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Journal of Differential Equations 415 (2025) 589–644

Journal of Differential Equations

www.elsevier.com/locate/jde

Boundedness for the chemotaxis system with logistic growth

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Received 10 December 2023; revised 20 August 2024; accepted 19 September 2024

Abstract

In this paper, we consider a mathematical model motivated by the studies of coral broadcast spawning

$$\begin{cases} \partial_t n + u \cdot \nabla n - \Delta n = -\chi \nabla \cdot (n \nabla c) + n - \epsilon n^q \\ \partial_t c + u \cdot \nabla c - \Delta c = -c + n \end{cases} \quad \text{in } \mathbb{R}^d \times \mathbb{R}^+,$$

where $d = 2, 3$, $\epsilon > 0$, and $q \geq 2$. We establish global-in-time well-posedness and boundedness of the solution to the Cauchy problem of this system by developing local-in-space estimates. The crux point of our proof depends intensely on localization in the space of solutions induced by "local effect" of the $L^\infty(\mathbb{R}^d)$-norm.

MSC: 35K55; 35Q92; 35Q35; 92C17

Keywords: Global-in-time boundedness; Logistic growth; Cauchy problem; Localization in space

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https://doi.org/10.1016/j.jde.2024.09.040

1. Introduction

Spawning involves aquatic organisms releasing eggs and sperm into the water, where fertilization occurs [6,7]. The mechanisms behind high fertilization rates in corals remain poorly understood, necessitating new mathematical studies and models to improve our comprehension of gamete fertilization in marine animals. In [9,10], the researchers examined the enhancement of biological reactions through vortex stirring. In the ocean, sperm and eggs (gametes) are initially dispersed, making biological mixing crucial for successful fertilization. Thus, to study broadcast spawning, it is important to consider the interplay of reactions, chemotaxis, diffusion, and fluid velocity transport.

Patlak [25] and Keller and Segel [16,17] employed a mathematical biology model of the dynamic response of cell density to chemical signals to explain the mechanism of chemotaxis

$$\begin{cases} \partial_t n - \Delta n = -\chi \nabla \cdot (n \nabla c), \\ \tau \partial_t c - \Delta c = -c + n, \end{cases} \tag{1.1}$$

where n indicates the density of cells and c denotes the concentration of chemical substances separately. The $\chi > 0$ and $\tau > 0$ are expressed by the strength of attractive chemotaxis and a relaxation time scale respectively. Many results have been obtained for system (1.1), primarily on the existence, boundedness and stability of solutions. A distinctive feature of equation (1.1) is the blow-up phenomenon, as demonstrated in many numerical experiments and in the strict mathematical framework of singularity formation.

Due to the significant biological implications and mathematical complexities, much of the literature focuses on the existence of globally bounded solutions and the conditions under which blow-up phenomena occur. Previous research on parabolic systems has shown that in one-dimensional cases, all solutions to system (1.1) are both global and bounded, thereby ruling out the possibility of convergent aggregation [13,23,24]. In $\mathbb{R}^2$, a threshold exists such that if the initial mass is below this threshold, the solution remains global and bounded in a bounded domain with homogeneous Neumann boundary conditions [12,21], as well as in $\mathbb{R}^2$ [5]. Conversely, if the initial mass (the L^1-norm of bacteria) exceeds this threshold in a smooth bounded domain with homogeneous Neumann boundary conditions [14,26], there are initial data configurations for which the solution will blow up, either in finite or infinite time. In higher dimension $d \geq 3$, the behavior is less understood. The construction of blow-up solutions within a ball [31,33] and in the entire space $\mathbb{R}^d$ [34] indicates that a small initial mass does not necessarily prevent chemotactic collapse. Nonetheless, evidence suggests that global existence can be achieved under certain conditions involving small initial data (n_0, c_0) in $L^{\frac{d}{2}} \times W^{1,d}$ within a bounded domain [3], and in $L^a \times W^{1,d}$ for $\frac{d}{2} < a \leq d$ in the whole space [8].

When accounting for the growth and death of bacteria, the logistic term is often included to prevent solutions to (1.1) from blowing up:

$$\begin{cases} \partial_t n - \Delta n = -\chi \nabla \cdot (n \nabla c) + an - \epsilon n^2, \\ \tau \partial_t c - \Delta c = n - c, \end{cases} \tag{1.2}$$

with a constant a and ϵ. It is well known that if $d = 2$ or $d = 3$ and ϵ is large enough, then there exists no blow-up [15,29,30,32]. If $\tau = 0$ and the second equation in (1.2) is changed into $-\Delta c = -c + n$, in the class $L^\infty_{x,t} \cap L^\infty_t W^{1,p}_x$ for any $p < \infty$, local well-posedness is proved in [15]; and the authors also obtain global existence for $\epsilon \geq 1$ and blow-up in finite time if $0 < \epsilon < 1$. Other interesting results can be found in [27,28].

Recently, Winkler [35] improved (1.2) to the following model defined in smoothly bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 1$,

$$\begin{cases} u_t = D\Delta u - \chi \nabla \cdot (u \nabla v) + \rho u - \mu u^{\alpha}, \\ v_t = d\Delta v - \kappa v + \lambda u. \end{cases} \tag{1.3}$$

Global existence is proved for any $\alpha > 1$, especially for widely arbitrary initial data with poor regularity properties. For $\alpha \geq 2 - \frac{2}{n}$, Winkler showed that any such solution approaches the non-trivial spatially homogeneous steady state $\left(\left(\frac{\rho}{\mu}\right)^{\frac{1}{\alpha-1}}, \frac{\lambda}{\kappa}\left(\frac{\rho}{\mu}\right)^{\frac{1}{\alpha-1}}\right)$ under an appropriate smallness assumption on χ. For other results, we refer the reader to [18,20].

In biology, the phenomenon of broadcast spawning is deeply studied. For example, for some invertebrates, males and females release sperm and eggs to the surrounding fluid flow, known as the radio egg. These chemicals are affected by the diffusion process and advected by the surrounding fluids, hence the following model is introduced

$$\begin{cases} \partial_t n + u \cdot \nabla n - \Delta n = -\chi \nabla \cdot (n \nabla c) - \epsilon n^q, \\ \partial_t c + u \cdot \nabla c - \Delta c = -c + n, \end{cases} \quad \text{in } \mathbb{R}^d \times \mathbb{R}^+, \tag{1.4}$$

where u denotes a given divergence-free velocity of fluid. The global well-posedness and stability are established in [1] for $q \geq 2$. Besides Cao and Winkler [4] study the large time behavior of the solutions to (1.4) with quadratic degradation in a liquid environment. They show that any non-trivial global bounded solution of (1.4) approaches the trivial equilibrium at a rate of $\frac{1}{1+t}$.

In this paper, we further consider the Cauchy problem of the chemotaxis system with logistic growth with $q \geq 2$

$$\begin{cases} \partial_t n + u \cdot \nabla n - \Delta n = -\chi \nabla \cdot (n \nabla c) + n - \epsilon n^q \\ \partial_t c + u \cdot \nabla c - \Delta c = -c + n, \\ n(0, x) = n_0(x), \quad n(0, x) = n_0(x). \end{cases} \tag{1.5}$$

The initial data (n_0, c_0) satisfies the following conditions

$$\begin{cases} n_0 \in L^{\infty}\left(\mathbb{R}^d\right) \text{ is nonnegative,} \\ c_0 \in W^{1,\infty}\left(\mathbb{R}^d\right) \text{ is nonnegative.} \end{cases} \tag{1.6}$$

Motivated by the location method introduced in [22], we aim to prove the global-in-time boundedness of the solutions of system (1.5). We assume that u is in $C^{\infty}\left(\mathbb{R}^d \times [0, \infty)\right)$ such that u and every spatial derivative of u are uniformly bounded for all $(x, t) \in \mathbb{R}^d \times [0, \infty)$ to facilitate the handling of the nonlinear terms $u \cdot \nabla n$ and $u \cdot \nabla c$. Our main results are as follows.

Theorem 1.1. *Let $\chi, \epsilon \geq 0$, $q \geq 2$ and $d = 2, 3$. Assume that the initial data (n_0, c_0) satisfies condition (1.6), and $u \in C^{\infty} \cap L^{\infty}\left(\mathbb{R}^d \times [0, \infty)\right)$ is divergence-free, and each of its spatial* *derivatives is uniformly bounded for all* $(x,t) \in \mathbb{R}^d \times [0,\infty)$. *Then there exists* $\epsilon_0 > 0$ *such that for all* $\epsilon \geq \epsilon_0$, *problem* (1.5) *possesses a unique nonnegative global weak solution in the sense of Definition* 1.1 *and*

$$(n,c) \in C_{\mathrm{w}}\left([0,\infty); L^\infty\left(\mathbb{R}^d\right)\right) \times C_{\mathrm{w}}\left([0,\infty); W^{1,\infty}\left(\mathbb{R}^d\right)\right).$$

Moreover the above solution is global-in-time bounded, namely, there exits a positive constant C *depending only on* χ, ϵ, q, d, $\|n_0\|_{L^\infty(\mathbb{R}^d)}$ *and* $\|c_0\|_{W^{1,\infty}(\mathbb{R}^d)}$ *such that for all* $t>0$,

$$\|n(t)\|_{L^\infty(\mathbb{R}^d)} + \|c(t)\|_{W^{1,\infty}(\mathbb{R}^d)} \leq C.$$

Remark 1.1. Compared to literature [22], our model has two nonlinear terms $u \cdot \nabla n$ and $u \cdot \nabla c$, which makes it difficult to investigate, and new method and estimates need to be developed to study the problem.

In the literatures [4,32], it is shown that in a smooth bounded domain Ω, system (1.5) has upper bounds on n and ∇c that depend on the volume $|\Omega|$. This suggests that techniques applicable to bounded domains may not directly translate to Cauchy problems. However, using the standard energy method to establish global well-posedness for certain chemotaxis models [36], it is observed that the $L^\infty(\mathbb{R}^d)$-norm of $(n,\nabla c)$ is bounded by a function of time t.

To establish global-in-time boundedness of the solution, we draw inspiration from [22] and note that the $L^\infty(\mathbb{R}^d)$-norm reflects a "local property" to some extent, specifically, $L^\infty(\mathbb{R}^d) \hookrightarrow L^p_{uloc}(\mathbb{R}^d)$ for every $1 \leq p < \infty$. Unfortunately, our model has more nonlinear terms $u \cdot \nabla n$ and $u \cdot \nabla c$, which increases the difficulties to tackle the problem, so we make certain assumptions about u and explore some new estimates. Inspired by [22,32], we utilize a coupling approach involving $\|n\|_{L^1_{uloc}(\mathbb{R}^d)}$ and $\|\nabla c\|_{L^2_{uloc}(\mathbb{R}^d)}$, and by selecting an appropriate test function, we successfully achieve the desired results.

Notation: Throughout the paper, $\mathbb{R}^+ = (0,\infty)$ and C stands for a "generic" constant which may change from line to line. For $p,q \in [1,\infty]$, the usual Lebesgue space is denoted by $L^p\left(\mathbb{R}^d\right)$ and $\|\cdot\|_{L^q_t L^p(\mathbb{R}^d)}$ defines the norm of $\left(\int_0^t \|\cdot\|^q_{L^p(\mathbb{R}^d)}\,\mathrm{d}s\right)^{\frac{1}{q}}$. We also denote

$$B_r(x_0) := \left\{x \in \mathbb{R}^d : |x - x_0| < r\right\}.$$

For $1 \leq p < \infty$, we denote

$$\|f\|_{p,\lambda} := \sup_{x\in\mathbb{R}^d} \left(\|f\|_{L^p(B_\lambda(x))}\right) = \sup_{x\in\mathbb{R}^d} \left(\int\limits_{|x-y|<\lambda} |f(y)|^p\,\mathrm{d}y\right)^{1/p}$$

$$L^p_{\mathrm{uloc}}\left(\mathbb{R}^d\right) := \left\{f \in L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right); \|f\|_{p,1} < \infty\right\}, \quad \|f\|_{L^p_{\mathrm{uloc}}(\mathbb{R}^d)} := \|f\|_{p,1}.$$

Finally, $\mathcal{D}\left(\mathbb{R}^d\right)$ is a space of compactly supported smooth functions on $\mathbb{R}^d$, and $W^{s,p}\left(\mathbb{R}^d\right)$ is the general Sobolev space with $1 \leq p \leq \infty$ and $s \geq 0$.

We also introduce a cut-off function which will be used throughout this paper. Set a function $\phi_{x_0}^R \in C_c^\infty\left(\mathbb{R}^d\right)$ by

$$\phi_{x_0}^R(x) = \begin{cases} \exp\left(\frac{4}{3} + \frac{4R^2}{|x-x_0|^2-4R^2}\right), & |x-x_0| < 2R \\ 0, & |x-x_0| \geq 2R. \end{cases} \tag{1.7}$$

From the definition of $\phi_{x_0}^R$, we get that

$$\text{(i) } 1 \leq \phi_{x_0}^R(x) < 2, \quad \forall x \in B_R(x_0).$$

$$\text{(ii) } \phi_{x_0}^R\Big|_{\partial B_{2R}(x_0)} = \frac{\partial \phi_{x_0}^R}{\partial \nu}\Bigg|_{\partial B_{2R}(x_0)} = 0.$$

$$\text{(iii) } \left|\nabla \phi_{x_0}^R\right| \leq \frac{C}{R}, \left|D^2 \phi_{x_0}^R\right| \leq \frac{C}{R^2},$$

where C is independent of R.

Definition 1.1. (n, c) is called a weak solution to the Cauchy problem of (1.5) if the following two conditions hold:

(i) $n(t,x) > 0,\ c(t,x) > 0,\ t \geq 0,\ x \in \mathbb{R}^d$,

$$\begin{cases} n \in \mathcal{C}^0([0,\infty); L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)), \\ c \in \mathcal{C}^0([0,\infty); H^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)). \end{cases} \tag{1.8}$$

(ii) $\forall\, \varphi \in C_0^\infty([0,\infty) \times \mathbb{R}^d)$,

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} n(\partial_t \varphi + \Delta \varphi) dx dt + \int_0^\infty \int_{\mathbb{R}^d} n u \cdot \nabla \varphi dx dt + \int_0^\infty \int_{\mathbb{R}^d} n \varphi dx dt \\ &+ \chi \int_0^\infty \int_{\mathbb{R}^d} n \nabla c \cdot \nabla \varphi dx dt - \int_0^\infty \int_{\mathbb{R}^d} n^q \varphi dx dt + \int_{\mathbb{R}^d} n_0(x) \varphi(0,x) dx = 0, \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} c(\partial_t \varphi + \Delta \varphi) dx dt + \int_0^\infty \int_{\mathbb{R}^d} c u \cdot \nabla \varphi dx dt \\ &- \int_0^\infty \int_{\mathbb{R}^d} c \varphi dx dt + \int_0^\infty \int_{\mathbb{R}^d} n \varphi dx dt + \int_{\mathbb{R}^d} c_0(x) \varphi(0,x) dx = 0. \end{aligned} \tag{1.10}$$

2. Local-in-time well-posedness for Cauchy problem

In this section, we aim to prove the local result of the system (1.5).

Theorem 2.1. *Let $\chi > 0$, $\epsilon > 0$, $q \geq 2$, $d \geq 1$ and the initial data (n_0, c_0) satisfy*

$$\begin{cases} n_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \text{ is nonnegative}, \\ c_0 \in H^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d) \text{ is nonnegative}. \end{cases} \tag{2.1}$$

Suppose n_0 and c_0 have a compact support. Assume $u \in C^\infty \cap L^\infty\left(\mathbb{R}^d \times [0, \infty)\right)$ is divergence-free, and its every spatial derivative is uniformly bounded for all $(x, t) \in \mathbb{R}^d \times [0, \infty)$. Then there exist a maximal $T_{max} \in (0, \infty]$ and a uniquely nonnegative classical solution (n, c) to system (1.5) satisfying

$$n \in C^0([0, T_{max}); L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \cap C^{2,1}(\mathbb{R}^d \times (0, T_{max})),$$
$$c \in C^0([0, T_{max}); H^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)) \cap C^{2,1}(\mathbb{R}^d \times (0, T_{max})).$$

Furthermore, if $T_{max} < \infty$, then

$$\limsup_{t \to T_{\max}-} \left(\|n(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} + \|c(\cdot, t)\|_{W^{1,\infty}(\mathbb{R}^d)} \right) = \infty. \tag{2.2}$$

We prove the above Theorem 2.1 in following 4 subsections.

2.1. Existence

Under the assumption (2.1), there exists a constant $M > 0$ such that the initial data fulfill

$$\|n_0\|_{L^1(\mathbb{R}^d)} + \|n_0\|_{L^\infty(\mathbb{R}^d)} \leq M,$$

and

$$\|c_0\|_{H^1(\mathbb{R}^d)} + \|c_0\|_{W^{1,\infty}(\mathbb{R}^d)} \leq M.$$

Using a standard contraction argument, for a small $T \in (0, 1)$, we select the following space X as the solution space:

$$X_T := C^0\left([0, T]; \left(L^1\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right)\right) \times \left(H^1\left(\mathbb{R}^d\right) \cap W^{1,\infty}\left(\mathbb{R}^d\right)\right)\right),$$

and the closed subset S as

$$S := \Big\{(n, c) \in X_T \mid \|n\|_{L^\infty([0,T];L^1(\mathbb{R}^d))} + \|n\|_{L^\infty([0,T];L^\infty(\mathbb{R}^d))} \leq 2M,$$
$$\|c\|_{L^\infty([0,T];H^1(\mathbb{R}^d))} + \|c\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))} \leq 2M\Big\}.$$

Then, we set a mapping $\Phi = (\Phi_1, \Phi_2)$ on S defined by

$$\Phi_1(n,c) = e^{t\Delta}n_0 - \int_0^t e^{(t-s)\Delta}\nabla(un)\,\mathrm{d}s - \chi\int_0^t e^{(t-s)\Delta}\nabla\cdot(n\nabla c)\,\mathrm{d}s + \int_0^t e^{(t-s)\Delta}n\,\mathrm{d}s$$
$$-\epsilon\int_0^t e^{(t-s)\Delta}n^q\,\mathrm{d}s$$

with

$$\Phi_2(n,c) = e^{t(-1+\Delta)}c_0 - \int_0^t e^{(t-s)(-1+\Delta)}\nabla(uc)\,\mathrm{d}s + \int_0^t e^{(t-s)(-1+\Delta)}n\,\mathrm{d}s$$

for each $t \in [0,T]$. Since n_0 and c_0 have a compact support, it is straightforward to show that $t \mapsto e^{t\Delta}n_0$ belongs to $C^0\left([0,\infty); L^1\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right)\right)$ and $t \mapsto e^{t(-1+\Delta)}c_0$ belongs to $C^0\left([0,\infty); H^1\left(\mathbb{R}^d\right) \cap W^{1,\infty}\left(\mathbb{R}^d\right)\right)$. According to estimates of $e^{t\Delta}\left(e^{t(-1+\Delta)}\right)$, such as those in [34, Lemma 2.1], we can conclude that Φ maps S into X_T. Using the standard L^p-L^q estimate for heat semigroups, it can be confirmed that for all $t \in [0,T]$,

$$\begin{aligned}
&\|\Phi_1(n,c)(t)\|_{L^1(\mathbb{R}^d)} \\
&\le \|n_0\|_{L^1(\mathbb{R}^d)} + C_1\chi\int_0^t (t-s)^{-\frac{1}{2}}\|n(s)\|_{L^2(\mathbb{R}^d)}\|\nabla c(s)\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}s \\
&\quad + C_2\int_0^t (t-s)^{-\frac{1}{2}}\|u(s)\|_{L^\infty(\mathbb{R}^d)}\|n(s)\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}s + \int_0^t \|n(s)\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}s + \epsilon\int_0^t \|n(s)\|^q_{L^q(\mathbb{R}^d)}\,\mathrm{d}s \\
&\le \|n_0\|_{L^1(\mathbb{R}^d)} + C_1\chi T^{\frac{1}{2}}\|n\|_{L^\infty_T(L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))}\|\nabla c\|_{L^\infty_T(L^2(\mathbb{R}^d))} + C_2T^{\frac{1}{2}}\|n\|_{L^\infty_T(L^1(\mathbb{R}^d))} \\
&\quad + T\|\phi n\|_{L^\infty_T(L^1(\mathbb{R}^d))} + \epsilon T\|\phi n\|^q_{L^\infty_T(L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))}.
\end{aligned}$$

Equally, via Hölder's and Young's inequalities, we get that for $0 \le t \le T$,

$$\begin{aligned}
&\|\Phi_1(n,c)(t)\|_{L^\infty(\mathbb{R}^d)} \\
&\le \|n_0\|_{L^\infty(\mathbb{R}^d)} + C_3\chi\int_0^t (t-s)^{-\frac{1}{2}}\|n(s)\|_{L^\infty(\mathbb{R}^d)}\|\nabla c(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s \\
&\quad + C_4\int_0^t (t-s)^{-\frac{1}{2}}\|u(s)\|_{L^\infty(\mathbb{R}^d)}\|n(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s \\
&\quad + \int_0^t \|n(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s + \epsilon\int_0^t \|n(s)\|^q_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s
\end{aligned}$$

$$\leq \|n_0\|_{L^\infty(\mathbb{R}^d)} + C_3\chi T^{\frac{1}{2}}\|n\|_{L^\infty_T(L^\infty(\mathbb{R}^d))}\|\nabla c\|_{L^\infty_T(L^\infty(\mathbb{R}^d))} + C_4 T^{\frac{1}{2}}\|n\|_{L^\infty_T(L^\infty(\mathbb{R}^d))}$$
$$+T\|n\|_{L^\infty_T(L^\infty(\mathbb{R}^d))} + \epsilon T\|n\|^q_{L^\infty_T(L^\infty(\mathbb{R}^d))}.$$

