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An Extension of De Giorgi Class and Applications

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Abstract. We present an extension of the classical De Giorgi class, and then we show that functions in this new class are locally bounded and locally Hölder continuous. Some applications are given. As a first application, we give a regularity result for local minimizers $u : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^4$ of a special class of polyconvex functionals with splitting form in four dimensional Euclidean spaces. Under some structural conditions on the energy density, we prove that each component u^α of the local minimizer u belongs to the generalized De Giorgi class, then one can derive that it is locally bounded and locally Hölder continuous. Our result can be applied to polyconvex integrals whose prototype is

$$\int_\Omega \Big(\sum_{\alpha=1}^{4} |Du^\alpha|^p + \sum_{\beta=1}^{6} |(\mathrm{adj}_2 Du)^\beta|^q + \sum_{\gamma=1}^{4} |(\mathrm{adj}_3 Du)^\gamma|^r + |\det Du|^s \Big) \mathrm{d}x$$

with suitable $p, q, r, s \geq 1$. As a second application, we consider a degenerate linear elliptic equation of the form

$$-\mathrm{div}(a(x)\nabla u) = -\mathrm{div} F,$$

with $0 < a(x) \leq \bar{\beta} < +\infty$. We prove, by virtue of the generalized De Giorgi class, that any weak solution is locally bounded and locally Hölder continuous provided that $\frac{1}{a(x)}$ and $F(x)$ belong to some suitable locally integrable function spaces. As a third application, we show that our theorem can be applied in dealing with regularity issues of elliptic equations with non-standard grow conditions.

AMS Subject Classification (2020): 35J20, 35J25, 35J47

Keywords: De Giorgi Class, extension, local minimizer, variational integral, locally bounded, locally Hölder continuous.

1 The classical De Giorgi class.

It is well-known (see, for example, Chapter 7 in [25]) that the quasi-minima and ω-minima of regular functionals of the calculus of variations are Hölder continuous functions. The main result is a version of the fundamental theorem of De Giorgi [15] and Nash [50] concerning the regularity of solutions of linear elliptic equations with discontinuous coefficients, a result that was later generalized among others by Ladyženskaya and Ural'ceva [35] to bounded solutions to nonlinear elliptic equations. De Giorgi's method relies on the fact that functions lie in the De Giorgi class are locally Hölder continuous.

We first recall the definition of De Giorgi class, see, for example, De Giorgi [15].

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Definition 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. We say that $u \in W^{1,p}_{loc}(\Omega)$, $p \leq n$, belongs to the De Giorgi class $DG^+_p(\Omega, p, y, y_*, \varepsilon, \kappa_0)$, $p > 1$, y and $\varepsilon > 0$, y_* and $\kappa_0 \geq 0$ if*

$$\int_{Q_{\sigma\rho}(x_0)} |D(u-k)_+|^p dx \leq y \int_{Q_\rho(x_0)} \left(\frac{u-k}{(1-\sigma)\rho}\right)^p_+ dx + y_* \big|\{u > k\} \cap \{Q_\rho(x_0)\}\big|^{1-\frac{p}{n}+\varepsilon}, \quad (1.1)$$

for all $k \geq \kappa_0$, $\sigma \in (0,1)$, and all pairs of concentric cubes $Q_{\sigma\rho}(x_0) \subset Q_\rho(x_0) \subset \Omega$ centered at x_0, where

$$(u-k)_+ = (u-k) \wedge 0 = \max\{u-k, 0\},$$

and $Q_\rho(x_0)$, $\rho > 0$, $x_0 \in \Omega$, be the cube with side length 2ρ, with sides parallel to the coordinate axis and centered at x_0.

In the sequel we denote $Q_\rho(x_0) = Q_\rho$ provided that no confusion may arise.
If we introduce

$$A_{k,\rho} = \{x \in Q_\rho : u(x) > k\},$$

then (1.1) is equivalent to

$$\int_{A_{k,\sigma\rho}} |Du|^p dx \leq y \int_{A_{k,\rho}} \left(\frac{u-k}{(1-\sigma)\rho}\right)^p dx + y_* |A_{k,\rho}|^{1-\frac{p}{n}+\varepsilon}. \quad (1.2)$$

One can define similarly $DG^-_p(\Omega, p, y, y_*, \varepsilon, \kappa_0)$ to be the class of functions u such that $-u \in DG^+_p(\Omega, p, y, y_*, \varepsilon, \kappa_0)$. More explicitly, they are the functions in $W^{1,p}_{loc}(\Omega)$ such that for all $k \leq -\kappa_0$, all $\sigma \in (0,1)$, and all pairs of concentric cubes $Q_{\sigma\rho} \subset Q_\rho \subset \Omega$,

$$\int_{B_{k,\sigma\rho}} |Du|^p dx \leq y \int_{B_{k,\rho}} \left(\frac{k-u}{(1-\sigma)\rho}\right)^p dx + y_* |B_{k,\rho}|^{1-\frac{p}{n}+\varepsilon}, \quad (1.3)$$

where

$$B_{k,\rho} = \{x \in Q_\rho : u(x) < k\}.$$

It is clear that if a function u satisfies (1.2) or (1.3) with some $\varepsilon > 0$, it will verify them with any positive $\varepsilon' \leq \varepsilon$. Consequently, we shall always assume $\varepsilon \leq \frac{p}{n}$.

We indicate by $DG_p(\Omega, p, y, y_*, \varepsilon, \kappa_0)$ the class of functions belonging both to $DG^+_p(\Omega, p, y, y_*, \varepsilon, \kappa_0)$ and $DG^-_p(\Omega, p, y, y_*, \varepsilon, \kappa_0)$:

$$DG_p(\Omega, p, y, y_*, \varepsilon, \kappa_0) = DG^+_p(\Omega, p, y, y_*, \varepsilon, \kappa_0) \cap DG^-_p(\Omega, p, y, y_*, \varepsilon, \kappa_0).$$

We notice that (1.2) and (1.3) are Caccioppoli type inequalities on super-/sub-level sets. A rather surprising characteristic of De Giorgi class is that (1.2) and (1.3) contain particularly much information deriving from the minimum properties of the function u, at least for what concerns its local boundedness and local Hölder continuity. The following proposition can be found, for example, in [11] Theorems 2.1 and 3.1 and [25] Chapter 7.

Proposition 1.1. *Let $u \in DG_p(\Omega, p, y, y_*, \varepsilon, \kappa_0)$ and $\sigma \in (0,1)$. There exist a constant $C > 1$ depending only upon the data and independent of u, such that for every pair of* *cubes* $Q_{\sigma\rho} \subset Q_\rho \subset\subset \Omega$*,*

$$\|u\|_{L^\infty(Q_{\sigma\rho})} \le \max\left\{ y_*\rho^{n\varepsilon}; \frac{C}{(1-\sigma)^{1/\varepsilon}} \left(\frac{1}{|Q_\rho|} \int_{Q_\rho} |u|^p \mathrm{d}x \right)^{\frac{1}{p}} \right\},$$

moreover, there exists $\tilde\alpha \in (0,1)$ *depending only upon the data and independent of* u*, such that*

$$\mathrm{osc}(u, Q_\rho) \le C \max\left\{ y_*\rho^{n\varepsilon}; \left(\frac{\rho}{R}\right)^{\tilde\alpha} \mathrm{osc}(u, Q_R) \right\},$$

where

$$\mathrm{osc}(u, Q_\rho) = \mathrm{esssup}_{Q_\rho} u - \mathrm{essinf}_{Q_\rho} u$$

is the oscillation of u *over* Q_ρ*. Therefore,* $u \in C^{0,\tilde\alpha_0}_{loc}(\Omega)$ *with* $\tilde\alpha_0 = \tilde\alpha \wedge (n\varepsilon)$*.*

The above proposition illustrates that functions in the De Giorgi class are locally bounded and locally Hölder continuous in Ω. This result was first proved by De Giorgi in his famous paper [15], which opened the way to the regularity of solutions of elliptic equations with bounded measurable coefficients, and for minima of regular functionals in the Calculus of Variations. De Giorgi's theorem was later generalized by various authors, so as to cover the most general solutions of nonlinear equations in divergence form. We note in particular the papers by Stampacchia [53–55] and the book by Ladyženskaya and Ural'ceva [35]. Almost at the same time, a different proof of the regularity of solutions to parabolic and elliptic equations was given by Nash [50]. Slightly later, Moser [49] proved Harnack's inequality, thus extending to solutions of linear equations in divergence form a classical result for harmonic functions. Starting from Harnack's inequality, Moser gave a new proof of the Hölder continuity of solutions of elliptic equations. The extension of the method of De Giorgi to minima (and quasi-minima) of functionals, independently of their Euler equation, was made by Giaquinta and Giusti [24], after Frehse [14] has studied a particular case, under rather restrictive hypothesis. Harnack's inequality was proved by Di Benedetto and Trudinger [10] for functions in De Giorgi class, and hence for quasi-minima of integral functionals

$$\mathcal{F}(u, \Omega) = \int_\Omega f(x, u(x), Du(x))dx.$$

For some recent developments of De Giorgi class and its applications, we refer the reader to [5, 6, 51, 52, 57].

2 An extension.

In this section, we shall give an extension of the classical De Giorgi class, and prove that it is closely related the regularity properties of the function u, including local boundedness and local Hölder continuity properties.

In the following, for $1 < p \le n$, we shall use the symbol p^* which is defined as: $p^* = \frac{np}{n-p}$ if $p < n$, and $p^* =$ any $\nu > p$ for $p = n$.

