DE GRUYTER Advances in Nonlinear Analysis 2025; 14: 20250108

Research Article

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Boundedness of solutions to quasilinear elliptic systems

https://doi.org/10.1515/anona-2025-0108
received October 29, 2024; accepted July 15, 2025

Abstract: This article deals with elliptic systems of the form

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^{N}\sum_{j=1}^{n}a_{i,j}^{\alpha,\beta}(x,u(x))\frac{\partial u^\beta(x)}{\partial x_j}\right)=f^\alpha(x),\quad \alpha=1,\dots,N.$$

Under ellipticity conditions of the diagonal coefficients and proportional conditions of the off-diagonal coefficients, we derive local and global boundedness results. Under ellipticity of all the coefficients and "butterfly" support of off-diagonal coefficients, we derive a global boundedness result. This article also considers regularizing effect of a lower-order term.

Keywords: quasilinear elliptic system, proportional condition, "butterfly" support, regularizing effect, boundedness

MSC 2020: 35J57

1 Introduction

Let $n > 2$, $N \geq 2$ be integers and Ω an open bounded subset of $\mathbb{R}^n$. We consider quasilinear elliptic systems involving N equations of the form

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^{N}\sum_{j=1}^{n}a_{i,j}^{\alpha,\beta}(x,u(x))\frac{\partial u^\beta(x)}{\partial x_j}\right)=f^\alpha(x),\quad \alpha=1,\dots,N, \tag{1.1}$$

where α is the equation index and $u(x) = (u^1(x), \dots, u^N(x)) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$. Denote

$$Du^\beta(x)=\left(\frac{\partial u^\beta(x)}{\partial x_1},\dots,\frac{\partial u^\beta(x)}{\partial x_n}\right),\quad \beta=1,\dots,N,$$

which is the βth row of the matrix

$$Du(x)=\left(\frac{\partial u^\beta(x)}{\partial x_j}\right)_{1\leq i\leq n}^{1\leq\beta\leq N}\in\mathbb{R}^{N\times n}.$$

We consider the following two sets of assumptions on the coefficients $a_{i,j}^{\alpha,\beta}(x,y)$, $i,j\in\{1,\dots,n\}$, $\alpha,\beta\in\{1,\dots,N\}$.

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The first set of assumptions is denoted by $(\mathcal{A})$: for all $i, j \in \{1, \ldots, n\}$ and all $\alpha, \beta \in \{1, \ldots, N\}$, we consider that $a_{i,j}^{\alpha,\beta}(x, y) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfying the following conditions:

$(\mathcal{A}_1)$ (Carathéodory condition) $x \mapsto a_{i,j}^{\alpha,\beta}(x, y)$ is measurable for all $y \in \mathbb{R}^N$ and $y \mapsto a_{i,j}^{\alpha,\beta}(x, y)$ is continuous for almost all $x \in \Omega$;

$(\mathcal{A}_2)$ (Boundedness of all the coefficients) there exists a positive constant $\tilde{c}$ such that for almost all $x \in \Omega$ and all $y \in \mathbb{R}^N$,

$$|a_{i,j}^{\alpha,\beta}(x, y)| \le \tilde{c};$$

$(\mathcal{A}_3)$ (Ellipticity of the diagonal coefficients) there exists a positive constant c_0 such that for almost all $x \in \Omega$, all $y \in \mathbb{R}^N$ and all $\lambda \in \mathbb{R}^n$,

$$c_0 \, |\lambda|^2 \le \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha}(x, y)\lambda_i\lambda_j;$$

$(\mathcal{A}_4)$ (Proportional condition of the off-diagonal coefficients) there exist constants $r^{\alpha,\beta}$, $\alpha, \beta \in \{1, \ldots, N\}$, such that for almost all $x \in \Omega$ and all $y \in \mathbb{R}^N$,

$$a_{i,j}^{\alpha,\beta}(x, y) = r^{\alpha,\beta} a_{i,j}^{\beta,\beta}(x, y),$$

the constants $r^{\alpha,\beta}$, $\alpha, \beta \in \{1, \ldots, N\}$, be such that $r^{\alpha,\alpha} = 1$ and

$$\det \mathcal{R} = \det \begin{pmatrix} 1 & r^{2,1} & r^{3,1} & \ldots & r^{N,1} \\ r^{1,2} & 1 & r^{3,2} & \ldots & r^{N,2} \\ r^{1,3} & r^{2,3} & 1 & \ldots & r^{N,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{1,N} & r^{2,N} & r^{3,N} & \ldots & 1 \end{pmatrix} \ne 0.$$

The second set of assumptions is denoted by $(\mathcal{A})'$: for all $i, j \in \{1, \ldots, n\}$ and all $\alpha, \beta \in \{1, \ldots, N\}$, we consider that $a_{i,j}^{\alpha,\beta}(x, y) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfying $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, and the following:

$(\mathcal{A}_3)'$ (ellipticity of all the coefficients) there exists a positive constant $\tilde{c}_0$ such that for almost all $x \in \Omega$, all $y \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}^{N\times n}$,

$$\tilde{c}_0 \, |\xi|^2 \le \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, y)\xi_i^\alpha \xi_j^\beta;$$

$(\mathcal{A}_4)'$ ("butterfly" support of off-diagonal coefficients) there exists $Q_0 \in (0, +\infty)$ such that for all $Q \ge Q_0$, when $\alpha \ne \beta$,

$$(a_{i,j}^{\alpha,\beta}(x, y) \ne 0 \quad \text{and} \quad |y^\alpha| > Q) \Rightarrow |y^\beta| > Q.$$

For the figure of "butterfly support," see [25, Figure 1].

The following example gives the coefficients $a_{i,j}^{\alpha,\beta}(x, y) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ with $i, j \in \{1, \ldots, n\}$ and $\alpha, \beta \in \{1, \ldots, N\}$, which satisfy the set of assumptions $(\mathcal{A})$.

Example 1.1. We let δ_{ij} be the Kronecker symbol and $\Omega = B_1(0)$, the unit ball in $\mathbb{R}^n$. For $i, j \in \{1, \ldots, n\}$ and $\alpha, \beta \in \{1, \ldots, N\}$, we define $a_{i,j}^{\alpha,\beta}(x, y)$ as follows: for $\alpha \in \{1, \ldots, N\}$,

$$a_{i,j}^{\alpha,\alpha}(x, y) = \left(1 + |x| + \frac{|y^\alpha|}{1 + |y^\alpha|}\right)\delta_{ij},$$

and for $\alpha, \beta \in \{1, \ldots, N\}$ with $\alpha \ne \beta$,

$$a_{i,j}^{\alpha,\beta}(x, y) = r^{\alpha,\beta} a_{i,j}^{\beta,\beta}(x, y) = r^{\alpha,\beta}\left(1 + |x| + \frac{|y^\beta|}{1 + |y^\beta|}\right)\delta_{ij}, \tag{1.2}$$

where the real numbers $r^{\alpha,\beta}$ are such that $\det\mathcal{R} = \det(r^{\alpha,\beta}) \neq 0$; thus, the condition $(\mathcal{A}_4)$ holds true naturally; moreover, the condition $(\mathcal{A}_1)$ is satisfied because $x \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is measurable and $y \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is continuous; we note from (1.2) that $|a_{i,j}^{\alpha,\beta}(x,y)| \leq 3|r^{\alpha,\beta}|$ for $x \in B_1(0)$, thus, the condition $(\mathcal{A}_2)$ is satisfied with $\tilde{c} = 3\|\mathcal{R}\| = 3\left(\sum_{\alpha,\beta=1}^{N}|r^{\alpha,\beta}|^2\right)^{1/2}$; the condition $(\mathcal{A}_3)$ is satisfied with $c_0 = 1$ since

$$\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha}(x,y)\lambda_i\lambda_j = \sum_{i=1}^{n}\left(1 + |x| + \frac{|y^\alpha|}{1 + |y^\alpha|}\right)\lambda_i^2 \geq |\lambda|^2.$$

We note that in this article, we consider the case $N \geq 2$, i.e., we deal with elliptic systems. For the case $N = 1$, (1.1) is only one single equation, existence and regularity results of distributional solutions $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ have been deeply studied, we refer the reader to [5,6,9,10,12,16,34] for existence results and [4,8,14,23,30–33] for regularity results.

For $N \geq 2$, one cannot expect, due to De Giorgi's counterexample, see [17], that weak solutions of (1.1) are locally and globally bounded if no additional assumptions are proposed. Quasilinear elliptic system (1.1) with the set of assumptions $(\mathcal{A})$ has been studied in [28, Theorem 2], where the authors considered the special case $N = 2$. The general case $N \geq 2$ may be found in [19, Theorem 2.1]. Quasilinear elliptic system (1.1) with the set of assumptions $(\mathcal{A})'$ has been studied in [25], where the authors give a local boundedness result. We refer the readers to [20,24,28,29] for some regularity results and estimates related to quasilinear elliptic systems under some staircase support conditions of the coefficients, [15,21] for some results related to nonlinear elliptic systems, and [18] for some local boundedness under nonstandard growth conditions.

In the next two sections, we shall give some boundedness results related to system (1.1) under the conditions $(\mathcal{A})$ or $(\mathcal{A})'$. In the sequel, we shall denote by c a generic constant, whose value, depending on the data, may vary from one line to another.

2 Boundedness under $(\mathcal{A})$

This section deals with local and global boundedness for solutions to elliptic systems (1.1) under the set of assumptions $(\mathcal{A})$.

2.1 Local boundedness result

In this section, we consider (1.1), i.e.,

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^{N}\sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u(x))\frac{\partial u^\beta(x)}{\partial x_j}\right) = f^\alpha(x), \quad \alpha = 1,\ldots,N. \tag{2.1}$$

Let

$$f = (f^1,\ldots,f^N) \in L_{\mathrm{loc}}^{(2^*)'}(\Omega;\mathbb{R}^N), \quad (2^*)' = \frac{2n}{n+2}. \tag{2.2}$$

We give the following:

Definition 2.1. A function $u \in W_{\mathrm{loc}}^{1,2}(\Omega;\mathbb{R}^N)$ is a local solution to (2.1) if

$$\int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u(x))D_j u^\beta(x) D_i\varphi^\alpha(x)\mathrm{d}x = \int_\Omega \sum_{\alpha=1}^{N} f^\alpha(x)\varphi^\alpha(x)\mathrm{d}x \tag{2.3}$$

holds true for all $\varphi \in W^{1,2}(\Omega;\mathbb{R}^N)$ with compact support.