Hence, combining the above two inequalities, we have

$$\begin{aligned}
&\|\Phi_1(n,c)(t)\|_{L^1(\mathbb{R}^d)} + \|\Phi_1(n,c)(t)\|_{L^\infty(\mathbb{R}^d)} \\
\leq\ & \|n_0\|_{L^1(\mathbb{R}^d\cap L^\infty(\mathbb{R}^d))} + C_1\chi T^{\frac{1}{2}}\|n\|_{L^\infty_T(L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))}\|\nabla c\|_{L^\infty_T(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))} \\
&+C_2\chi T^{\frac{1}{2}}\|n\|_{L^\infty_T(L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))} + C_3 T\|n\|_{L^\infty_T(L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))} \\
&+\epsilon T\|n\|^q_{L^\infty_T(L^1(\mathbb{R}^d\cap L^\infty(\mathbb{R}^d)))} \\
\leq\ & M + 4C_1\chi T^{\frac{1}{2}}M^2 + 2C_2\chi T^{\frac{1}{2}}M + 2C_3 TM + \epsilon C_5 T(2M)^q.
\end{aligned}$$

Choosing a small enough $T_1 \in (0, 1)$ to satisfy the following condition

$$\max\left\{4C_1\chi T_1^{\frac{1}{2}}M, 2C_2\chi T_1^{\frac{1}{2}}M, 2C_3T_1, \epsilon C_5 2^q T_1 M^{q-1}\right\} \leq \frac{1}{5},$$

then for $0 \leq t \leq T_1$, we have

$$\|\Phi_1(n,c)(t)\|_{L^1(\mathbb{R}^d)} + \|\Phi_1(n,c)(t)\|_{L^\infty(\mathbb{R}^d)} \leq 2M.$$

Using the identical method as above, we can get

$$\begin{aligned}
&\|\Phi_2(n,c)(t)\|_{H^1(\mathbb{R}^d)} + \|\Phi_2(n,c)(t)\|_{W^{1,\infty}(\mathbb{R}^d)} \\
\leq\ & e^{-t}\left(\|c_0\|_{H^1(\mathbb{R}^d)} + \|c_0\|_{W^{1,\infty}(\mathbb{R}^d)}\right) + \int_0^t e^{-(t-s)}\left(\|n(s)\|_{L^2(\mathbb{R}^d)} + \|n(s)\|_{L^\infty(\mathbb{R}^d)}\right)\,\mathrm{d}s \\
&+\int_0^t e^{-(t-s)}\|u(s)\|_{L^\infty(\mathbb{R}^d)}\left(\|\nabla c(s)\|_{L^2(\mathbb{R}^d)} + \|\nabla c(s)\|_{L^\infty(\mathbb{R}^d)}\right)\,\mathrm{d}s \\
&+C_6\int_0^t (t-s)^{-\frac{1}{2}}e^{-(t-s)}\left(\|n(s)\|_{L^2(\mathbb{R}^d)} + \|n(s)\|_{L^\infty(\mathbb{R}^d)}\right)\,\mathrm{d}s \\
&+C_7\int_0^t (t-s)^{-\frac{1}{2}}e^{-(t-s)}\left(\|\nabla c(s)\|_{L^2(\mathbb{R}^d)} + \|\nabla c(s)\|_{L^\infty(\mathbb{R}^d)}\right)\,\mathrm{d}s \\
\leq\ & \left(\|c_0\|_{H^1(\mathbb{R}^d)} + \|c_0\|_{W^{1,\infty}(\mathbb{R}^d)}\right) + 2\left(T + C_6T^{\frac{1}{2}}\right)\left(\|n\|_{L^\infty_T(L^1(\mathbb{R}^d))} + \|n\|_{L^\infty_T(L^\infty(\mathbb{R}^d))}\right) \\
&+2\left(T + C_7T^{\frac{1}{2}}\right)\left(\|\nabla c\|_{L^\infty_T(L^2(\mathbb{R}^d))} + \|\nabla c\|_{L^\infty_T(L^\infty(\mathbb{R}^d))}\right)
\end{aligned}$$

$$\leq M + 4M\left(T + C_6 T^{\frac{1}{2}}\right) + 4M\left(T + C_7 T^{\frac{1}{2}}\right).$$

We choose $T_2 \in (0, 1)$ satisfying the following condition

$$\max\left\{4\left(T_2 + C_6 T_2^{\frac{1}{2}}\right), 4\left(T_2 + C_7 T_2^{\frac{1}{2}}\right)\right\} \leq \frac{1}{2},$$

from which it follows

$$\|\Phi_2(n,c)(t)\|_{H^1(\mathbb{R}^d)} + \|\Phi_2(n,c)(t)\|_{W^{1,\infty}(\mathbb{R}^d)} \leq 2M.$$

Therefore $\Phi(t)$ maps S into itself for $t \leq T_0 := \min\{T_1, T_2\}$. Then, we testify that Φ is a contraction mapping in a short time. For any couple $(n_1, c_1), (n_2, c_2) \in S$, a simple calculation gives

$$\begin{aligned}
&\|\Phi_1(n_1,c_1)(t) - \Phi_1(n_2,c_2)(t)\|_{L^1(\mathbb{R}^d)} \\
\leq\ & \chi \int_0^t \left\| e^{(t-s)\Delta}\nabla\cdot(n_1(s)\nabla c_1(s) - n_2(s)\nabla c_2(s))\right\|_{L^1(\mathbb{R}^d)} \mathrm{d}s \\
&+ \int_0^t \left\| e^{(t-s)\Delta}\nabla\cdot(u(s)n_1(s) - u(s)n_2(s))\right\|_{L^1(\mathbb{R}^d)} \mathrm{d}s \\
&+ \int_0^t \left\| e^{(t-s)\Delta}(n_1(s) - n_2(s))\right\|_{L^1(\mathbb{R}^d)} \mathrm{d}s \\
&+ \epsilon \int_0^t \left\| e^{(t-s)\Delta}\left(n_1^q(s) - n_2^q(s)\right)\right\|_{L^1(\mathbb{R}^d)} \mathrm{d}s \\
\leq\ & \chi \int_0^t \left\| e^{(t-s)\Delta}\nabla\cdot((n_1(s) - n_2(s))\nabla c_1(s))\right\|_{L^1(\mathbb{R}^d)} \mathrm{d}s \\
&+ \chi \int_0^t \| e^{(t-s)\Delta}\nabla\cdot\left(n_2(s)\nabla(c_1(s) - c_2(s))\right\|_{L^1(\mathbb{R}^d)} \mathrm{d}s \\
&+ \int_0^t \left\| e^{(t-s)\Delta}\nabla\cdot((n_1(s) - n_2(s))u(s))\right\|_{L^1(\mathbb{R}^d)} \mathrm{d}s \\
&+ \left(T + \epsilon q T\left(\|n_1\|_{L_T^\infty(L^\infty(\mathbb{R}^d))} + \|n_2\|_{L_T^\infty(L^\infty(\mathbb{R}^d))}\right)^{q-1}\right)\|n_1 - n_2\|_{L_T^\infty(L^1(\mathbb{R}^d))} \\
\leq\ & C_8 \chi T^{\frac{1}{2}} \|n_1 - n_2\|_{L_T^\infty(L^1(\mathbb{R}^d))} \|c_2\|_{L_T^\infty(\dot{W}^{1,\infty}(\mathbb{R}^d))} \\
&+ C_8 \chi T^{\frac{1}{2}} \|n_2\|_{L_T^\infty(L^1(\mathbb{R}^d))} \|c_1 - c_2\|_{L_T^\infty(\dot{W}^{1,\infty}(\mathbb{R}^d))}
\end{aligned}$$

$$
\begin{aligned}
&+C_9 T^{\frac{1}{2}}\|u(s)\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}\|n_1-n_2\|_{L_T^{\infty}(L^1(\mathbb{R}^d))}\\
&+\big(T+\epsilon q C_{10} T\left(\|n_1\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}+\|n_2\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}\right)^{q-1}\big)\|n_1-n_2\|_{L_T^{\infty}(L^1(\mathbb{R}^d))}\\
\leq\ &\left(4C_8 M\chi T^{\frac{1}{2}}+C_9 T^{\frac{1}{2}}+T+\epsilon q C_{10}(4M)^{q-1}T\right)\|(n_1,c_1)-(n_2,c_2)\|_{X_T}.
\end{aligned}
$$

Using a similar approach, we have

$$
\begin{aligned}
&\|\Phi_1(n_1,c_1)(t)-\Phi_1(n_2,c_2)(t)\|_{L^{\infty}(\mathbb{R}^d)}\\
\leq\ &C_{11}\chi T^{\frac{1}{2}}\|n_1-n_2\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}\|\nabla c_2\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}\\
&+C_{11}\chi T^{\frac{1}{2}}\|n_2\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}\|\nabla(c_1-c_2)\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}\\
&+C_{12} T^{\frac{1}{2}}\|u(s)\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}\|n_1-n_2\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}\\
&+\big(T+\epsilon q C_{13} T\left(\|n_1\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}+\|n_2\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}\right)^{q-1}\big)\|n_1-n_2\|_{L_T^{\infty}(L^{\infty}(\mathbb{R}^d))}\\
\leq\ &\left(4C_{11} M\chi T^{\frac{1}{2}}+C_{12} T^{\frac{1}{2}}+T+\epsilon C_{13} q(4M)^{q-1}T\right)\|(n_1,c_1)-(n_2,c_2)\|_{X_T},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\Phi_2(n_1,c_1)(t)-\Phi_2(n_2,c_2)(t)\|_{H^1(\mathbb{R}^d)}+\|\Phi_2(n_1,c_1)(t)-\Phi_2(n_2,c_2)(t)\|_{W^{1,\infty}(\mathbb{R}^d)}\\
\leq\ &C_{14}\int_0^t\left(1+(t-s)^{-\frac{1}{2}}\right)\left(\|n_1(s)-n_2(s)\|_{L^2(\mathbb{R}^d)}+\|n_1(s)-n_2(s)\|_{L^{\infty}(\mathbb{R}^d)}\right)\mathrm{d}s\\
&+C_{15}\int_0^t\left(1+(t-s)^{-\frac{1}{2}}\right)\|u(s)\|_{L^{\infty}(\mathbb{R}^d)}\left(\|\nabla c_1(s)-\nabla c_2(s)\|_{L^2(\mathbb{R}^d)}\right.\\
&\left.+\|\nabla c_1(s)-\nabla c_2(s)\|_{L^{\infty}(\mathbb{R}^d)}\right)\mathrm{d}s\\
\leq\ &(C_{14}+C_{15})\left(T+T^{\frac{1}{2}}\right)\|(n_1,c_1)-(n_2,c_2)\|_{X_T}.
\end{aligned}
$$

We set T_3 satisfying the following condition

$$
\begin{aligned}
\max\Big\{&4C_8 M\chi T_3^{\frac{1}{2}}+C_9 T_3^{\frac{1}{2}}+T_3+\epsilon C_{10} q(4M)^{q-1}T_3,\\
&4C_{11} M\chi T_3^{\frac{1}{2}}+C_{12} T_3^{\frac{1}{2}}+T_3+\epsilon C_{13} q(4M)^{q-1}T_3, C_{14}\left(T_3+T_3^{\frac{1}{2}}\right), C_{15}\left(T_3+T_3^{\frac{1}{2}}\right)\Big\}\leq\frac{1}{4}.
\end{aligned}
$$

Let

$$
\widetilde{T}=\min\{T_0,T_3\},
$$

then Φ is a contractive mapping on S. We utilize the Banach fixed point theorem to acquire that there exists $(n, c) \in S$ such that $\Phi(n, c) = (n, c)$. In addition, via standard arguments of semi-group estimates (cf. Lemma 3.3 in [11]), we deduce that (n, c) belongs to $C^{2,1}\left(\mathbb{R}^d \times (0, \widetilde{T})\right)^2$ and is a local solution of the problem (1.5) in $(0, \widetilde{T}) \times \mathbb{R}^d$. Now the conclusion (2.2) is an immediate consequence of the observation that our above choice of T actually depends on $\|n_0\|_{L^\infty(\mathbb{R}^d)}$ and $\|c_0\|_{W^{1,\infty}(\mathbb{R}^d)}$ only.

2.2. *Nonnegativity*

We now need to show that $n, c \geq 0$ if the initial data $n_0, c_0 \geq 0$. First, select a smooth cut-off function ϕ satisfying

$$\phi(x) = \begin{cases} 1, & x \in B_R(0) \\ 0, & x \in \mathbb{R}^d \backslash B_{2R}(0), \end{cases} \tag{2.3}$$

where $R > 0$ is a real number.

On the basis of the local property $(n, c) \in C^{2,1}\left(\mathbb{R}^d \times (0, \widetilde{T})\right)$, the global property

$$\partial_t(\phi n), \quad \phi \nabla n \in L^2\left(\mathbb{R}^d\right)$$

can be accomplished by importing the cut-off function ϕ. Set

$$n^- = \max\{0, -n\}.$$

Multiplying Eq. $(1.5)_1$ by $\phi^2 n^-$ and integrating, it results in

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi n^-(t)\right\|_{L^2(\mathbb{R}^d)}^2 + \left\|\phi \nabla n^-(t)\right\|_{L^2(\mathbb{R}^d)}^2 + \epsilon\left\|\phi^{\frac{2}{q+1}} n^-(t)\right\|_{L^{q+1}(\mathbb{R}^d)}^{q+1} \\
\leq\ & \|\nabla c\|_{L^\infty(\mathbb{R}^d)}\left\|\phi \nabla n^-\right\|_{L^2(\mathbb{R}^d)}\left\|\phi n^-\right\|_{L^2(\mathbb{R}^d)} \\
& + 2\|n\|_{L^2(\mathbb{R}^d)}\left\|\phi n^-\right\|_{L^2(\mathbb{R}^d)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)}\|\nabla c\|_{L^\infty(\mathbb{R}^d)} \\
& + \|u\|_{L^\infty(\mathbb{R}^d)}\left\|\phi \nabla n^-\right\|_{L^2(\mathbb{R}^d)}\left\|\phi n^-\right\|_{L^2(\mathbb{R}^d)} \\
& + 2\|n\|_{L^2(\mathbb{R}^d)}\left\|\phi n^-\right\|_{L^2(\mathbb{R}^d)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)}\|u\|_{L^\infty(\mathbb{R}^d)} \\
& + 2\|n\|_{L^2(\mathbb{R}^d)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)}\left\|\phi \nabla n^-\right\|_{L^2(\mathbb{R}^d)} + \left\|\phi n^-\right\|_{L^2(\mathbb{R}^d)}^2 \\
\leq\ & C\left(\|\nabla c\|_{L^\infty(\mathbb{R}^d)}^2 + \|u\|_{L^\infty(\mathbb{R}^d)}^2 + 1\right)\left\|\phi n^-\right\|_{L^2(\mathbb{R}^d)}^2 + \frac{C}{R^2}\|n\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2}\left\|\phi \nabla n^-\right\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned}$$

For $(n, c) \in X_{\widetilde{T}}$, the Grönwall inequality implies that for any $t < \widetilde{T}$,

$$\left\|\phi n^-(t)\right\|_{L^2(\mathbb{R}^d)}^2 \leq \exp\left(\int_0^t C\left(\|\nabla c(s)\|_{L^\infty(\mathbb{R}^d)}^2 + \|u(s)\|_{L^\infty(\mathbb{R}^d)}^2\right) \mathrm{d}s + 1\right)\left\|n_0^-\right\|_{L^2(\mathbb{R}^d)}^2 + \frac{C}{R^2}\widetilde{T}.$$

Letting R go to infinite, we obtain that $n(x,t) \geq 0$ in the domain $\mathbb{R}^d \times (0, \widetilde{T})$. Similarly, we possess $c(x,t) \geq 0$ in the domain $\mathbb{R}^d \times (0, \widetilde{T})$.

2.3. Uniqueness

We now show the uniqueness of the solution of (1.5) in the framework

$$(n, c) \in C^0\left([0, T_{\max}); \left(L^1\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right)\right) \times H^1 \cap W^{1,\infty}\left(\mathbb{R}^d\right)\right).$$

Let (n, c) and $(\widetilde{n}, \widetilde{c})$ be two solutions to system (1.5) in $\mathbb{R}^d \times (0, T_{\max})$ with the same initial data. Denote $\delta n = n - \widetilde{n}$ and $\delta c = c - \widetilde{c}$, we get that the couple $(\delta n, \delta c)$ in $\mathbb{R}^d \times (0, T_{\max})$ fulfills

$$\begin{cases} \partial_t \delta n + u \cdot \nabla \delta n - \Delta \delta n = -\chi \nabla \cdot (\delta n \nabla c) - \chi \nabla \cdot (\widetilde{n} \nabla \delta c) + \delta n - \epsilon (n^q - \widetilde{n}^q), \\ \partial_t \delta c + u \cdot \nabla \delta c - \Delta \delta c = -\delta c + \delta n. \end{cases} \tag{2.4}$$

Applying the energy method and the cut-off function ϕ introduced in (2.3), we can show that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi \delta n(t)\|_{L^2(\mathbb{R}^d)}^2 + \|\phi \nabla \delta n(t)\|_{L^2(\mathbb{R}^d)}^2 \\
\leq\ & \int_{\mathbb{R}^d} u \cdot \nabla \delta n \phi^2 \delta n \, \mathrm{d}x + \chi \int_{\mathbb{R}^d} \phi^2 \delta n \nabla c \cdot \nabla \delta n \, \mathrm{d}x + 2\chi \int_{\mathbb{R}^d} \phi \delta n \nabla c \cdot \nabla \phi \delta n \, \mathrm{d}x \\
& + \chi \int_{\mathbb{R}^d} \phi \tilde{n} \nabla \delta c \cdot \phi \nabla \delta n \, \mathrm{d}x + 2\chi \int_{\mathbb{R}^d} \phi \tilde{n} \nabla \delta c \cdot \nabla \phi \delta n \, \mathrm{d}x - 2 \int_{\mathbb{R}^d} \phi \nabla \delta n \cdot \nabla \phi \delta n \, \mathrm{d}x \\
& + \int_{\mathbb{R}^d} |\phi \delta n|^2 dx + \epsilon q \int_{\mathbb{R}^d} |\phi \delta n|^2 \cdot (\|n\|_{L^\infty} + \|\tilde{n}\|_{L^\infty})^{q-1} \, \mathrm{d}x \\
\leq\ & \|u\|_{L^\infty(\mathbb{R}^d)} \|\phi \delta n\|_{L^2(\mathbb{R}^d)} \|\phi \nabla \delta n\|_{L^2(\mathbb{R}^d)} + \chi \|\nabla c\|_{L^\infty(\mathbb{R}^d)} \|\phi \delta n\|_{L^2(\mathbb{R}^d)} \|\phi \nabla \delta n\|_{L^2(\mathbb{R}^d)} \\
& + \chi \|\tilde{n}\|_{L^\infty(\mathbb{R}^d)} \|\phi \nabla \delta c\|_{L^2(\mathbb{R}^d)} \|\phi \nabla \delta n\|_{L^2(\mathbb{R}^d)} + 2\|\nabla \phi\|_{L^\infty(\mathbb{R}^d)} \|\phi \nabla \delta n\|_{L^2(\mathbb{R}^d)} \|\delta n\|_{L^2(\mathbb{R}^d)} \\
& + 2\chi \|\nabla \phi\|_{L^\infty(\mathbb{R}^d)} \|\delta n\|_{L^2(\mathbb{R}^d)} \left(\|\nabla c\|_{L^\infty(\mathbb{R}^d)} \|\phi \delta n\|_{L^2(\mathbb{R}^d)} + \|\tilde{n}\|_{L^\infty(\mathbb{R}^d)} \|\phi \nabla \delta c\|_{L^2(\mathbb{R}^d)}\right) \\
& + \|\phi \delta n\|_{L^2(\mathbb{R}^d)}^2 (1 + \epsilon q \left(\|n\|_{L^\infty(\mathbb{R}^d)} + \|\tilde{n}\|_{L^\infty(\mathbb{R}^d)}\right)^{q-1}) \\
\leq\ & \frac{1}{2}\|\phi \nabla \delta n\|_{L^2(\mathbb{R}^d)}^2 + C\chi^2 \|\nabla c\|_{L^\infty(\mathbb{R}^d)}^2 \|\phi \delta n\|_{L^2(\mathbb{R}^d)}^2 + C\chi^2 \|\widetilde{n}\|_{L^\infty(\mathbb{R}^d)}^2 \|\phi \nabla \delta c\|_{L^2(\mathbb{R}^d)}^2 \\
& + C\|u\|_{L^\infty(\mathbb{R}^d)}^2 \|\phi \delta n\|_{L^2(\mathbb{R}^d)}^2 + \|\phi \delta n\|_{L^2(\mathbb{R}^d)}^2 (1 + \epsilon q \left(\|n\|_{L^\infty(\mathbb{R}^d)} + \|\tilde{n}\|_{L^\infty(\mathbb{R}^d)}\right)^{q-1}) + \frac{C}{R}.
\end{aligned}$$

Likewise, making use of Hölder's and Young's inequalities, we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi \nabla \delta c(t)\|_{L^2(\mathbb{R}^d)}^2 + \|\phi \Delta \delta c(t)\|_{L^2(\mathbb{R}^d)}^2 + \|\phi \nabla \delta c(t)\|_{L^2(\mathbb{R}^d)}^2$$

$$
\begin{aligned}
&= \int\limits_{\mathbb{R}^d} u\phi\nabla\delta c\cdot\phi\Delta\delta c\,\mathrm{d}x + 2\int\limits_{\mathbb{R}^d} u\phi\nabla\delta c\cdot\nabla\phi\nabla\delta c\,\mathrm{d}x + \int\limits_{\mathbb{R}^d} \phi^2\nabla\delta n\cdot\nabla\delta c\,\mathrm{d}x\\
&\quad -2\int\limits_{\mathbb{R}^d} \phi\Delta\delta c\nabla\phi\cdot\nabla\delta c\,\mathrm{d}x + \int\limits_{\mathbb{R}^d} \nabla\delta n\phi^2\nabla\delta c\,\mathrm{d}x\\
&\le C\|u\|^2_{L^\infty(\mathbb{R}^d)}\|\phi\nabla\delta c\|^2_{L^2(\mathbb{R}^d)} + \|\phi\delta n\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{4}\|\phi\Delta\delta c\|^2_{L^2(\mathbb{R}^d)}\\
&\quad +C\|\nabla\phi\|^2_{L^\infty(\mathbb{R}^d)}\|\nabla\delta c\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{4}\|\phi\nabla\delta n\|^2_{L^2(\mathbb{R}^d)} + C\|\phi\nabla\delta c\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

Summing up the above two estimates, we get that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\phi\delta n(t)\|^2_{L^2(\mathbb{R}^d)} + \|\phi\nabla\delta c(t)\|^2_{L^2(\mathbb{R}^d)}\right)\\
&\le \|\phi\delta n\|^2_{L^2(\mathbb{R}^d)}\left(C\chi^2\|\nabla c\|^2_{L^\infty(\mathbb{R}^d)} + C\|u\|^2_{L^\infty(\mathbb{R}^d)} + \epsilon q\left(\|n\|_{L^\infty(\mathbb{R}^d)} + \|\tilde n\|_{L^\infty(\mathbb{R}^d)}\right)^{q-1} + 1\right)\\
&\quad +\|\phi\nabla\delta c\|^2_{L^2(\mathbb{R}^d)}\left(C\chi^2\|\tilde n\|^2_{L^\infty(\mathbb{R}^d)} + C\|u\|^2_{L^\infty(\mathbb{R}^d)} + C\right) + \frac{C}{R} + \frac{C}{R^2}.
\end{aligned}
$$

Integrating in t and letting R go to infinite produces

$$
\begin{aligned}
&\|\delta n(t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla\delta c(t)\|^2_{L^2(\mathbb{R}^d)}\\
&\le \int\limits_0^t \|\delta n\|^2_{L^2(\mathbb{R}^d)}\left(C\chi^2\|\nabla c\|^2_{L^\infty(\mathbb{R}^d)}+C\|u\|^2_{L^\infty(\mathbb{R}^d)}+\epsilon q\left(\|n\|_{L^\infty(\mathbb{R}^d)}+\|\tilde n\|_{L^\infty(\mathbb{R}^d)}\right)^{q-1}+1\right)\mathrm{d}x\\
&\quad +\int\limits_0^t \|\nabla\delta c\|^2_{L^2(\mathbb{R}^d)}\left(C\chi^2\|\tilde n\|^2_{L^\infty(\mathbb{R}^d)} + C\|u\|^2_{L^\infty(\mathbb{R}^d)} + C\right)\mathrm{d}s.
\end{aligned}
$$

The Grönwall inequality implies $(n,c)\equiv(\widetilde n,\widetilde c)$ in $\mathbb{R}^d\times(0,T_{\max})$.