We give the following

Definition 2.1. *We say that* $u \in W^{1,p}_{loc}(\Omega)$ *belongs to the generalized De Giorgi class*

$GDG_p^+(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)$, $1<p\le n$, $p\le Q<p^*$, y and $\varepsilon>0$, y_* and $\kappa_0\ge 0$, if

$$\int_{A_{k,\sigma\rho}} |Du|^p dx \le y\int_{A_{k,\rho}} \left(\frac{u-k}{(1-\sigma)\rho}\right)^Q dx + y_*\left|A_{k,\rho}\right|^{1-\frac{p}{n}+\varepsilon}, \tag{2.1}$$

for all $k\ge\kappa_0$, $\sigma\in(0,1)$, and all pairs of concentric cubes $Q_{\sigma\rho}(x_0)\subset Q_\rho(x_0)\subset\Omega$ centered at x_0.

We can define similarly $GDG_p^-(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)$ to be the class of functions u such that $-u\in GDG_p^+(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)$. More explicitly, they are the functions in $W^{1,p}_{loc}(\Omega)$ such that for all $k\le-\kappa_0$, all $\sigma\in(0,1)$, and all pairs of concentric cubes $Q_{\sigma\rho}\subset Q_\rho\subset\Omega$,

$$\int_{B_{k,\sigma\rho}} |Du|^p dx \le y\int_{B_{k,\rho}} \left(\frac{k-u}{(1-\sigma)\rho}\right)^Q dx + y_*\left|B_{k,\rho}\right|^{1-\frac{p}{n}+\varepsilon}. \tag{2.2}$$

We shall indicate by $GDG_p(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)$ the class of functions belonging both to GDG_p^+ $(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)$ and $GDG_p^-(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)$:

$$\begin{aligned} & GDG_p(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)\\ = \;& GDG_p^+(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)\cap GDG_p^-(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0). \end{aligned} \tag{2.3}$$

It is clear that $GDG_p(\Omega,p,p,y,y_*,\varepsilon,\kappa_0)=DG_p(\Omega,p,y,y_*,\varepsilon,\kappa_0)$. If no confusion may arise, $GDG_p(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)$ will be abbreviated as GDG_p.

We remark that the difference between (1.2) and (2.1) (similarly (1.3) and (2.2)) is that the power p in the first integral in the right hand side is replaced by a power Q which is greater than or equals to p but smaller than p^*. It is obvious that (2.1) is weaker than (1.2) (similarly (2.2) is slightly weaker than (1.3)).

The main result of this paper is the following

Theorem 2.1. *Let $u\in GDG_p(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)$ for $1<p\le n$ and some $Q\in[p,p^*)$, then u is locally bounded and locally Hölder continuous in Ω.*

The above theorem illustrates that, functions in the generalized De Giorgi class GDG_p $(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)$ has also remarkable regularity properties as De Giorgi class does, in particular, they are locally bounded and locally Hölder continuous.

We notice that in the above Theorem 2.1 we have restricted ourselves to the case $1<p\le n$, since if $p>n$, then any function $u\in W^{1,p}_{loc}(\Omega)$ is trivially in $C^{0,\tilde\alpha}_{loc}(\Omega)$ for some $0<\tilde\alpha<1$ by Sobolev Imbedding Theorem, i.e., it is automatically a Hölder continuous function.

We should mention that in (2.1) and (2.2) we have restricted ourselves to $p\le Q<p^*$. One may wonder if $u\in GDG_p(\Omega,p,Q,y,y_*,\varepsilon,\kappa_0)$ is also locally bounded and locally Hölder continuous for the case $Q=p^*$. Despite some efforts, we can not prove this result. Fortunately, the case $Q<p^*$ is enough for our purposes to derive some regularity results of minimizers of some variational integrals, as well as weak solutions of some elliptic equations and systems, see Sections 3, 4 and 5.

We remark that, in the proof of Theorem 2.1 we borrow some ideas from [15, 25].

In the following we shall denote by $c(\cdots)$ a constant depending only on the quantities, whose value may vary from line to line.

We divide the proof of Theorem 2.1 into several lemmas.

Lemma 2.1. *Let $u \in GDG_p(\Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$ for $1 < p \le n$ and some $Q \in [p, p^*]$. Then u is locally bounded in Ω.*

Proof. When $Q = p^*$ in (2.1), the local boundedness is known to be true. see Fusco-Sbordone [20], Moscariello-Nania [47] and Cupini-Leonetti-Mascolo [7]. In the following we assume $Q < p^*$. We notice that if $u \in GDG_p^+(\Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$, then Young inequality with exponents $\frac{p^*}{Q}$ and $\frac{p^*}{p^*-Q}$ allows us to estimate

$$\begin{aligned}
&\int_{A_{k,\sigma\rho}} |Du|^p dx \\
\le\ & y \int_{A_{k,\rho}} \left(\frac{u-k}{(1-\sigma)\rho}\right)^Q dx + y_* \big|A_{k,\rho}\big|^{1-\frac{p}{n}+\varepsilon} \\
\le\ & c \left(\int_{A_{k,\rho}} \left(\frac{u-k}{(1-\sigma)\rho}\right)^{p^*} dx + |A_{k,\rho}| + \big|A_{k,\rho}\big|^{1-\frac{p}{n}+\varepsilon} \right) \\
\le\ & c \left(\int_{A_{k,\rho}} \left(\frac{u-k}{(1-\sigma)\rho}\right)^{p^*} dx + \big|A_{k,\rho}\big|^{1-\frac{p}{n}+\varepsilon} \right),
\end{aligned}$$

here we have used the facts $\varepsilon \le \frac{p}{n}$ and $|A_{k,\rho}| \le |\Omega|$, which imply

$$|A_{k,\rho}| = |A_{k,\rho}|^{\frac{p}{n}-\varepsilon} |A_{k,\rho}|^{1-\frac{p}{n}+\varepsilon} \le |\Omega|^{\frac{p}{n}-\varepsilon} |A_{k,\rho}|^{1-\frac{p}{n}+\varepsilon}.$$

We shall use this fact repeatedly in the sequel. Thus u satisfies the inequality (2.6) in [7] (the only difference between the above inequality and (2.6) in [7] is that cubes in place of balls. In the following we will not distinguish between cubes and balls). It has been proved in [7], by using De Giorgi's iteration method, that any function satisfying the above inequality is locally bounded from above.

Analogously, if $u \in GDG_p^-(\Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$, then it is locally bounded from below.

The locally boundedness result follows since (2.3). □

The following is a technical lemma, which can be found, for example, in Lemma 7.1 in [25].

Lemma 2.2. *Let $\alpha > 0$ and let $\{x_i\}$ be a sequence of real positive numbers such that*

$$x_{i+1} \le C B^i x_i^{1+\alpha}$$

with $C > 0$ and $B > 1$. If

$$x_0 \le C^{-\frac{1}{\alpha}} B^{-\frac{1}{\alpha^2}},$$

then we have

$$x_i \le B^{-\frac{i}{\alpha}} x_0,$$

and hence in particular

$$\lim_{i\to\infty} x_i = 0.$$

We next prove

Lemma 2.3. *Let $u \in W^{1,p}_{loc}(\Omega)$. Suppose that there exists $c > 0$ and $Q \in [p, p^*)$, such that for every $0 < r < \rho \le R$ and $k \ge \kappa_0 \ge 0$,*

$$\int_{A_{k,r}} |Du|^p dx \le c \int_{A_{k,\rho}} \left(\frac{u-k}{\rho - r}\right)^Q dx + c|A_{k,\rho}|^{1-\frac{p}{n}+\varepsilon}. \tag{2.4}$$

If

$$\int_{Q_R} |u|^Q dx \le 1, \quad |Q_R| \le 1, \tag{2.5}$$

then

$$\sup_{Q_{\frac{R}{2}}}(u-\kappa_0) \le c\left(\frac{1}{R^{\frac{(np-Q)n}{p(n-1)}}} \int_{A_{\kappa_0,R}} (u-\kappa_0)^Q\, dx\right)^{\frac{1}{Q}} \left(\frac{|A_{\kappa_0,R}|}{R^{\frac{(np-Q)n}{p(n-1)}}}\right)^{\frac{\gamma}{Q}} + cR^\tau, \tag{2.6}$$

where γ is the positive solution of the equation

$$\gamma^2 + \gamma = \varepsilon, \tag{2.7}$$

and τ is a small positive number depending on n, p, Q.

Proof. We can suppose $\kappa_0 = 0$, and thus (2.4) is satisfied for every $k \ge 0$. For $\frac{1}{2} \le r < \rho \le R \le 1$, we let $\eta(x)$ be a function of class $C_0^\infty\left(Q_{\frac{r+\rho}{2}}\right)$ with $\eta = 1$ on Q_r and $|D\eta| \le \frac{4}{\rho - r}$. Setting $\zeta = \eta(u-k)_+$. By the Sobolev Imbedding Theorem, we have

$$\begin{aligned}
&\int_{A_{k,r}} (u-k)^Q \mathrm{d}x \le \int_{A_{k,\frac{\rho+r}{2}}} \zeta^Q \mathrm{d}x \\
\le&\left(\int_{A_{k,\frac{\rho+r}{2}}} |D(\eta(u-k))|^{Q_*} \mathrm{d}x\right)^{\frac{Q}{Q_*}} \\
\le&2^Q\left(\int_{A_{k,\frac{\rho+r}{2}}} |(u-k)D\eta|^{Q_*} \mathrm{d}x + \int_{A_{k,\frac{\rho+r}{2}}} |\eta Du|^{Q_*} \mathrm{d}x\right)^{\frac{Q}{Q_*}},
\end{aligned} \tag{2.8}$$

where $Q_* = \frac{nQ}{n+Q}$. By Hölder inequality with exponents $\frac{Q}{Q_*}$ and $\frac{Q}{Q-Q_*}$ we obtain

$$\begin{aligned}
&\int_{A_{k,\frac{\rho+r}{2}}} |(u-k)D\eta|^{Q_*} \mathrm{d}x \\
\le& c\int_{A_{k,\rho}} \left(\frac{u-k}{\rho-r}\right)^{Q_*} \mathrm{d}x \\
\le& c\left(\int_{A_{k,\rho}} \left(\frac{u-k}{\rho-r}\right)^{Q} \mathrm{d}x\right)^{\frac{Q_*}{Q}} |A_{k,\rho}|^{1-\frac{Q_*}{Q}}.
\end{aligned} \tag{2.9}$$