We note that (2.2) is added in order to make finite the right-hand integral in (2.3).

The main result of this section is the following theorem.

Theorem 2.1. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local solution to* (2.1). *Under the set of assumptions* $(\mathcal{A})$, *if $f \in L^m_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $m > \frac{n}{2}$, then u is locally bounded in Ω.*

In order to prove Theorem 2.1, we need the following Caccioppoli inequality.

Lemma 2.1. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local solution to* (2.1) *under the set of assumptions* $(\mathcal{A})$. *Let $B_R(x_0) \Subset \Omega$ be the ball centered at $x_0 \in \Omega$ with radius R, $|B_R(x_0)| < 1$. For $k \geq 0$, $0 < s < t \leq R$, denote*

$$A^\beta_k = \{x \in \Omega : |u^\beta(x)| > k\}, \quad A^\beta_{k,t} = A^\beta_k \cap B_t(x_0).$$

If $f \in L^m_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $m > (2^*)'$, then

$$\sum_{\beta=1}^N \int_{A^\beta_{k,s}} |D|u^\beta||^2 \mathrm{d}x \leq c \sum_{\beta=1}^N \left(\int_{A^\beta_{k,t}} \left(\frac{|u^\beta| - k}{t-s} \right)^2 \mathrm{d}x + |A^\beta_{k,t}|^\theta \right), \tag{2.4}$$

where c is a constant depending upon $n, N, m, \|f\|_{L^m(B_R)}, \tilde{c}, c_0$ and $r^{\alpha,\beta}$, $\alpha, \beta = 1,\ldots,N$, and

$$\theta = 2\left(\frac{1}{(2^*)'} - \frac{1}{m} \right) > 0.$$

Proof. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local solution to (2.1). Let $B_R(x_0) \Subset \Omega$ with $|B_R(x_0)| < 1$ (which implies $R < 1$). For $0 < s < t \leq R$, let us consider a smooth cut-off function $\eta \in C^\infty_0(B_t(x_0))$ satisfying

$$0 \leq \eta \leq 1, \quad \eta \equiv 1, \quad \text{in } B_s(x_0) \quad \text{and} \quad |D\eta| \leq \frac{2}{t-s}.$$

Let us take $\varphi = (\varphi^1, \ldots, \varphi^N)$ with

$$\varphi^\alpha = \sum_{\gamma=1}^N C^\gamma_\alpha \eta^2 G_k(u^\gamma), \quad \alpha \in \{1,\ldots,N\}, \tag{2.5}$$

here and in what follows, for $s \in \mathbb{R}$,

$$G_k(s) = s - T_k(s) = s - \min\left\{1, \frac{k}{|s|}\right\} s, \tag{2.6}$$

and C^γ_α, $\alpha, \gamma \in \{1,\ldots,N\}$, are the real constants to be chosen later. (2.5) yields

$$D_i\varphi^\alpha = \sum_{\gamma=1}^N C^\gamma_\alpha [\eta^2 D_i u^\gamma + 2\eta D_i\eta G_k(u^\gamma)]\chi_{A^\gamma_k}, \quad i = 1,\ldots,n,$$

where, for a set E, $\chi_E(x)$ is the characteristic function of the set E, i.e., $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ otherwise. Such a function φ is admissible for Definition 2.1 since it is obvious that $\varphi \in W^{1,2}(\Omega;\mathbb{R}^N)$ and $\operatorname{supp}\varphi \Subset \Omega$. We use such a φ in (2.3), and we have

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a^{\alpha,\beta}_{i,j}(x,u) D_j u^\beta \sum_{\gamma=1}^N C^\gamma_\alpha [\eta^2 D_i u^\gamma + 2\eta D_i \eta G_k(u^\gamma)]\chi_{A^\gamma_k} \mathrm{d}x = \int_\Omega \sum_{\alpha=1}^N f^\alpha \sum_{\gamma=1}^N C^\gamma_\alpha \eta^2 G_k(u^\gamma) \mathrm{d}x,$$

from which we derive

$$\begin{aligned} &\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a^{\alpha,\beta}_{i,j}(x,u) D_j u^\beta \sum_{\gamma=1}^N C^\gamma_\alpha \eta^2 D_i u^\gamma \chi_{A^\gamma_k} \mathrm{d}x \\ &= -\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a^{\alpha,\beta}_{i,j}(x,u) D_j u^\beta \sum_{\gamma=1}^N C^\gamma_\alpha 2\eta D_i\eta G_k(u^\gamma)\chi_{A^\gamma_k} \mathrm{d}x + \int_\Omega \sum_{\alpha=1}^N f^\alpha \sum_{\gamma=1}^N C^\gamma_\alpha \eta^2 G_k(u^\gamma) \mathrm{d}x. \end{aligned} \tag{2.7}$$

For the left-hand side of (2.7), we use the proportional condition ($\mathcal{A}_4$) and we have

$$
\begin{aligned}
&\int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u)D_ju^\beta \sum_{\gamma=1}^{N} C_\alpha^\gamma \eta^2 D_iu^\gamma \chi_{A_k^\gamma}\mathrm{d}x\\
&= \int_\Omega \sum_{\alpha=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha}(x,u)D_ju^\alpha \sum_{\gamma=1}^{N} C_\alpha^\gamma \eta^2 D_iu^\gamma \chi_{A_k^\gamma}\mathrm{d}x \quad (\text{terms for } \beta=\alpha)\\
&\quad+ \int_\Omega \sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u)D_ju^\beta \sum_{\gamma=1}^{N} C_\alpha^\gamma \eta^2 D_iu^\gamma \chi_{A_k^\gamma}\mathrm{d}x \quad (\text{terms for } \beta\neq\alpha)\\
&= \int_\Omega \sum_{\alpha=1}^{N}\sum_{i,j=1}^{n} C_\alpha^\alpha a_{i,j}^{\alpha,\alpha}(x,u)D_ju^\alpha D_iu^\alpha \chi_{A_k^\alpha}\eta^2\mathrm{d}x \quad (\text{terms for } \gamma=\beta=\alpha)\\
&\quad+ \int_\Omega \sum_{\alpha=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha}(x,u)D_ju^\alpha \sum_{\gamma=1,\gamma\neq\alpha}^{N} C_\alpha^\gamma D_iu^\gamma \chi_{A_k^\gamma}\eta^2\mathrm{d}x \quad (\text{terms for } \gamma\neq\beta=\alpha)\\
&\quad+ \int_\Omega \sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{i,j=1}^{n} r^{\alpha,\beta} a_{i,j}^{\beta,\beta}(x,u)D_ju^\beta C_\alpha^\beta D_iu^\beta \chi_{A_k^\beta}\eta^2\mathrm{d}x \quad (\text{terms for } \gamma=\beta\neq\alpha)\\
&\quad+ \int_\Omega \sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{i,j=1}^{n} r^{\alpha,\beta} a_{i,j}^{\beta,\beta}(x,u)D_ju^\beta \sum_{\gamma=1,\gamma\neq\beta}^{N} C_\alpha^\gamma D_iu^\gamma \chi_{A_k^\gamma}\eta^2\mathrm{d}x \quad (\text{terms for } \gamma\neq\beta,\beta\neq\alpha)\\
&= I_1+I_2+I_3+I_4.
\end{aligned}
\tag{2.8}
$$

It is obvious that, recalling that $r^{\alpha,\alpha}=1$ for $\alpha\in\{1,\dots,N\}$,

$$
\begin{aligned}
I_1+I_3 &= \int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} r^{\alpha,\beta} a_{i,j}^{\beta,\beta}(x,u)D_ju^\beta C_\alpha^\beta D_iu^\beta \chi_{A_k^\beta}\eta^2\mathrm{d}x\\
&= \int_\Omega \sum_{\beta=1}^{N}\sum_{i,j=1}^{n}\left(\sum_{\alpha=1}^{N} r^{\alpha,\beta}C_\alpha^\beta\right) a_{i,j}^{\beta,\beta}(x,u)D_ju^\beta D_iu^\beta \chi_{A_k^\beta}\eta^2\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
I_2+I_4 &= \int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} r^{\alpha,\beta} a_{i,j}^{\beta,\beta}(x,u)D_ju^\beta \sum_{\gamma=1,\gamma\neq\beta}^{N} C_\alpha^\gamma D_iu^\gamma \chi_{A_k^\gamma}\eta^2\mathrm{d}x\\
&= \int_\Omega \sum_{\beta,\gamma=1,\beta\neq\gamma}^{N}\sum_{i,j=1}^{n}\left(\sum_{\alpha=1}^{N} r^{\alpha,\beta}C_\alpha^\gamma\right) a_{i,j}^{\beta,\beta}(x,u)D_ju^\beta D_iu^\gamma \chi_{A_k^\gamma}\eta^2\mathrm{d}x.
\end{aligned}
$$

If one can choose

$$\sum_{\alpha=1}^{N} r^{\alpha,\beta}C_\alpha^\beta = 1, \quad \text{for } \beta\in\{1,\dots,N\}, \tag{2.9}$$

and

$$\sum_{\alpha=1}^{N} r^{\alpha,\beta}C_\alpha^\gamma = 0, \quad \text{for } \beta,\gamma\in\{1,\dots,N\}, \quad \beta\neq\gamma, \tag{2.10}$$

then the assumption ($\mathcal{A}_3$) allows us to estimate

$$\sum_{j=1}^{4} I_j = \int_\Omega \sum_{\beta=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\beta,\beta}(x,u)D_ju^\beta D_iu^\beta \chi_{A_k^\beta}\eta^2\mathrm{d}x \geq c_0\sum_{\beta=1}^{N}\int_{A_{k,t}^\beta}\eta^2\,|Du^\beta|^2\mathrm{d}x. \tag{2.11}$$

Now, we prove that equations (2.9) and (2.10) are valid for appropriate choice of the constants C_α^γ, $\alpha,\gamma\in\{1,\dots,N\}$. In fact, (2.9) and (2.10) have the form

$$\sum_{\alpha=1}^{N} r^{\alpha,\beta} C_{\alpha}^{\gamma} = \delta_{\beta\gamma}, \quad \text{for } \beta, \gamma \in \{1, \ldots, N\}. \tag{2.12}$$