According to our definition of T, it is obvious that (n,c) can be extended up to a maximal $T_{\max}\in(0,\infty]$ satisfying

$$
\|n(\cdot,t)\|_{L^1(\mathbb{R}^d)} + \|n(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|c(\cdot,t)\|_{H^1(\mathbb{R}^d)} + \|c(\cdot,t)\|_{W^{1,\infty}(\mathbb{R}^d)} \to \infty \text{ as } t\nearrow T_{\max}. \tag{2.5}
$$

Assuming (2.5) is false, then there exists a constant $L>0$ such that

$$
\|n(\cdot,t)\|_{L^1(\mathbb{R}^d)} + \|n(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|c(\cdot,t)\|_{H^1(\mathbb{R}^d)} + \|c(\cdot,t)\|_{W^{1,q}(\mathbb{R}^d)} \le L.
$$

From the method of choosing $\widetilde T$, we take

$$
\varepsilon = \min\{T_1, T_2, T_3\}.
$$

According to the above existence proof, we summarize that there exists a unique solution $(\bar{n}, \bar{c})$ with initial data $(n\,(T_{\max} - \varepsilon/2)\,, c\,(T_{\max} - \varepsilon/2))$ on $[0, \varepsilon]$. In addition, we obtain by uniqueness that

$$\bar{n}(t) = n\,(t + T_{\max} - \varepsilon/2) \quad \text{and} \quad \bar{c}(t) = c\,(t + T_{\max} - \varepsilon/2)$$

on $[0, \varepsilon/2)$, which means that $(\bar{n}, \bar{c})$ extends solution (n, c) beyond $T_{\max}$. This contradicts the definition of $T_{\max}$. Consequently, (2.5) is true.

2.4. Continuation

We need to guarantee that the quantities $\|n(t)\|_{L^1(\mathbb{R}^d)}$ and $\|c(t)\|_{H^1(\mathbb{R}^d)}$ do not blow up at finite time.

Proposition 2.1. Suppose (n, c) is the solution of system (1.5) and $q \geq 2$. Then, for all $t \in (0, T_{max})$, we have the following estimates:

$$\|n(t)\|_{L^1(\mathbb{R}^d)} + \int_0^t \|n(s)\|^q_{L^q(\mathbb{R}^d)}\, \mathrm{d}s \leq Ce^t, \tag{2.6}$$

$$\|c(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|c(s)\|^2_{H^1(\mathbb{R}^d)} \mathrm{d}s \leq Ce^{Ct}, \tag{2.7}$$

and

$$\|\nabla c(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla c(s)\|^2_{H^1(\mathbb{R}^d)} \mathrm{d}s \leq Ce^{Ct}. \tag{2.8}$$

Proof. For $n \geq 0$, multiplying Eq. $(1.5)_1$ by ϕ and then integrating in x, we get that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\phi n(t)\|_{L^1(\mathbb{R}^d)} + \epsilon\|\phi^{\frac{1}{q}} n(t)\|^q_{L^q(\mathbb{R}^d)} \\
\leq\ & \|u\|_{L^\infty(\mathbb{R}^d)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)}\|n\|_{L^1(\mathbb{R}^d)} + \|\Delta\phi\|_{L^\infty(\mathbb{R}^d)}\|n\|_{L^1(\mathbb{R}^d)} \\
&+\|\nabla c\|_{L^\infty(\mathbb{R}^d)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)}\|n\|_{L^1(\mathbb{R}^d)} + \|\phi n(t)\|_{L^1(\mathbb{R}^d)} \\
\leq\ & \frac{C}{R}\|u\|_{L^\infty(\mathbb{R}^d)}\|n\|_{L^1(\mathbb{R}^d)} + \frac{C}{R}\|\nabla c\|_{L^\infty(\mathbb{R}^d)}\|n\|_{L^1(\mathbb{R}^d)} + \frac{C}{R^2}\|n\|_{L^1(\mathbb{R}^d)} + \|\phi n(t)\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Applying Grönwall's inequality and taking R approaching infinite, we have

$$\|n(t)\|_{L^1(\mathbb{R}^d)} + \int_0^t \|n(s)\|^q_{L^q(\mathbb{R}^d)}\, \mathrm{d}s \leq Ce^t\, \|n_0\|_{L^1(\mathbb{R}^d)}\,.$$

Similarly, performing $\phi^2 c$ to Eq. $(1.5)_2$ yields that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi c(t)\|_{L^2(\mathbb{R}^d)}^2+\|\phi\nabla c(t)\|_{L^2(\mathbb{R}^d)}^2+\|\phi c(t)\|_{L^2(\mathbb{R}^d)}^2\\
\le\ &\|u\|_{L^\infty(\mathbb{R}^d)}\|\phi\nabla c\|_{L^2(\mathbb{R}^d)}\|\phi c\|_{L^2(\mathbb{R}^d)}+2\|\phi\nabla c\|_{L^2(\mathbb{R}^d)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)}\|c\|_{L^2(\mathbb{R}^d)}\\
&+\|\phi n\|_{L^2(\mathbb{R}^d)}\|\phi c\|_{L^2(\mathbb{R}^d)}\\
\le\ &\frac{1}{2}\|\phi\nabla c\|_{L^2(\mathbb{R}^d)}^2+\|\phi c\|_{L^2(\mathbb{R}^d)}^2(\|u\|_{L^\infty(\mathbb{R}^d)}^2+1)+\|\phi n\|_{L^2(\mathbb{R}^d)}^2+\frac{C}{R^2}\|c\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned}
$$

Using Gronwall's inequality and taking R to infinite, we achieve that

$$
\begin{aligned}
&\|c(t)\|_{L^2(\mathbb{R}^d)}^2+\int_0^t\left(\|\nabla c(s)\|_{L^2(\mathbb{R}^d)}^2+\|c(s)\|_{L^2(\mathbb{R}^d)}^2\right)\mathrm{d}s\\
&\le Ce^{\int_0^t(\|u(s)\|_{L^\infty(\mathbb{R}^d)}^2+1)\,\mathrm{d}s}\left(\|c_0\|_{L^2(\mathbb{R}^d)}^2+\int_0^t\|n(s)\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s\right).
\end{aligned}
$$

Owing to the estimate (2.6) and $q\ge 2$, we can easily obtain that

$$
\|c(t)\|_{L^2(\mathbb{R}^d)}^2+\int_0^t\|c(s)\|_{H^1(\mathbb{R}^d)}^2\,\mathrm{d}s\le Ce^{Ct}.
$$

By the same process, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\nabla c(t)\|_{L^2(\mathbb{R}^d)}^2+\|\phi\Delta c(t)\|_{L^2(\mathbb{R}^d)}^2+\|\phi\nabla c(t)\|_{L^2(\mathbb{R}^d)}^2\\
=&\int_{\mathbb{R}^d}u\phi\nabla c\cdot\phi\Delta c\,\mathrm{d}x+2\int_{\mathbb{R}^d}u\nabla c\nabla\phi\cdot\phi\nabla c\,\mathrm{d}x-\int_{\mathbb{R}^d}\phi^2 n\Delta c\,\mathrm{d}x\\
&-2\int_{\mathbb{R}^d}\phi\nabla c\cdot(\nabla\phi\cdot\Delta c)\mathrm{d}x-2\int_{\mathbb{R}^d}\phi n\nabla\phi\cdot\nabla c\,\mathrm{d}x\\
\le\ &2\|u\|_{L^\infty(\mathbb{R}^d)}^2\|\phi\nabla c\|_{L^2(\mathbb{R}^d)}^2+2\|\phi n\|_{L^2(\mathbb{R}^d)}^2+\frac{1}{2}\left\|\phi\nabla^2 c\right\|_{L^2(\mathbb{R}^d)}^2+\frac{C}{R^2}\|\nabla c\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned}
$$

Therefore, we can get

$$
\|\nabla c(t)\|_{L^2(\mathbb{R}^d)}^2+\int_0^t\|\nabla c(s)\|_{H^1(\mathbb{R}^d)}^2\mathrm{d}s\le Ce^{Ct}.\quad\square
$$

3. Proof of main result

In this section, we prove Theorem 1.1. First construct the approximate schemes for the Cauchy problem (1.5)-(1.6) as follows:

$$\begin{cases} \partial_t n_M - u \cdot \nabla n_M - \Delta n_M = -\chi \nabla \cdot (n_M \nabla c_M) + n_M - \epsilon n_M^q \\ \partial_t c_M - u \cdot \nabla c_M - \Delta c_M = -c_M + n_M \end{cases} \quad \text{in } \mathbb{R}^d \times \mathbb{R}^+. \tag{3.1}$$

$$n_M(0,x) = \psi(x/M) n_0(x), \quad c_M(0,x) = \psi(x/M) c_0(x) \quad \text{in } \mathbb{R}^d. \tag{3.2}$$

Here ψ is a smooth function satisfying

$$\psi(x) = \begin{cases} 1, & x \in B_1(0) \\ 0, & x \in \mathbb{R}^d \backslash B_2(0). \end{cases}$$

Due to $n_0 \in L^\infty\left(\mathbb{R}^d\right)$ and $c_0 \in W^{1,\infty}\left(\mathbb{R}^d\right)$, the initial data in (3.2) satisfy

$$n_M(0,x) \in L^1\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right),$$

and

$$c_M(0,x) \in H^1\left(\mathbb{R}^d\right) \cap W^{1,\infty}\left(\mathbb{R}^d\right).$$

In addition, we know from Theorem 2.1 that the Cauchy problem (3.1)-(3.2) admits a unique nonnegative solution $(n_M, c_M) \in \left(C^{2,1}\left(\mathbb{R}^d \times (0, T_{\max})\right)\right)^2$ such that

$$\begin{aligned} & n_M \in C^0\left([0, T_{\max}); L^1\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right)\right) \cap C^{2,1}\left(\mathbb{R}^d \times (0, T_{\max})\right), \\ & c_M \in C^0\left([0, T_{\max}); H^1\left(\mathbb{R}^d\right) \cap W^{1,\infty}\left(\mathbb{R}^d\right)\right) \cap C^{2,1}\left(\mathbb{R}^d \times (0, T_{\max})\right), \end{aligned} \tag{3.3}$$

if $T_{\max} < \infty$, then

$$\limsup_{T \to T_{\max}-} \left(\|n_M(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|c_M(\cdot,t)\|_{W^{1,\infty}(\mathbb{R}^d)}\right) = \infty. \tag{3.4}$$

This indicates that the solutions of the regularized problem (3.1)-(3.2) are smooth and decay rapidly fast at infinity, allowing us to carry out the following calculations using integration by parts without boundary terms. Moreover, we denote by $L^{p,R}_{\text{uloc}}\left(\mathbb{R}^d\right)$ the uniformly local space composed of the local integral function f fulfilling

$$\|f\|_{p,R} < +\infty.$$

What we need to explain here is that $L^{p,R}_{\text{uloc}}\left(\mathbb{R}^d\right)$ coincides with space $L^{p}_{\text{uloc}}\left(\mathbb{R}^d\right)$ defined in Notation for arbitrary $R > 0$. Actually, it is explicit that for $R \geq 1$,

$$\|f\|_{p,1} \leq \|f\|_{p,R}.$$

Furthermore, through the covering theorem, we know that for $R \geq 1$,

$$\|f\|_{p,R} \leq \left(CR^d\right)^{\frac{1}{p}} \|f\|_{p,1}.$$

To simplify notation, we will use (n, c) to refer to (n_M, c_M) and $L^p_{\mathrm{uloc}}\left(\mathbb{R}^d\right)$ to denote $L^{p,R}_{\mathrm{uloc}}\left(\mathbb{R}^d\right)$ for the remainder of this section. Next, we will establish key a priori estimates for (n_M, c_M) by employing the uniformly local space $L^{p,R}_{\mathrm{uloc}}\left(\mathbb{R}^d\right)$.

Proposition 3.1. Suppose $\epsilon, \chi > 0, q \geq 2$. Let the couple (n, c) be the solution of system (3.1)-(3.2), then there exists a $R_0 > 1$ such that for all $R \geq R_0$ and $T \in (0, T_{\max})$,

$$\begin{aligned}
&\|n\|_{L^\infty_T L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + \frac{\chi}{2}\|\nabla c\|^2_{L^\infty_T L^2_{\mathrm{uloc}}(\mathbb{R}^d)} \\
&\leq 4\|n_0\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + 2\chi\|\nabla c_0\|^2_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)} + C(\epsilon, \chi, R, d).
\end{aligned}$$

Proof. Using the fact

$$\nabla\Delta c \cdot \nabla c = \frac{1}{2}\Delta|\nabla c|^2 - \left|D^2 c\right|^2,$$

from the system (3.1), we can deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(n + \frac{\chi}{2}|\nabla c|^2\right) - \Delta\left(n + \frac{\chi}{2}|\nabla c|^2\right) + \chi\left(|D^2 c|^2 + |\nabla c|^2\right) + \epsilon n^q \\
= &-\chi n\Delta c - u\cdot\nabla n - \chi\nabla(u\cdot\nabla c)\nabla c + n.
\end{aligned}$$

Multiplying the above equation by $\phi^R_{x_0}$ introduced in (1.7) and taking the integral over $\mathbb{R}^d$ give

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\left(n + \frac{\chi}{2}|\nabla c|^2\right)\phi^R_{x_0}\,\mathrm{d}x - \int_{\mathbb{R}^d}\Delta\left(n + \frac{\chi}{2}|\nabla c|^2\right)\phi^R_{x_0}\,\mathrm{d}x + \int_{\mathbb{R}^d}\chi\left(|D^2 c|^2 + |\nabla c|^2\right)\phi^R_{x_0}\,\mathrm{d}x \\
&+\epsilon\int_{\mathbb{R}^d} n^q\phi^R_{x_0}\,\mathrm{d}x \\
&= -\chi\int_{\mathbb{R}^d} n\Delta c\phi^R_{x_0}\,\mathrm{d}x - \int_{\mathbb{R}^d} u\cdot\nabla n\phi^R_{x_0}\,\mathrm{d}x - \chi\int_{\mathbb{R}^d}\nabla(u\cdot\nabla c)\nabla c\phi^R_{x_0}\,\mathrm{d}x + \int_{\mathbb{R}^d} n\phi^R_{x_0}\,\mathrm{d}x.
\end{aligned}$$

Integrating by parts, together with the definition of function $\phi^R_{x_0}$, we get

$$\int_{\mathbb{R}^d}\Delta\left(n + \frac{\chi}{2}|\nabla c|^2\right)\phi^R_{x_0}\,\mathrm{d}x = \int_{B_{2R}(x_0)}\left(n + \frac{\chi}{2}|\nabla c|^2\right)\Delta\phi^R_{x_0}\,\mathrm{d}x.$$

For a ball family $\{B_R(y_i)\}_{i=1}^{3^d}$, we have

$$B_{2R}(x_0) \subseteq \bigcup_{i=1}^{3^d} B_R(y_i).$$

Then we can infer that

$$\int_{B_{2R}(x_0)} \left(n + \frac{\chi}{2}|\nabla c|^2\right) \Delta\phi_{x_0}^R \, \mathrm{d}x \le \frac{C}{R^2} \int_{B_{2R}(x_0)} \left(n + \frac{\chi}{2}|\nabla c|^2\right) \mathrm{d}x$$
$$\le \frac{C}{R^2} \sum_{i=1}^{3^d} \int_{B_R(y_i)} \left(n + \frac{\chi}{2}|\nabla c|^2\right) \mathrm{d}x$$
$$\le \frac{C3^d}{R^2} \left(\|n\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + \frac{\chi}{2}\|\nabla c\|^2_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}\right).$$

$$\|n\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} \le \int_{\mathbb{R}^d} \phi_{x_0}^R n dx = \int_{\mathbb{R}^d} (\phi_{x_0}^R)^{1-\frac{1}{q}} (\phi_{x_0}^R)^{\frac{1}{q}} n dx \le C(d,q) R^{\frac{d(q-1)}{q}} \left(\int_{\mathbb{R}^d} \phi_{x_0}^R n^q dx\right)^{\frac{1}{q}}$$
$$\le C(d,q) R^d + \frac{\epsilon R^2}{C8 \cdot 3^d} \int_{\mathbb{R}^d} n^q \phi_{x_0}^R dx.$$

Because $|\Delta c|^2 \le d|D^2 c|^2$, using Hölder's inequality and Young's inequality, we obtain the following result

$$-\chi \int_{\mathbb{R}^d} n \Delta c \phi_{x_0}^R \, \mathrm{d}x \le \frac{\chi}{d} \int_{\mathbb{R}^d} |\Delta c|^2 \phi_{x_0}^R \, \mathrm{d}x + \frac{d\chi}{4} \int_{\mathbb{R}^d} n^2 \phi_{x_0}^R \, \mathrm{d}x$$
$$\le \chi \int_{\mathbb{R}^d} |D^2 c|^2 \phi_{x_0}^R \, \mathrm{d}x + \frac{d\chi}{4} \int_{\mathbb{R}^d} n^2 \phi_{x_0}^R \, \mathrm{d}x,$$

where

$$\int_{\mathbb{R}^d} \phi_{x_0}^R n^2 dx = \int_{\mathbb{R}^d} (\phi_{x_0}^R)^{1-\frac{2}{q}} (\phi_{x_0}^R)^{\frac{2}{q}} n^2 dx \le C(d,q) R^{\frac{d(q-2)}{q}} \left(\int_{\mathbb{R}^d} \phi_{x_0}^R n^q dx\right)^{\frac{2}{q}}$$
$$\le C(d,q) R^d + \frac{\epsilon 4}{8 d\chi} \int_{\mathbb{R}^d} n^q \phi_{x_0}^R dx.$$

Similarly, we have

$$-\int_{\mathbb{R}^d} u \cdot \nabla n \phi_{x_0}^R \, \mathrm{d}x \le \int_{\mathbb{R}^d} u \cdot n \nabla \phi_{x_0}^R \, \mathrm{d}x \le \frac{\|u\|^2_{L^\infty}}{2} \int_{\mathbb{R}^d} |\nabla \phi_{x_0}^R|^2 (\phi_{x_0}^R)^{-1} \, \mathrm{d}x + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} n^2 \phi_{x_0}^R \, \mathrm{d}x$$
$$\le \frac{C}{R^2} + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} n^2 \phi_{x_0}^R \, \mathrm{d}x \le C(R,d,q) + + \frac{\epsilon}{8} \int_{\mathbb{R}^d} n^q \phi_{x_0}^R dx$$

and

$$-\chi \int_{\mathbb{R}^d} \nabla(u \cdot \nabla c)\nabla c \phi_{x_0}^R \, \mathrm{d}x = \chi \int_{\mathbb{R}^d} (u \cdot \nabla c)\nabla^2 c \phi_{x_0}^R \, \mathrm{d}x + \chi \int_{\mathbb{R}^d} (u \cdot \nabla c)\nabla c \nabla \phi_{x_0}^R \, \mathrm{d}x$$
$$= -\frac{\chi}{2} \int_{B_{2R}(x_0)} u \cdot |\nabla c|^2 \nabla \phi_{x_0}^R \, \mathrm{d}x + \chi \int_{B_{2R}(x_0)} (u \cdot \nabla c)\nabla c \nabla \phi_{x_0}^R \, \mathrm{d}x$$
$$\leq \frac{C3^d \chi}{R} \|\nabla c\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^2.$$