The condition $Q < p^*$ is equivalent to $Q_* < p$. Using Hölder inequality with exponents

$\frac{p}{Q_*}$ and $\frac{p}{p-Q_*}$ and taking into account the inequality (2.4) we obtain

$$
\begin{aligned}
&\int_{A_{k,\frac{\rho+r}{2}}} |\eta Du|^{Q_*}\mathrm{d}x \\
\leq & c\Big(\int_{A_{k,\frac{\rho+r}{2}}} |Du|^p \mathrm{d}x\Big)^{\frac{Q_*}{p}} |A_{k,\rho}|^{1-\frac{Q_*}{p}} \\
\leq & c\Big(\int_{A_{k,\rho}} \Big(\frac{u-k}{\rho-r}\Big)^Q \mathrm{d}x + |A_{k,\rho}|^{1-\frac{p}{n}+\varepsilon}\Big)^{\frac{Q_*}{p}} |A_{k,\rho}|^{1-\frac{Q_*}{p}} \\
\leq & c\Big(\int_{A_{k,\rho}} \Big(\frac{u-k}{\rho-r}\Big)^Q \mathrm{d}x\Big)^{\frac{Q_*}{p}} |A_{k,\rho}|^{1-\frac{Q_*}{p}} + |A_{k,\rho}|^{1-(\frac{p}{n}-\varepsilon)\frac{Q_*}{p}}.
\end{aligned}
\tag{2.10}
$$

By (2.8), (2.9) and (2.10) we get for every $r < \rho \leq R$,

$$
\begin{aligned}
&\int_{A_{k,r}} (u-k)^Q \mathrm{d}x \\
\leq & c\Bigg\{\Big(\int_{A_{k,\rho}} \Big(\frac{u-k}{\rho-r}\Big)^Q \mathrm{d}x\Big)^{\frac{Q_*}{Q}} |A_{k,\rho}|^{1-\frac{Q_*}{Q}} \\
&+\Big(\int_{A_{k,\rho}} \Big(\frac{u-k}{\rho-r}\Big)^Q \mathrm{d}x\Big)^{\frac{Q_*}{p}} |A_{k,\rho}|^{1-\frac{Q_*}{p}} + |A_{k,\rho}|^{1-(\frac{p}{n}-\varepsilon)\frac{Q_*}{p}}\Bigg\}^{\frac{Q}{Q_*}} \\
\leq & c\Big(\frac{1}{\rho-r}\Big)^Q \Big(\int_{A_{k,\rho}} (u-k)^Q \mathrm{d}x\Big)|A_{k,\rho}|^{\frac{Q}{n}} \\
&+ c\Big(\frac{1}{\rho-r}\Big)^{\frac{Q^2}{p}} \Big(\int_{A_{k,\rho}} (u-k)^Q \mathrm{d}x\Big)^{\frac{Q}{p}} |A_{k,\rho}|^{1+\frac{Q}{n}-\frac{Q}{p}} + c|A_{k,\rho}|^{1+\frac{Q}{p}\varepsilon}.
\end{aligned}
\tag{2.11}
$$

Denote

$$
U(k,t) = \int_{A_{k,t}} (u-k)^Q \mathrm{d}x.
$$

It is obvious that $U(\cdot,\rho)$ is non-increasing and $U(k,\cdot)$ is non-decreasing. For each $h < k$ and $r < \rho$, one has

$$
\begin{aligned}
U(h,\rho) &= \int_{A_{h,\rho}} (u-h)^Q \mathrm{d}x \geq \int_{A_{k,\rho}} (u-h)^Q \mathrm{d}x \\
&\geq (k-h)^Q |A_{k,\rho}| \geq (k-h)^Q |A_{k,r}|.
\end{aligned}
\tag{2.12}
$$

From (2.5), it is easy to get

$$
U(k,t) \leq 1 \quad \text{and} \quad |A_{k,t}| \leq 1. \tag{2.13}
$$

We take ε small enough such that

$$
\varepsilon < \frac{p}{Q}\left(1 - \frac{Q}{p^*}\right),
$$

this implies

$$\frac{Q}{n} > 1 + \frac{Q}{n} - \frac{Q}{p} > \frac{Q}{p}\varepsilon. \tag{2.14}$$

Let τ satisfy

$$\frac{np-Q}{p(n-1)} n\varepsilon = Q\tau.$$

We get from (2.11) that

$$\begin{aligned} U(k,r) &\le c\Big(\frac{1}{\rho-r}\Big)^{Q} U(k,\rho)|A_{k,\rho}|^{\frac{Q}{n}} \\ &\quad + c\Big(\frac{1}{\rho-r}\Big)^{\frac{Q^2}{p}} U(k,\rho)^{\frac{Q}{p}}|A_{k,\rho}|^{1+\frac{Q}{n}-\frac{Q}{p}} + c|A_{k,\rho}|^{1+\frac{Q}{p}\varepsilon} \\ &\quad := I_1 + I_2 + I_3. \end{aligned} \tag{2.15}$$

I_1 can be estimated by using Young inequality with exponents $\frac{Q}{p}$ and $\frac{Q}{Q-p}$:

$$\begin{aligned} I_1 &= c\left(\frac{1}{\rho-r}\right)^{Q} U(k,\rho)\,|A_{k,\rho}|^{\frac{Q}{n}} \\ &= c\left\{\left(\frac{1}{\rho-r}\right)^{Q} U(k,\rho)\,|A_{k,\rho}|^{\left(1+\frac{Q}{n}-\frac{Q}{p}\right)\frac{p}{Q}}\right\}|A_{k,\rho}|^{\frac{Q}{n}\left(1-\frac{p}{Q}\right)}\,|A_{k,\rho}|^{-\left(1-\frac{Q}{p}\right)\frac{p}{Q}} \\ &\le c\left(\frac{1}{\rho-r}\right)^{Q^2/p} U(k,\rho)^{Q/p}\,|A_{k,\rho}|^{1+\frac{Q}{n}-\frac{Q}{p}} + c\,|A_{k,\rho}|^{\frac{Q}{n}\left(1-\frac{p}{Q}\right)\frac{Q}{Q-p}}\,|A_{k,\rho}|^{-\left(1-\frac{Q}{p}\right)\frac{p}{Q}\frac{Q}{Q-p}} \\ &= c\left(\frac{1}{\rho-r}\right)^{Q^2/p} U(k,\rho)^{Q/p}\,|A_{k,\rho}|^{1+\frac{Q}{n}-\frac{Q}{p}} + c\,|A_{k,\rho}|^{\frac{Q}{n}}\,|A_{k,\rho}|\,, \end{aligned}$$

which together with the facts $|A_{k,\rho}| \le |A_{h,\rho}|$ and (2.13) implies

$$I_1 \le c\left(\frac{1}{\rho-r}\right)^{Q^2/p} U(h,\rho)\,|A_{k,\rho}|^{1+\frac{Q}{n}-\frac{Q}{p}} + c\,|A_{h,\rho}|^{\frac{Q}{n}}\,|A_{k,\rho}|\,.$$

Using (2.14) and the fact $|A_{h,\rho}| < |Q_R| \le 1$, one gets

$$|A_{h,\rho}|^{\frac{Q}{n}} \le |A_{h,\rho}|^{\frac{Q}{p}\varepsilon}\,,$$

thus

$$I_1 \le c\left(\frac{1}{\rho-r}\right)^{Q^2/p} U(h,\rho)\,|A_{h,\rho}|^{1+\frac{Q}{n}-\frac{Q}{p}} + c\,|A_{h,\rho}|^{\frac{Q}{p}\varepsilon}\,|A_{k,\rho}|\,. \tag{2.16}$$

Using (2.13), we get

$$\begin{aligned} I_2 &= c\Big(\frac{1}{\rho-r}\Big)^{\frac{Q^2}{p}} U(k,\rho)^{\frac{Q}{p}}|A_{k,\rho}|^{1+\frac{Q}{n}-\frac{Q}{p}} \\ &\le c\Big(\frac{1}{\rho-r}\Big)^{\frac{Q^2}{p}} U(k,\rho)|A_{k,\rho}|^{1+\frac{Q}{n}-\frac{Q}{p}}. \end{aligned} \tag{2.17}$$

Using $|A_{k,\rho}| \le |A_{h,\rho}|$ again we derive

$$I_3 = c|A_{k,\rho}|^{1+\frac{Q}{p}\varepsilon} \le c|A_{h,\rho}|^{\frac{Q}{p}\varepsilon} \cdot |A_{k,\rho}|. \tag{2.18}$$

Substituting (2.16), (2.17) and (2.18) into (2.15), using (2.12) and (2.14) one gets

$$\begin{aligned} U(k,r) &\le c\Big(\frac{1}{\rho - r}\Big)^{\frac{Q^2}{p}} U(h,\rho)|A_{h,\rho}|^{1+\frac{Q}{n}-\frac{Q}{p}} + c|A_{h,\rho}|^{\frac{Q}{p}\varepsilon} \cdot |A_{k,\rho}| \\ &\le c\left(\frac{1}{\rho - r}\right)^{\frac{Q^2}{p}} U(h,\rho)|A_{h,\rho}|^{1+\frac{Q}{n}-\frac{Q}{p}} + c|A_{h,\rho}|^{\frac{Q}{p}\varepsilon} U(h,\rho)\Big(\frac{1}{k-h}\Big)^{Q} \\ &\le c\left[\left(\frac{1}{\rho - r}\right)^{\frac{Q^2}{p}} + \left(\frac{1}{k-h}\right)^{Q}\right] U(h,\rho)|A_{h,\rho}|^{\frac{Q}{p}\varepsilon} \\ &\le c\left[\left(\frac{\rho^{\frac{p\tau}{Q}}}{\rho - r}\right)^{\frac{Q^2}{p}} + \left(\frac{\rho^{\tau}}{k-h}\right)^{Q}\right] \rho^{-\frac{np-Q}{p(n-1)}n\varepsilon} U(h,\rho)|A_{h,\rho}|^{\varepsilon}. \end{aligned} \tag{2.19}$$