We note that the aforementioned system has N^2 equations with N^2 unknowns C_{α}^{γ}, $\alpha, \gamma \in \{1, \ldots, N\}$ and can be rewritten as the form

$$\begin{pmatrix} \mathcal{R} & 0 & \ldots & 0 \\ 0 & \mathcal{R} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathcal{R} \end{pmatrix} \begin{pmatrix} C^1 \\ C^2 \\ \vdots \\ C^N \end{pmatrix} = \begin{pmatrix} e^1 \\ e^2 \\ \vdots \\ e^N \end{pmatrix}, \tag{2.13}$$

where

$$\mathcal{R} = \begin{pmatrix} 1 & r^{2,1} & r^{3,1} & \ldots & r^{N,1} \\ r^{1,2} & 1 & r^{3,2} & \ldots & r^{N,2} \\ r^{1,3} & r^{2,3} & 1 & \ldots & r^{N,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{1,N} & r^{2,N} & r^{3,N} & \ldots & 1 \end{pmatrix}, \qquad C^j = \begin{pmatrix} C_1^j \\ C_2^j \\ C_3^j \\ \vdots \\ C_N^j \end{pmatrix},$$

and e^j is the unit vector of $\mathbb{R}^N$, $j \in \{1, \ldots, N\}$. By assumption $(\mathcal{A}_4)$, $\det\mathcal{R} \neq 0$; thus, the determinant of the left-hand side square matrix in (2.13) is nonzero, and noting the right-hand side of system (2.13) is nonzero, then there exists a unique nonzero solution to (2.13). We choose C_{α}^{γ} to be the unique nonzero solution to (2.13) and we have (2.9) and (2.10). Note that the values of C_{α}^{γ} rely on the values of $r^{\alpha,\beta}$, $\alpha, \beta = 1, \ldots, N$.

We now use (2.12) again, $(\mathcal{A}_2)$ and the proportional condition of the off-diagonal coefficients in $(\mathcal{A}_4)$ to estimate the first term of the right-hand side of (2.7):

$$\begin{aligned}
&-\int_{\Omega} \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u) D_j u^{\beta} \sum_{\gamma=1}^{N} C_{\alpha}^{\gamma} 2\eta D_i \eta G_k(u^{\gamma}) \chi_{A_k^{\gamma}} \mathrm{d}x \\
&= -\int_{\Omega} \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} r^{\alpha,\beta} a_{i,j}^{\beta,\beta}(x,u) D_j u^{\beta} \sum_{\gamma=1}^{N} C_{\alpha}^{\gamma} 2\eta D_i \eta G_k(u^{\gamma}) \chi_{A_k^{\gamma}} \mathrm{d}x \\
&= -\int_{\Omega} \sum_{\gamma,\beta=1}^{N} \sum_{i,j=1}^{n} \left(\sum_{\alpha=1}^{N} r^{\alpha,\beta} C_{\alpha}^{\gamma} \right) a_{i,j}^{\beta,\beta}(x,u) D_j u^{\beta} 2\eta D_i \eta G_k(u^{\gamma}) \chi_{A_k^{\gamma}} \mathrm{d}x \\
&= -\int_{\Omega} \sum_{\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\beta,\beta}(x,u) D_j u^{\beta} 2\eta D_i \eta G_k(u^{\beta}) \chi_{A_k^{\beta}} \mathrm{d}x \\
&\leq 2\tilde{c}n^2 \int_{\Omega} \sum_{\beta=1}^{N} |Du^{\beta}| \eta |D\eta| |G_k(u^{\beta})| \chi_{A_k^{\beta}} \mathrm{d}x \\
&\leq \tilde{c}n^2 \int_{\Omega} \sum_{\beta=1}^{N} \left(\varepsilon \eta^2 |Du^{\beta}|^2 + \frac{|D\eta|^2 |G_k(u^{\beta})|^2}{\varepsilon} \right) \chi_{A_k^{\beta}} \mathrm{d}x \\
&\leq \varepsilon \tilde{c}n^2 \sum_{\beta=1}^{N} \int_{A_{k,t}^{\beta}} \eta^2 |Du^{\beta}|^2 \mathrm{d}x + \frac{4\tilde{c}n^2}{\varepsilon} \sum_{\beta=1}^{N} \int_{A_{k,t}^{\beta}} \left(\frac{|u^{\beta}| - k}{t - s} \right)^2 \mathrm{d}x,
\end{aligned} \tag{2.14}$$

where we used Young's inequality

$$2ab \leq \varepsilon a^2 + b^2/\varepsilon, \quad \varepsilon > 0,$$

and the fact

$$|G_k(u^{\beta})| = |u^{\beta}| - k, \quad x \in A_k^{\beta}.$$

We next estimate the second term of the right-hand side of (2.7). Sobolev embedding theorem, Young's and Hölder's inequalities allow us to obtain

$$\begin{aligned}
&\int_\Omega \sum_{\alpha=1}^N f^\alpha \sum_{\gamma=1}^N C_\alpha^\gamma \eta^2 G_k(u^\gamma)\mathrm{d}x \\
&\leq \bar{c} \int_{A_{k,t}^\beta} \sum_{\alpha=1}^N |f^\alpha| \sum_{\beta=1}^N |\eta^2 G_k(u^\beta)|\mathrm{d}x \\
&\leq \bar{c} \sum_{\beta=1}^N \left(\int_{A_{k,t}^\beta} \left(\sum_{\alpha=1}^N |f^\alpha| \right)^{(2^*)'} \mathrm{d}x \right)^{\frac{1}{(2^*)'}} \left(\int_{B_t} |\eta^2 G_k(u^\beta)|^{2^*} \mathrm{d}x \right)^{\frac{1}{2^*}} \\
&\leq \bar{c} \sum_{\beta=1}^N \left[\left(\int_{A_{k,t}^\beta} \left(\sum_{\alpha=1}^N |f^\alpha| \right)^m \mathrm{d}x \right)^{\frac{(2^*)'}{m}} |A_{k,t}^\beta|^{1-\frac{(2^*)'}{m}} \right]^{\frac{1}{(2^*)'}} c(n) \left(\int_{B_t} |D(\eta^2 G_k(u^\beta))|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq \bar{c} N \|f\|_{L^m(B_t)} \sum_{\beta=1}^N |A_{k,t}^\beta|^{\frac{\theta}{2}} \left(\int_{A_{k,t}^\beta} |2\eta D\eta G_k(u^\beta) + \eta^2 Du^\beta|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq \bar{c} N \|f\|_{L^m(B_t)} \sum_{\beta=1}^N |A_{k,t}^\beta|^{\frac{\theta}{2}} \left[\left(\int_{A_{k,t}^\beta} |2\eta D\eta G_k(u^\beta)|^2 \mathrm{d}x \right)^{\frac{1}{2}} + \left(\int_{A_{k,t}^\beta} |\eta^2 Du^\beta|^2 \mathrm{d}x \right)^{\frac{1}{2}} \right] \\
&\leq \bar{c} N \|f\|_{L^m(B_t)} \sum_{\beta=1}^N |A_{k,t}^\beta|^{\frac{\theta}{2}} \left[4 \left(\int_{A_{k,t}^\beta} \left(\frac{|u^\beta| - k}{t-s} \right)^2 \mathrm{d}x \right)^{\frac{1}{2}} + \left(\int_{A_{k,t}^\beta} \eta^2 \, |Du^\beta|^2 \mathrm{d}x \right)^{\frac{1}{2}} \right] \\
&\leq \bar{c} N \|f\|_{L^m(B_t)} \left[\varepsilon \sum_{\beta=1}^N \int_{A_{k,t}^\beta} \eta^2 \, |Du^\beta|^2 \mathrm{d}x + 4\varepsilon \sum_{\beta=1}^N \int_{A_{k,t}^\beta} \left(\frac{|u^\beta| - k}{t-s} \right)^2 \mathrm{d}x + c(\varepsilon) \sum_{\beta=1}^N |A_{k,t}^\beta|^\theta \right],
\end{aligned} \tag{2.15}$$

where $\bar{c} = c(n)\sum_{\alpha,\gamma=1}^N |C_\alpha^\gamma|$ and $\theta = 2\left(\frac{1}{(2^*)'} - \frac{1}{m} \right)$.

Substituting (2.8), (2.11), (2.14), (2.15) into (2.7), and choosing ε small enough such that

$$(\bar{c} N \|f\|_{L^m(B_t)} + \tilde{c} n^2)\varepsilon = \frac{c_0}{2},$$

we then derive

$$\sum_{\beta=1}^N \int_{A_{k,s}^\beta} |Du^\beta|^2 \mathrm{d}x \leq c \left[\sum_{\beta=1}^N \int_{A_{k,t}^\beta} \left(\frac{|u^\beta| - k}{t-s} \right)^2 \mathrm{d}x + \sum_{\beta=1}^N |A_{k,t}^\beta|^\theta \right],$$

where c is a constant depending upon $n, N, m, \|f\|_{L^m(B_R)}, \tilde{c}, c_0$, and $\bar{c}$. Note that

$$|D_i|u^\beta|| = |D_i u^\beta|,$$

then (2.1) reduces to (2.4), completing the proof of Lemma 2.1. □

In the next lemma, we state and prove a general result that holds true for some general vectorial function $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. Eventually, we will use such a result with $v = (|u^1|, \dots, |u^N|)$ and $p = 2$. Note that this lemma is a generalization of [25, Lemma 3.2].