Collecting the above estimates, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \left(n + \frac{\chi}{2}|\nabla c|^2\right) \phi_{x_0}^R \, \mathrm{d}x + 2\int_{\mathbb{R}^d} \left(n + \frac{\chi}{2}|\nabla c|^2\right) \phi_{x_0}^R \, \mathrm{d}x \leq \frac{C3^d \chi}{2R^2} \|\nabla c\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^2.$$

For any $t \in [0, T]$, we have the following estimates

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \left(n + \frac{\chi}{2}|\nabla c|^2\right) \phi_{x_0}^R \, \mathrm{d}x + 2\int_{\mathbb{R}^d} \left(n + \frac{\chi}{2}|\nabla c|^2\right) \phi_{x_0}^R \, \mathrm{d}x \leq \frac{C3^d \chi}{2R^2} \|\nabla c\|_{L^\infty_T L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^2.$$

We know that $y'(t) + cy(t) \leq C$ in $(0, T]$ implies for all $t \in (0, T]$,

$$y \leq \max\left\{y(0), \frac{C}{c}\right\},$$

then we get the following relation, for any $t \in (0, T]$,

$$\int_{\mathbb{R}^d} \left(n(t) + \frac{\chi}{2}|\nabla c(t)|^2\right) \phi_{x_0}^R \, \mathrm{d}x$$
$$\leq \max\left\{\int_{\mathbb{R}^d} n_0 \phi_{x_0}^R \, \mathrm{d}x + \frac{\chi}{2}\int_{\mathbb{R}^d} |\nabla c_0|^2 \phi_{x_0}^R \, \mathrm{d}x, \frac{C3^d \chi}{4R^2} \|\nabla c\|_{L^\infty_T L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^2\right\}.$$

According to the definition of $L^p_{uloc}(\mathbb{R}^d)$ and the properties of $\phi_{x_0}^R$, we obtain from the above inequality that for any $0 < t \leq T$,

$$\|n(t)\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + \frac{\chi}{2}\|\nabla c(t)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^2$$
$$\leq \max\left\{2\|n_0\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + \chi\|\nabla c_0\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^2, \frac{C3^d \chi}{4R^2} \|\nabla c\|_{L^\infty_T L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^2\right\}.$$

Taking sup for the time variable t, we have that

$$
\begin{aligned}
&\|n\|_{L_T^\infty L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + \frac{\chi}{2}\|\nabla c\|^2_{L_T^\infty L^2_{\mathrm{uloc}}(\mathbb{R}^d)} \\
\le\ &\max\left\{4\|n_0\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + 2\chi\|\nabla c_0\|^2_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}, \frac{\chi C 3^d}{2R^2}\|\nabla c\|^2_{L_T^\infty L^2_{\mathrm{uloc}}(\mathbb{R}^d)}\right\}.
\end{aligned}
$$

Taking some $R_0 \ge 1$, for any $R \ge R_0$, we get

$$
\frac{C3^d}{R^2} \le \frac{1}{8}.
$$

Then we deduce that

$$
\begin{aligned}
&\|n\|_{L_T^\infty L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + \frac{\chi}{2}\|\nabla c\|^2_{L_T^\infty L^2_{\mathrm{uloc}}(\mathbb{R}^d)} \\
\le\ &4\|n_0\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + 2\chi\|\nabla c_0\|^2_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)} + C(\epsilon, \chi, R, d).
\end{aligned}
$$

Hence the proof is completed. □

Proposition 3.2. Assume $\chi > 0$, $R \ge 1$ and $\epsilon \ge 0$. Then for any $k \in \mathbb{N}$ with $k \ge 2, q \ge 2$, there exist constant C and C_k depending on k, χ such that the solution (n, c) of equations (3.1)-(3.2) satisfies

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} n^k \phi^R_{x_0}\,\mathrm{d}x + \frac{k(k-1)}{4}\int_{\mathbb{R}^d} |\nabla n|^2 n^{k-2}\phi^R_{x_0}\,\mathrm{d}x \\
\le\ &\frac{C3^d k}{2(k-1)R^2}\|n\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)} + \frac{C3^d k}{R^{2k}}\|\nabla c\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)} + \frac{3(q-1)}{2(k+q-1)}CR^d + CR^{\frac{k-1}{2k(k+1)}} \\
&+ (C_k - \epsilon k)\int_{\mathbb{R}^d} n^{k+q-1}\phi^R_{x_0}\,\mathrm{d}x + k\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x.
\end{aligned}
\tag{3.5}
$$

Proof. Multiplying the first equation of (3.1) by $n^{k-1}\phi^R_{x_0}$ and integrating in x give

$$
\begin{aligned}
&\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} n^k\phi^R_{x_0}\,\mathrm{d}x - \int_{\mathbb{R}^d} \Delta n n^{k-1}\phi^R_{x_0}\,\mathrm{d}x \\
=\ &-\chi\int_{\mathbb{R}^d} \nabla\cdot(n\nabla c)n^{k-1}\phi^R_{x_0}\,\mathrm{d}x + \frac{1}{k}\int_{\mathbb{R}^d} u\cdot n^k\nabla\phi^R_{x_0}\,\mathrm{d}x + \int_{\mathbb{R}^d} n^k\phi^R_{x_0}\,\mathrm{d}x - \epsilon\int_{\mathbb{R}^d} n^{k+q-1}\phi^R_{x_0}\,\mathrm{d}x.
\end{aligned}
$$

Using integration by parts, one gets

$$
-\int_{\mathbb{R}^d} \Delta n n^{k-1}\phi^R_{x_0}\,\mathrm{d}x = (k-1)\int_{\mathbb{R}^d} |\nabla n|^2 n^{k-2}\phi^R_{x_0}\,\mathrm{d}x + \int_{\mathbb{R}^d} n^{k-1}\nabla n\cdot\nabla\phi^R_{x_0}\,\mathrm{d}x,
$$

and

$$-\chi \int_{\mathbb{R}^d} \nabla \cdot (n\nabla c) n^{k-1} \phi_{x_0}^R \,\mathrm{d}x = \chi \int_{\mathbb{R}^d} n^k \nabla c \cdot \nabla \phi_{x_0}^R \,\mathrm{d}x + \chi(k-1) \int_{\mathbb{R}^d} n^{k-1} \nabla c \cdot \nabla n \phi_{x_0}^R \,\mathrm{d}x.$$

Then we have the following result

$$\begin{aligned} &\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} n^k \phi_{x_0}^R \,\mathrm{d}x + (k-1) \int_{\mathbb{R}^d} |\nabla n|^2 n^{k-2} \phi_{x_0}^R \,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d} n^{k-1} \nabla n \cdot \nabla \phi_{x_0}^R \,\mathrm{d}x + \chi \int_{\mathbb{R}^d} n^k \nabla c \cdot \nabla \phi_{x_0}^R \,\mathrm{d}x + \frac{1}{k} \int_{\mathbb{R}^d} u \cdot n^k \nabla \phi_{x_0}^R \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^d} n^k \phi_{x_0}^R \,\mathrm{d}x - \epsilon \int_{\mathbb{R}^d} n^{k+q-1} \phi_{x_0}^R \,\mathrm{d}x + \chi(k-1) \int_{\mathbb{R}^d} n^{k-1} \nabla c \cdot \nabla n \phi_{x_0}^R \,\mathrm{d}x. \end{aligned} \tag{3.6}$$

Combining with Hölder's inequality and Young's inequality, we get

$$-\int_{\mathbb{R}^d} n^{k-1} \nabla n \cdot \nabla \phi_{x_0}^R \,\mathrm{d}x \le \frac{k-1}{2} \int_{\mathbb{R}^d} |\nabla n|^2 n^{k-2} \phi_{x_0}^R \,\mathrm{d}x + \frac{C3^d k}{2(k-1)R^2} \|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k. \tag{3.7}$$

Similarly, we have

$$\begin{aligned} &\chi \int_{\mathbb{R}^d} n^k \nabla c \cdot \nabla \phi_{x_0}^R \,\mathrm{d}x \\ &\le \chi^{\frac{k+1}{k}} \int_{\mathbb{R}^d} n^{k+1} \phi_{x_0}^R \,\mathrm{d}x + \int_{\mathbb{R}^d} |\nabla c|^{2k} \left(\phi_{x_0}^R\right)^{-\frac{k}{k+1}\cdot 2k} \left|\nabla \phi_{x_0}^R\right|^{2k} \,\mathrm{d}x + \left(\int_{B_{2R}(x_0)} 1 \,\mathrm{d}x \right)^{\frac{k-1}{2k(k+1)}} \\ &\le \chi^{\frac{k+1}{k}} \int_{\mathbb{R}^d} n^{k+1} \phi_{x_0}^R \,\mathrm{d}x + \frac{C}{R^{2k}} \int_{B_{2R}(x_0)} |\nabla c|^{2k} \,\mathrm{d}x + CR^{\frac{k-1}{2k(k+1)}} \\ &\le \chi^{\frac{k+1}{k}} \int_{\mathbb{R}^d} n^{k+1} \phi_{x_0}^R \,\mathrm{d}x + \frac{C3^d}{R^{2k}} \|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} + CR^{\frac{k-1}{2k(k+1)}}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\chi(k-1) \int_{\mathbb{R}^d} n^{k-1} \nabla c \cdot \nabla n \phi_{x_0}^R \,\mathrm{d}x \\ &\le \frac{k-1}{4} \int_{\mathbb{R}^d} |\nabla n|^2 n^{k-2} \phi_{x_0}^R \,\mathrm{d}x + \chi^2 (k-1) \int_{\mathbb{R}^d} n^k |\nabla c|^2 \phi_{x_0}^R \,\mathrm{d}x \qquad (3.9) \\ &\le \frac{k-1}{4} \int_{\mathbb{R}^d} |\nabla n|^2 n^{k-2} \phi_{x_0}^R \,\mathrm{d}x + \int_{\mathbb{R}^d} n^2 |\nabla c|^{2k-2} \phi_{x_0}^R \,\mathrm{d}x + \chi^{\frac{2(k-1)}{k-2}} (k-1)^{\frac{k-1}{k-2}} \int_{\mathbb{R}^d} n^{k+1} \phi_{x_0}^R \,\mathrm{d}x. \end{aligned}$$

Then we conclude

$$\begin{aligned}\frac{1}{k}\int_{\mathbb{R}^d} u\cdot n^k\nabla\phi_{x_0}^R\,\mathrm{d}x &\le \frac{C}{2k^2}\int_{\mathbb{R}^d} n^k|\nabla\phi_{x_0}^R|^2(\phi_{x_0}^R)^{-1}\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^d} n^k\phi_{x_0}^R\,\mathrm{d}x \\ &\le \frac{C3^d}{2k^2R^2}\|n\|_{L^k_{\mathrm{uloc}}}^k+\frac{k}{2(k+q-1)}\int_{\mathbb{R}^d} n^{k+q-1}\phi_{x_0}^R\,\mathrm{d}x+\frac{q-1}{2(k+q-1)}CR^d,\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}\chi^{\frac{k+1}{k}}\int_{\mathbb{R}^d} n^{k+1}\phi_{x_0}^R\,\mathrm{d}x &\le \frac{k+1}{k+q-1}\chi^{\frac{k+1}{k}\cdot\frac{k+q-1}{k+1}}\int_{\mathbb{R}^d} n^{k+q-1}\phi_{x_0}^R\,\mathrm{d}x+\frac{q-2}{k+q-1}CR^d \\ &\le \frac{k+1}{k+q-1}\chi^{\frac{k+q-1}{k}}\int_{\mathbb{R}^d} n^{k+q-1}\phi_{x_0}^R\,\mathrm{d}x+\frac{q-2}{k+q-1}CR^d,\end{aligned}$$

and

$$\begin{aligned}&\chi^{\frac{2(k-1)}{k-2}}(k-1)^{\frac{k-1}{k-2}}\int_{\mathbb{R}^d} n^{k+1}\phi_{x_0}^R\,\mathrm{d}x \\ &\le \frac{k+1}{k+q-1}\chi^{\frac{2(k-1)}{k-2}\cdot\frac{k+q-1}{k+1}}(k-1)^{\frac{k-1}{k-2}\cdot\frac{k+q-1}{k+1}}\int_{\mathbb{R}^d} n^{k+q-1}\phi_{x_0}^R\,\mathrm{d}x+\frac{q-2}{k+q-1}CR^d \\ &\le \frac{k+1}{k+q-1}\chi^{\frac{2(k-1)(k+q-1)}{k^2-k-2}}(k-1)^{\frac{(k-1)(k+q-1)}{k^2-k-2}}\int_{\mathbb{R}^d} n^{k+q-1}\phi_{x_0}^R\,\mathrm{d}x+\frac{q-2}{k+q-1}CR^d.\end{aligned}$$

Substituting (3.7)-(3.10) into (3.6) implies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} n^k\phi_{x_0}^R\,\mathrm{d}x+\frac{k(k-1)}{4}\int_{\mathbb{R}^d}|\nabla n|^2 n^{k-2}\phi_{x_0}^R\,\mathrm{d}x \\ &\le \frac{C3^dk}{2(k-1)R^2}\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k+\frac{C3^d}{2k^2R^2}\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k \\ &\quad+\frac{C3^dk}{R^{2k}}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}+\frac{3(q-1)}{2(k+q-1)}CR^d+CR^{\frac{k-1}{2k(k+1)}} \\ &\quad+k\left(\frac{k}{2(k+q-1)}+\frac{k+1}{k+q-1}(\chi^{\frac{k+q-1}{k}}+(\chi^{\frac{2(k-1)}{k-2}}(k-1)^{\frac{k-1}{k-2}})^{\frac{k+q-1}{k+1}})-\epsilon\right)\int_{\mathbb{R}^d} n^{k+q-1}\phi_{x_0}^R\,\mathrm{d}x \\ &\quad+k\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x.\end{aligned}$$

This completes the proof of Proposition 3.2. □

Then we need to estimate $\|\nabla c\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}$ and $\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x$.

Proposition 3.3. Assume $R \geq 1$ and (n, c) is the solution of system (3.1)-(3.2). For any $k \in \mathbb{N}$ with $k \geq 2$, $q \geq 2$, one can find an absolute constant C such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |\nabla c|^{2k}\phi^R_{x_0}\,\mathrm{d}x + \frac{k(k-1)}{4}\int_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi^R_{x_0}\,\mathrm{d}x \\
&+k\int_{\mathbb{R}^d} \left|D^2 c\right|^2 |\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x + k\int_{\mathbb{R}^d} |\nabla c|^{2k}\phi^R_{x_0}\,\mathrm{d}x \\
\leq &\left(d+1+4(k-1)\right)k\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x + \frac{C3^d k}{R^2}\|\nabla c\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}.
\end{aligned} \tag{3.11}$$

Proof. Applying ∇ on the second equation of (3.1) and multiplying by $\nabla c|\nabla c|^{2k-2}\phi^R_{x_0}$, we get

$$\begin{aligned}
&\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |\nabla c|^{2k}\phi^R_{x_0}\,\mathrm{d}x - \int_{\mathbb{R}^d} \nabla\Delta c\cdot\nabla c|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x + \int_{\mathbb{R}^d} |\nabla c|^{2k}\phi^R_{x_0}\,\mathrm{d}x \\
= &\int_{\mathbb{R}^d} \nabla n\cdot\nabla c|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x - \int_{\mathbb{R}^d} \nabla(u\cdot\nabla c)\nabla c|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x.
\end{aligned}$$

Then by taking advantage of

$$\nabla\Delta c\cdot\nabla c = \frac{1}{2}\Delta|\nabla c|^2 - \left|D^2 c\right|^2,$$

it follows that

$$\begin{aligned}
&\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |\nabla c|^{2k}\phi^R_{x_0}\,\mathrm{d}x + \int_{\mathbb{R}^d} \left|D^2 c\right|^2 |\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x + \int_{\mathbb{R}^d} |\nabla c|^{2k}\phi^R_{x_0}\,\mathrm{d}x \\
= &\int_{\mathbb{R}^d} \nabla n\cdot\nabla c|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} \Delta|\nabla c|^2|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x - \int_{\mathbb{R}^d} \nabla(u\cdot\nabla c)\nabla c|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x.
\end{aligned}$$

Integrating by parts implies

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}^d} \Delta|\nabla c|^2|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x \\
= &\ \frac{k-1}{2}\int_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi^R_{x_0}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} \nabla|\nabla c|^2\cdot\nabla\phi^R_{x_0}|\nabla c|^{2k-2}\,\mathrm{d}x.
\end{aligned}$$

By virtue of Young's inequality, we have

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\mathbb{R}^d} \nabla|\nabla c|^2 \cdot \nabla\phi_{x_0}^R |\nabla c|^{2k-2}\,\mathrm{d}x \\
\le\ & \frac{k-1}{4}\int\limits_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x + \frac{1}{4(k-1)}\int\limits_{\mathbb{R}^d} |\nabla c|^{2k}\left(\phi_{x_0}^R\right)^{-1}\left|\nabla\phi_{x_0}^R\right|^2\,\mathrm{d}x \\
\le\ & \frac{k-1}{4}\int\limits_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x + \frac{C3^d}{4(k-1)R^2}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}.
\end{aligned}
$$

Using Hölder's inequality and Young's inequality, we get

$$
\begin{aligned}
&-\int\limits_{\mathbb{R}^d} \nabla(u\cdot\nabla c)\nabla c|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
=\ & \int\limits_{\mathbb{R}^d} (u\cdot\nabla c)\nabla c|\nabla c|^{2k-2}\nabla\phi_{x_0}^R\,\mathrm{d}x + (2k-1)\int\limits_{\mathbb{R}^d} u\cdot|\nabla c|^{2k-1}\nabla|\nabla c|\phi_{x_0}^R\,\mathrm{d}x \\
\le\ & 2\|u\|_{L^\infty}^2\int\limits_{\mathbb{R}^d} |\nabla c|^{2k}(\phi_{x_0}^R)^{-1}|\nabla\phi_{x_0}^R|^2\,\mathrm{d}x + \frac{1}{2}\int\limits_{\mathbb{R}^d} |\nabla c|^{2k}\phi_{x_0}^R\,\mathrm{d}x \\
\le\ & \frac{C3^d}{R^2}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} + \frac{1}{2}\int\limits_{\mathbb{R}^d} |\nabla c|^{2k}\phi_{x_0}^R\,\mathrm{d}x.
\end{aligned}
$$

Using integrating by parts, we get

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^d} \nabla n\cdot\nabla c|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
=\ & -\int\limits_{\mathbb{R}^d} n\Delta c|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x - (k-1)\int\limits_{\mathbb{R}^d} n\nabla c\cdot\nabla|\nabla c|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x \\
& -\int\limits_{\mathbb{R}^d} n\nabla c\cdot\nabla\phi_{x_0}^R|\nabla c|^{2k-2}\,\mathrm{d}x \\
\le\ & \frac{1}{2d}\int\limits_{\mathbb{R}^d} |\Delta c|^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + \frac{d}{2}\int\limits_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
& + \frac{k-1}{8}\int\limits_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x + 2(k-1)\int\limits_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
& + \frac{1}{2}\int\limits_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + \frac{1}{2}\int\limits_{\mathbb{R}^d} |\nabla c|^{2k}\left(\phi_{x_0}^R\right)^{-1}\left|\nabla\phi_{x_0}^R\right|^2\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int\limits_{\mathbb{R}^d} \left|D^2 c\right|^2 |\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{d+1+4(k-1)}{2}\int\limits_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x \\
&\quad + \frac{C3^d}{R^2}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} + \frac{k-1}{8}\int\limits_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x.
\end{aligned}
$$

Summing up the above inequalities, we obtain

$$
\begin{aligned}
&\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^d} |\nabla c|^{2k}\phi_{x_0}^R \,\mathrm{d}x + \frac{k-1}{8}\int\limits_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x \\
&+\frac{1}{2}\int\limits_{\mathbb{R}^d} \left|D^2 c\right|^2 |\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{1}{2}\int\limits_{\mathbb{R}^d} |\nabla c|^{2k}\phi_{x_0}^R \,\mathrm{d}x \\
&\leq \frac{d+1+4(k-1)}{2}\int\limits_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{C3^d}{R^2}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}.
\end{aligned}
$$

This completes the proof for the proposition. $\square$

We now need to establish the following proposition to absorb the right-hand integral term of inequality (3.5) and (3.11).

