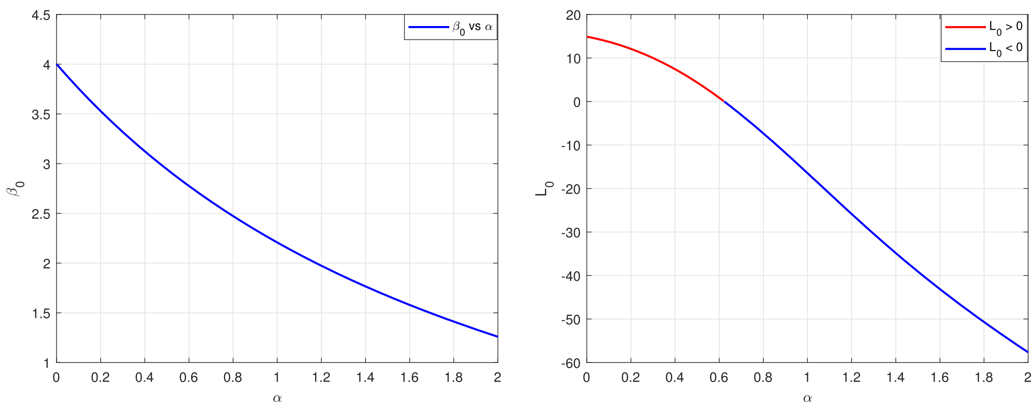
Fig. 1. Eq. (3) undergoes Hopf bifurcation under parameter (P_1) with $L_0 > 0$.



(a) u_λ is locally asymptotically stable at $\tau = 1$

(b) u_λ remains locally asymptotically stable at $\tau = 2$

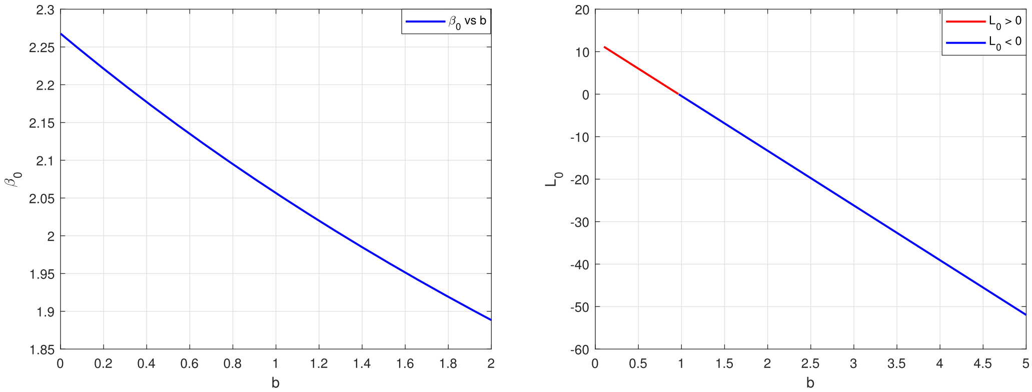
Fig. 2. The solution to Eq. (3) under parameter (P_1) with $L_0 < 0$.



(a) (b)

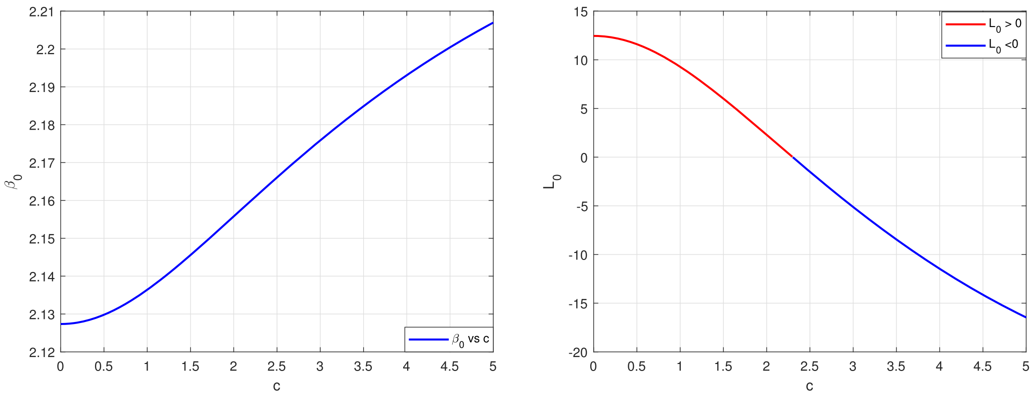
Fig. 3. (a) Relationship between convection parameter α and β_0; (b) Relationship between convection parameter α and L_0.