Lemma 2.2. *Let $v = (v^1, \dots, v^N) \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ with $1 < p < n$. Suppose that there exist constants $c_1 > 0$, $L_0 \geq 0$ and $\theta > 1 - \frac{p}{n}$, such that*

$$\sum_{\beta=1}^{N} \int_{A^\beta_{L,s}} |Dv^\beta|^p \mathrm{d}x \leq c_1 \sum_{\beta=1}^{N} \left(\int_{A^\beta_{L,t}} \left(\frac{v^\beta - L}{t - s} \right)^p \mathrm{d}x + |A^\beta_{L,t}|^\theta \right), \tag{2.16}$$

for every s, t, L, where $0 < s < t$, $B_t(x_0) \Subset \Omega$ and $L > L_0$, where

$$A^\beta_{L,s} = \{x \in B_s(x_0) : v^\beta(x) > L\}.$$

Then, for every $\beta = 1, \dots, N$, v^β is locally bounded from above, and for every r, R with $0 < r < R$, $B_R(x_0) \Subset \Omega$ and $|B_R(x_0)| < 1$,

$$\sup_{B_r(x_0)} v^\beta \leq \hat{c},$$

where $\hat{c}$ is a constant depending only upon $n, p, \theta, c_1, |\Omega|, L_0, \frac{1}{R-r}$ and

$$\sum_{\beta=1}^{N} \int_{B_R(x_0)} \max\{v^\beta; 0\}^p \mathrm{d}x.$$

Remark 2.1. Without loss of generality, we assume $\theta \leq 1$ in (2.16) since otherwise,

$$|A^\beta_{L,t}|^\theta = |A^\beta_{L,t}|^{\theta-1} |A^\beta_{L,t}|^1 \leq |\Omega|^{\theta-1} |A^\beta_{L,t}|,$$

then (2.16) holds true with c_1 replaced by $c_1 N \max\{|\Omega|^{\theta-1}, 1\}$ and θ replaced by 1.

We shall need the following preliminary lemma in order to prove Lemma 2.2, see [22, Lemma 7.1].

Lemma 2.3. *Let $\alpha > 1$ and let $\{J_i\}$ be a sequence of real positive numbers, such that*

$$J_{i+1} \leq C B^i J_i^\alpha,$$

with $C > 0$ and $B > 1$. If

$$J_0 \leq C^{-\frac{1}{\alpha-1}} B^{-\frac{1}{(\alpha-1)^2}},$$

we have

$$J_i \leq B^{-\frac{i}{\alpha-1}} J_0,$$

and hence, in particular, $\lim_{i\to+\infty} J_i = 0$.

Proof of Lemma 2.2. Let us consider balls $B_{r_1}(x_0)$ and $B_{r_2}(x_0)$ with $0 < r_1 < r_2 < R$, $B_R(x_0) \Subset \Omega$ and $|R_R(x_0)| < 1$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a cut-off function such that

$$0 \leq \eta \leq 1, \quad \eta \in C^1_0(B_{\frac{r_1+r_2}{2}}(x_0)), \quad \eta = 1 \quad \text{in } B_{r_1}(x_0) \quad \text{and} \quad |D\eta| \leq \frac{4}{r_2 - r_1}.$$

Then, using Hölder's inequality, Sobolev embedding theorem, and the properties of the cut-off function η, one obtains

$$
\begin{aligned}
\int_{A^{\beta}_{L,r_1}} (v^\beta - L)^p \mathrm{d}x &\le \left(\int_{A^{\beta}_{L,r_1}} (v^\beta - L)^{p*} \mathrm{d}x \right)^{\frac{p}{p*}} |A^{\beta}_{L,r_1}|^{1-\frac{p}{p*}} \\
&= \left(\int_{A^{\beta}_{L,r_1}} [\eta (v^\beta - L)]^{p*} \mathrm{d}x \right)^{\frac{p}{p*}} |A^{\beta}_{L,r_1}|^{1-\frac{p}{p*}} \\
&= \left(\int_{B_{r_1}} [\eta \max\{v^\beta - L;\ 0\}]^{p*}\, \mathrm{d}x \right)^{\frac{p}{p*}} |A^{\beta}_{L,r_1}|^{1-\frac{p}{p*}} \\
&\le \left(\int_{B_{\frac{r_1+r_2}{2}}} [\eta \max\{v^\beta - L;\ 0\}]^{p*}\, \mathrm{d}x \right)^{\frac{p}{p*}} |A^{\beta}_{L,r_1}|^{1-\frac{p}{p*}} \\
&\le c \left(\int_{B_{\frac{r_1+r_2}{2}}} |D[\eta \max\{v^\beta - L;\ 0\}]|^p \ \mathrm{d}x \right) |A^{\beta}_{L,r_1}|^{1-\frac{p}{p*}} \\
&= c \left(\int_{B_{\frac{r_1+r_2}{2}}} |\max\{v^\beta - L;\ 0\} D\eta + \eta D \max\{v^\beta - L;\ 0\}|^p \ \mathrm{d}x \right) |A^{\beta}_{L,r_1}|^{1-\frac{p}{p*}} \\
&= c \left(\int_{A^{\beta}_{L,\frac{r_1+r_2}{2}}} |(v^\beta - L) D\eta + \eta D v^\beta|^p \mathrm{d}x \right) |A^{\beta}_{L,r_1}|^{1-\frac{p}{p*}} \\
&\le c \left(\int_{A^{\beta}_{L,\frac{r_1+r_2}{2}}} |(v^\beta - L) D\eta|^p \mathrm{d}x + \int_{A^{\beta}_{L,\frac{r_1+r_2}{2}}} |\eta D v^\beta|^p \mathrm{d}x \right) |A^{\beta}_{L,r_1}|^{1-\frac{p}{p*}} \\
&\le c \left(\int_{A^{\beta}_{L,\frac{r_1+r_2}{2}}} \left(\frac{v^\beta - L}{r_2 - r_1} \right)^p \mathrm{d}x + \int_{A^{\beta}_{L,\frac{r_1+r_2}{2}}} |D v^\beta|^p \mathrm{d}x \right) |A^{\beta}_{L,r_1}|^{1-\frac{p}{p*}},
\end{aligned}
$$

where c is a constant depending upon n and p. Now, we sum upon β from 1 to N obtaining

$$
\begin{aligned}
\sum_{\beta=1}^{N} \int_{A^{\beta}_{L,r_1}} (v^\beta - L)^p \mathrm{d}x &\le c \sum_{\beta=1}^{N} \left(\int_{A^{\beta}_{L,\frac{r_1+r_2}{2}}} \left(\frac{v^\beta - L}{r_2 - r_1} \right)^p \mathrm{d}x + \int_{A^{\beta}_{L,\frac{r_1+r_2}{2}}} |D v^\beta|^p \mathrm{d}x \right) |A^{\beta}_{L,r_1}|^{1-\frac{p}{p*}} \\
&\le c \left(\sum_{\beta=1}^{N} \int_{A^{\beta}_{L,\frac{r_1+r_2}{2}}} \left(\frac{v^\beta - L}{r_2 - r_1} \right)^p \mathrm{d}x + \sum_{\beta=1}^{N} \int_{A^{\beta}_{L,\frac{r_1+r_2}{2}}} |D v^\beta|^p \mathrm{d}x \right) \left(\sum_{\beta=1}^{N} |A^{\beta}_{L,r_1}| \right)^{1-\frac{p}{p*}}.
\end{aligned}
$$

In order to control the second term in the aforementioned bracket, we use our assumption (2.16) with $s = \frac{r_1 + r_2}{2}$ and $t = r_2$. From the aforementioned inequality, one obtains

$$\sum_{\beta=1}^{N}\int_{A^{\beta}_{L,r_1}}(v^{\beta}-L)^{p}\mathrm{d}x \le c\left[\sum_{\beta=1}^{N}\int_{A^{\beta}_{L,\frac{r_1+r_2}{2}}}\left(\frac{v^{\beta}-L}{r_2-r_1}\right)^{p}\mathrm{d}x+\sum_{\beta=1}^{N}\int_{A^{\beta}_{L,r_2}}\left(\frac{v^{\beta}-L}{r_2-r_1}\right)^{p}\mathrm{d}x+\sum_{\beta=1}^{N}|A^{\beta}_{L,r_2}|^{\theta}\right]\left(\sum_{\beta=1}^{N}|A^{\beta}_{L,r_1}|\right)^{1-\frac{p}{p*}}$$
$$\le c\left(\sum_{\beta=1}^{N}\int_{A^{\beta}_{L,r_2}}\left(\frac{v^{\beta}-L}{r_2-r_1}\right)^{p}\mathrm{d}x\right)\left(\sum_{\beta=1}^{N}|A^{\beta}_{L,r_1}|\right)^{1-\frac{p}{p*}}+c\left(\sum_{\beta=1}^{N}|A^{\beta}_{L,r_2}|^{\theta}\right)\left(\sum_{\beta=1}^{N}|A^{\beta}_{L,r_1}|\right)^{1-\frac{p}{p*}}, \tag{2.17}$$

where c is a constant depending upon n, p, and c_1.

We are able to estimate $|A^{\beta}_{L,r_1}|$ and $|A^{\beta}_{L,r_2}|$ by means of $\int_{A^{\beta}_{\tilde{L},r_2}}(v^{\beta}-\tilde{L})^{p}\mathrm{d}x$, where $L > \tilde{L} \ge L_0$. In fact,

$$|A^{\beta}_{L,r_1}| \le |A^{\beta}_{L,r_2}| = \frac{1}{(L-\tilde{L})^{p}}(L-\tilde{L})^{p}|A^{\beta}_{L,r_2}| = \frac{1}{(L-\tilde{L})^{p}}\int_{A^{\beta}_{L,r_2}}(L-\tilde{L})^{p}\mathrm{d}x$$
$$\le \frac{1}{(L-\tilde{L})^{p}}\int_{A^{\beta}_{L,r_2}}(v^{\beta}-\tilde{L})^{p}\mathrm{d}x \le \frac{1}{(L-\tilde{L})^{p}}\int_{A^{\beta}_{\tilde{L},r_2}}(v^{\beta}-\tilde{L})^{p}\mathrm{d}x. \tag{2.18}$$

In the mean time,

$$\int_{A^{\beta}_{L,r_2}}(v^{\beta}-L)^{p}\mathrm{d}x \le \int_{A^{\beta}_{L,r_2}}(v^{\beta}-\tilde{L})^{p}\mathrm{d}x \le \int_{A^{\beta}_{\tilde{L},r_2}}(v^{\beta}-\tilde{L})^{p}\mathrm{d}x. \tag{2.19}$$