Since

$$\rho^{\frac{p\tau}{Q}} < \rho + 1,$$

then

$$\left(\frac{\rho^{\frac{p\tau}{Q}}}{\rho - r}\right)^{\frac{Q^2}{p}} \le c\left\{\left(\frac{\rho}{\rho - r}\right)^{\frac{Q^2}{p}} + \left(\frac{1}{\rho - r}\right)^{\frac{Q^2}{p}}\right\},$$

thus (2.19) implies

$$U(k,r) \le c\left[\left(\frac{\rho}{\rho - r}\right)^{\frac{Q^2}{p}} + \left(\frac{1}{\rho - r}\right)^{\frac{Q^2}{p}} + \left(\frac{\rho^{\tau}}{k-h}\right)^{Q}\right] \rho^{-\frac{np-Q}{p(n-1)}n\varepsilon} U(h,\rho)|A_{h,\rho}|^{\varepsilon}. \tag{2.20}$$

From (2.12) and (2.7) we know that

$$|A_{k,r}|^{\gamma} \le \frac{U(h,\rho)^{\gamma}}{(k-h)^{Q\gamma}},$$

which together with (2.20) yields

$$\begin{aligned} &U(k,r)|A_{k,r}|^{\gamma} \\ \le & c\left[\left(\frac{\rho}{\rho - r}\right)^{\frac{Q^2}{p}} + \left(\frac{1}{\rho - r}\right)^{\frac{Q^2}{p}} + \left(\frac{\rho^{\tau}}{k-h}\right)^{Q}\right] \rho^{-\frac{np-Q}{p(n-1)}n\varepsilon} U(h,\rho)^{1+\gamma} \frac{1}{(k-h)^{Q\gamma}} |A_{h,\rho}|^{\varepsilon}. \end{aligned}$$

According to (2.7), let us define

$$\varphi(k,t) = U(k,t)|A_{k,t}|^{\gamma}.$$

For $r<\rho\le R$ and $h<k$ we have

$$\varphi(k,r)\le c\left[\left(\frac{\rho}{\rho-r}\right)^{\frac{Q^2}{p}}+\left(\frac{1}{\rho-r}\right)^{\frac{Q^2}{p}}+\left(\frac{\rho^\tau}{k-h}\right)^{Q}\right]\rho^{-\frac{np-Q}{p(n-1)}n\varepsilon}\frac{\varphi(h,\rho)^{1+\gamma}}{(k-h)^{Q\gamma}}. \qquad (2.21)$$

Let now $d\ge CR^\tau$ be a constant that we shall fix later, and define

$$k_i=d\left(1-2^{-i}\right)$$

and

$$r_i=\frac{R}{2}\left(1+2^{-i}\right).$$

In (2.21) we choose

$$r=r_{i+1},\ \rho=r_i,\ k=k_{i+1},\ h=k_i,$$

then

$$\rho-r=r_i-r_{i+1}=\frac{R}{2}\frac{1}{2^{i+1}},$$

$$k-h=k_{i+1}-k_i=d\frac{1}{2^{i+1}}.$$

Let us define $\varphi_i=\varphi(k_i,r_i)$, then

$$\begin{aligned}\varphi_{i+1}&\le c\left\{\left[\frac{\frac{R}{2}(1+2^{-i})}{\frac{R}{2}\frac{1}{2^{i+1}}}\right]^{\frac{Q^2}{p}}+\left[\frac{1}{\frac{R}{2}\frac{1}{2^{i+1}}}\right]^{\frac{Q^2}{p}}+\left[\frac{(\frac{R}{2}(1+2^{-i}))^\tau}{d\frac{1}{2^{i+1}}}\right]^{Q}\right\}\times\\&\quad\times\left[\frac{R}{2}(1+2^{-i})\right]^{-\frac{np-Q}{p(n-1)}n\varepsilon}\frac{\varphi_i^{1+\gamma}}{(\frac{d}{2^{i+1}})^{Q\gamma}}\\&\le cd^{-Q\gamma}2^{Qi(\frac{Q}{p}+\gamma)}R^{-\frac{np-Q}{p(n-1)}n\varepsilon}\varphi_i^{1+\gamma}.\end{aligned}$$

We can now apply Lemma 2.2 with

$$C=cd^{-Q\gamma}R^{-\frac{np-Q}{p(n-1)}n\varepsilon}>0,\ B=2^{Q(\frac{Q}{p}+\gamma)}>1,\ \alpha=\gamma.$$

Choosing

$$d\ge cR^{-\frac{np-Q}{p(n-1)}n\varepsilon\frac{1}{Q\gamma}}\varphi_0^{\frac{1}{Q}}$$

with the constant c large enough, we can conclude that the sequence φ_i tends to zero, and hence

$$\varphi\left(d,\frac{R}{2}\right)=0.$$

The condition imposed on d will be satisfied by taking

$$d=cR^\tau+cR^{-\frac{np-Q}{p(n-1)}n\varepsilon\frac{1}{Q\gamma}}\varphi_0^{\frac{1}{Q}},$$

and hence, recalling the choice of γ, we arrive at

$$\sup_{B_{\frac{Q}{2}}} u \leq d = c\left(\frac{1}{R^{\frac{(np-Q)n}{p(n-1)}}}\int_{A_{0,R}} u^Q dx\right)^{\frac{1}{Q}}\left(\frac{|A_{0,R}|}{R^{\frac{(np-Q)n}{p(n-1)}}}\right)^{\frac{\gamma}{Q}} + cR^{\tau}.$$

The conclusion follows at once writing $u - \kappa_0$ instead of u. □

Lemma 2.4. *Let u be a locally bounded function, satisfying (2.4) (with $p > 1$) for every k and every κ_0 such that $k \geq \kappa_0 \geq 0$, let $R \leq 1$ and*

$$2k_0 = M(2R) + m(2R) =: \sup_{Q_{2R}} u + \inf_{Q_{2R}} u.$$

Assume that $|A_{k_0,R}| \leq \lambda\,|Q_R|$ for some $\lambda < 1$. If for an integer ν, it holds that

$$\operatorname{osc}(u, 2R) \geq c2^{\nu+1}R^{\tau}, \tag{2.22}$$

where $\operatorname{osc}(u, 2R)$ is the oscillation of the function u over Q_{2R}, then, setting

$$k_\nu = M(2R) - 2^{-\nu-1}\operatorname{osc}(u, 2R), \tag{2.23}$$

we have

$$|A_{k_\nu,R}| \leq c\nu^{-\frac{n(p-1)}{p(n-1)}} R^{\frac{(np-Q)n}{p(n-1)}}. \tag{2.24}$$

In the proof of Lemma 2.4 we shall need the following Sobolev inequality, which can be found, for example, in Proposition 37 of [30].

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $E \subset \Omega$, $|E| > 0$, $1 \leq p < n$. Then*

$$\left(\int_\Omega |u|^{p^*}|dx\right)^{1/p^*} \leq c(\Omega, E, p)\left(\int_\Omega |\nabla u|^p dx\right)^{1/p}$$

for all $u \in W^{1,p}(\Omega)$ with $u|_E = 0$.

Proof of Lemma 2.4. For $k_0 < h < k$ let us define

$$v(x) = \begin{cases} k-h & \text{if} \quad u \geq k, \\ u-h & \text{if} \quad h < u < k, \\ 0 & \text{if} \quad u \leq h. \end{cases}$$

We have $v = 0$ in $B_R \setminus A_{k_0,R}$, and since the measure of this set is greater than $(1-\lambda)\,|B_R|$, we may use Lemma 2.5 with $E = B_R \setminus A_{k_0,R}$, obtaining

$$\left(\int_{Q_R} v^{\frac{n}{n-1}} dx\right)^{1-\frac{1}{n}} \leq c\int_\Delta |Dv|dx = c\int_\Delta |Du|dx$$

in which

$$\Delta = A_{h,R} \setminus A_{k,R}.$$

We therefore have

$$(k-h)|A_{k,R}|^{1-\frac{1}{n}} \leq \left(\int_{Q_R} v^{\frac{n}{n-1}}dx\right)^{1-\frac{1}{n}} \leq c|\Delta|^{1-\frac{1}{p}}\left(\int_{A_{h,R}}|Du|^p dx\right)^{\frac{1}{p}}. \tag{2.25}$$

On the other hand, from (2.4) when $r=R, \rho=2R, k=h$, we deduce

$$\begin{aligned}\int_{A_{h,R}}|Du|^p dx \leq &\frac{c}{R^Q}\int_{A_{h,2R}}(u-h)^Q dx + c|A_{h,2R}|^{1-\frac{p}{n}+\varepsilon}\\ \leq & cR^{n-Q}(M(2R)-h)^Q + cR^{n-p+n\varepsilon}.\end{aligned} \tag{2.26}$$

For $h \leq k_\nu$, (2.22) and (2.23) merge into

$$M(2R)-h \geq M(2R)-k_\nu \geq 2^{-\nu-1}\operatorname{osc}(u,2R) \geq cR^\tau,$$

therefore

$$\begin{gathered}(M(2R)-h)^Q \geq cR^{Q\tau} = cR^{n\varepsilon\frac{(np-Q)}{p(n-1)}} \geq cR^{n\varepsilon},\\ R^{n-Q} \geq R^{n-p},\\ R^{n-Q}(M(2R)-h)^Q \geq cR^{n-p+n\varepsilon},\end{gathered}$$

which combining with (2.26) one obtains

$$\int_{A_{h,R}}|Du|^p dx \leq 2cR^{n-Q}(M(2R)-h)^Q. \tag{2.27}$$

Substituting (2.27) into (2.25), one has

$$(k-h)|A_{k,R}|^{1-\frac{1}{n}} \leq c|\Delta|^{1-\frac{1}{p}}R^{\frac{n-Q}{p}}(M(2R)-h)^{\frac{Q}{p}}.$$