Proposition 3.4. Suppose that $\chi > 0, \epsilon \geq 0$ and $R \geq 1$. For any $k \in \mathbb{N}$ with $k \geq 2, q \geq 2$, one can find a constant $C_1(\chi, k)$ such that the solution (n, c) of equations (3.1)-(3.2) satisfies the following estimate

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^d} n|\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{(k-1)(k-2)}{2}\int\limits_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-6}n\phi_{x_0}^R \,\mathrm{d}x \\
&+(2k-2)\int\limits_{\mathbb{R}^d} \left|D^2 c\right|^2 |\nabla c|^{2k-4}n\phi_{x_0}^R \,\mathrm{d}x \\
&\leq C_1\int\limits_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x + \int\limits_{\mathbb{R}^d} |\nabla n|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x \\
&\quad + (C_1-\epsilon)\int\limits_{\mathbb{R}^d} n^q|\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{C3^d}{R^2}\left(\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} + \|n\|_{L^{k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{k}\right) \\
&\quad +\left(\frac{1}{4}+\frac{2q-3}{q}\right)\int\limits_{\mathbb{R}^d} |\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x.
\end{aligned}
$$

Proof. According to equation (3.1), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} n|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + (2k-2)\int_{\mathbb{R}^d} n|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
= &-\int_{\mathbb{R}^d} u\cdot\nabla n|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x - (2k-2)\int_{\mathbb{R}^d} \nabla(u\cdot\nabla c)n\nabla c|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x \\
&+\int_{\mathbb{R}^d} \Delta n|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x - \chi\int_{\mathbb{R}^d} \nabla\cdot(n\nabla c)|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
&-\epsilon\int_{\mathbb{R}^d} n^q|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + (2k-2)\int_{\mathbb{R}^d} \nabla\Delta c\cdot\nabla c|\nabla c|^{2k-4}n\phi_{x_0}^R\,\mathrm{d}x \\
&+(2k-2)\int_{\mathbb{R}^d} \nabla n\cdot\nabla c|\nabla c|^{2k-4}n\phi_{x_0}^R\,\mathrm{d}x + \int_{\mathbb{R}^d} n|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x.
\end{aligned}
\tag{3.12}
$$

By applying Hölder's inequality and Young's inequality, we get

$$
\begin{aligned}
&-\int_{\mathbb{R}^d} u\cdot\nabla n|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x - (2k-2)\int_{\mathbb{R}^d} \nabla(u\cdot\nabla c)n\nabla c|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x \\
= &-\int_{\mathbb{R}^d} u\cdot\nabla n|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x - (2k-2)\int_{\mathbb{R}^d} \nabla u\cdot n|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
&-\int_{\mathbb{R}^d} u\cdot\nabla\left(|\nabla c|^{2k-2}\right)n\phi_{x_0}^R\,\mathrm{d}x \\
\le\ &(2k-2)^2\|\nabla u\|_{L^\infty}^2\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^d} |\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + \int_{\mathbb{R}^d} u|\nabla c|^{2k-2}n\nabla\phi_{x_0}^R\,\mathrm{d}x \\
\le\ &(2k-2)^2\|\nabla u\|_{L^\infty}^2\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^d} |\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
&+\frac{1}{2}\|u\|_{L^\infty}^2\int_{\mathbb{R}^d} |\nabla c|^{2k-2}|\nabla\phi_{x_0}^R|^2(\phi_{x_0}^R)^{-1}\,\mathrm{d}x \\
\le\ &C(2k-2)^2\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^d} |\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + \frac{C}{R^2}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}.
\end{aligned}
\tag{3.13}
$$

Using integration by parts, we have

$$
-\chi\int_{\mathbb{R}^d} \nabla\cdot(n\nabla c)|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x
$$

$$
\begin{aligned}
&= \chi(k-1)\int_{\mathbb{R}^d} n\nabla c\cdot\nabla|\nabla c|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x + \chi\int_{\mathbb{R}^d} n\nabla c\cdot\nabla\phi_{x_0}^R|\nabla c|^{2k-2}\,\mathrm{d}x\\
&\le \frac{(k-1)^2}{2}\int_{\mathbb{R}^d}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x + \frac{\chi^2}{2}\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x + \frac{\chi^2}{2}\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d}|\nabla c|^{2k}(\phi_{x_0}^R)^{-1}|\nabla\phi_{x_0}^R|^2\,\mathrm{d}x\\
&\le \frac{(k-1)^2}{2}\int_{\mathbb{R}^d}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x + \chi^2\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x\\
&\quad + \frac{C3^d}{R^2}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}.
\end{aligned}
\tag{3.14}
$$

Similarly, we know

$$
\begin{aligned}
&(2k-2)\int_{\mathbb{R}^d}\nabla\Delta c\cdot\nabla c|\nabla c|^{2k-4}n\phi_{x_0}^R\,\mathrm{d}x\\
&= (k-1)\int_{\mathbb{R}^d}\Delta|\nabla c|^2|\nabla c|^{2k-4}n\phi_{x_0}^R\,\mathrm{d}x - (2k-2)\int_{\mathbb{R}^d}\left|D^2c\right|^2|\nabla c|^{2k-4}n\phi_{x_0}^R\,\mathrm{d}x,
\end{aligned}
$$

where

$$
\begin{aligned}
&(k-1)\int_{\mathbb{R}^d}\Delta|\nabla c|^2|\nabla c|^{2k-4}n\phi_{x_0}^R\,\mathrm{d}x\\
&= -(k-1)(k-2)\int_{\mathbb{R}^d}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-6}n\phi_{x_0}^R\,\mathrm{d}x - (k-1)\int_{\mathbb{R}^d}\nabla|\nabla c|^2\cdot\nabla n|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x\\
&\quad -(k-1)\int_{\mathbb{R}^d}\nabla|\nabla c|^2\cdot\nabla\phi_{x_0}^R|\nabla c|^{2k-4}n\,\mathrm{d}x\\
&\le -(k-1)(k-2)\int_{\mathbb{R}^d}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-6}n\phi_{x_0}^R\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}|\nabla n|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x\\
&\quad + \frac{(k-1)^2}{2}\int_{\mathbb{R}^d}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x + \frac{k-1}{2(k-2)}\int_{\mathbb{R}^d}|\nabla c|^{2k-2}n(\phi_{x_0}^R)^{-1}|\nabla\phi_{x_0}^R|^2\,\mathrm{d}x\\
&\quad + \frac{(k-1)(k-2)}{2}\int_{\mathbb{R}^d}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-6}n\phi_{x_0}^R\,\mathrm{d}x\\
&\le -\frac{(k-1)(k-2)}{2}\int_{\mathbb{R}^d}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-6}n\phi_{x_0}^R\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}|\nabla n|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x
\end{aligned}
$$

$$+\frac{(k-1)^2}{2}\int_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x + \frac{C3^d}{R^2}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} + \frac{C3^d}{R^2}\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^{k}.$$

After arranging the above inequalities, we get

$$\begin{aligned}
&(2k-2)\int_{\mathbb{R}^d} \nabla\Delta c\cdot\nabla c|\nabla c|^{2k-4}n\phi_{x_0}^R \,\mathrm{d}x \\
\leq& -\frac{(k-1)(k-2)}{2}\int_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-6}n\phi_{x_0}^R \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla n|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x \\
&+\frac{(k-1)^2}{2}\int_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x - (2k-2)\int_{\mathbb{R}^d} \left|D^2c\right|^2|\nabla c|^{2k-4}n\phi_{x_0}^R \,\mathrm{d}x \\
&+\frac{C3^d}{R^2}\left(\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} + \|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^{k}\right).
\end{aligned} \tag{3.15}$$

Similarly, we can get the following result

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Delta n|\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x + (2k-2)\int_{\mathbb{R}^d} \nabla n\cdot\nabla c|\nabla c|^{2k-4}n\phi_{x_0}^R \,\mathrm{d}x \\
\leq& -\int_{\mathbb{R}^d} \nabla n\cdot\nabla\phi_{x_0}^R|\nabla c|^{2k-2} \,\mathrm{d}x - (k-1)\int_{\mathbb{R}^d} \nabla n\cdot\nabla|\nabla c|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x \\
&+\frac{1}{4}\int_{\mathbb{R}^d} |\nabla n|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x + 2(2k-2)^2\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x \\
\leq& \frac{1}{2}\int_{\mathbb{R}^d} |\nabla n|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x + 2(k-1)^2\int_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x \\
&+2(2k-2)^2\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{C3^d}{R^2}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}.
\end{aligned} \tag{3.16}$$

Combining the above estimates with inequalities (3.12)-(3.16), we infer that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} n|\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{(k-1)(k-2)}{2}\int_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-6}n\phi_{x_0}^R \,\mathrm{d}x \\
&+(2k-2)\int_{\mathbb{R}^d} \left|D^2c\right|^2|\nabla c|^{2k-4}n\phi_{x_0}^R \,\mathrm{d}x \\
\leq& \; 3(k-1)^2\int_{\mathbb{R}^d} |\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x + \int_{\mathbb{R}^d} |\nabla n|^2|\nabla c|^{2k-4}\phi_{x_0}^R \,\mathrm{d}x
\end{aligned}$$

$$+\left(\frac{1}{4}+\frac{2q-3}{q}\right)\int\limits_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x$$

$$+\left(\frac{2}{q}(\chi^2+C(2k-2)^2)^{\frac{q}{2}}+\frac{1}{q}-\epsilon\right)\int\limits_{\mathbb{R}^d}n^q|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x$$

$$+\frac{C3^d}{R^2}\left(\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k+\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}\right),$$

where we have used the following estimates

$$\begin{aligned}\int\limits_{\mathbb{R}^d}n|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x&=\int\limits_{\mathbb{R}^d}n(\phi_{x_0}^R)^{\frac{1}{q}}|\nabla c|^{\frac{2k-2}{q}}|\nabla c|^{2k-2-\frac{2k-2}{q}}(\phi_{x_0}^R)^{1-\frac{1}{q}}\,\mathrm{d}x\\&\le\left(\int\limits_{\mathbb{R}^d}n^q\phi_{x_0}^R|\nabla c|^{2k-2}\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int\limits_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x\right)^{\frac{q-1}{q}}\\&\le\frac{1}{q}\int\limits_{\mathbb{R}^d}n^q\phi_{x_0}^R|\nabla c|^{2k-2}\,\mathrm{d}x+\frac{q-1}{q}\int\limits_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x\end{aligned}$$

and

$$\left(\chi^2+C(2k-2)^2\right)\int\limits_{\mathbb{R}^d}n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x$$

$$\le\frac{2}{q}\left(\chi^2+C(2k-2)^2\right)^{\frac{q}{2}}\int\limits_{\mathbb{R}^d}n^q|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x+\frac{q-2}{q}\int\limits_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x.$$

This completes the proof of proposition. □

To absorb the integral term in the right-hand side of the inequality in Proposition 3.4, by combining the coupling structure of the system of equations (3.15)-(3.16), the following integral term will now be estimated

$$\int\limits_{\mathbb{R}^d}n^j|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x\,(j=2,3,k-1).$$

Proposition 3.5. Suppose $\chi,\epsilon>0$ and $R\ge 1$. Let $k\in\mathbb{N}$ with $k\ge 3, q\ge 2$ and $j\in\mathbb{N}$ with $2\le j\le k-1$. Then there exists a constant C_j depending on j,ϵ,χ and C such that the solution (n,c) of system (3.1)-(3.2) satisfies the following inequalities

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^d}n^j|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x+\frac{j(j-1)}{4}\int\limits_{\mathbb{R}^d}n^{j-2}|\nabla n|^2|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x + C_j \int_{\mathbb{R}^d} |\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R \,\mathrm{d}x + C_j R^d \\
&\quad + \big(C_j - \epsilon j\big)\int_{\mathbb{R}^d} n^{q+j-1}|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x + C_j \int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x + j\int_{\mathbb{R}^d} |\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x \\
&\quad + \left((2k-2j+1) + (k-j) + 2j^2 + \frac{j^2}{2k} + \frac{j(k-j)}{2k}\right)\frac{C3^d}{R^2}\left(\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} + \|n\|_{L^{k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{k}\right).
\end{aligned}
$$

Proof. From system (3.1), it can be obtained that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x \\
&= j\frac{d}{dt}\int_{\mathbb{R}^d} n^{j-1}|\nabla c|^{2k-2j} n\phi_{x_0}^R \,\mathrm{d}x + (2k-2j)\frac{d}{dt}\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j-1}|\nabla c|\phi_{x_0}^R \,\mathrm{d}x,
\end{aligned}
$$

then we can infer that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x + 2(k-j)\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x \\
&= j\int_{\mathbb{R}^d} \Delta n|\nabla c|^{2k-2j}n^{j-1}\phi_{x_0}^R \,\mathrm{d}x - \chi j\int_{\mathbb{R}^d} \nabla\cdot(n\nabla c)|\nabla c|^{2k-2j}n^{j-1}\phi_{x_0}^R \,\mathrm{d}x \\
&\quad -j\int_{\mathbb{R}^d} u\cdot\nabla n|\nabla c|^{2k-2j}n^{j-1}\phi_{x_0}^R \,\mathrm{d}x - (2k-2j)\int_{\mathbb{R}^d} \nabla(u\cdot\nabla c)\nabla c|\nabla c|^{2k-2j-2}n^j\phi_{x_0}^R \,\mathrm{d}x \\
&\quad +(2k-2j)\int_{\mathbb{R}^d} \nabla\Delta c\cdot\nabla c n^j|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x + (2k-2j)\int_{\mathbb{R}^d} n^j\nabla n\cdot\nabla c|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x \\
&\quad +j\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x - \epsilon j\int_{\mathbb{R}^d} n^{q+j-1}|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x.
\end{aligned}
$$

Calculating the terms on the right-hand side of the above equation yields the following results,

$$
\begin{aligned}
&-j\int_{\mathbb{R}^d} u\cdot\nabla n|\nabla c|^{2k-2j}n^{j-1}\phi_{x_0}^R \,\mathrm{d}x - (2k-2j)\int_{\mathbb{R}^d} \nabla(u\cdot\nabla c)\nabla c|\nabla c|^{2k-2j-2}n^j\phi_{x_0}^R \,\mathrm{d}x \\
&= -j\int_{\mathbb{R}^d} u\cdot\nabla n|\nabla c|^{2k-2j}n^{j-1}\phi_{x_0}^R \,\mathrm{d}x - (2k-2j)\int_{\mathbb{R}^d} \nabla u\cdot|\nabla c|^{2k-2j}n^j\phi_{x_0}^R \,\mathrm{d}x \\
&\quad -(2k-2j)\int_{\mathbb{R}^d} u\cdot\nabla^2 c\nabla c|\nabla c|^{2k-2j-2}n^j\phi_{x_0}^R \,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= -j\int_{\mathbb{R}^d} u\cdot\nabla n|\nabla c|^{2k-2j}n^{j-1}\phi_{x_0}^R\,\mathrm{d}x-(2k-2j)\int_{\mathbb{R}^d}\nabla u\cdot|\nabla c|^{2k-2j}n^j\phi_{x_0}^R\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^d} u\nabla|\nabla c|^{2k-2j}n^j\phi_{x_0}^R\,\mathrm{d}x\\
&= -(2k-2j)\int_{\mathbb{R}^d}\nabla u|\nabla c|^{2k-2j}n^j\phi_{x_0}^R\,\mathrm{d}x+\int_{B_{2R}(x_0)} u|\nabla c|^{2k-2j}n^j\nabla\phi_{x_0}^R\,\mathrm{d}x\\
&\le (2k-2j)\frac{C3^d}{R}\left(\frac{j}{k}\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k+\frac{k-j}{k}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}\right)\\
&\quad+\frac{C3^d}{R}\left(\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k+\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}\right),
\end{aligned}
$$

where we have used the following estimates

$$
\begin{aligned}
&\int_{B_{2R}(x_0)} u\cdot|\nabla c|^{2k-2j}n^j\nabla\phi_{x_0}^R\,\mathrm{d}x\\
&\le \|u\|_{L^\infty}\left(\frac{k-j}{k}\int_{B_{2R}(x_0)}|\nabla c|^{2k}\nabla\phi_{x_0}^R\,\mathrm{d}x+\frac{j}{k}\int_{B_{2R}(x_0)} n^k\nabla\phi_{x_0}^R\,\mathrm{d}x\right)\\
&\le \frac{C3^d}{R}\left(\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k+\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&(2k-2j)\|\nabla u\|_{L^\infty}\left(\frac{j}{k}\int_{\mathbb{R}^d} n^k\phi_{x_0}^R\,\mathrm{d}x+\frac{k-j}{k}\int_{\mathbb{R}^d}|\nabla c|^{2k}\phi_{x_0}^R\,\mathrm{d}x\right)\\
&= (2k-2j)\|\nabla u\|_{L^\infty}\left(\frac{j}{k}\int_{B_{2R}(x_0)} n^k\nabla\phi_{x_0}^R\phi_{x_0}^R|\nabla\phi_{x_0}^R|^{-1}\,\mathrm{d}x\right.\\
&\quad\left.+\frac{k-j}{k}\int_{B_{2R}(x_0)}|\nabla c|^{2k}\nabla\phi_{x_0}^R\phi_{x_0}^R|\nabla\phi_{x_0}^R|^{-1}\,\mathrm{d}x\right)\\
&\le (2k-2j)\frac{C3^d}{R}\left(\frac{j}{k}\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k+\frac{k-j}{k}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}\right).
\end{aligned}
$$

For

$$\nabla\Delta c\cdot\nabla c=\frac{1}{2}\Delta|\nabla c|^2-\left|D^2c\right|^2,$$

we have

$$
\begin{aligned}
& 2(k-j)\int_{\mathbb{R}^d} \nabla\Delta c\cdot\nabla c n^j|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x \\
=\ & (k-j)\int_{\mathbb{R}^d} \Delta|\nabla c|^2 n^j|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x - 2(k-j)\int_{\mathbb{R}^d}\left|D^2c\right|^2 n^j|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x \\
=\ & -(k-j)j\int_{\mathbb{R}^d} n^{j-1}\nabla n\cdot\nabla|\nabla c|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x \\
& -(k-j)\int_{\mathbb{R}^d} n^j\nabla|\nabla c|^2\cdot\nabla\phi_{x_0}^R|\nabla c|^{2k-2j-2}\,\mathrm{d}x \\
& -(k-j)(k-j-1)\int_{\mathbb{R}^d} n^j|\nabla|\nabla c|^2|^2|\nabla c|^{2k-2j-4}\phi_{x_0}^R\,\mathrm{d}x \\
& -2(k-j)\int_{\mathbb{R}^d} n^j\left|D^2c\right|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x,
\end{aligned}
$$

from which and Hölder's inequality, we have

$$
\begin{aligned}
& -(k-j)j\int_{\mathbb{R}^d} n^{j-1}\nabla n\cdot\nabla|\nabla c|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x \\
\le\ & \frac{j^2(k-j)^2}{2}\int_{\mathbb{R}^d} n^{j-1}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x \\
=\ & \frac{j^2(k-j)^2}{2}\int_{\mathbb{R}^d}\left(n^j|\nabla c|^{2k-2j-4}\right)^{\frac{j-1}{j}}\left(|\nabla c|^{2k-4}\right)^{\frac{1}{j}}|\nabla|\nabla c|^2|^2\phi_{x_0}^R\,\mathrm{d}x \\
& +\frac{1}{2}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x \\
\le\ & \frac{1}{2}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x + \frac{k-j}{8}\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R\,\mathrm{d}x \\
& +(8^{j-1}j^{2j}(k-j)^{j+1})\int_{\mathbb{R}^d}|\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R\,\mathrm{d}x.
\end{aligned}
$$

Further as

$$
|\nabla|\nabla c|^2|^2 = \left|2D^2c\cdot\nabla c\right|^2 \le 4\left|D^2c\right|^2|\nabla c|^2, \tag{3.17}
$$

we have

$$
\begin{aligned}
&-(k-j)j\int_{\mathbb{R}^d} n^{j-1}\nabla n\cdot\nabla|\nabla c|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x\\
\le\ &\frac{1}{2}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x+\frac{k-j}{2}\int_{\mathbb{R}^d} n^j\left|D^2c\right|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x\\
&+(8^{j-1}j^{2j}(k-j)^{j+1})\int_{\mathbb{R}_d}|\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R\,\mathrm{d}x.
\end{aligned}
$$

Using the Hölder's inequality and Young's inequality, we get

$$
\begin{aligned}
&-(k-j)\int_{\mathbb{R}^d} n^j\nabla|\nabla c|^2\cdot\nabla\phi_{x_0}^R|\nabla c|^{2k-2j-2}\,\mathrm{d}x\\
\le\ &\frac{k-j}{4}\int_{\mathbb{R}^d} n^j|\nabla|\nabla c|^2|^2|\nabla c|^{2k-2j-4}\phi_{x_0}^R\,\mathrm{d}x+(k-j)\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\left(\phi_{x_0}^R\right)^{-1}\left|\nabla\phi_{x_0}^R\right|^2\,\mathrm{d}x\\
\le\ &(k-j)\int_{\mathbb{R}^d} n^j\left|D^2c\right|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x+(k-j)\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}(\phi_{x_0,R}^R)^{-1}\left|\nabla\phi_{x_0}^R\right|^2\,\mathrm{d}x.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&2(k-j)\int_{\mathbb{R}^d}\nabla\Delta c\cdot\nabla c n^j|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x\\
\le\ &-(k-j)(k-j-1)\int_{\mathbb{R}^d} n^j|\nabla|\nabla c|^2|^2|\nabla c|^{2k-2j-4}\phi_{x_0}^R\,\mathrm{d}x\\
&-\frac{k-j}{2}\int_{\mathbb{R}^d} n^j\left|D^2c\right|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x\\
&+8^{j-1}j^{2j}(k-j)^{j+1}\int_{\mathbb{R}^d}|\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R\,\mathrm{d}x\\
&+(k-j)\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\left(\phi_{x_0}^R\right)^{-1}\left|\nabla\phi_{x_0}^R\right|^2\,\mathrm{d}x.
\end{aligned}
\tag{3.18}
$$

By a similar calculation, it can be obtained that

$$
2(k-j)\int_{\mathbb{R}^d} n^j\nabla n\cdot\nabla c|\nabla c|^{2k-2j-2}\phi_{x_0}^R\,\mathrm{d}x
$$

$$\leq \frac{1}{4}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x + 4(k-j)^2\int_{\mathbb{R}^d} n^{j+1}|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x. \tag{3.19}$$

Evaluating the following integral, we get

$$\begin{aligned}
& j\int_{\mathbb{R}^d} \Delta n|\nabla c|^{2k-2j} n^{j-1}\phi_{x_0}^R \,\mathrm{d}x \\
= & -j(j-1)\int_{\mathbb{R}^d} n^{j-2}|\nabla n|^2|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x - j\int_{\mathbb{R}^d} n^{j-1}\nabla n\cdot\nabla\phi_{x_0}^R|\nabla c|^{2k-2j} \,\mathrm{d}x \\
& -j(j-k)\int_{\mathbb{R}^d} n^{j-1}\nabla n\cdot\nabla|\nabla c|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x \\
\triangleq & \; I_1 + I_2 + I_3.
\end{aligned}$$

Applying Hölder's inequality to I_2, it is easy to get

$$I_2 \leq \frac{1}{8}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x + 2j^2\int_{\mathbb{R}^d} n^{j-1}|\nabla c|^{2k-2j+2}\left(\phi_{x_0}^R\right)^{-1}\left|\nabla\phi_{x_0}^R\right|^2 \,\mathrm{d}x.$$