Substituting (2.18) and (2.19) into (2.17), and noting that $1-\frac{p}{p^*}=\frac{p}{n}$, we then arrive at

$$\sum_{\beta=1}^{N}\int_{A^{\beta}_{L,r_1}}(v^{\beta}-L)^{p}\mathrm{d}x$$
$$\le \frac{c}{(r_2-r_1)^{p}(L-\tilde{L})^{pp/n}}\left(\sum_{\beta=1}^{N}\int_{A^{\beta}_{L,r_2}}(v^{\beta}-L)^{p}\mathrm{d}x\right)\left(\sum_{\beta=1}^{N}\int_{A^{\beta}_{\tilde{L},r_2}}(v^{\beta}-\tilde{L})^{p}\mathrm{d}x\right)^{\frac{p}{n}}$$
$$+\frac{c}{(L-\tilde{L})^{pp/n+p\theta}}\sum_{\beta=1}^{N}\left(\int_{A^{\beta}_{\tilde{L},r_2}}(v^{\beta}-\tilde{L})^{p}\mathrm{d}x\right)^{\theta}\left(\sum_{\beta=1}^{N}\int_{A^{\beta}_{\tilde{L},r_2}}(v^{\beta}-\tilde{L})^{p}\mathrm{d}x\right)^{\frac{p}{n}}$$
$$\le \frac{c}{(r_2-r_1)^{p}(L-\tilde{L})^{pp/n}}\left(\sum_{\beta=1}^{N}\int_{A^{\beta}_{\tilde{L},r_2}}(v^{\beta}-\tilde{L})^{p}\mathrm{d}x\right)^{1+\frac{p}{n}}$$
$$+\frac{c}{(L-\tilde{L})^{pp/n+p\theta}}\left(\sum_{\beta=1}^{N}\int_{A^{\beta}_{\tilde{L},r_2}}(v^{\beta}-\tilde{L})^{p}\mathrm{d}x\right)^{\theta+\frac{p}{n}}. \tag{2.20}$$

Now, we fix $0 < r < R$, with $B_R(x_0) \Subset \Omega$ and $|B_R(x_0)| < 1$, and we take the following sequence of radii:

$$\rho_i = r + \frac{R-r}{2^i}, \quad i = 0, 1, 2, \ldots,$$

then $\rho_0 = R$ and

$$\rho_i - \rho_{i+1} = \frac{R-r}{2^{i+1}} > 0,$$

so ρ_i strictly decreases and $r < \rho_i \le R$. Let us fix a level $d > \max\{L_0, 1\}$ and we take the following sequence of levels:

$$k_i = 2d\left(1 - \frac{1}{2^{i+1}}\right), \quad i = 0, 1, 2, \ldots,$$

then $k_0 = d$ and $k_{i+1} - k_i = \frac{d}{2^{i+1}} > 0$, so k_i strictly increases and $L_0 < d \le k_i < 2d$. We can use (2.20) with levels $L = k_{i+1} > k_i = \tilde{L}$ and radii $r_1 = \rho_{i+1} < \rho_i = r_2$:

$$\sum_{\beta=1}^{N} \int_{A^{\beta}_{k_{i+1},\rho_{i+1}}} (v^\beta - k_{i+1})^p \mathrm{d}x \le \frac{c2^{(i+1)p(1+p/n)}}{(R-r)^p d^{pp/n}} \left(\sum_{\beta=1}^{N} \int_{A^{\beta}_{k_i,\rho_i}} (v^\beta - k_i)^p \mathrm{d}x \right)^{1+\frac{p}{n}} + \frac{c2^{(i+1)(pp/n+p\theta)}}{d^{pp/n+p\theta}} \left(\sum_{\beta=1}^{N} \int_{A^{\beta}_{k_i,\rho_i}} (v^\beta - k_i)^p \mathrm{d}x \right)^{\theta+\frac{p}{n}}. \tag{2.21}$$

Let us set

$$J_i = \sum_{\beta=1}^{N} \int_{A^{\beta}_{k_i,\rho_i}} (v^\beta - k_i)^p \mathrm{d}x, \quad i = 0, 1, 2, \ldots.$$

Since

$$\begin{aligned} J_{i+1} &= \sum_{\beta=1}^{N} \int_{A^{\beta}_{k_{i+1},\rho_{i+1}}} (v^\beta - k_{i+1})^p \mathrm{d}x \\ &\le \sum_{\beta=1}^{N} \int_{A^{\beta}_{k_i,\rho_{i+1}}} (v^\beta - k_{i+1})^p \mathrm{d}x \\ &\le \sum_{\beta=1}^{N} \int_{A^{\beta}_{k_i,\rho_i}} (v^\beta - k_{i+1})^p \mathrm{d}x \\ &\le \sum_{\beta=1}^{N} \int_{A^{\beta}_{k_i,\rho_i}} (v^\beta - k_i)^p \mathrm{d}x = J_i, \end{aligned}$$

$\{J_i\}$ is a decreasing sequence. Note that $d > L_0 \ge 0$, so when $v^\beta > d$, we have $v^\beta - d \le v^\beta = \max\{v^\beta;\, 0\}$; then

$$J_0 = \sum_{\beta=1}^{N} \int_{A^{\beta}_{d,R}} (v^\beta - d)^p \mathrm{d}x \le \sum_{\beta=1}^{N} \int_{A^{\beta}_{d,R}} (\max\{v^\beta;\, 0\})^p \,\mathrm{d}x \le \sum_{\beta=1}^{N} \int_{B_R(x_0)} \max\{v^\beta;\, 0\}^p \mathrm{d}x \coloneqq T. \tag{2.22}$$

We use the aforementioned number T, and we keep in mind that $\theta \le 1$, $R - r < 1$ and $d > 1$, then (2.21) yields

$$\begin{aligned} J_{i+1} &\le \frac{c}{(R-r)^p d^{pp/n}} \left(2^{\left(1+\frac{p}{n}\right)p}\right)^i J_i^{1+\frac{p}{n}} + \frac{c}{d^{pp/n+p\theta}} \left(2^{\left(\frac{p}{n}+\theta\right)p}\right)^i J_i^{\theta+\frac{p}{n}} \\ &\le \frac{c}{(R-r)^p d^{pp/n}} \left(2^{\left(1+\frac{p}{n}\right)p}\right)^i J_i^{\theta+\frac{p}{n}} T^{1-\theta} + \frac{c}{(R-r)^p d^{pp/n}} \left(2^{\left(1+\frac{p}{n}\right)p}\right)^i J_i^{\theta+\frac{p}{n}} \\ &\le \frac{c}{(R-r)^p d^{pp/n}} \left(2^{\left(1+\frac{p}{n}\right)p}\right)^i J_i^{\theta+\frac{p}{n}}, \end{aligned}$$

where c is a constant depending upon $n, N, p, \theta, c_1, |\Omega|$, and T. We would like to use Lemma 2.3 to obtain

$$\lim_{i\to\infty} J_i = 0, \tag{2.23}$$

this is true provided that

$$\theta + \frac{p}{n} > 1 \tag{2.24}$$

and

$$J_0 \leq \left(\frac{c}{(R-r)^p d^{pp/n}}\right)^{-\frac{1}{\theta+\frac{p}{n}-1}} \left(2^{\left(1+\frac{p}{n}\right)p}\right)^{-\frac{1}{\left(\theta+\frac{p}{n}-1\right)^2}}. \tag{2.25}$$

(2.24) is easy since $\theta > 1 - \frac{p}{n}$. Let us try to check (2.25). Since $J_0 \leq T$ by (2.22), we obtain the following sufficient condition when checking (2.25):

$$\sum_{\beta=1}^{N} \int_{B_R(x_0)} \max\{v^\beta;\ 0\}^p \mathrm{d}x \leq \left(\frac{c}{(R-r)^p d^{pp/n}}\right)^{-\frac{1}{\theta+\frac{p}{n}-1}} \left(2^{\left(1+\frac{p}{n}\right)p}\right)^{-\frac{1}{\left(\theta+\frac{p}{n}-1\right)^2}}. \tag{2.26}$$

Then, we fix d verifying (2.26) and $d > \max\{L_0, 1\}$; then, (2.25) is satisfied and (2.23) holds true. It is obvious that such a constant d depends upon $n, N, p, \theta, c_1, |\Omega|, L_0, \frac{1}{R-r}$, and T.

We keep in mind that $r < \rho_i$ and $k_i < 2d$, so we can use (2.19) with $r_2 = r < \rho_i$, $L = 2d$, and $\tilde{L} = k_i$:

$$\int_{A^\beta_{2d,r}} (v^\beta - 2d)^p \mathrm{d}x \leq \int_{A^\beta_{k_i,r}} (v^\beta - k_i)^p \mathrm{d}x \leq \int_{A^\beta_{k_i,\rho_i}} (v^\beta - k_i)^p \mathrm{d}x, \tag{2.27}$$

so that

$$0 \leq \sum_{\beta=1}^{N} \int_{A^\beta_{2d,r}} (v^\beta - 2d)^p \mathrm{d}x \leq \sum_{\beta=1}^{N} \int_{A^\beta_{k_i,\rho_i}} (v^\beta - k_i)^p \mathrm{d}x = J_i.$$

Since (2.23) holds true, we have, by Lemma 2.3, $\lim_{i\to\infty} J_i = 0$, so

$$\sum_{\beta=1}^{N} \int_{A^\beta_{2d,r}} (v^\beta - 2d)^p \mathrm{d}x = 0,$$

this mean that $|A^\beta_{2d,r}| = 0$, so that

$$v^\beta \leq 2d, \quad \text{a.e. in} \quad B_r(x_0).$$

This completes the proof of Lemma 2.2. □

Proof of Theorem 2.1. Caccioppoli inequality proved in Lemma 2.1 with $v^\beta = |u^\beta|$ and $p = 2$ allows us to use Lemma 2.2 to derive local boundedness of u^α (note that $\theta = 2\left(\frac{1}{(2^*)'} - \frac{1}{m}\right) > 1 - \frac{2}{n}$ since $m > \frac{n}{2}$). The arbitrariness of α implies that u is locally bounded in Ω. □

2.2 Global boundedness result

Let $n > 2$, $N \geq 2$ be integers and Ω an open bounded subset of $\mathbb{R}^n$. Consider Dirichlet problem of the following quasilinear elliptic systems involving N equations:

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^{N}\sum_{j=1}^{n} a^{\alpha,\beta}_{i,j}(x, u(x)) \frac{\partial u^\beta(x)}{\partial x_j}\right) = f^\alpha(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.28}$$

where $\alpha \in \{1, 2, \ldots, N\}$ is the equation index.