Writing the above inequality for the levels,

$$k = k_i = M(2R) - 2^{-i-1}\operatorname{osc}(u,2R)$$

and

$$h = k_{i-1} = M(2R) - 2^{-i}\operatorname{osc}(u,2R),$$

where $i=\{1,2,\cdots,\nu\}$, we get

$$\begin{aligned}&\frac{1}{2}\frac{1}{2^i}\operatorname{osc}(u,2R)|A_{k_{i+1},R}|^{1-\frac{1}{n}}\\ \leq\ & \frac{1}{2^{i+1}}\operatorname{osc}(u,2R)|A_{k_i,R}|^{1-\frac{1}{n}}\\ \leq\ & c|\Delta_i|^{1-\frac{1}{p}}R^{\frac{n-Q}{p}}\left(\frac{1}{2^i}\operatorname{osc}(u,2R)\right)^{\frac{Q}{p}},\end{aligned}$$

the inequality above implies

$$|A_{k_\nu,R}|^{1-\frac{1}{n}} \leq |A_{k_i,R}|^{1-\frac{1}{n}} \leq c|\Delta_i|^{1-\frac{1}{p}}R^{\frac{n-Q}{p}}\left(\frac{1}{2^i}\operatorname{osc}(u,2R)\right)^{\frac{Q}{p}-1},$$

where $\Delta_i = A_{k_{i-1},R} \setminus A_{k_i,R}$. Raising both sides of the above inequality to the power $\frac{p}{p-1}$ one gets

$$|A_{k_\nu,R}|^{\frac{p(n-1)}{n(p-1)}} \le cR^{\frac{n-Q}{p-1}} |\Delta_i| \left(\frac{1}{2^i}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}}.$$

We now sum over i from 1 to ν,

$$\begin{aligned}
&\sum_{i=1}^{\nu} |\Delta_i| \left(\frac{1}{2^i}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} \\
\le& |A_{k_0,R}| \left(\frac{1}{2}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} - |A_{k_1,R}| \left(\frac{1}{2}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} \\
&+ |A_{k_1,R}| \left(\frac{1}{2^2}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} - |A_{k_2,R}| \left(\frac{1}{2^2}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} \\
&+ \cdots \\
&+ |A_{k_{\nu-1},R}| \left(\frac{1}{2^\nu}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} - |A_{k_\nu,R}| \left(\frac{1}{2^\nu}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} \\
\le& |A_{k_0,R}| \left(\frac{1}{2}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} - |A_{k_1,R}| \left(\frac{1}{2^2}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} \\
&+ |A_{k_1,R}| \left(\frac{1}{2^2}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} - |A_{k_2,R}| \left(\frac{1}{2^3}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} \\
&+ \cdots \\
&+ |A_{k_{\nu-1},R}| \left(\frac{1}{2^\nu}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} - |A_{k_\nu,R}| \left(\frac{1}{2^\nu}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} \\
\le& |A_{k_0,R}| \left(\frac{1}{2}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} - |A_{k_\nu,R}| \left(\frac{1}{2^\nu}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} \\
\le& |A_{k_0,R}| \left(\frac{1}{2}\operatorname{osc}(u,2R)\right)^{\frac{Q-p}{p-1}} \\
\le& c\,|A_{k_0,R}|,
\end{aligned}$$

where we have used the fact

$$\operatorname{osc}(u,2R) \le \operatorname{esssup}_{B_{2R}} u < \infty.$$

Thus

$$\nu\,|A_{k_\nu,R}|^{\frac{p(n-1)}{n(p-1)}} \le cR^{\frac{n-Q}{p-1}} |A_{k_0,R}| \le cR^{\frac{pn-Q}{(p-1)}},$$

and so

$$|A_{k_\nu,R}| \le c\nu^{-\frac{n(p-1)}{p(n-1)}} R^{\frac{(np-Q)n}{p(n-1)}},$$

as desired.

In the proof of Theorem 2.1 we shall need the following algebraic lemma, which comes from Lemma 7.3 in [25].

Lemma 2.6. *Let $\varphi(t)$ be a positive function, and assume that there exists a constant q and a number $\tilde{\tau}$, $0<\tilde{\tau}<1$ such that for every $R<R_0$*

$$\varphi(\tilde{\tau}R)\leq \tilde{\tau}^{\delta}\varphi(R)+BR^{\beta}$$

with $0<\beta<\delta$, and

$$\varphi(t)\leq q\varphi\left(\tilde{\tau}^{k}R\right)$$

for every t in the interval $\left(\tilde{\tau}^{k+1}R,\tilde{\tau}^{k}R\right)$. Then, for every $\varrho<R\leq R_0$ we have

$$\varphi(\varrho)\leq C\left\{\left(\frac{\varrho}{R}\right)^{\beta}\varphi(R)+B\varrho^{\beta}\right\},$$

where C is a constant depending only on $q,\tilde{\tau},\delta$ and β.

With the above Lemmas in hands, we can now prove Theorem 2.1.

Proof. By lemma 2.1 u is locally bounded. Let, as above, $2k_0=M(2R)+m(2R)$. We can assume without loss of generality that

$$\left|A_{k_0,R}\right|\leq \frac{1}{2}\left|Q_R\right|,$$

since otherwise we would have

$$\left|B_{k_0,R}\right|=\left|Q_R\right|-\left|A_{k_0,R}\right|\leq \frac{1}{2}\left|Q_R\right|,$$

and it will be sufficient to write $-u$ instead of u.

Setting $k_\nu=M(2R)-2^{-\nu-1}\operatorname{osc}(u,2R)$, we have $k_\nu>\kappa_0$. Write (2.6) with k_ν instead of κ_0

$$\begin{aligned}\sup_{B_{\frac{R}{2}}}u-k_\nu&\leq c\left(\frac{1}{R^{\frac{(np-Q)n}{p(n-1)}}}\int_{A_{k_\nu,R}}(u-k_\nu)^{Q}\,dx\right)^{\frac{1}{Q}}\left(\frac{\left|A_{k_\nu,R}\right|}{R^{\frac{(np-Q)n}{p(n-1)}}}\right)^{\frac{\gamma}{Q}}+cR^{\tau}\\&\leq c\sup_{B_R}(u-k_\nu)\left(\frac{\left|A_{k_\nu,R}\right|}{R^{\frac{(np-Q)n}{p(n-1)}}}\right)^{\frac{\gamma+1}{Q}}+cR^{\tau}.\end{aligned}$$

Let us now choose the integer ν in such a way that

$$c\nu^{-\frac{n(p-1)}{p(n-1)}}\leq\frac{1}{2}.$$

If $\operatorname{osc}(u,2R)\geq c2^{\nu+1}R^{\tau}$, we deduce from (2.24) that

$$\sup_{B_{\frac{R}{2}}}u-k_\nu\leq\frac{1}{2}\sup_{B_R}(u-k_\nu)+cR^{\tau},$$

therefore

$$M\left(\frac{R}{2}\right)-k_\nu\leq\frac{1}{2}\left(M(2R)-k_\nu\right)+cR^{\tau},$$

so that, subtracting from both members the quantity $m\left(\frac{R}{2}\right)$,

$$M\left(\frac{R}{2}\right)-m\left(\frac{R}{2}\right)-k_\nu\le\frac{1}{2}\left(M(2R)-k_\nu\right)-m\left(\frac{R}{2}\right)+cR^\tau,$$

then

$$M\left(\frac{R}{2}\right)-m\left(\frac{R}{2}\right)\le\frac{1}{2}M(2R)-m\left(\frac{R}{2}\right)+\frac{1}{2}k_\nu+cR^\tau,$$

by (2.23) we have

$$\begin{aligned}\operatorname{osc}\left(u,\frac{R}{2}\right)&\le\frac{1}{2}M(2R)-m\left(\frac{R}{2}\right)+\frac{1}{2}\left[M(2R)-\frac{1}{2^{\nu+1}}\operatorname{osc}(u,2R)\right]+cR^\tau\\&\le M(2R)-m(2R)-\frac{1}{2^{\nu+2}}\operatorname{osc}(u,2R)+cR^\tau\\&\le\left(1-\frac{1}{2^{\nu+2}}\right)\operatorname{osc}(u,2R)+cR^\tau.\end{aligned}$$

In conclusion, either the function $\operatorname{osc}(u,R)$ satisfies the above relation, or else

$$\operatorname{osc}(u,2R)\le c2^{\nu+1}R^\tau.$$

In any case, we have

$$\operatorname{osc}\left(u,\frac{R}{2}\right)\le\left(1-\frac{1}{2^{\nu+2}}\right)\operatorname{osc}(u,2R)+c2^\nu R^\tau.$$

We can now apply the preceding Lemma 2.6 with $\tilde{\tau}=1/4$, $\beta=\tau$ and $\delta=\log_{\tilde{\tau}}(1-(2^{-\nu-2}))$. Decreasing if necessary the value of β, we can assume that $\beta<\delta$. We therefore have

$$\operatorname{osc}(u,\varrho)\le c\left\{\left(\frac{\varrho}{R}\right)^\beta\operatorname{osc}(u,R)+c\varrho^\beta\right\},$$

for every $\varrho<R\le\min\left(R_0,\operatorname{dist}(x_0,\partial\Omega)\right)$. The above inequality shows that u is locally Hölder continuous. This ends the proof of Theorem 2.1. □

In the next three sections we shall give applications of Theorems 2.1. In Section 3 we shall consider a polyconvex integral functional in four-dimensional Euclidean spaces with the integrand has splitting form. Under some structural conditions on the energy density, we prove that all local minimizers are locally bounded and locally Hölder continuous. In Section 4 we shall consider a special type of linear elliptic equation with degenerate coercivity, under suitable integrability assumption on the coefficient, we derive that any of its weak solutions is locally bounded and locally Hölder continuous. In Section 5, we shall show that our Theorem 2.1 can be applied in dealing with regularity issues of elliptic equations with non-standard grow conditions.