By the same process, we obtain

$$\begin{aligned}
I_3 \leq & \; 2j^2(j-k)^2\int_{\mathbb{R}^d} n^{j-1}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-2j+2}\phi_{x_0}^R \,\mathrm{d}x \\
& + \frac{1}{8}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x \\
= & \; 2j^2(j-k)^2\int_{\mathbb{R}^d}\left(n^j|\nabla c|^{2k-2j-4}\right)^{\frac{j-1}{j}}\left(|\nabla c|^{2k-4}\right)^{\frac{1}{j}}|\nabla|\nabla c|^2|^2\phi_{x_0}^R \,\mathrm{d}x \\
& + \frac{1}{8}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x \\
\leq & \; \frac{1}{8}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{k-j}{16}\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R \,\mathrm{d}x \\
& + \frac{\left(32j^2\right)^j(k-j)^{j+1}}{16}\int_{\mathbb{R}^d}|\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R \,\mathrm{d}x.
\end{aligned}$$

Combining with (3.17), we have the following result

$$
\begin{aligned}
I_3 \le\ & \frac{1}{8}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{k-j}{4}\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j-2}\left|D^2c\right|^2\phi_{x_0}^R \,\mathrm{d}x \\
& + \frac{\left(32j^2\right)^j (k-j)^{j+1}}{16}\int_{\mathbb{R}^d} |\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R \,\mathrm{d}x.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& j\int_{\mathbb{R}^d} \Delta n|\nabla c|^{2k-2j}n^{j-1}\phi_{x_0}^R \,\mathrm{d}x \\
\le\ & -(j-1)\int_{\mathbb{R}^d} n^{j-2}|\nabla n|^2|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x \\
& +2j^2\int_{\mathbb{R}^d} n^{j-1}|\nabla c|^{2k-2j-2}|\nabla\phi_{x_0}^R|^2(\phi_{x_0}^R)^{-1} \,\mathrm{d}x + \frac{k-j}{4}\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j-2}\left|D^2c\right|^2\phi_{x_0}^R \,\mathrm{d}x \\
& + \frac{(32j^2)^j(k-j)^{j+1}}{16}\int_{\mathbb{R}^d} |\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R \,\mathrm{d}x.
\end{aligned}
\tag{3.20}
$$

By integrating by parts, the following integral terms can be split, resulting in the following results

$$
\begin{aligned}
& -\chi j\int_{\mathbb{R}^d} \nabla\cdot(n\nabla c)|\nabla c|^{2k-2j}n^{j-1}\phi_{x_0}^R \,\mathrm{d}x \\
=\ & \chi j\int_{\mathbb{R}^d} n^j\nabla c\cdot\nabla|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x + \chi j\int_{\mathbb{R}^d} n\nabla c\cdot\nabla n^{j-1}|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x \\
& +\chi j\int_{\mathbb{R}^d} n^j\nabla c\cdot\nabla\phi_{x_0}^R|\nabla c|^{2k-2j} \\
\triangleq\ & I_4+I_5+I_6.
\end{aligned}
$$

Applying Young's inequality to I_4 and combining (3.17) yield

$$
\begin{aligned}
I_4 \le\ & \frac{k-j}{32}\int_{\mathbb{R}^d} n^j|\nabla|\nabla c|^2|^2|\nabla c|^{2k-2j-4}\phi_{x_0}^R \,\mathrm{d}x + 8\chi^2j^2(k-j)\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j+2}\phi_{x_0}^R \,\mathrm{d}x \\
\le\ & \frac{k-j}{8}\int_{\mathbb{R}^d} n^j\left|D^2c\right|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x + 8\chi^2j^2(k-j)\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j+2}\phi_{x_0}^R \,\mathrm{d}x.
\end{aligned}
$$

Similarly,

$$I_5 \leq \frac{j(j-1)}{2} \int_{\mathbb{R}^d} n^{j-2} |\nabla n|^2 |\nabla c|^{2k-2j} \phi_{x_0}^R \, \mathrm{d}x + \frac{\chi^2 j(j-1)}{2} \int_{\mathbb{R}^d} n^j |\nabla c|^{2k-2j+2} \phi_{x_0}^R \, \mathrm{d}x,$$

and

$$\begin{aligned} I_6 \leq & \frac{\chi^2 j}{2} \int_{\mathbb{R}^d} n^j |\nabla c|^{2k-2j+2} \phi_{x_0}^R \, \mathrm{d}x + \frac{j}{2} \int_{\mathbb{R}^d} n^j |\nabla c|^{2k-2j} \left(\phi_{x_0}^R\right)^{-1} \left|\nabla \phi_{x_0}^R\right|^2 \, \mathrm{d}x \\ \leq & \frac{\chi^2 j}{2} \int_{\mathbb{R}^d} n^j |\nabla c|^{2k-2j+2} \phi_{x_0}^R \, \mathrm{d}x + \frac{j^2}{2k} \int_{\mathbb{R}^d} n^k \left(\phi_{x_0}^R\right)^{-1} \left|\nabla \phi_{x_0}^R\right|^2 \, \mathrm{d}x \\ & + \frac{j(k-j)}{2k} \int_{\mathbb{R}^d} |\nabla c|^{2k} \left(\phi_{x_0}^R\right)^{-1} \left|\nabla \phi_{x_0}^R\right|^2 \, \mathrm{d}x. \end{aligned}$$

Arranging the inequalities I_4, I_5, I_6 and adding them together yield

$$\begin{aligned} & -\chi j \int_{\mathbb{R}^d} \nabla \cdot (n \nabla c) |\nabla c|^{2k-2j} n^{j-1} \phi_{x_0}^R \, \mathrm{d}x \\ \leq & \frac{k-j}{8} \int_{\mathbb{R}^d} n^j \left|D^2 c\right|^2 |\nabla c|^{2k-2j-2} \phi_{x_0}^R \, \mathrm{d}x + \frac{j(j-1)}{2} \int_{\mathbb{R}^d} n^{j-2} |\nabla n|^2 |\nabla c|^{2k-2j} \phi_{x_0}^R \, \mathrm{d}x \\ & + \chi^2 \left(8 j^2 (k-j) + \frac{j(j-1)}{2} + \frac{j}{2}\right) \int_{\mathbb{R}^d} n^j |\nabla c|^{2k-2j+2} \phi_{x_0}^R \, \mathrm{d}x \\ & + \frac{j^2}{2k} \int_{\mathbb{R}^d} n^k \left(\phi_{x_0}^R\right)^{-1} \left|\nabla \phi_{x_0}^R\right|^2 \, \mathrm{d}x + \frac{j(k-j)}{2k} \int_{\mathbb{R}^d} |\nabla c|^{2k} \left(\phi_{x_0}^R\right)^{-1} \left|\nabla \phi_{x_0}^R\right|^2 \, \mathrm{d}x. \end{aligned}$$

Using Hölder's inequality and Young's inequality, we have that

$$\begin{aligned} & \int_{\mathbb{R}^d} n^j |\nabla c|^{2k-2j+2} \phi_{x_0}^R \, \mathrm{d}x \\ = & \int_{\mathbb{R}^d} \left(n^2 |\nabla c|^{2k-2}\right)^{\frac{1}{j-1}} \left(n^{j+1} |\nabla c|^{2k-2j}\right)^{\frac{j-2}{j-1}} \phi_{x_0}^R \, \mathrm{d}x \\ \leq & \frac{1}{j-1} \int_{\mathbb{R}^d} n^2 |\nabla c|^{2k-2} \phi_{x_0}^R \, \mathrm{d}x + \frac{j-2}{j-1} \int_{\mathbb{R}^d} n^{j+1} |\nabla c|^{2k-2j} \phi_{x_0}^R \, \mathrm{d}x. \end{aligned}$$

Then we can get

$$-\chi j \int_{\mathbb{R}^d} \nabla \cdot (n \nabla c) |\nabla c|^{2k-2j} n^{j-1} \phi_{x_0}^R \, \mathrm{d}x$$

$$
\begin{aligned}
&\le \frac{k-j}{8}\int_{\mathbb{R}^d} n^j \left|D^2 c\right|^2 |\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{j(j-1)}{2}\int_{\mathbb{R}^d} n^{j-2}|\nabla n|^2|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x \\
&\quad +\chi^2 j\left(8j(k-j)+\frac{j}{2}\right)\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x + \frac{j^2}{2k}\int_{\mathbb{R}^d} n^k\left(\phi_{x_0}^R\right)^{-1}\left|\nabla\phi_{x_0}^R\right|^2 \,\mathrm{d}x \\
&\quad +\chi^2 j\left(8j(k-j)+\frac{j}{2}\right)\int_{\mathbb{R}^d} n^{j+1}|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x \\
&\quad +\frac{j(k-j)}{2k}\int_{\mathbb{R}^d} |\nabla c|^{2k}\left(\phi_{x_0}^R\right)^{-1}\left|\nabla\phi_{x_0}^R\right|^2 \,\mathrm{d}x.
\end{aligned}
\tag{3.21}
$$

Adding inequalities (3.18)-(3.19) gives the following result

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x + 2(k-j)\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x \\
&+\frac{k-j}{8}\int_{\mathbb{R}^d} n^j\left|D^2 c\right|^2|\nabla c|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x \\
&+\frac{j(j-1)}{2}\int_{\mathbb{R}^d} n^{j-2}|\nabla n|^2|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x \\
&+(k-j)(k-j-1)\int_{\mathbb{R}^d} n^j|\nabla|\nabla c|^2|^2|\nabla c|^{2k-2j-4}\phi_{x_0}^R \,\mathrm{d}x \\
\le &\int_{\mathbb{R}^d} n^{j-1}|\nabla n|^2|\nabla|^{2k-2j-2}\phi_{x_0}^R \,\mathrm{d}x \\
&+(2k-2j)\frac{C3^d}{R}\left(\frac{j}{k}\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k + \frac{k-j}{k}\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}\right) \\
&+\frac{C3^d}{R}\left(\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k + \|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}\right) \\
&+(k-j)^{j+1}\left(2^{2j-3}j^{2j}+\frac{(32j^2)^j}{16}\right)\int_{\mathbb{R}^d} |\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R \,\mathrm{d}x \\
&+(k-j)\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}(\phi_{x_0}^R)^{-1}\left|\nabla\phi_{x_0}^R\right|^2 \,\mathrm{d}x + 2j^2\int_{\mathbb{R}^d} n^{j-1}|\nabla c|^{2k-2j+2}(\phi_{x_0}^R)^{-1}\left|\nabla\phi_{x_0}^R\right|^2 \,\mathrm{d}x \\
&+\left(\chi^2 j\left(8j(k-j)+\frac{j}{2}\right)+4(k-j)^2\right)\int_{\mathbb{R}^d} n^{j+1}|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x
\end{aligned}
$$

$$+\chi^2 j\left(8j(k-j)+\frac{j}{2}\right)\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x+\frac{j^2}{2k}\int_{\mathbb{R}^d} n^k\left(\phi_{x_0}^R\right)^{-1}\left|\nabla\phi_{x_0}^R\right|^2\,\mathrm{d}x$$
$$+\frac{j(k-j)}{2k}\int_{\mathbb{R}^d}|\nabla c|^{2k}\left(\phi_{x_0}^R\right)^{-1}\left|\nabla\phi_{x_0}^R\right|^2\,\mathrm{d}x+\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x$$
$$-\epsilon j\int_{\mathbb{R}^d} n^{q+j-1}|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x,$$

where

$$\begin{aligned}&\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x\\ \le\ &\frac{j}{q+j-1}\int_{\mathbb{R}^d} n^{q+j-1}|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x+\frac{q-1}{q+j-1}\int_{\mathbb{R}^d}|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x\\ \le\ &\frac{j}{q+j-1}\int_{\mathbb{R}^d} n^{q+j-1}|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x+\frac{q-1}{q+j-1}\left(\int_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x+CR^d\right)\end{aligned}$$

and

$$\begin{aligned}&\left(\chi^2 j\left(8j(k-j)+\frac{j}{2}\right)+4(k-j)^2\right)\int_{\mathbb{R}^d} n^{j+1}|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x\\ \le\ &\frac{j+1}{q+j-1}\left(\chi^2 j\left(8j(k-j)+\frac{j}{2}\right)+4(k-j)^2\right)^{\frac{q+j-1}{j+1}}\int_{\mathbb{R}^d} n^{q+j-1}|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x\\ &+\frac{q-2}{q+j-1}\int_{\mathbb{R}^d}|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x\\ \le\ &\frac{j+1}{q+j-1}\left(\chi^2 j\left(8j(k-j)+\frac{j}{2}\right)+4(k-j)^2\right)^{\frac{q+j-1}{j+1}}\int_{\mathbb{R}^d} n^{q+j-1}|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x\\ &+\frac{q-2}{q+j-1}\left(\int_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x+CR^d\right).\end{aligned}$$

Combining with Hölder's inequality and Young's inequality, we have the following result

$$\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\left(\phi_{x_0}^R\right)^{-1}\left|\nabla\phi_{x_0}^R\right|^2\,\mathrm{d}x$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}^d} n^k \left(\phi_{x_0}^R\right)^{-1} \left|\nabla \phi_{x_0}^R\right|^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} |\nabla c|^{2k} \left(\phi_{x_0}^R\right)^{-1} \left|\nabla \phi_{x_0}^R\right|^2 \,\mathrm{d}x \\
&\leq \frac{C3^d}{R^2} \|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k + \frac{C3^d}{R^2} \|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}
\end{aligned}
$$

and

$$
\int_{\mathbb{R}^d} n^{j-1} |\nabla c|^{2k-2j+2} \left(\phi_{x_0}^R\right)^{-1} \left|\nabla \phi_{x_0}^R\right|^2 \,\mathrm{d}x \leq \frac{C3^d}{R^2} \|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k + \frac{C3^d}{R^2} \|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}.
$$

Therefore, it can be further obtained that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} n^j |\nabla c|^{2k-2j} \phi_{x_0}^R \,\mathrm{d}x + \frac{j(j-1)}{4} \int_{\mathbb{R}^d} n^{j-2} |\nabla n|^2 |\nabla c|^{2k-2j} \phi_{x_0}^R \,\mathrm{d}x \\
\leq &\int_{\mathbb{R}^d} n^{j-1} |\nabla n|^2 |\nabla c|^{2k-2j-2} \phi_{x_0}^R \,\mathrm{d}x \\
&+ (k-j)^{j+1} \left(2^{2j-3} j^{2j} + \frac{\left(32 j^2\right)^j}{16}\right) \int_{\mathbb{R}^d} |\nabla c|^{2k-4} |\nabla |\nabla c|^2|^2 \phi_{x_0}^R \,\mathrm{d}x \\
&+ \frac{j+1}{q+j-1} \left(\chi^2 j \left(8j(k-j) + \frac{j}{2}\right) + \frac{4(k-j)^2}{2} - \epsilon j\right)^{\frac{q+j-1}{j-1}} \int_{\mathbb{R}^d} n^{j+q-1} |\nabla c|^{2k-2j} \phi_{x_0}^R \,\mathrm{d}x \\
&+ (\chi^2 j \left(8j(k-j) + \frac{j}{2}\right) \int_{\mathbb{R}^d} n^2 |\nabla c|^{2k-2} \phi_{x_0}^R \,\mathrm{d}x + j \int_{\mathbb{R}^d} |\nabla c|^{2k-2} \phi_{x_0}^R \,\mathrm{d}x + C_j R^d \\
&+ \left((2k-2j+1) + (k-j) + 2j^2 + \frac{j^2}{2k} + \frac{j(k-j)}{2k}\right) \frac{C3^d}{R^2} \left(\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} + \|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k\right).
\end{aligned}
$$

This means that we have completed the proof. □

Proposition 3.6. Suppose that $\chi > 0$ and $R \geq 1$. Let $k \in \mathbb{N}$ with $k \geq 3, q \geq 2$. Then there exist constants $b_1, \cdots, b_k, \epsilon_0 = \epsilon_0(\chi, k)$ and an absolute constant C such that, for $\epsilon \geq \epsilon_0$, the solution (n, c) of equations (3.1)-(3.2) satisfies

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{\mathbb{R}^d} |\nabla c|^{2k} \phi_{x_0}^R \,\mathrm{d}x + \sum_{j=1}^{k} b_j \int_{\mathbb{R}^d} n^j |\nabla c|^{2k-2j} \phi_{x_0}^R \,\mathrm{d}x \right) \\
\leq &\left(b_1 \left(\frac{9q-12}{4q} \right) + \frac{k(q-2)}{q} + \sum_{j=2}^{k-1} j b_j \right) \int_{\mathbb{R}^d} |\nabla c|^{2k-2} \phi_{x_0}^R \,\mathrm{d}x
\end{aligned}
$$

$$
+\left(\frac{C3^d b_1}{R^2}+\sum_{j=2}^{k-1}\frac{b_j C_j}{R^2}+\frac{b_k C3^d k}{2(k-1)R^2}+\frac{b_k C3^d}{2k^2R^2}\right)\|n\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}
$$

$$
+\left(\frac{C3^d k}{R^2}+\frac{C3^d b_1}{R^2}+\sum_{j=2}^{k-1}\frac{b_j C_j}{R^2}+\frac{b_k C3^d k}{R^{2k}}\right)\|\nabla c\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}
$$

$$
+\left(\sum_{j=2}^{k-1}C_j b_j R^d+\frac{b_k C3(q-1)}{2(k+q-1)}\right)R^d+CR^{\frac{k-1}{2k(k+1)}}.
$$

Proof. By sorting and adding the results from Proposition 3.2 to Proposition 3.5, we can easily obtain the following results

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d}|\nabla c|^{2k}\phi^R_{x_0}\,\mathrm{d}x+\sum_{j=1}^{k}b_j\int_{\mathbb{R}^d}n^j|\nabla c|^{2k-2j}\phi^R_{x_0}\,\mathrm{d}x\right)\\
&+\frac{k(k-1)}{16}\int_{\mathbb{R}^d}|\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi^R_{x_0}\,\mathrm{d}x+\frac{b_k}{8}\int_{\mathbb{R}^d}n^{k-2}|\nabla n|^2\phi^R_{x_0}\,\mathrm{d}x\\
\le\ &-\frac{k(k-1)}{4}\int_{\mathbb{R}^d}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi^R_{x_0}\,\mathrm{d}x+\big(d+1+4(k-1)\big)k\int_{\mathbb{R}^d}n^2|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x\\
&+\frac{C3^d k}{R^2}\|\nabla c\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}+b_1\left(3(k-1)^2\int_{\mathbb{R}^d}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi^R_{x_0}\,\mathrm{d}x\right.\\
&\left.+\int_{\mathbb{R}^d}|\nabla n|^2|\nabla c|^{2k-4}\phi^R_{x_0}\,\mathrm{d}x+\left(\frac{9q-12}{4q}\right)\int_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x\right)\\
&+b_1\left(\frac{2}{q}(\chi^2+C(2k-2)^2)^{\frac{q}{2}}-\frac{1}{q}-\epsilon\right)\int_{\mathbb{R}^d}n^q|\nabla c|^{2k-2}\phi^R_{x_0}\,\mathrm{d}x\\
&+\frac{C3^d b_1}{R^2}\left(\|n\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}+\|\nabla c\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}\right)\\
&+\sum_{j=2}^{k-1}b_j\left\{-\frac{j(j-1)}{4}\int_{\mathbb{R}^d}n^{j-2}|\nabla n|^2|\nabla c|^{2k-2j}\phi^R_{x_0}\,\mathrm{d}x+\int_{\mathbb{R}^d}n^{j-1}|\nabla n|^2|\nabla c|^{2k-2j-2}\phi^R_{x_0}\,\mathrm{d}x\right.\\
&+(k-j)^{j+1}\left(2^{2j-3}j^{2j}+\frac{(32j^2)^j}{16}\right)\int_{\mathbb{R}^d}|\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi^R_{x_0}\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{j+1}{q+j-1}\left(\chi^2 j\left(8j(k-j)+\frac{j}{2}\right)+\frac{4(k-j)^2}{2}-\epsilon j\right)^{\frac{q+j-1}{j-1}}\int_{\mathbb{R}^d} n^{j+q-1}|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x\\
&+\chi^2 j\left(8j(k-j)+\frac{j}{2}\right)\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x+j\int_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x+C_jR^d\\
&+\left((2k-2j+1)+(k-j)+2j^2+\frac{j^2}{2k}+\frac{j(k-j)}{2k}\right)\frac{C3^d}{R^2}\left(\|\nabla c\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}+\|n\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}\right)\Bigg\}\\
&+b_k\Bigg(-\frac{k(k-1)}{4}\int_{\mathbb{R}^d}|\nabla n|^2n^{k-2}\phi_{x_0}^R\,\mathrm{d}x+\frac{C3^dk}{2(k-1)R^2}\|n\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}\\
&+\frac{C3^d}{2k^2R^2}\|n\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}+\frac{C3^dk}{R^{2k}}\|\nabla c\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}+\frac{3(q-1)}{2(k+q-1)}CR^d+CR^{\frac{k-1}{2k(k+1)}}\\
&+k\left(\frac{k}{2(k+q-1)}+\frac{k+1}{k+q-1}(\chi^{\frac{k+q-1}{k}}+(\chi^{\frac{2(k-1)}{k-2}}(k-1)^{\frac{k-1}{k-2}})^{\frac{k+q-1}{k+1}})-\epsilon\right)\int_{\mathbb{R}^d} n^{k+q-1}\phi_{x_0}^R\,\mathrm{d}x\\
&+k\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x\Bigg)\\
&+\frac{k(k-1)}{16}\int_{\mathbb{R}^d}|\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2\phi_{x_0}^R\,\mathrm{d}x+\frac{b_k}{8}\int_{\mathbb{R}^d} n^{k-2}|\nabla n|^2\phi_{x_0}^R\,\mathrm{d}x.
\end{aligned}
$$