We let the coefficients $a^{\alpha,\beta}_{i,j}(x, y)$, $i, j \in \{1, \ldots, n\}$, $\alpha, \beta \in \{1, \ldots, N\}$, satisfying the set of assumptions $(\mathcal{A})$ and

$$f(x) = (f^1(x), \ldots, f^N(x)) \in L^m(\Omega; \mathbb{R}^N), \quad m \geq (2^*)'. \tag{2.29}$$

Definition 2.2. A function $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ is a global solution to (2.28), if

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x) D_i \varphi^\alpha(x)\mathrm{d}x = \int_\Omega \sum_{\alpha=1}^N f^\alpha(x)\varphi^\alpha(x)\mathrm{d}x \tag{2.30}$$

holds true for all $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^N)$.

Next we prove that if the right-hand side function f is good enough, then the global solution to (2.28) is globally bounded.

Theorem 2.2. *Suppose that u is a weak solution of* (2.28). *Under the set of assumptions* ($\mathcal{A}$), *if* $f \in L^m(\Omega; \mathbb{R}^N)$, $m > \frac{n}{2}$, *then any global solution* $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ *to* (2.28) *is globally bounded.*

Proof. We take for any $k > 0$, $\varphi = (\varphi^1, \dots, \varphi^N)$ with

$$\varphi^\alpha = \sum_{\gamma=1}^N C_\alpha^\gamma G_k(u^\gamma), \quad \alpha \in \{1, \dots, N\},$$

where $G_k(s)$ is defined in (2.6), and C_α^γ, $\alpha, \gamma \in \{1, \dots, N\}$, are the real constants satisfying (2.12). It is obvious that

$$D_i\varphi^\alpha = \sum_{\gamma=1}^N C_\alpha^\gamma D_i u^\gamma \chi_{A_k^\gamma}, \quad i = 1, \dots, n.$$

Such a function φ is admissible for Definition 2.2 since it belongs to $W_0^{1,2}(\Omega; \mathbb{R}^N)$. We use φ in (2.30) and we have

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x) \sum_{\gamma=1}^N C_\alpha^\gamma D_i u^\gamma \chi_{A_k^\gamma}\mathrm{d}x = \int_\Omega \sum_{\alpha=1}^N f^\alpha \sum_{\gamma=1}^N C_\alpha^\gamma G_k(u^\gamma)\mathrm{d}x. \tag{2.31}$$

We compare the left-hand side of (2.31) with the left-hand side of (2.7), and we find that the only difference between them is a function η^2. We use the method from (2.8) to (2.11) and we have that

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x) \sum_{\gamma=1}^N C_\alpha^\gamma D_i u^\gamma \chi_{A_k^\gamma}\mathrm{d}x \ge c_0 \sum_{\beta=1}^N \int_{A_k^\beta} |Du^\beta|^2 \mathrm{d}x. \tag{2.32}$$

In order to estimate the right-hand side of (2.31), we use Hölder's inequality and Sobolev embedding theorem to derive

$$\begin{aligned} \int_\Omega \sum_{\alpha=1}^N f^\alpha \sum_{\gamma=1}^N C_\alpha^\gamma G_k(u^\gamma)\mathrm{d}x &\le \bar{c} \int_\Omega \sum_{\alpha=1}^N |f^\alpha| \sum_{\beta=1}^N |G_k(u^\beta)|\mathrm{d}x \\ &\le \bar{c} \sum_{\beta=1}^N \left(\int_{A_k^\beta} \left(\sum_{\alpha=1}^N |f^\alpha| \right)^{(2^*)'} \mathrm{d}x \right)^{\frac{1}{(2^*)'}} \left(\int_{A_k^\beta} |G_k(u^\beta)|^{2^*} \mathrm{d}x \right)^{\frac{1}{2^*}} \\ &\le c \sum_{\beta=1}^N \left(\int_{A_k^\beta} \left(\sum_{\alpha=1}^N |f^\alpha| \right)^{(2^*)'} \mathrm{d}x \right)^{\frac{1}{(2^*)'}} \left(\int_{A_k^\beta} |Du^\beta|^2 \mathrm{d}x \right)^{\frac{1}{2}}, \end{aligned} \tag{2.33}$$

where C_α^γ, $\alpha, \gamma = 1, \dots, N$, are solutions to (2.13) and $\bar{c} = \sum_{\alpha,\gamma=1}^N |C_\alpha^\gamma|$ is a constant.

Substituting (2.32) and (2.33) into (2.31), we arrive at

$$\sum_{\beta=1}^N \int_{A_k^\beta} |Du^\beta|^2 \mathrm{d}x \le c \sum_{\beta=1}^N \left(\int_{A_k^\beta} \left(\sum_{\alpha=1}^N |f^\alpha| \right)^{(2^*)'} \mathrm{d}x \right)^{\frac{2}{(2^*)'}}.$$

Hölder's inequality gives

$$\sum_{\beta=1}^{N}\int_{A_k^\beta}|Du^\beta|^2\mathrm{d}x \le c\sum_{\beta=1}^{N}\left[\left(\int_{A_k^\beta}\left(\sum_{\alpha=1}^{N}|f^\alpha|\right)^m\mathrm{d}x\right)^{\frac{(2^*)'}{m}}|A_k^\beta|^{1-\frac{(2^*)'}{m}}\right]^{\frac{2}{(2^*)'}} \le c\left(\sum_{\beta=1}^{N}|A_k^\beta|\right)^{\frac{2}{(2^*)'}-\frac{2}{m}}, \tag{2.34}$$

where c is a constant depending upon $n, N, m, c_0, \tilde{c}, \|f\|_{L^m(\Omega)}$ and $r^{\alpha,\beta}, \alpha, \beta = 1,\ldots, N$. The left-hand side of the aforementioned inequality can be estimated by Sobolev embedding theorem, for any $L > k$,

$$\begin{aligned}\sum_{\beta=1}^{N}\int_{A_k^\beta}|Du^\beta|^2\mathrm{d}x &= \sum_{\beta=1}^{N}\int_{\Omega}|DG_k(u^\beta)|^2\mathrm{d}x\\ &\ge c\sum_{\beta=1}^{N}\left(\int_{\Omega}|G_k(u^\beta)|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \ge c\sum_{\beta=1}^{N}\left(\int_{A_L^\beta}|G_k(u^\beta)|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}\\ &\ge c(L-k)^2\sum_{\beta=1}^{N}|A_L^\beta|^{\frac{2}{2^*}} \ge c(L-k)^2\left(\sum_{\beta=1}^{N}|A_L^\beta|\right)^{\frac{2}{2^*}}.\end{aligned} \tag{2.35}$$

(2.34) and (2.35) merge into

$$\sum_{\beta=1}^{N}|A_L^\beta| \le \frac{c}{(L-k)^{2_*}}\left(\sum_{\beta=1}^{N}|A_k^\beta|\right)^{\left(\frac{2}{(2^*)'}-\frac{2}{m}\right)\frac{2^*}{2}},$$

for every L, k with $L > k > 0$. We let

$$\psi(t) = \sum_{\beta=1}^{N}|A_t^\beta| = \sum_{\beta=1}^{N}|\{|u^\beta(x)|>t\}|.$$

We use the following Stampacchia Lemma, see [35, Lemma 4.1], which we provide below for the convenience of the reader.

Lemma 2.4. *Let $\psi(k) : [k_0, +\infty) \to [0, +\infty)$ be decreasing. We assume that there exists $c, \alpha \in (0, +\infty)$ and $\beta \in (1, +\infty)$ such that*

$$L > k \ge k_0 \Rightarrow \psi(L) \le \frac{c}{(L-k)^\alpha}[\psi(k)]^\beta.$$

Then, it results that $\psi(k_0 + d) = 0$, where

$$d = \left[c(\psi(k_0))^{\beta-1}2^{\frac{\alpha\beta}{\beta-1}}\right]^{\frac{1}{\alpha}}.$$

Since $m > \frac{n}{2}$ implies $\beta = \left(\frac{2}{(2^*)'} - \frac{2}{m}\right)\frac{2^*}{2} > 1$, we use Lemma 2.4 and we have

$$\sum_{\beta=1}^{N}|\{|u^\beta(x)|>d\}| = 0,$$

almost everywhere in Ω, which implies the desired result $|u^\beta(x)| \le d$, a.e. Ω, $\beta \in \{1, \ldots, N\}$. □

3 Boundedness under $(\mathcal{A})'$

This section deals with global boundedness for solutions to elliptic systems (1.1) under the set of assumptions $(\mathcal{A})'$. We also consider regularizing effect of a lower-order term.

3.1 Global boundedness result

In this section, we also consider Dirichlet problem of quasilinear elliptic systems involving N equations of the form (2.28). We let the coefficients $a_{i,j}^{\alpha,\beta}(x,y)$, $i,j \in \{1,\dots,n\}$, $\alpha,\beta \in \{1,\dots,N\}$, satisfying the set of assumptions $(\mathcal{A})'$ and f satisfying (2.29). The definition for a global solution $u \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ to (2.28) is the same as Definition 2.2.

Next we prove that, if the right-hand side function f is good enough, then the global solution to (2.28) is globally bounded.