3 A polyconvex integral functional.

In this section we give an application of Theorem 2.1 to regularity property for minimizers of some polyconvex integrals in four dimensional Euclidean spaces with the integrand has splitting structure. More precisely, let Ω be an open bounded subset of $\mathbb{R}^4$ and let us consider the variational integral

$$\mathcal{F}(u,\Omega) = \int_\Omega f(x, Du(x))dx, \tag{3.1}$$

where $f : \Omega \times \mathbb{R}^{4\times 4} \to \mathbb{R}$ is a Carathéodory function (that is, measurable with respect to x for every $\xi \in \mathbb{R}^{4\times 4}$ and continuous with respect to ξ for almost every $x \in \Omega$),

$$u = (u^1, u^2, u^3, u^4)^t : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^4$$

is a vector-valued map, and Du is the 4×4 Jacobian matrix of its partial derivatives, i.e.,

$$Du = \begin{pmatrix} Du^1 \\ Du^2 \\ Du^3 \\ Du^4 \end{pmatrix} = \begin{pmatrix} D_1u^1 & D_2u^1 & D_3u^1 & D_4u^1 \\ D_1u^2 & D_2u^2 & D_3u^2 & D_4u^2 \\ D_1u^3 & D_2u^3 & D_3u^3 & D_4u^3 \\ D_1u^4 & D_2u^4 & D_3u^4 & D_4u^4 \end{pmatrix}, \; D_\beta u^\alpha = \frac{\partial u^\alpha}{\partial x_\beta}, \; \alpha, \beta \in \{1,2,3,4\}.$$

We assume that $f : \Omega \times \mathbb{R}^{4\times 4} \to \mathbb{R}$ has splitting form:

$$f(x,\xi) = \sum_{\alpha=1}^{4} F_\alpha\big(x, \xi^\alpha\big) + \sum_{\beta=1}^{6} G_\beta\big(x, (\mathrm{adj}_2\xi)^\beta\big) + \sum_{\gamma=1}^{4} H_\gamma\big(x, (\mathrm{adj}_3\xi)^\gamma\big) + I(x, \det \xi), \tag{3.2}$$

where

$$F_\alpha(x,\lambda) : \Omega \times \mathbb{R}^4 \to \mathbb{R}, \quad \alpha = 1,2,3,4,$$
$$G_\beta(x,\eta) : \Omega \times \mathbb{R}^6 \to \mathbb{R}, \quad \beta = 1,2,3,4,5,6,$$
$$H_\gamma(x,\lambda) : \Omega \times \mathbb{R}^4 \to \mathbb{R}, \quad \gamma = 1,2,3,4,$$

and

$$I(x,t) : \Omega \times \mathbb{R} \to \mathbb{R},$$

are Carathéodory functions such that $\lambda \mapsto F_\alpha(x,\lambda)$, $\eta \mapsto G_\beta(x,\eta)$, $\lambda \mapsto H_\gamma(x,\lambda)$ and $t \mapsto I(x,t)$ are convex.

In (3.2),

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \xi^4 \end{pmatrix} = \begin{pmatrix} \xi^1_1 & \xi^1_2 & \xi^1_3 & \xi^1_4 \\ \xi^2_1 & \xi^2_2 & \xi^2_3 & \xi^2_4 \\ \xi^3_1 & \xi^3_2 & \xi^3_3 & \xi^3_4 \\ \xi^4_1 & \xi^4_2 & \xi^4_3 & \xi^4_4 \end{pmatrix} \in \mathbb{R}^{4\times 4},$$

$\xi^\alpha = (\xi^\alpha_1, \xi^\alpha_2, \xi^\alpha_3, \xi^\alpha_4)$ is the αth row of ξ, $\alpha = 1,2,3,4$; $\mathrm{adj}_2\xi \in \mathbb{R}^{6\times 6}$ denote the adjugate matrix of order 2 whose components are

$$(\mathrm{adj}_2\xi)^j_k = (-1)^{j_1+j_2+k_1+k_2} \det \begin{pmatrix} \xi^{j_1}_{k_1} & \xi^{j_1}_{k_2} \\ \xi^{j_2}_{k_1} & \xi^{j_2}_{k_2} \end{pmatrix}, \; j,k = 1,2,3,4,5,6, \tag{3.3}$$

where we have denoted

$$\begin{cases} 1_1 = 1 \\ 1_2 = 2 \end{cases}, \begin{cases} 2_1 = 1 \\ 2_2 = 3 \end{cases}, \begin{cases} 3_1 = 1 \\ 3_2 = 4 \end{cases}, \begin{cases} 4_1 = 2 \\ 4_2 = 3 \end{cases}, \begin{cases} 5_1 = 2 \\ 5_2 = 4 \end{cases}, \begin{cases} 6_1 = 3 \\ 6_2 = 4 \end{cases},$$

and

$$(\mathrm{adj}_2\xi)^\beta = \big((\mathrm{adj}_2\xi)^\beta_1, \cdots, (\mathrm{adj}_2\xi)^\beta_6\big) \in \mathbb{R}^6, \quad \beta = 1,2,3,4,5,6,$$

is the βth row of $\mathrm{adj}_2\xi \in \mathbb{R}^{6\times 6}$. We note that, for $j = 1,2,3$, since $j_1 = 1$, then $(\mathrm{adj}_2\xi)^j$ depends on the entries of the first row of ξ, while for $j = 4,5,6$, since $j_1 \neq 1$ and $j_1 < j_2$, then $(\mathrm{adj}_2\xi)^j$ does not depend on the entries of the first row of ξ, we shall use these facts in the sequel; $\mathrm{adj}_3\xi \in \mathbb{R}^{4\times 4}$ denotes the adjugate matrix of order 3 whose components are

$$(\mathrm{adj}_3\xi)^\gamma_j = (-1)^{\gamma+j} \det \begin{pmatrix} \xi^\varepsilon_k & \xi^\varepsilon_\ell & \xi^\varepsilon_m \\ \xi^\delta_k & \xi^\delta_\ell & \xi^\delta_m \\ \xi^\tau_k & \xi^\tau_\ell & \xi^\tau_m \end{pmatrix}, \quad \gamma, j \in \{1,2,3,4\}, \tag{3.4}$$

where $\varepsilon, \delta, \tau \in \{1,2,3,4\}\backslash\{\gamma\}$, $\varepsilon < \delta < \tau$, $k, \ell, m \in \{1,2,3,4\}\backslash\{j\}$, $k < \ell < m$, and

$$(\mathrm{adj}_3\xi)^\gamma = ((\mathrm{adj}_3\xi)^\gamma_1, \cdots, (\mathrm{adj}_3\xi)^\gamma_4) \in \mathbb{R}^4, \quad \gamma = 1,2,3,4,$$

is the γth row of $\mathrm{adj}_3\xi \in \mathbb{R}^{4\times 4}$; moreover, $\mathrm{adj}_4\xi = \det\xi$ denotes the adjugate matrix of order 4, i.e., the determinant of the square matrix $\xi \in \mathbb{R}^{4\times 4}$.

We assume that there exist exponents $p \in (1,4]$, $q > 1$, $r > 1$, $s \geq 1$, constants $c_1, c_3 > 0$, $c_2 \geq 0$, and nonnegative functions

$$a(x), b(x), c(x), d(x) \in L^\sigma_{loc}(\Omega), \ \sigma > \frac{4}{p},$$

such that for $\alpha \in \{1,2,3,4\}$, $\beta \in \{1,2,3,4,5,6\}$ and $\gamma \in \{1,2,3,4\}$,

$$c_1|\lambda|^p - c_2 \leq F_\alpha(x,\lambda) \leq c_3(|\lambda|^p + 1) + a(x), \quad \forall \lambda \in \mathbb{R}^4, \tag{3.5}$$

$$c_1|\eta|^q - c_2 \leq G_\beta(x,\eta) \leq c_3(|\eta|^q + 1) + b(x), \quad \forall \eta \in \mathbb{R}^6, \tag{3.6}$$

$$c_1|\lambda|^r - c_2 \leq H_\gamma(x,\lambda) \leq c_3(|\lambda|^r + 1) + c(x), \quad \forall \lambda \in \mathbb{R}^4, \tag{3.7}$$

$$0 \leq I(x,t) \leq c_3(|t|^s + 1) + d(x), \quad \forall t \in \mathbb{R}. \tag{3.8}$$

Note that we have assumed that the integrand $f(x,\xi)$ has splitting form, and the functions $F_\alpha(x,\cdot)$, $G_\beta(x,\cdot)$, $H_\gamma(x,\cdot)$ and $I(x,\cdot)$ are convex, thus the function $f(x,\xi)$ defined in (3.2) is polyconvex. Recall that a function $f = f(\xi) : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be polyconvex if there exists a convex function $g : \mathbb{R}^{\tau(m,n)} \to \mathbb{R}$ such that

$$f(\xi) = g(T(\xi)),$$

where

$$\tau(m,n) = \sum_{i=1}^{\min\{m,n\}} \binom{n}{i}\binom{m}{i},$$

and $T(\xi)$ is the vector defined as follows:

$$T(\xi) = (\xi, \mathrm{adj}_2\xi, \cdots, \mathrm{adj}_i\xi, \cdots, \mathrm{adj}_{\min\{m,n\}}\xi),$$

here $\mathrm{adj}_i\xi$ denotes the adjugate matrix of order i. In particular, if $m = n$, then $\mathrm{adj}_n\xi = \det\xi$.

Definition 3.1. *A function $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^4)$ is a local minimizer of $\mathcal{F}(u,\Omega)$ in (3.1) with*

$f(x,\xi)$ be as in (3.2) if $f(x,Du(x)) \in L^1_{loc}(\Omega)$ and

$$\mathcal{F}(u, \mathrm{supp}\varphi) \le \mathcal{F}(u+\varphi, \mathrm{supp}\varphi)$$

for all $\varphi \in W^{1,1}(\Omega,\mathbb{R}^4)$ with $\mathrm{supp}\varphi \Subset \Omega$.