Rearranging the above formula, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d}|\nabla c|^{2k}\phi_{x_0}^R\,\mathrm{d}x+\sum_{j=1}^k b_j\int_{\mathbb{R}^d} n^j|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x\right)\\
\le\ &\left(-\frac{k(k-1)}{4}+\sum_{j=1}^{k-1}b_jC_j+\frac{k(k-1)}{16}\right)\int_{\mathbb{R}^d}|\nabla|\nabla c|^2|^2|\nabla c|^{2k-4}\phi_{x_0}^R\,\mathrm{d}x\\
&+\left(\big(d+1+4(k-1)\big)k+\sum_{j=2}^{k-1}b_jC_j+kb_k\right)\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x\\
&+\left(\sum_{j=2}^{k}\left(b_{j-1}-\frac{j(j-1)b_j}{4}\right)+\frac{b_k}{8}\right)\int_{\mathbb{R}^d} n^{j-2}|\nabla n|^2|\nabla c|^{2k-2j}\phi_{x_0}^R\,\mathrm{d}x\\
&+\left(b_1\left(\frac{9q-12}{4q}\right)+\sum_{j=2}^{k-1}jb_j\right)\int_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x+b_1\left(C_1-\epsilon\right)\int_{\mathbb{R}^d} n^q|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\sum_{j=2}^{k-1} b_j\left(C_j-\epsilon j\right)\right)^{\frac{q+j-1}{j-1}} \int_{\mathbb{R}^d} n^{q+j-1}|\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x \\
&+\left(\frac{C3^d b_1}{R^2}+\sum_{j=2}^{k-1}\frac{b_j C_j}{R^2}+\frac{b_k C3^d k}{2(k-1)R^2}+\frac{b_k C3^d}{2k^2R^2}\right)\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k \\
&+\left(\frac{C3^d k}{R^2}+\frac{C3^d b_1}{R^2}+\sum_{j=2}^{k-1}\frac{b_j C_j}{R^2}+\frac{b_k C3^d k}{R^{2k}}\right)\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} \\
&+\left(\sum_{j=2}^{k-1} C_j b_j R^d+\frac{b_k C3(q-1)}{2(k+q-1)}\right)R^d+CR^{\frac{k-1}{2k(k+1)}}.
\end{aligned}
$$

Considering the integral term in the above inequality and applying Hölder's inequality, we get

$$
\begin{aligned}
&\left(\big(d+1+4(k-1)\big)k+\sum_{j=2}^{k-1} b_j C_j+kb_k\right)\int_{\mathbb{R}^d} n^2|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
&+b_1\left(\frac{5q-8}{4q}\right)\int_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x+b_1\left(C_1-\epsilon\right)\int_{\mathbb{R}^d} n^q|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
\le&\left(\frac{2}{q}\left(\left(\big(d+1+4(k-1)\big)k\right)^{\frac{q}{2}}+\sum_{j=1}^{k-1}(b_jC_j)^{\frac{q}{2}}+(kb_k)^{\frac{q}{2}}\right)-(b_1+1)\epsilon\right)\int_{\mathbb{R}^d} n^q|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x \\
&+\left(b_1\frac{5q-8}{4q}+\frac{(q-2)}{q}\right)\int_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x.
\end{aligned}
$$

For a fixed χ, we can easily observe that there is a constant $C_0=C_0(\chi)$ that satisfies

$$C_j\le C_0k^{3k+1},\quad (j=1,\dots,k-1).$$

Then for any $j=1,2,\cdots,k$, we assume that

$$b_j=\frac{k^{2-5k}(k-1)}{16C_0}k^{2j}.$$

From this it can be calculated that

$$\sum_{j=1}^{k-1} b_jC_j\le\frac{k^{3-2k}(k-1)}{16}\sum_{j=1}^{k-1}k^{2j}<\frac{k(k-1)}{8} \tag{3.22}$$

and

$$\frac{b_{j-1}}{b_j} = k^{-2}, \quad j = 2, \cdots, k.$$

This means

$$-\frac{k(k-1)}{4} + \sum_{j=1}^{k-1} b_j C_j + \frac{k(k-1)}{16} < 0, \tag{3.23}$$

and

$$\sum_{j=2}^{k}\left(b_{j-1} - \frac{j(j-1)b_j}{4}\right) + \frac{b_k}{8} \leq b_k \sum_{j=2}^{k}\left(k^{-2} - \frac{j(j-1)}{4} + \frac{1}{8}\right) < 0. \tag{3.24}$$

We set $\epsilon_0 \geq C_0 k^{3k+1} \geq C_j$, then we have

$$C_j - \epsilon_0 j \leq 0, \quad j = 2, 3, \cdots, k. \tag{3.25}$$

Combining with (3.22), we get

$$\begin{aligned}
&\frac{2}{q}\left(((d+4k-3)k)^{\frac{q}{2}} + \sum_{j=1}^{k-1}(b_j C_j)^{\frac{q}{2}} + (kb_k)^{\frac{q}{2}}\right) - (b_1+1)\epsilon \\
&< \frac{2}{q}\left(((d+4k-3)k)^{\frac{q}{2}} + \left(\frac{k(k-1)}{8}\right)^{\frac{q}{2}} + \left(\frac{1}{16C_0}\right)^{\frac{q}{2}}\right) - (b_1+1)\epsilon_0 < 0,
\end{aligned} \tag{3.26}$$

and

$$\sum_{j=2}^{k}(C_j - \epsilon_0 j) \leq 0. \tag{3.27}$$

From inequalities (3.23)-(3.27), when we choose $\epsilon \geq \epsilon_0$, the following results can be obtained

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} |\nabla c|^{2k}\phi_{x_0}^R \,\mathrm{d}x + \sum_{j=1}^{k} b_j \int_{\mathbb{R}^d} n^j |\nabla c|^{2k-2j}\phi_{x_0}^R \,\mathrm{d}x\right) \\
&\leq \left(b_1\left(\frac{5q-8}{4q}\right) + \frac{(q-2)}{q} + \frac{1}{8C_0}\right)\int_{\mathbb{R}^d} |\nabla c|^{2k-2}\phi_{x_0}^R \,\mathrm{d}x \\
&\quad + \left(\frac{C3^d b_1}{R^2} + \sum_{j=2}^{k-1}\frac{b_j C_j}{R^2} + \frac{b_k C3^d k}{2(k-1)R^2} + \frac{b_k C3^d}{2k^2R^2}\right)\|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k \\
&\quad + \left(\frac{C3^d k}{R^2} + \frac{C3^d b_1}{R^2} + \sum_{j=2}^{k-1}\frac{b_j C_j}{R^2} + \frac{b_k C3^d k}{R^{2k}}\right)\|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k}
\end{aligned}$$

$$+\left(\sum_{j=2}^{k-1} C_j b_j R^d + \frac{b_k C3(q-1)}{2(k+q-1)}\right) R^d + C R^{\frac{k-1}{2k(k+1)}}.$$

This means that we complete the proof. □

Next we introduce an interpolation lemma, so that it is convenient to establish a local $L^p(\mathbb{R}^d)$ estimate of n later.

Lemma 3.1. *[22] We assume that $\omega \in H^1(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} |\omega(x)|^{\frac{2}{k}} dx < \infty$. Then there is a constant C independent of d and*

$$\|\omega\|^2_{L^2(\mathbb{R}^2)} \le C\|\nabla\omega\|^2_{L^2(\mathbb{R}^d)} + C^k \left(\int_{\mathbb{R}^d} |\omega|^{\frac{2}{k}} dx\right)^k.$$

Proposition 3.7. Assume $\chi > 0$, $k \in \mathbb{N}$ with $k \ge 3$, $j > 0$, $q \ge 2$ and $\epsilon \ge \epsilon_0(\chi, k)$ as given in Proposition 3.6. Then there exist some $R > 1$ and $C(\chi, k, d, \epsilon)$ such that the solution of problem (3.1)-(3.2) obeys, for all $t \in (0, T_{\max})$,

$$\begin{aligned}
&\|n\|^k_{L^\infty_t L^k_{\mathrm{uloc}}(\mathbb{R}^d)} + \|\nabla c\|^{2k}_{L^\infty_t L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)} \\
&\le C(k, d, \epsilon, \chi, j)\left(1 + \|n_0\|^2_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + \|\nabla c\|^2_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)} + \|n_0\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)} + \|\nabla c_0\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}\right).
\end{aligned}$$

Proof. Combining with the Lemma 3.1 above, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} n^k \phi^R_{x_0}\, \mathrm{d}x = \left\| n^{\frac{k}{2}} \left(\phi^R_{x_0}\right)^{\frac{1}{2}} \right\|^2_{L^2(\mathbb{R}^d)} \\
&\le C \int_{\mathbb{R}^d} \left|\nabla\left(n^{\frac{k}{2}}\left(\phi^R_{x_0}\right)^{\frac{1}{2}}\right)\right|^2 \mathrm{d}x + C^k \left(\int_{B_{2R}(x_0)} n \left(\phi^R_{x_0}\right)^{\frac{1}{k}} \mathrm{d}x\right)^2 \\
&\le Ck^2 \int_{\mathbb{R}^d} n^{k-2}|\nabla n|^2 \phi^R_{x_0}\, \mathrm{d}x + C\int_{\mathbb{R}^d} n^k \left|\nabla\left(\phi^R_{x_0}\right)^{\frac{1}{2}}\right|^2 \mathrm{d}x + C^k 3^d \|n\|^2_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} \\
&\le Ck^2 \int_{\mathbb{R}^d} n^{k-2}|\nabla n|^2 \phi^R_{x_0}\, \mathrm{d}x + \frac{C}{R^2}\|n\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)} + C^k 3^d \|n\|^2_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)}.
\end{aligned}$$

Using the same calculation method, we have

$$\int_{\mathbb{R}^d} |\nabla c|^{2k} \phi^R_{x_0}\, \mathrm{d}x$$

$$\leq Ck^2 \int_{\mathbb{R}^d} |\nabla c|^{2k-4} |\nabla |\nabla c|^2|^2 \phi_{x_0}^R \, \mathrm{d}x + \frac{C}{R^2} \|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} + C^k 3^d \|\nabla c\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^2.$$

By analyzing the above two items, it can be concluded that

$$\begin{aligned}
&\frac{k(k-1)}{16} \int_{\mathbb{R}^d} n^{k-2} |\nabla n|^2 \phi_{x_0}^R \, \mathrm{d}x + \frac{b_k}{8} \int_{\mathbb{R}^d} |\nabla c|^{2k-4} |\nabla |\nabla c|^2|^2 \phi_{x_0}^R \, \mathrm{d}x \\
\geq\ & \frac{k(k-1)}{16} \left(\frac{C}{k^2} \int_{\mathbb{R}^d} n^k \phi_{x_0}^R \, \mathrm{d}x - \frac{C}{k^2 R^2} \|n\|_{L^k_{\mathrm{uloc}}}^k - \frac{C^k 3^d}{k^2} \|n\|_{L^1_{\mathrm{uloc}}}^2 \right) \\
&+ \frac{b_k}{8} \left(\frac{C}{k^2} \int_{\mathbb{R}^d} |\nabla c|^{2k} \phi_{x_0}^R \, \mathrm{d}x - \frac{C}{k^2 R^2} \|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} - \frac{C^k 3^d}{k^2} \|\nabla c\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^2 \right).
\end{aligned}$$

Define

$$y(t) := \int_{\mathbb{R}^d} |\nabla c|^{2k} \phi_{x_0}^R \, \mathrm{d}x + \sum_{j=1}^{k} b_j \int_{\mathbb{R}^d} n^j |\nabla c|^{2k-2j} \phi_{x_0}^R \, \mathrm{d}x.$$

Then we get that there exists an constant C_1 only depending on d and satisfying the follow inequality

$$\begin{aligned}
&y'(t) + \frac{C_1 k(k-1)}{16\tau k^2} \int_{\mathbb{R}^d} n^k \phi_{x_0}^R \, \mathrm{d}x + \frac{C_1 b_k}{k^2} \int_{\mathbb{R}^d} |\nabla c|^{2k} \phi_{x_0}^R \, \mathrm{d}x \\
\leq\ & \frac{3^d C^k k(k-1)}{\tau k^2} \|n\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)}^2 + \frac{C^k b_k 3^d}{k^2} \|\nabla c\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^2 \\
&+ \left(b_1 \left(\frac{5q-8}{4q} \right) + \frac{(q-2)}{q} + \frac{1}{8C_0} \right) \int_{\mathbb{R}^d} |\nabla c|^{2k-2} \phi_{x_0}^R \, \mathrm{d}x \\
&+ \left(\frac{C3^d b_1}{R^2} + \sum_{j=2}^{k-1} \frac{b_j C_j}{R^2} + \frac{b_k C 3^d k}{2(k-1)R^2} + \frac{b_k C 3^d}{2k^2 R^2} + \frac{Ck(k-1)}{k^2 R^2} \right) \|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}^k \\
&+ \left(\frac{C3^d k}{R^2} + \frac{C3^d b_1}{R^2} + \sum_{j=2}^{k-1} \frac{b_j C_j}{R^2} + \frac{b_k C 3^d k}{R^{2k}} + \frac{C b_k}{k^2 R^2} \right) \|\nabla c\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}^{2k} \\
&+ \left(\sum_{j=2}^{k-1} C_j b_j R^d + \frac{b_k C 3(q-1)}{2(k+q-1)} \right) R^d + C R^{\frac{k-1}{2k(k+1)}}.
\end{aligned}$$

Using the Hölder inequality, we have

$$\left(b_1\left(\frac{5q-8}{4q}\right)+\frac{(q-2)}{q}+\frac{1}{8C_0}\right)\int\limits_{\mathbb{R}^d}|\nabla c|^{2k-2}\phi_{x_0}^R\,\mathrm{d}x\le\frac{C_1b_k}{2k^2}\int\limits_{\mathbb{R}^d}|\nabla c|^{2k}\phi_{x_0}^R\,\mathrm{d}x$$
$$+C(\epsilon,k,q,\chi)R^d.$$

We choose for any k satisfying

$$C_0=\min\left\{\frac{C_1k(k-1)}{16k^2},\frac{C_1b_k}{2k^2}\right\},$$

then there exists a constant $C_2=C_2(\chi,\epsilon,k)$ such that

$$y(t)\le C_2\left(\int\limits_{\mathbb{R}^d}n^k\phi_{x_0}^R\,\mathrm{d}x+\int\limits_{\mathbb{R}^d}|\nabla c|^{2k}\phi_{x_0}^R\,\mathrm{d}x\right),$$

from which it follows that

$$\begin{aligned}y'(t)+\frac{c_0}{C_2}y(t)\le\ &\frac{3^dC^kk(k-1)}{k^2}\|n\|^2_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)}+\frac{C^kb_k3^d}{k^2}\|\nabla c\|^2_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}\\&+\left(\frac{C3^db_1}{R^2}+\sum_{j=2}^{k-1}\frac{b_jC_j}{R^2}+\frac{b_kC3^dk}{2(k-1)R^2}+\frac{b_kC3^d}{2k^2R^2}+\frac{Ck(k-1)}{k^2R^2}\right)\|n\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}\\&+\left(\frac{C3^dk}{R^2}+\frac{C3^db_1}{R^2}+\sum_{j=2}^{k-1}\frac{b_jC_j}{R^2}+\frac{b_kC3^dk}{R^{2k}}+\frac{Cb_k}{k^2R^2}\right)\|\nabla c\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}\\&+C(k,d,q,\epsilon,\chi,R)R^d.\end{aligned}$$

Using the method of Proposition 3.1, we can get

$$\begin{aligned}y(t)\le\ \max\Bigg\{&y(0),\frac{C(k,d,\epsilon,\chi)}{R^2}\left(\|n\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}+\|\nabla c\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}\right)\\&+C(k,d,q,\epsilon,\chi,R)\left(1+\|n_0\|^2_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)}+\|\nabla c_0\|^2_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}\right)\Bigg\}.\end{aligned}$$

Similar to Proposition 3.1, we choose $R\ge R_0$ satisfying that

$$\frac{C(k,d,\epsilon,\chi)}{R^2}\le\min\left\{\frac{1}{2},\frac{b_k}{2}\right\}.$$

Then we choose a constant R independent of d,k,ϵ,χ, which satisfies the following relationship

$$\begin{aligned}&\frac{1}{2}\|\nabla c(t)\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}+\frac{b_k}{2}\|n(t)\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}\\&\le\ y(0)+C(\epsilon,q,\chi,d,R)\left(1+\|n_0\|^2_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)}+\|\nabla c_0\|^2_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}\right)\end{aligned}$$

$$\leq C(k,d,q,\epsilon,\chi,R)\left(1+\|n_0\|^2_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)}+\|\nabla c_0\|^2_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}+\|\nabla c_0\|^{2k}_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}+\|n_0\|^k_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)}\right),$$

here we end the proof. □

Proposition 3.8. Let $\chi > 0$ and $q \geq 2$. Suppose $k \in \mathbb{N}$ with $k \geq 3$ and $\epsilon \geq \epsilon_0(\chi,k,q)$ as given in Proposition 3.7. Then there exist some $R > 1$ and $C(\chi,k,\epsilon,q,d)$, such that the solution of problem (3.1)-(3.2) satisfies

$$\|n(t)\|_{L^\infty(\mathbb{R}^d)}+\|c(t)\|_{W^{1,\infty}(\mathbb{R}^d)} \leq C, \quad \forall t \in (0,T_{\max}).$$

Before proving Proposition 3.8, we first need to state the following lemma, which is known as the generalized inequality.

Lemma 3.2. *[22] Let $\varphi \in \mathcal{D}\left(\mathbb{R}^d\right)$ and $f \in L^p_{\mathrm{uloc}}\left(\mathbb{R}^d\right)$ with $1 \leq p < \infty$. Then there exists a constant C such that*

$$\|\varphi * f\|_{L^\infty(\mathbb{R}^d)} \leq C\|f\|_{1,1}. \tag{3.28}$$

Particularly, for $j \geq 0$, we have

$$\left\|\dot{\Delta}_j f\right\|_{L^\infty(\mathbb{R}^d)}+\left\|\dot{S}_j f\right\|_{L^\infty(\mathbb{R}^d)} \leq C2^{\frac{d}{p}j}\|f\|_{p,1}. \tag{3.29}$$

Then, we prove Proposition 3.8.

Proof. According to system (3.1), we apply the semigroup as

$$\nabla c(t)=e^{t(-1+\Delta)}\nabla\left(\psi(x/M)c_0\right)-\int_0^t e^{(t-s)(-1+\Delta)}\nabla(u\cdot\nabla c)\,\mathrm{d}s+\int_0^t e^{(t-s)(-1+\Delta)}\nabla n\,\mathrm{d}s.$$

Due to low-high frequency decomposition, we divide ∇c into two parts as follows

$$\nabla c=\dot{S}_0\nabla c+\sum_{j\geq 0}\dot{\Delta}_j\nabla c:=\nabla c^L+\nabla c^H.$$

For the low frequency regime, using the generalized Young inequality (3.29), we get

$$\left\|\nabla c^L\right\|_{L^\infty(\mathbb{R}^d)} \leq C\,\|\nabla c_0\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}.$$

For the high frequency regime, we can constrain it via making use of [2, Lemma 2.4] and (3.29) such that for $k > d$,

$$
\begin{aligned}
\left\|\nabla c^H\right\|_{L^\infty(\mathbb{R}^d)} &\le C\|c_0\|_{W^{1,\infty}(\mathbb{R}^d)} + \sum_{j\ge 0}\int_0^t \left\|\dot{\Delta}_j e^{(t-s)(-1+\Delta)}\nabla(u\cdot\nabla c)n\right\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}s \\
&\quad + \sum_{j\ge 0}\int_0^t \left\|\dot{\Delta}_j e^{(t-s)(-1+\Delta)}\nabla n\right\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}s \\
&\le C\|c_0\|_{W^{1,\infty}(\mathbb{R}^d)} + C\sum_{j\ge 0}\int_0^t e^{-c(t-s)2^{2j}}\|u\cdot\nabla c\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}s \\
&\quad + C\sum_{j\ge 0}\int_0^t e^{-c(t-s)2^{2j}}\|\nabla n\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}s \\
&\le C\|c_0\|_{W^{1,\infty}(\mathbb{R}^d)} + C\sum_{j\ge 0} C2^{j\left(1+\frac{d}{k}\right)}\int_0^t e^{-c(t-s)2^{2j}}\|n(s)\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)} \mathrm{d}s \\
&\quad + C\sum_{j\ge 0} C2^{j\left(1+\frac{d}{2k}\right)}\int_0^t e^{-c(t-s)2^{2j}}\|u\|_{L^\infty(\mathbb{R}^d)}\|\nabla c(s)\|_{L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)} \mathrm{d}s \\
&\le C\|c_0\|_{W^{1,\infty}(\mathbb{R}^d)} + C\|n\|_{L^\infty_T L^k_{\mathrm{uloc}}(\mathbb{R}^d)}\sum_{j\ge 0} C2^{j\left(-1+\frac{d}{k}\right)} \\
&\quad + C\|\nabla c\|_{L^\infty_T L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}\sum_{j\ge 0} C2^{j\left(-1+\frac{d}{2k}\right)} \\
&\le C\|c_0\|_{W^{1,\infty}(\mathbb{R}^d)} + C\|n\|_{L^\infty_T L^k_{\mathrm{uloc}}(\mathbb{R}^d)} + C\|\nabla c\|_{L^\infty_T L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}.
\end{aligned}
$$

Summing up the above two estimates means

$$
\|\nabla c\|_{L^\infty(\mathbb{R}^d)} \le C\left(\|\nabla c_0\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)} + \|c_0\|_{W^{1,\infty}(\mathbb{R}^d)} + \|n\|_{L^\infty_T L^k_{\mathrm{uloc}}(\mathbb{R}^d)} + \|\nabla c\|_{L^\infty_T L^{2k}_{\mathrm{uloc}}(\mathbb{R}^d)}\right).
$$