Theorem 3.1. *Suppose that $u \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ is a weak solution of* (2.28). *Under the set of assumptions $(\mathcal{A})'$, if $f = (f^1,\dots,f^N) \in L^m(\Omega;\mathbb{R}^N)$, $m > \frac{n}{2}$, then any global solution $u \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ to* (2.28) *is globally bounded.*

Proof. Let $u \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ be a global solution to (2.28). For every $L \geq Q_0$, we define $\varphi = (\varphi^1,\dots,\varphi^N)$ with

$$\varphi^\alpha = G_L(u^\alpha),$$

then

$$D_i\varphi^\alpha = D_iu^\alpha\chi_{A_L^\alpha}.$$

Such a φ is admissible for Definition 2.2 since $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^N)$. We use φ in (2.30) and we have

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x,u)D_ju^\beta(x)D_iu^\alpha\chi_{A_L^\alpha}\mathrm{d}x = \int_\Omega \sum_{\alpha=1}^N f^\alpha G_L(u^\alpha)\mathrm{d}x.$$

Now, assumption $(\mathcal{A}_4)'$ guarantees that

$$a_{i,j}^{\alpha,\beta}(x,u)\chi_{A_L^\alpha} = a_{i,j}^{\alpha,\beta}(x,u)\chi_{A_L^\alpha}\chi_{A_L^\beta}$$

when $\beta \neq \alpha$ and $L \geq Q_0$. It is worthwhile to note that (3.1) holds true when $\alpha = \beta$ as well; then,

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x,u)D_ju^\beta\chi_{A_L^\beta}D_iu^\alpha\chi_{A_L^\alpha}\mathrm{d}x = \int_\Omega \sum_{\alpha=1}^N f^\alpha G_L(u^\alpha)\mathrm{d}x.$$

Now, we can use ellipticity assumption $(\mathcal{A}_3)'$ with $\xi_i^\alpha = D_iu^\alpha\chi_{A_L^\alpha}$ and we obtain

$$\tilde{c}_0 \sum_{\alpha=1}^N \int_{A_L^\alpha} |Du^\alpha|^2\mathrm{d}x \leq \int_\Omega \sum_{\alpha=1}^N f^\alpha G_L(u^\alpha)\mathrm{d}x. \tag{3.1}$$

In order to estimate the right-hand side of (3.1), we use Hölder's inequality and Sobolev embedding theorem to derive

$$\begin{aligned}
\int_\Omega \sum_{\alpha=1}^N f^\alpha G_L(u^\alpha)\mathrm{d}x &\leq \sum_{\alpha=1}^N \int_\Omega |f^\alpha||G_L(u^\alpha)|\mathrm{d}x \\
&\leq \sum_{\alpha=1}^N \left(\int_{A_L^\alpha} |f^\alpha|^{(2^*)'}\mathrm{d}x\right)^{\frac{1}{(2^*)'}} \left(\int_\Omega |G_L(u^\alpha)|^{2^*}\mathrm{d}x\right)^{\frac{1}{2^*}} \\
&\leq c\sum_{\alpha=1}^N \left(\int_{A_L^\alpha} |f^\alpha|^{(2^*)'}\mathrm{d}x\right)^{\frac{1}{(2^*)'}} \left(\int_{A_L^\alpha} |Du^\alpha|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq c\sum_{\alpha=1}^N \left(\int_{A_L^\alpha} |f^\alpha|^{(2^*)'}\mathrm{d}x\right)^{\frac{1}{(2^*)'}} \left(\sum_{\beta=1}^N \int_{A_L^\beta} |Du^\beta|^2\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), we arrive at

$$\sum_{\alpha=1}^{N}\int_{A_L^\alpha}|Du^\alpha|^2\mathrm{d}x \le c\sum_{\alpha=1}^{N}\left(\int_{A_L^\alpha}|f^\alpha|^{(2^*)'}\mathrm{d}x\right)^{\frac{2}{(2^*)'}}.$$

Hölder's inequality gives

$$\sum_{\alpha=1}^{N}\int_{A_L^\alpha}|Du^\alpha|^2\mathrm{d}x \le c\sum_{\alpha=1}^{N}\left[\left(\int_{A_L^\alpha}|f^\alpha|^m\mathrm{d}x\right)^{\frac{(2^*)'}{m}}|A_L^\alpha|^{1-\frac{(2^*)'}{m}}\right]^{\frac{2}{(2^*)'}} \le c\left(\sum_{\alpha=1}^{N}|A_L^\alpha|\right)^{\frac{2}{(2^*)'}-\frac{2}{m}}, \tag{3.3}$$

where c is a constant depending upon $n, N, \tilde{c}_0$, and $\|f\|_{L^m(\Omega)}$. The left-hand side of the aforementioned inequality can be estimated by Sobolev embedding theorem: for any $\tilde{L} > L$,

$$\begin{aligned}\sum_{\alpha=1}^{N}\int_{A_L^\alpha}|Du^\alpha|^2\mathrm{d}x &= \sum_{\alpha=1}^{N}\int_{\Omega}|DG_L(u^\alpha)|^2\mathrm{d}x\\ &\ge c\sum_{\alpha=1}^{N}\left(\int_{\Omega}|G_L(u^\alpha)|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \ge c\sum_{\alpha=1}^{N}\left(\int_{A_{\tilde{L}}^\alpha}|G_L(u^\alpha)|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}\\ &\ge c(\tilde{L}-L)^2\sum_{\alpha=1}^{N}|A_{\tilde{L}}^\alpha|^{\frac{2}{2^*}} \ge c(\tilde{L}-L)^2\left(\sum_{\alpha=1}^{N}|A_{\tilde{L}}^\alpha|\right)^{\frac{2}{2^*}}.\end{aligned} \tag{3.4}$$

(3.3) and (3.4) merge into

$$\sum_{\alpha=1}^{N}|A_{\tilde{L}}^\alpha| \le \frac{c}{(\tilde{L}-L)^{2^*}}\left(\sum_{\alpha=1}^{N}|A_L^\alpha|\right)^{\left(\frac{2}{(2^*)'}-\frac{2}{m}\right)\frac{2^*}{2}},$$

for every $\tilde{L}, L$ with $\tilde{L} > L \ge Q_0$. We let

$$\psi(t) = \sum_{\alpha=1}^{N}|A_t^\alpha| = \sum_{\alpha=1}^{N}|\{|u^\alpha(x)| > t\}|.$$

We use the Stampacchia Lemma 2.4 and we keep in mind that $m > \frac{n}{2}$ implies $\beta = \left(\frac{2}{(2^*)'}-\frac{2}{m}\right)\frac{2^*}{2} > 1$, then

$$\sum_{\alpha=1}^{N}|\{|u^\alpha(x)| > Q_0 + d\}| = 0,$$

which implies the desired result $|u^\alpha(x)| \le Q_0 + d$, a.e. Ω, $\alpha \in \{1, \dots, N\}$. □

Remark 3.1. We note that, in [27], the authors considered the elliptic system (2.28) with $f^\alpha(x) = 0$, a.e. Ω, $\alpha = 1, \dots, N$. Under the assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)'$ and support of off-diagonal coefficients (see [27, Figure 1]), the authors derives a local boundedness result by using De Giorgi's iterative method. We mention that the support of off-diagonal coefficients in [27] is contained in the "butterfly support" (compare [27, Figure 1] with [27, Figure 1]); thus, the condition $(\mathcal{A})'$ in this article is weaker than the one proposed in [27]. Of course, generally, dealing with local boundedness requires more skills than global ones. Existence and boundedness results of weak solutions to some vectorial Dirichlet problems of elliptic systems can be found in the recent article [13].

3.2 Regularizing effect

In this section, we concentrate ourselves to regularizing effect of a lower-order term. A good reference in this field is the article [1] by Arcoya and Boccardo, where the authors studied the regularizing effect of the interaction between the coefficient of the zero-order term and the datum in some linear, semilinear and nonlinear Dirichlet problems. For other results related to regularizing effect, we refer to [7, Section 11.8] and the recent articles [2,3,11].

We next show that there is also regularizing effect of a lower-order term for elliptic systems. More precisely, let $n > 2$, $N \geq 2$ be integers and Ω an open bounded subset of $\mathbb{R}^n$. We consider quasilinear elliptic systems involving N equations of the form

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^{N}\sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x, u(x))\frac{\partial u^\beta(x)}{\partial x_j}\right) + b^\alpha(x)u^\alpha(x) = f^\alpha, & \text{in} \quad \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $\alpha \in \{1, 2, \ldots, N\}$ is the equation index. The difference between (3.5) and (2.28) is that there is a lower-order term $b^\alpha(x)u^\alpha(x)$ in the left-hand side of (3.5).

We assume that the coefficients $a_{i,j}^{\alpha,\beta}$ satisfy $(\mathcal{A})'$. For the functions $f^\alpha(x)$ and $b^\alpha(x)$, $\alpha = 1, \ldots, N$, we assume

$$0 \leq b^\alpha(x) \in L^{\frac{n}{2}}(\Omega), \tag{3.6}$$

$$|f^\alpha(x)| \leq Q b^\alpha(x), \quad \text{for some } Q \geq Q_0. \tag{3.7}$$

Definition 3.1. We say that a function $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ is a global solution with respect to (3.5), if

$$\int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x) D_i \varphi^\alpha(x)\mathrm{d}x + \int_\Omega \sum_{\alpha=1}^{N} b^\alpha(x)u^\alpha(x)\varphi^\alpha(x) = \int_\Omega \sum_{\alpha=1}^{N} f^\alpha(x)\varphi^\alpha(x)\mathrm{d}x \tag{3.8}$$

holds true for all $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^N)$.

We remark that conditions (3.6) and (3.7) guarantee integrability of the second and third integrands in (3.8).

Theorem 3.1 tells us that, in order to guarantee boundedness of solutions to (2.28), we need $f \in L^m(\Omega)$, $m > \frac{n}{2}$. From (3.6) and (3.7), we know that, $f \in L^{\frac{n}{2}}(\Omega)$. The next theorem shows that there is a regularizing effect of (3.7), which forces global boundedness of solutions to (3.5).