The main result in this section is the following

Theorem 3.1. *Let f has splitting structure as in (3.2), and satisfy the growth conditions (3.5)-(3.8). Let $u \in W^{1,1}_{loc}(\Omega,\mathbb{R}^4)$ be a local minimizer of $\mathcal{F}$ and $1 \le s < r < q < p \le 4$. Assume*

$$\frac{p}{p^*} < 1 - \max\left\{\frac{qp^*}{p(p^*-q)}, \frac{rp^*}{q(p^*-r)}, \frac{sp^*}{r(p^*-s)}, \frac{1}{\sigma}\right\}, \tag{3.9}$$

where $p^* = \frac{4p}{4-p}$, if $p < 4$, and, if $p = 4$, then p^* is any $\nu > 4$.

Then all the local minimizers $u \in W^{1,1}_{loc}(\Omega,\mathbb{R}^4)$ of $\mathcal{F}$ are locally bounded and locally Hölder continuous.

For related results to Theorem 3.1, we should mention Cupini-Leonetti-Mascolo [7], where the authors considered a special class of polyconvex functionals from $\Omega \subset \mathbb{R}^3$ to $\mathbb{R}^3$:

$$f(x,\xi) = \sum_{\alpha=1}^{3} \left\{F_\alpha(x,\xi^\alpha) + G_\alpha(x,(\mathrm{adj}_2\xi)^\alpha)\right\} + H(x,\det\xi). \tag{3.10}$$

Under some structural assumptions on the energy density, the authors proved that local minimizers u are locally bounded by using De Giorgi's iteration method. Some related results can be found in [4, 9, 21] and [22], where in [4] the authors found conditions on the structure of some integral functional which forces minimizers to be locally bounded; in [9], the authors proved local Hölder continuity of vectorial local minimizers of special classes of integral functionals with rank-one and polyconvex integrands. The regularity of minimizers was obtained by proving that each component stays in a suitable De Giorgi class; in [21] the authors gave regularity results for minimizers of two special cases of polyconvex functionals, under some structural assumptions on the energy density, the authors proved that the minimizers are either bounded, or have suitable integrability properties by using Stampacchia Lemma; and in [22], the authors obtained regularity properties for minimizing sequences of some integral functionals in 3-dimensional spaces related to nonlinear elasticity theory. Under some structural conditions on the energy density, the authors derived that the minimizing sequences and the derivatives of the sequences have some regularity properties by using the Ekeland variational principle. Partial regularity for minimizers of degenerate polyconvex energies can be found in [13]. Partial regularity results related to nonlinear elasticity theory can be found in [16–19, 31]. For some other contributions related to polyconvex functionals, we refer to [34, 56].

We now present some preliminary lemmas that will be used in the proof of Theorem 3.1. Proof the Theorem 3.1 is done reasoning as in Cupini-Leonetti-Mascolo [7]. Indeed, Lemmas 3.1, 3.2, 3.3 and 3.4 are a generalization in four dimension of what has been proved in [7] in three dimension.

Lemma 3.1. *Let $\xi \in \mathbb{R}^{4\times 4}$ be a square matrix, $\mathrm{adj}_2\xi \in \mathbb{R}^{6\times 6}$ be the adjugate matrix of order 2 whose components are as in (3.3), $\mathrm{adj}_3\xi \in \mathbb{R}^{4\times 4}$ be the adjugate matrix of order*

3 whose components are as in (3.4), and $\det \xi$ *be the adjugate matrix of order 4 (that is, the determinant of* ξ*). Then*

(i)

$$|(\mathrm{adj}_2\xi)^\beta| \le 12|\xi^1| \sum_{\alpha=2}^{4} |\xi^\alpha|, \quad \beta = 1,2,3,$$

where $(\mathrm{adj}_2\xi)^\beta$ *is the* β*th row of* $\mathrm{adj}_2\xi \in \mathbb{R}^{6\times 6}$*;*

(ii)

$$|(\mathrm{adj}_3\xi)^\gamma| \le 12|\xi^1| \sum_{\beta=4}^{6} |(\mathrm{adj}_2\xi)^\beta|, \quad \gamma = 2,3,4,$$

where $(\mathrm{adj}_3\xi)^\gamma$ *is the* γ*th row of* $\mathrm{adj}_3\xi \in \mathbb{R}^{4\times 4}$*;*

(iii)

$$|\det \xi| \le 4|\xi^1||(\mathrm{adj}_3\xi)^1|.$$

Proof. From the definition of the adjugate matrix of order 2, we know that $(\mathrm{adj}_2\xi)^\beta$ is a 6-dimensional vector whose components are $(\mathrm{adj}_2\xi)^\beta_j$, $j = 1,2,3,4,5,6$. For $\beta = 1,2,3$, $(\mathrm{adj}_2\xi)^\beta_j$ is, up to a sign, a determinant of a 2×2 matrix with the 2 entries on the first row come from ξ^1, thus

$$|(\mathrm{adj}_2\xi)^\beta_j| \le 2|\xi^1| \sum_{k=2}^{4} |\xi^k|, \quad \beta = 1,2,3, \ j = 1,2,3,4,5,6,$$

from which we derive

$$|(\mathrm{adj}_2\xi)^\beta| \le \sum_{j=1}^{6} |(\mathrm{adj}_2\xi)^\beta_j| \le 12|\xi^1| \sum_{k=2}^{4} |\xi^k|, \ \beta = 1,2,3.$$

From the definition of the adjugate matrix of order 3, we know that $(\mathrm{adj}_3\xi)^\gamma$ is a 4-dimensional vector whose components are $(\mathrm{adj}_3\xi)^\gamma_j$, $j = 1,2,3,4$. For $\gamma = 2,3,4$, $(\mathrm{adj}_3\xi)^\gamma_j$ is, up to a sign, a determinant of a 3×3 matrix with the 3 entries on the first row come from ξ^1. We use the cofactor expansion formula for $(\mathrm{adj}_3\xi)^\gamma_j$ and we have

$$|(\mathrm{adj}_3\xi)^\gamma_j| \le 3|\xi^1| \sum_{\beta=4}^{6} |(\mathrm{adj}_2\xi)^\beta|, \quad \gamma = 2,3,4, \ j = 1,2,3,4,$$

from which we derive

$$|(\mathrm{adj}_3\xi)^\gamma| \le \sum_{j=1}^{4} |(\mathrm{adj}_2\xi)^\gamma_j| \le 12|\xi^1| \sum_{\beta=4}^{6} |(\mathrm{adj}_2\xi)^\beta|, \ \gamma \in \{2,3,4\}.$$

We use the cofactor expansion formula again to derive

$$|\det \xi| \le \sum_{j=1}^{4} |\xi^1_j||(\mathrm{adj}_3\xi)^1_j| \le 4|\xi^1||(\mathrm{adj}_3\xi)^1|.$$

This ends the proof of Lemma 3.1. □

Lemma 3.2. *Let f be as in (3.2) and satisfy the growth conditions (3.5)-(3.8). Let $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^4)$ be such that*

$$f(x, Du(x)) \in L^1_{loc}(\Omega).$$

Fix $\eta \in C^1_0(\Omega), \eta \geq 0$ and $k \in \mathbb{R}$ and denote for almost every $x \in \{u^1 > k\} \cap \{\eta > 0\}$,

$$A = \begin{pmatrix} p^* \eta^{-1}(k - u^1) D\eta \\ Du^2 \\ Du^3 \\ Du^4 \end{pmatrix}, \tag{3.11}$$

If

$$q < \frac{pp^*}{p + p^*}, \ r < \frac{qp^*}{q + p^*}, \ s < \frac{rp^*}{r + p^*}, \tag{3.12}$$

then

$$\eta^{p^*} f(x, A) \in L^1\left(\{u^1 > k\} \cap \{\eta > 0\}\right).$$

Proof. Denote

$$\hat{u} := \begin{pmatrix} 0 \\ u^2 \\ u^3 \\ u^4 \end{pmatrix}, \quad D\hat{u} := \begin{pmatrix} 0 \\ Du^2 \\ Du^3 \\ Du^4 \end{pmatrix}.$$

By the definition of $f(x, \xi)$ in (3.2) and the growth conditions (3.5)-(3.8) we have, almost everywhere in $\{u^1 > k\} \cap \{\eta > 0\}$,

$$\begin{aligned} |f(x, A)| \leq & c\Big\{ |A|^p + \sum_{\beta=1}^{3} \left|(\mathrm{adj}_2\, A)^\beta\right|^q + \sum_{\beta=4}^{6} \left|(\mathrm{adj}_2\, A)^\beta\right|^q + \left|(\mathrm{adj}_3\, Du)^1\right|^r \\ & + \sum_{\gamma=2}^{4} |(\mathrm{adj}_3\, A)^\gamma|^r + |\det A|^s + \omega(x)\Big\}, \end{aligned} \tag{3.13}$$

where c is a constant depending upon c_1, c_2, c_3 and

$$\omega(x) = a(x) + b(x) + c(x) + d(x) + 1.$$

It is obvious that, almost everywhere in $\{u^1 > k\} \cap \{\eta > 0\}$,

$$|A|^p \leq c\left((p^* \eta^{-1}(u^1 - k)|D\eta|)^p + |D\hat{u}|^p\right), \tag{3.14}$$

$$\left|(\mathrm{adj}_2 A)^\beta\right|^q = \left|(\mathrm{adj}_2 Du)^\beta\right|^q, \quad \beta = 4, 5, 6, \tag{3.15}$$