By using the same method, for $k > \frac{qd}{2}$, we have

$$
\begin{aligned}
\|n(t)\|_{L^\infty(\mathbb{R}^d)} &\le C\|n\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + C\|n_0\|_{L^\infty(\mathbb{R}^d)} + \sum_{j\ge 0}\int_0^t \left\|\dot{\Delta}_j e^{(t-s)\Delta}\nabla(n\nabla c)\right\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}s \\
&\quad + \sum_{j\ge 0}\int_0^t \left\|\dot{\Delta}_j e^{(t-s)\Delta}\nabla(un)\right\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}s + \epsilon\sum_{j\ge 0}\int_0^t \left\|\dot{\Delta}_j e^{(t-s)\Delta} n^q\right\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}s \\
&\le C\|n\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + C\|n_0\|_{L^\infty(\mathbb{R}^d)} + C\epsilon\sum_{j\ge 0} 2^{j\frac{qd}{k}}\int_0^t e^{-c(t-s)2^{2j}}\|n\|^q_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)} \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ C \sum_{j \geq 0} 2^{j\left(1+\frac{d}{k}\right)} \int_0^t e^{-c(t-s)2^{2j}} \|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)} \|\nabla c\|_{L^\infty(\mathbb{R}^d)} \,\mathrm{d}s \\
&+ C \sum_{j \geq 0} 2^{j\left(1+\frac{d}{k}\right)} \int_0^t e^{-c(t-s)2^{2j}} \|n\|_{L^k_{\mathrm{uloc}}(\mathbb{R}^d)} \|u\|_{L^\infty(\mathbb{R}^d)} \,\mathrm{d}s \\
\leq\ & C\big(\|n\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^d)} + \|c_0\|_{L^\infty(\mathbb{R}^d)} + \epsilon \|n\|^q_{L^\infty_T L^k_{\mathrm{uloc}}(\mathbb{R}^d)}\big) \\
&+ C\big(\|n\|^2_{L^\infty_T L^k_{\mathrm{uloc}}(\mathbb{R}^d)} + \|\nabla c\|^2_{L^\infty_T L^\infty(\mathbb{R}^d)}\big).
\end{aligned} \tag{3.30}
$$

Next, we need to prove another estimate, namely the uniformly local $L^2_{\mathrm{uloc}}\left(\mathbb{R}^d\right)$-estimate for c. By the same method as above for ∇c, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \left(\phi^R_{x_0} c\right)^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} \left(\phi^R_{x_0} c\right)^2 \,\mathrm{d}x - \int_{\mathbb{R}^d} \Delta c c \left(\phi^R_{x_0}\right)^2 \,\mathrm{d}x \\
= & -\int_{\mathbb{R}^d} u \cdot \nabla c c \left(\phi^R_{x_0}\right)^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} n c \left(\phi^R_{x_0}\right)^2 \,\mathrm{d}x.
\end{aligned}
$$

Integrating by parts

$$
-\int_{\mathbb{R}^d} \Delta c c \left(\phi^R_{x_0}\right)^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} \left(\phi^R_{x_0} \nabla c\right)^2 \,\mathrm{d}x + 2\int_{\mathbb{R}^d} \phi^R_{x_0} c \nabla c \cdot \nabla \phi^R_{x_0} \,\mathrm{d}x.
$$

According to the Cauchy-Schwarz inequality, we conclude that

$$
2\int_{\mathbb{R}^d} \phi^R_{x_0} c \nabla c \cdot \nabla \phi^R_{x_0} \,\mathrm{d}x \leq \frac{C}{R} \left\|\phi^R_{x_0} c\right\|_{L^2(\mathbb{R}^d)} \|\nabla c\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}.
$$

Hence, we have

$$
\begin{aligned}
-\int_{\mathbb{R}^d} u \cdot \nabla c c \left(\phi^R_{x_0}\right)^2 \,\mathrm{d}x &\leq C \|u\|_{L^\infty(\mathbb{R}^d)} \|\nabla c\|_{L^\infty(\mathbb{R}^d)} \left\|\phi^R_{x_0} c\right\|_{L^2(\mathbb{R}^d)} \|\phi^R_{x_0}\|_{L^2(\mathbb{R}^d)} \\
&\leq C R^{\frac{d}{2}} \|\nabla c\|_{L^\infty(\mathbb{R}^d)} \left\|\phi^R_{x_0} c\right\|_{L^2(\mathbb{R}^d)},
\end{aligned}
$$

and by the Hölder inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^d} n c \left(\phi^R_{x_0}\right)^2 \,\mathrm{d}x &\leq \left\|\phi^R_{x_0}\right\|_{L^2(\mathbb{R}^d)} \|n\|_{L^\infty(\mathbb{R}^d)} \left\|\phi^R_{x_0} c\right\|_{L^2(\mathbb{R}^d)} \\
&\leq C R^{\frac{d}{2}} \|n\|_{L^\infty(\mathbb{R}^d)} \left\|\phi^R_{x_0} c\right\|_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

Summarizing the above estimates, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi_{x_0}^R c\right\|_{L^2(\mathbb{R}^d)}^2+\left\|\phi_{x_0}^R c\right\|_{L^2(\mathbb{R}^d)}^2\\
\le\ &\frac{C}{R}\|\nabla c\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}\left\|\phi_{x_0}^R c\right\|_{L^2(\mathbb{R}^d)}+CR^{\frac{d}{2}}\|\nabla c\|_{L^\infty(\mathbb{R}^d)}\left\|\phi_{x_0}^R c\right\|_{L^2(\mathbb{R}^d)}\\
&+CR^{\frac{d}{2}}\|n\|_{L^\infty(\mathbb{R}^d)}\left\|\phi_{x_0}^R c\right\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

By calculation, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi_{x_0}^R c\right\|_{L^2(\mathbb{R}^d)}+\left\|\phi_{x_0}^R c\right\|_{L^2(\mathbb{R}^d)}\le\frac{C}{R}\|\nabla c\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}+CR^{\frac{d}{2}}\|\nabla c\|_{L^\infty(\mathbb{R}^d)}+CR^{\frac{d}{2}}\|n\|_{L^\infty(\mathbb{R}^d)}.$$

Integrating the above inequality with respect to time t, we readily obtain

$$\begin{aligned}
&\left\|\phi_{x_0}^R c(t)\right\|_{L^2(\mathbb{R}^d)}\le e^{-t}\left\|\phi_{x_0}^R\psi(x/M)c_0\right\|_{L^2(\mathbb{R}^d)}\\
&+\left(\frac{C}{R}\|\nabla c\|_{L^\infty_T L^2_{\mathrm{uloc}}(\mathbb{R}^d)}+CR^{\frac{d}{2}}\|\nabla c\|_{L^\infty_T L^\infty(\mathbb{R}^d)}+CR^{\frac{d}{2}}\|n\|_{L^\infty_T L^\infty(\mathbb{R}^d)}\right)\int_0^t e^{-(t-s)}\mathrm{d}s.
\end{aligned}$$

Owning to

$$\|\nabla c\|_{L^\infty_T L^2_{\mathrm{uloc}}(\mathbb{R}^d)}+\|\nabla c\|_{L^\infty_T L^\infty(\mathbb{R}^d)}+\|n\|_{L^\infty_T L^\infty(\mathbb{R}^d)}\le C(\epsilon,\chi,q,d,R),$$

we can easily conclude that

$$\|c\|_{L^\infty_T L^2_{\mathrm{uloc}}(\mathbb{R}^d)}\le\|c_0\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}+C(\epsilon,\chi,q,d,R).$$

In the light of the sharp interpolation inequality, we get

$$\|c\|_{L^\infty(\mathbb{R}^d)}\le C\|c\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^d)}^{\frac{2}{d+2}}\|\nabla c\|_{L^\infty(\mathbb{R}^d)}^{\frac{d}{d+2}}.$$

Summarizing the above estimates, we ultimately conclude that for $0<t<T_{\max}$,

$$\|n(t)\|_{L^\infty(\mathbb{R}^d)}+\|c(t)\|_{W^{1,\infty}(\mathbb{R}^d)}\le C\left(\chi,k,\epsilon,q,d,\|n_0\|_{L^\infty(\mathbb{R}^d)},\|c_0\|_{W^{1,\infty}(\mathbb{R}^d)}\right),$$

and hence the proof of the proposition is completed. □

On the basis of Proposition 3.8 and the continuation criterion (3.4), we see that $T_{\max}=\infty$, and more accurately, the couple (n_M,c_M) is the global-in-time solution to problem (3.1)-(3.2) satisfying

$$\|n_M(t)\|_{L^\infty(\mathbb{R}^d)}+\|c_M(t)\|_{W^{1,\infty}(\mathbb{R}^d)}\le C\left(\chi,k,\epsilon,q,d,\|n_0\|_{L^\infty(\mathbb{R}^d)},\|c_0\|_{W^{1,\infty}(\mathbb{R}^d)}\right),$$

for all $t \geq 0$. Therefore, we have

$$
\begin{aligned}
& n_M \in C^0\left([0, \infty)) ; L^1\left(\mathbb{R}^d\right) \cap L^{\infty}\left(\mathbb{R}^d\right)\right) \cap C^{2,1}\left(\mathbb{R}^d \times(0, \infty)\right), \\
& c_M \in C^0\left([0, \infty) ; H^1\left(\mathbb{R}^d\right) \cap W^{1, \infty}\left(\mathbb{R}^d\right)\right) \cap C^{2,1}\left(\mathbb{R}^d \times(0, \infty)\right).
\end{aligned}
$$

Then we will show the convergence. The uniform boundedness for $\partial_t n^M$ and $\partial_t c^M$ is needed. By the first equation of (3.1), we deduce

$$
\begin{aligned}
\|\partial_t n_M\|_{L_t^2 H^{-1}} \leq & \|\Delta n_M\|_{L_t^2 H^{-1}}+\|u \cdot \nabla n_M\|_{L_t^2 H^{-1}}+\|\nabla \cdot(n_M \nabla c_M)\|_{L_t^2 H^{-1}} \\
& +\|n_M\|_{L_t^2 H^{-1}}+\|(n_M)^q\|_{L_t^2 H^{-1}} \\
\leq & \|n_M\|_{L_t^2 H^1}+\|u\|_{L_t^{\infty} L^{\infty}}\|n_M\|_{L_t^2 L^2}+\|n_M\|_{L_t^{\infty} L^{\infty}}\|c_M\|_{L_t^2 H^1} \\
& +\|n_M\|_{L_t^2 L^2}+\|n_M\|_{L_t^{\infty} L^{\infty}}^{q-1}\|n_M\|_{L_t^2 L^2} \\
\leq & C .
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
\|\partial_t c_M\|_{L_t^2 H^{-1}} & \leq\|\Delta c_M\|_{L_t^2 H^{-1}}+\|u \cdot \nabla c_M\|_{L_t^2 H^{-1}}+\|c_M\|_{L_t^2 H^{-1}}+\|n_M\|_{L_t^2 H^{-1}} \\
& \leq\|c_M\|_{L_t^2 H^1}+\|u\|_{L_t^{\infty} L^{\infty}}\|c_M\|_{L_t^2 L^2}+\|c_M\|_{L_t^2 L^2}+\|n_M\|_{L_t^2 L^2} \\
& \leq C .
\end{aligned}
$$

We know L^2 is locally compactly embedded in H^s and H^s continuously embedded in H^{-1} with $s \in(-1,0)$, the classical Aubin-Lions argument guarantees that, up to an extraction of subsequence, the approximate solution sequence (n_M, c_M) strongly converges in $\mathcal{C}(\mathbb{R}^{+} ; H^s)$ with $s<0$ to some function (n, c) such that

$$
\begin{aligned}
& n \in C^0\left([0, \infty)) ; L^1\left(\mathbb{R}^d\right) \cap L^{\infty}\left(\mathbb{R}^d\right)\right), \\
& c \in C^0\left([0, \infty) ; H^1\left(\mathbb{R}^d\right) \cap W^{1, \infty}\left(\mathbb{R}^d\right)\right)
\end{aligned}
$$

and

$$
\|n(t)\|_{L^{\infty}(\mathbb{R}^d)}+\|c(t)\|_{W^{1, \infty}(\mathbb{R}^d)} \leq C\left(\chi, k, \epsilon, q, d,\|n_0\|_{L^{\infty}(\mathbb{R}^d)},\|c_0\|_{W^{1, \infty}(\mathbb{R}^d)}\right),
$$

for all $t \geq 0$, and it is the global-in-time solution to problem (1.5)-(1.6).

All we have to do now is to prove that the uniqueness of solution (n, c) to problem (1.5)-(1.6). Suppose (n_1, c_1) and (n_2, c_2) are two solutions of Eqs. (1.5) with the same initial data and satisfy (1.6). Let $\delta n=n_1-n_2$ and $\delta c=c_1-c_2$, then the difference equations are as follows

$$
\begin{cases}\partial_t \delta n+u \cdot \nabla \delta n=\Delta \delta n-\nabla \cdot(\delta n \nabla c_1)-\nabla \cdot(n_2 \nabla \delta c)+\delta n-\epsilon(n_1^q-n_2^q), \\ \partial_t \delta c+u \cdot \nabla \delta c=\Delta \delta c-\delta c+\delta n .\end{cases} \tag{3.31}
$$

Multiplying the first equation of (3.31) with δn and integrating in space, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\delta n(t)\|_2^2 + \|\nabla \delta n(t)\|_2^2 \\
&= -\int_{\mathbb{R}^d} \nabla \cdot (\delta n \nabla c_1)\delta n dx - \int_{\mathbb{R}^d} \nabla \cdot (n_2 \nabla \delta c)\delta n dx \\
&\quad + \int_{\mathbb{R}^d} \delta n \delta n dx - \epsilon \int_{\mathbb{R}^d} (n_1^q - n_2^q)\delta n dx \\
&\le C\|\delta n\|_2^2 \|\nabla c_1\|_{L^\infty}^2 + \frac{1}{4}\|\nabla \delta n\|_2^2 \\
&\quad + C\|n_2\|_{L^\infty}^2 \|\nabla \delta c\|_2^2 + \frac{1}{4}\|\nabla \delta n\|_2^2 + \|\delta n\|_2^2,
\end{aligned}$$

from which we can have

$$\frac{d}{dt}\|\delta n(t)\|_2^2 + \|\nabla \delta n(t)\|_2^2 \le C(\|\delta n\|_2^2 \|\nabla c_1\|_{L^\infty}^2 + \|n_2\|_{L^\infty}^2 \|\nabla \delta c\|_2^2 + \|\delta n\|_2^2). \tag{3.32}$$

Similarly, operating the L^2-inner product of the second equation of (3.31) with δc yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\delta c(t)\|_2^2 + \|\nabla \delta c(t)\|_2^2 &= -\int_{\mathbb{R}^d} \delta c \delta c dx + \int_{\mathbb{R}^d} \delta n \delta c dx \\
&\le C(\|\delta c\|_2^2 + \|\delta n\|_2^2).
\end{aligned} \tag{3.33}$$

Applying ∂_i on both sides of the second equation of (3.31), we have

$$\partial_t \partial_i \delta c + u \cdot \nabla \partial_i \delta c - \Delta \partial_i \delta c = -\partial_i u \cdot \nabla \delta c - \partial_i \delta c + \partial_i \delta n.$$

Taking the L^2-inner product with the above equality by $\partial_i \delta c$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla \delta c(t)\|_2^2 + \|\Delta \delta c(t)\|_2^2 &= -\sum_i \int_{\mathbb{R}^d} \partial_i u \cdot \nabla \delta c \partial_i \delta c dx \\
&\quad - \sum_i \int_{\mathbb{R}^d} \partial_i \delta c \partial_i \delta c dx + \sum_i \int_{\mathbb{R}^d} \partial_i \delta n \partial_i \delta c dx \\
&= -\int_{\mathbb{R}^d} (\nabla \delta c \cdot \nabla) u \cdot \nabla \delta c dx \\
&\quad + \int_{\mathbb{R}^d} \delta c \Delta \delta c dx - \int_{\mathbb{R}^d} \delta n \Delta \delta c dx \\
&\le C\|\nabla \delta c\|_2^2 \|\nabla u\|_{L^\infty}
\end{aligned}$$

$$+C\|\delta c\|_2^2+\frac{1}{4}\|\Delta\delta c\|_2^2+C\|\delta n\|_2^2+\frac{1}{4}\|\Delta\delta c\|_2^2.$$

Hence we conclude

$$\frac{d}{dt}\|\nabla\delta c(t)\|_2^2+\|\Delta\delta c(t)\|_2^2\le C(\|\nabla\delta c\|_2^2+\|\delta c\|_2^2+\|\delta n\|_2^2). \tag{3.34}$$

Summing up (3.32)-(3.34), we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\delta n(t)\|_2^2+\|\delta c(t)\|_2^2+\|\nabla\delta c(t)\|_2^2)+\|\nabla\delta n\|_2^2+\|\nabla\delta c\|_2^2+\|\Delta\delta c\|_2^2\\&\le C(\|\delta n\|_2^2\|\nabla c_1\|_{L^\infty}^2+\|n_2\|_{L^\infty}^2\|\nabla\delta c\|_2^2+\|\delta n\|_2^2+\|\delta c\|_2^2+\|\nabla\delta c\|_2^2).\end{aligned}$$

Then we conclude

$$\frac{d}{dt}(\|\delta n(t)\|_2^2+\|\delta c(t)\|_2^2+\|\nabla\delta c(t)\|_2^2)\le CF(t)(\|\delta n\|_2^2+\|\delta c\|_2^2+\|\nabla\delta c\|_2^2),$$

with

$$F(t)=1+\|n_2\|_{L^\infty}^2+\|\nabla c_1\|_{L^\infty}^2.$$

Since $F(t)$ is integrable, we can apply the Gronwall inequality to derive the uniqueness. Consequently, we complete the proof of Theorem 1.1.

Data availability

No data was used for the research described in the article.

Acknowledgments

Q. Zhang is supported by the National Natural Science Foundation of China [grant number 12326416] and the Natural Science Foundation of Hebei Province [grant number A2024201008], Y. Wu is supported by the Australian Research Council, and P. Wang is supported by the National Natural Science Foundation of China [grant number 12171135].

Appendix A

In appendix, we shall prove the unique classical solution for (1.5) using energy estimates and classical compactness arguments.

We begin by introducing the dynamic partition of unity to define Besov spaces. For more details, refer to [19]. Let $\varphi\in C_0^\infty(\mathbb{R}^d)$ be supported in $\mathcal{C}=\{\xi\in\mathbb{R}^d,\ \frac{3}{4}\le|\xi|\le\frac{8}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1,\quad \text{for}\quad \xi\neq0.$$

Set $\chi(\xi)=1-\sum\limits_{j\in\mathbb{N}}\varphi(2^{-j}\xi)$. For $f\in\mathcal{S}'$, we define Littlewood-Paley operators as follows

$$\Delta_{-1} f = \chi(D) f;\ \forall j \in \mathbb{N},\ \Delta_j f = \varphi(2^{-j} D) f \quad \text{and} \quad \forall j \in \mathbb{Z},\ \dot{\Delta}_j f = \varphi(2^{-j} D) f.$$

The following low-frequency cut-off will be also used:

$$S_j f = \sum_{-1 \le j' \le j-1} \Delta_{j'} f \quad \text{and} \quad \dot{S}_j f = \sum_{j' \le j-1} \dot{\Delta}_{j'} f.$$

Now, let us recall the definition of the Besov space. For $s \in \mathbb{R}$, $1 \le p, r \le \infty$, we define the homogenous Besov space $\dot{B}^s_{p,r}$ as the set of tempered distributions of $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{B}^s_{p,r}} := (\sum_{j \in \mathbb{Z}} 2^{jsr} \|\dot{\Delta}_j f\|_p^r \Big)^{\frac{1}{r}} < \infty,$$

where $\mathcal{P}$ is the polynomial space. The inhomogeneous space $B^s_{p,r}$ is the set of tempered distribution f such that

$$\|f\|_{B^s_{p,r}} := \Big(\sum_{j \ge -1} 2^{jsr} \|\Delta_j f\|_p^r \Big)^{\frac{1}{r}} < \infty.$$

It is worthwhile to remark that $B^s_{2,2}$ and $B^s_{\infty,\infty}$ coincide with the usual Sobolev spaces H^s and the usual Hölder space C^s for $s \in \mathbb{R} \setminus \mathbb{Z}$, respectively. One can refer to [19] for more details.

In our study, we also use the space-time Besov spaces $L^q_T B^s_{p,r}$. For $T > 0$ and $q > 1$, the space $L^q_T B^s_{p,r}$ is the set of all tempered distribution f satisfying

$$\|f\|_{L^q_T B^s_{p,r}} \triangleq \|(\sum_{j \ge -1} 2^{2jsr} \|\Delta_j f\|_{L^p(R^d)})^{\frac{1}{r}} \|_{L^q_T} < \infty.$$

Lemma A.1. *[19] Let $1 \le p \le q \le \infty$. Assume that $f \in L^p$, then there exists a constant C independent of f, j such that*

$$\operatorname{supp} \hat{f} \subset \{|\xi| \le C 2^j\} \Longrightarrow \|\partial^\alpha f\|_q \le C 2^{j|\alpha| + dj(\frac{1}{p} - \frac{1}{q})} \|f\|_p,$$

$$\operatorname{supp} \hat{f} \subset \Big\{ \frac{1}{C} 2^j \le |\xi| \le C 2^j \Big\} \Longrightarrow \|f\|_p \le C 2^{-j|\alpha|} \sup_{|\beta| = |\alpha|} \|\partial^\beta f\|_p.$$

Lemma A.2. *[19] For all $s \in \mathbb{Z}^+ \cap \{0\}$, there exists a constant $C > 0$ such that for all u, $v \in L^\infty \cap H^s(\mathbb{R}^2)$,*

$$\|fg\|_{H^s} \le C(\|f\|_\infty \|g\|_{H^s} + \|g\|_\infty \|f\|_{H^s}).$$

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