Theorem 3.2. *Assume* $(\mathcal{A})'$, (3.6), *and* (3.7), *then a solution* $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ *of system* (3.5) *is bounded. Moreover,*

$$\|u\|_{L^\infty(\Omega;\, \mathbb{R}^N)} \leq Q\sqrt{N}.$$

Proof. Let $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ be a global solution of system (3.5). We take a test function $\varphi = (\varphi^1, \ldots, \varphi^N)$ with

$$\varphi^\alpha = G_Q(u^\alpha), \quad \alpha \in \{1, \ldots, N\}. \tag{3.9}$$

Such a function φ is admissible for Definition 3.1 since it belongs to $W_0^{1,2}(\Omega; \mathbb{R}^N)$. We use such a test function in the weak formulation (3.8) and we have

$$\int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, u) D_j u^\beta D_i G_Q(u^\alpha)\mathrm{d}x + \sum_{\alpha=1}^{N}\int_\Omega b^\alpha u^\alpha G_Q(u^\alpha)\mathrm{d}x = \sum_{\alpha=1}^{N}\int_\Omega f^\alpha G_Q(u^\alpha)\mathrm{d}x. \tag{3.10}$$

For the first integral in the right-hand side of (3.10), we use $(\mathcal{A}_4)'$ and $(\mathcal{A}_3)'$ to derive

$$\begin{aligned}&\int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u)D_ju^\beta D_iG_Q(u^\alpha)\mathrm{d}x\\ &= \int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u)D_ju^\beta D_iu^\alpha\chi_{A_Q^\alpha}\mathrm{d}x\\ &= \int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u)D_ju^\beta\chi_{A_Q^\beta} D_iu^\alpha\chi_{A_Q^\alpha}\mathrm{d}x\\ &\geq c_0\sum_{\alpha=1}^{N}\int_{A_Q^\alpha}|Du^\alpha|^2\mathrm{d}x \geq 0.\end{aligned} \tag{3.11}$$

Using (3.7), we obtain

$$\begin{aligned}\sum_{\alpha=1}^{N}\int_\Omega f^\alpha(x)G_Q(u^\alpha)\mathrm{d}x &\leq \sum_{\alpha=1}^{N}\int_\Omega |f^\alpha(x)||G_Q(u^\alpha)|\mathrm{d}x\\ &\leq \sum_{\alpha=1}^{N}\int_\Omega Qb^\alpha(x)|G_Q(u^\alpha)|\mathrm{d}x.\end{aligned} \tag{3.12}$$

Combining (3.10)–(3.12), and noting $u^\alpha(x)G_Q(u^\alpha) = |u^\alpha(x)||G_Q(u^\alpha)|$, one obtains

$$\sum_{\alpha=1}^{N}\int_\Omega b^\alpha(x)|G_Q(u^\alpha)|(|u^\alpha(x)| - Q)\mathrm{d}x \leq 0, \tag{3.13}$$

from which we derive

$$|u^\alpha(x)| \leq Q, \quad \text{a.e. } \Omega,$$

we thus have derived that $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \leq \sqrt{N}Q.$$

This completes the proof of Theorem 3.2. □

Acknowledgments: All authors would like to thank the anonymous referees for their detailed comments and helpful suggestions.

Funding information: The corresponding author thanks NSFC (Grant No. 12071021) and the Innovation Capacity Enhancement Program-Science and Technology Platform Project, Hebei Province (Grant No. 22567623H) for the support.

Author contributions: The authors contributed equally to the preparation, the revision, and the writing of the manuscript.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: No data were used for the research described in the article.

References

[1] D. Arcoya and L. Boccardo, *Regularizing effect of the interplay between coefficients in some elliptic equations*, J. Funct. Anal. **268** (2015), 1153–1166, DOI: https://doi.org/10.1016/j.jfa.2014.11.011.

[2] D. Arcoya and L. Boccardo, *Regularizing effect of L^q interplay between coefficients in some elliptic equations*, J. Math. Pures Appl. **111** (2018), 106–125, DOI: https://doi.org/10.1016/j.matpur.2017.08.001.

[3] D. Arcoya, L. Boccardo, and L. Orsina, *Regularizing effect of the interplay between coefficients in some nonlinear Dirichlet problems with distributional data*, Annali di Matematica Pura ed Applicata **199** (2020), 1909–1921, DOI: https://doi.org/10.1007/s10231-020-00949-8.

[4] P. Baroni, *Riesz potential estimates for a general class of quasilinear equations*, Calc. Var. Partial Differential Equations **53** (2015), no. 3–4, 803–846, DOI: https://doi.org/10.1007/s00526-014-0768-z.

[5] M. F. Betta, T. Del Vecchio, and M. R. Posteraro, *Existence and regularity results for nonlinear degenerate elliptic equations with measure data*, Ricerche Mat. **47** (1998), 277–295.

[6] L. Boccardo, *Problemi differenziali ellittici e parabolici con dati misure*, Boll. Unione Mat. Ital. Sez. A **11** (1997), 439–461.

[7] L. Boccardo and G. Croce, *Elliptic Partial Differential Equations*, De Gruyter, Berlin/Boston, 2014 DOI: https://doi.org/10.1080/03605309208820857.

[8] L. Boccardo and G. Croce, *Existenza e Regolarita di Soluzioni di Alcuni Problemi Ellittici*, in: Quaderni dell UMI, vol. 51, Pitagora Editrice, Bologna, 2010.

[9] L. Boccardo and T. Gallouet, *Non-linear elliptic and parabolic equations involving measure data*, J. Funct. Anal. **87** (1989), 149–169, DOI: https://doi.org/10.1016/0022-1236(89)90005-0.

[10] L. Boccardo, T. Gallouet, and P. Marcellini, *Anisotropic equations in L^1*, Differ. Integral Equ. **9** (1996), 209–212. DOI: https://doi.org/10.57262/die/1367969997.

[11] L. Boccardo and L. Orsina, *Regularizing effects in some elliptic systems with a convection term*, J. Math. Anal. Appl. **521** (2023), 126963, DOI: https://doi.org/10.1016/j.jmaa.2022.126963.

[12] G. R. Cirmi, *On the existence of solutions to non-linear degenerate elliptic equations with measure data*, Ricerche Mat. **42** (1993), 315–329.

[13] G. Cirmi, S. D'Asero, S. Leonardi, F. Leonetti, E. Rocha, and V. Staicu, *Existence and boundedness of weak solutions to some vectorial Dirichlet problems*, Nonlinear Anal. **254** (2025), 113751, DOI: https://doi.org/10.1016/j.na.2025.113751.

[14] G. R. Cirmi and S. Leonardi, *Regularity results for the gradient of solutions linear elliptic equations with $L^{1,\lambda}$-data*, Ann. Mat. Pura Appl. **185** (2006), 537–553, DOI: https://doi.org/10.1007/s10231-005-0167-3.

[15] G. Cupini, F. Leonetti, and E. Mascolo, *Local boundedness for solutions of a class of nonlinear elliptic systems*, Calc. Var. **61** (2022), 94, DOI: https://doi.org/10.1007/s00526-022-02204-9.

[16] A. Dall Aglio, *Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations*. Ann. Mat. Pura Appl. **185** (1996), 207–240, DOI: https://doi.org/10.1007/BF01758989.

[17] E. De Giorgi, *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*, Boll. Un. Mat. Ital. **1** (1968), no. 4, 135–137.

[18] Z. Feng, A. Zhang, and H. Gao, *Local boundedness under nonstandard growth conditions*, J. Math. Anal. Appl. **526** (2023), 127280, DOI: https://doi.org/10.1016/j.jmaa.2023.127280.

[19] H. Gao, H. Deng, M. Huang, and W. Ren, *Generalizations of Stampacchia Lemma and applications to quasilinear elliptic systems*, Nonlinear Analysis **208** (2021), 112297, DOI: https://doi.org/10.1016/j.na.2021.112297.

[20] H. Gao, M. Huang, H. Deng, and W. Ren, *Global integrability for solutions to quasilinear elliptic systems*, Manuscript. Math. **164** (2021), 23–37, DOI: https://doi.org/10.1007/s00229-020-01183-5.

[21] P. Di Gironimo, F. Leonetti, M. Macrí, and P. V. Petricca, Existence of solutions to some quasilinear degenerate elliptic systems when the datum has an intermediate degree of integrability, *Complex Variables and Elliptic Equations*, 68 (2023), 1566–1580, DOI: https://doi.org/10.1080/17476933.2022.2069753.

[22] E. Giusti, *Direct methods in the calculus of variations*, World Scientic Publishing Co., Inc., River Edge, NJ, 2003. DOI: https://doi.org/10.1142/9789812795557.

[23] T. Kuusi and G. Mingione, *Universal potential estimates*, J. Funct. Anal., **262** (2012), 4205–4269, DOI: https://doi.org/10.1016/j.jfa.2012.02.018.

[24] S. Leonardi, F. Leonetti, C. Pignotti, E. Rocha, and V. Staicu, *Maximum principles for some quasilinear elliptic systems*, Nonlinear Anal., **194** (2020), 111377, DOI: https://doi.org/10.1016/j.na.2018.11.004.

[25] S. Leonardi, F. Leonetti, E. Rocha, and V. Staicu, *Butterfly support for off diagonal coefficients and boundedness of solutions to quasilinear elliptic systems*, Adv. Nonlinear Anal. **11** (2021), 672–683, DOI: https://doi.org/10.1515/anona-2021-0205.

[26] S. Leonardi, F. Leonetti, and C. Pignotti, *Local boundedness for weak solutions to some quasilinear elliptic systems*, 2019, arXiv: http://arXiv.org/abs/arXiv:1911.05987.

[27] S. Leonardi, F. Leonetti, C. Pignotti, E. Rocha, and V. Staicu, *Local boundedness for weak solutions to some quasilinear elliptic systems*, Minimax Theory Appl. **6** (2021), no. 1, 365–372, DOI: https://doi.org/10.48550/arXiv.1911.05987.

[28] F. Leonetti, E. Rocha, and V. Staicu, *Quasilinear elliptic systems with measure data*, Nonlinear Anal. **154** (2017), 210–224, DOI: https://doi.org/10.1016/j.na.2016.04.002.

[29] F. Leonetti, E. Rocha, and V. Staicu, *Smallness and cancellation in some elliptic systems with measure data*, J. Math. Anal. Appl. **465** (2018), 885–902, DOI: https://doi.org/10.1016/j.jmaa.2018.05.047.

[30] G. Mingione, *The Calderon-Zygmund theory for elliptic problems with measure data*, Ann. Sc. Norm. Super. Pisa **6** (2007), 195–261, DOI: https://doi.org/10.48550/arXiv.math/0609670.

[31] G. Mingione, *Gradient estimates below the duality exponent*, Math. Ann. **346** (2010), 571–627, DOI: https://doi.org/10.1007/s00208-009-0411-z.

[32] G. Mingione, *Nonlinear measure data problems*, Milan J. Math. **79** (2011), 429–496, DOI: https://doi.org/10.1007/s00032-011-0168-1.

[33] G. Mingione, *La teoria di Calderon-Zygmund dal caso lineare a quello non lineare*, Boll. Unione Mat. Ital. **9** (2013), 269–297.

[34] P. Oppezzi and A. M. Rossi, *Esistenza di soluzioni per problemi unilateri con dato misura o in L^1*, Ricerche Mat. **45** (1996), 491–513.

[35] G. Stampacchia, *Equations Elliptiques du Second Ordre a Coefficientes Discontinus*, Semin. de Math. Superieures, Univ. de Montreal, 1966, p. 16.