Based on the numerical simulation results, as observed from the left graph in Fig. 3, the population density gradually decreases as α increases. From the right graph, it can be seen that when α is small, $L_0 > 0$ satisfies the condition for a Hopf bifurcation. However, when α is sufficiently large, $L_0 < 0$, and u_λ becomes locally asymptotically stable. This suggests that in a lower convection environment, individuals can better utilize resources, while a high convection rate may lead to excessive migration, thereby affecting



(a) (b)

Fig. 4. (a) Relationship between predation rate b and β_0; (b) Relationship between predation rate b and L_0.



(a) (b)

Fig. 5. (a) Relationship between half-saturation constant c and β_0; (b) Relationship between half-saturation constant c and L_0.

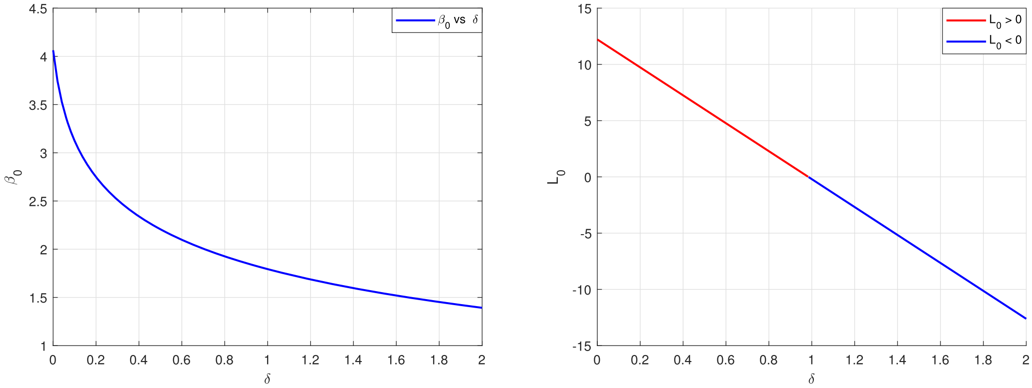
population stability. Therefore, a moderate increase in the convection rate can reduce population density and eliminate population oscillations.

5.3. Effect of reaction term parameters

To investigate the impact of predation, we first conduct numerical simulations in this section to analyze the effects of the saturation predation rate b and the half-saturation constant c on the fly population. From the left panel of Fig. 4, we observe that as the saturation predation rate b increases, the population density gradually decreases. When b is relatively small, a lower predation rate leads to an increase in population density, and the model is more likely to undergo a Hopf bifurcation due to the directed movement of the population. From the left panel of Fig. 5, we observe that as the half-saturation constant c increases, the population density gradually increases. From the right panel, we see that when c is small, $L_0 > 0$, which satisfies the condition for a Hopf bifurcation to occur. However, when c is sufficiently large, $L_0 < 0$, and u_λ becomes locally asymptotically stable. Therefore, when c is large, the population density is higher, and oscillatory behavior no longer occurs.

Subsequently, we analyzed the impact of the mortality rate δ on the population's survival through numerical simulations. As observed from the left graph in Fig. 6, as the mortality rate δ increases, the population density gradually decreases. When δ is small, a lower mortality rate increases the population density, and the model becomes more susceptible to a Hopf bifurcation due to the directional movement of the population. This suggests that populations with lower mortality rates are more likely to undergo periodic oscillations.

These findings provide theoretical support for ecosystem management. For example, in environments with strong convective effects (such as rivers or oceans), populations may require higher survival rates or lower predation pressure to remain stable. Furthermore, in population control or resource management, adjusting predation rates and the half-saturation constant may be significant methods for influencing population dynamics. Overall, the numerical simulations in this paper not only validate the correctness of the theoretical analysis but also reveal how key ecological factors affect population stability and cyclic behavior through nonlinear dynamical



(a) (b)

Fig. 6. (a) Relationship between mortality rate δ and β_0; (b) Relationship between mortality rate δ and L_0.

mechanisms. This provides important references for subsequent ecological management, species conservation, and biological invasion control.

CRediT authorship contribution statement

Jianzhi Cao: Writing – review & editing, Writing – original draft, Validation, Supervision, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization; **Kaipeng Mu:** Writing – review & editing, Writing – original draft, Validation, Software, Methodology, Formal analysis; **Pengmiao Hao:** Writing – review & editing, Validation, Software, Methodology, Funding acquisition.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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