Thanks to Lemma 3.1, we get a.e. in $\{u^1 > k\} \cap \{\eta > 0\}$,

$$\begin{aligned}
&\left|(\mathrm{adj}_2\, A)^\beta\right|^q \le 12^q \Big(p^*\eta^{-1}(u^1-k)|D\eta| \sum_{\alpha=2}^4 |Du^\alpha|\Big)^q, \quad \beta = 1,2,3,\\
&|(\mathrm{adj}_3\, A)^\gamma|^r \le 12^r \Big(p^*\eta^{-1}(u^1-k)|D\eta| \sum_{\beta=4}^6 |(\mathrm{adj}_2 Du)^\beta|\Big)^r, \quad \gamma = 2,3,4,\\
&|\det A|^s \le 4^s \left((p^*\eta^{-1}(u^1-k)|D\eta|)|(\mathrm{adj}_3 Du)^1|\right)^s.
\end{aligned}$$

We use Young inequality and we derive from the above inequalities that, almost everywhere in $\{u^1 > k\} \cap \{\eta > 0\}$,

$$\begin{aligned}
&\sum_{\beta=1}^3 \left|(\mathrm{adj}_2\, A)^\beta\right|^q + \sum_{\gamma=2}^4 |(\mathrm{adj}_3\, A)^\gamma|^r + |\det A|^s\\
\le& c\Big\{\Big(p^*\eta^{-1}(u^1-k)|D\eta| \sum_{\alpha=2}^4 |Du^\alpha|\Big)^q + \Big(p^*\eta^{-1}(u^1-k)|D\eta| \sum_{\beta=4}^6 |(\mathrm{adj}_2 Du)^\beta|\Big)^r\\
&+ \left((p^*\eta^{-1}(u^1-k)|D\eta|)|(\mathrm{adj}_3 Du)^1|\right)^s\Big\}\\
\le& c\Big\{(p^*\eta^{-1}(u^1-k)|D\eta|)^{\frac{pq}{p-q}} + \sum_{\alpha=2}^4 |Du^\alpha|^p + (p^*\eta^{-1}(u^1-k)|D\eta|)^{\frac{qr}{q-r}}\\
&+ \sum_{\beta=4}^6 |(\mathrm{adj}_2 Du)^\beta|^q + (p^*\eta^{-1}(u^1-k)|D\eta|)^{\frac{rs}{r-s}} + |(\mathrm{adj}_3 Du)^1|^r\Big\},
\end{aligned} \tag{3.16}$$

where c is a constant depending upon p, q, r, s. Denote

$$\tilde{q} := \max\left\{\frac{pq}{p-q}, \frac{qr}{q-r}, \frac{rs}{r-s}\right\},$$

then

$$\begin{aligned}
&(p^*\eta^{-1}(u^1-k)|D\eta|)^{\frac{pq}{p-q}} + (p^*\eta^{-1}(u^1-k)|D\eta|)^{\frac{qr}{q-r}} + (p^*\eta^{-1}(u^1-k)|D\eta|)^{\frac{rs}{r-s}}\\
\le\;& c(p^*\eta^{-1}(u^1-k)|D\eta|)^{\tilde{q}} + 1),
\end{aligned}$$

with $c = c(p,q,r,s)$ a positive constant. (3.12) ensures $\tilde{q} < p^*$. The above inequality together with (3.16) implies

$$\begin{aligned}
&\sum_{\beta=1}^3 \left|(\mathrm{adj}_2\, A)^\beta\right|^q + \sum_{\gamma=2}^4 |(\mathrm{adj}_3\, A)^\gamma|^r + |\det A|^s\\
\le& c\Big\{(p^*\eta^{-1}(u^1-k)|D\eta|)^{\tilde{q}} + \sum_{\alpha=2}^4 |Du^\alpha|^p + \sum_{\beta=4}^6 |(\mathrm{adj}_2 Du)^\beta|^q + |(\mathrm{adj}_3 Du)^1|^r + 1\Big\}.
\end{aligned} \tag{3.17}$$

Substituting (3.14), (3.15) and (3.17) into (3.13), one has, almost everywhere in $\{u^1 > k\} \cap$

$\{\eta>0\}$,

$$
\begin{aligned}
\eta^{p^*}|f(x,A)| \leq & c\Big\{\eta^{p^*-p}(p^*(u^1-k)|D\eta|)^p+\eta^{p^*}|D\hat{u}|^p \\
& +(p^*)^{\tilde{q}}\eta^{p^*-\tilde{q}}\left(u^1-k\right)^{\tilde{q}}|D\eta|^{\tilde{q}}+\eta^{p^*}\sum_{\alpha=2}^{4}|Du^\alpha|^p \\
& +\eta^{p^*}\sum_{\beta=4}^{6}|(\mathrm{adj}_2 Du)^\beta|^q+\eta^{p^*}|\left(\mathrm{adj}_3\, Du\right)^1|^r+\eta^{p^*}\omega(x)\Big\}.
\end{aligned} \tag{3.18}
$$

By the growth conditions (3.5)-(3.8) and the fact $f(x,Du(x))\in L^1_{\mathrm{loc}}(\Omega)$ we obtain

$$
\begin{aligned}
&\eta^{p^*}|D\hat{u}|^p+\eta^{p^*}\sum_{\alpha=2}^{4}|Du^\alpha|^p+\eta^{p^*}\sum_{\beta=4}^{6}|(\mathrm{adj}_2 Du)^\beta|^q+\eta^{p^*}|\left(\mathrm{adj}_3\, Du\right)^1|^r \\
\leq & c\eta^{p^*}(f(x,Du)+1)\in L^1(\{u^1>k\}\cap\{\eta>0\}).
\end{aligned} \tag{3.19}
$$

The definition for $f(x,\xi)$ in (3.1), the conditions (3.5)-(3.8) and $f(x,Du(x))\in L^1_{loc}(\Omega)$ imply $u\in W^{1,p}_{loc}(\Omega)$, thus $u\in L^{p^*}_{\mathrm{loc}}\left(\Omega;\mathbb{R}^4\right)$ by Sobolev Embedding Theorem. Since $p^*-\tilde{q}>0$, we then have

$$
\begin{aligned}
&\eta^{p^*-p}(p^*(u^1-k)|D\eta|)^p+(p^*)^{\tilde{q}}\eta^{p^*-\tilde{q}}\left(u^1-k\right)^{\tilde{q}}|D\eta|^{\tilde{q}}+\eta^{p^*}\omega(x) \\
&\in L^1\left(\left\{u^1>k\right\}\cap\{\eta>0\}\right).
\end{aligned} \tag{3.20}
$$

(3.18) together with (3.19) and (3.20) implies $\eta^{p^*}|f(x,A)|\in L^1\left(\left\{u^1>k\right\}\cap\{\eta>0\}\right)$, and the proof of Lemma 3.2 has been finished. □

For a local minimizer $u=\left(u^1,u^2,u^3,u^4\right)^t$ of $\mathcal{F}(u,\Omega)$ in (3.1), the particular structure (3.2) of the variational integral $f(x,Du)$ guarantee an extension of De Giorgi Class for any component u^α of u on every superlevel set $\{u^\alpha>k\}$ and every sublevel set $\{u^\alpha<k\}$, and we then use Theorem 2.1 to derive that it is locally bounded and locally Hölder continuous. In the following we consider the first component u^1 (we can argue similarly for the other components u^2, u^3, u^4).

Lemma 3.3. *Let f be as in (3.2) satisfying the growth conditions (3.5)-(3.8). Suppose $u\in W^{1,1}_{loc}(\Omega,\mathbb{R}^4)$ be a local minimizer of $\mathcal{F}$. Let $1<p\leq 4$ and q,r,s satisfy the relations (3.12). Let $B_{R_0}(x_0)\Subset\Omega$ with $|B_{R_0}|<1$. For $k\in\mathbb{R}$, denote*

$$A^1_{k,t}:=\{x\in B_t(x_0):u^1(x)>k\},\quad 0<t\leq R_0.$$

Then there exists $c>0$, independent of k, such that for every $0<\rho<R\leq R_0$ and every $p\leq Q<p^*$ (where we recall $p^*=\frac{4p}{4-p}$ if $p<4$, and $p^*=$ any $\nu>p$ for $p=4$),

$$\int_{A^1_{k,\rho}}|Du^1|^p dx\leq\int_{A^1_{k,R}}\left(\frac{u^1-k}{R-\rho}\right)^Q dx+c|A^1_{k,R}|^\theta, \tag{3.21}$$

where

$$\theta=1-\max\left\{\frac{qQ}{p(Q-q)},\frac{Qr}{q(Q-r)},\frac{Qs}{r(Q-s)},\frac{1}{\sigma}\right\}. \tag{3.22}$$

Proof. Let $B_{R_0}(x_0) \Subset \Omega$, $|B_{R_0}| < 1$ (which obviously implies $R_0 < 1$). Let ρ, R be such that $0 < \rho < R \le R_0$. Consider a cut-off function $\eta \in C_0^\infty(B_R)$ satisfying the following assumptions

$$0 \le \eta \le 1,\ \eta \equiv 1 \text{ in } B_\rho(x_0),\ |D\eta| \le \frac{2}{R-\rho}. \tag{3.23}$$

Fixing $k \in \mathbb{R}$, define $w \in W^{1,1}_{\mathrm{loc}}\left(\Omega; \mathbb{R}^4\right)$ as

$$w^1 := (u^1 - k) \vee 0 = \max\left\{u^1 - k, 0\right\}, \quad w^2 := 0, \quad w^3 := 0, \quad w^4 := 0,$$

and

$$\varphi := -\eta^{p^*} w.$$

We have $\varphi = 0$ a.e in $\Omega \backslash \left(\{\eta > 0\} \cap \left\{u^1 > k\right\}\right)$, thus

$$f(x, Du + D\varphi) = f(x, Du) \text{ almost everywhere in } \Omega \backslash \left(\{\eta > 0\} \cap \left\{u^1 > k\right\}\right). \tag{3.24}$$

Denote A by (3.11). For almost every $x \in \{\eta > 0\} \cap \left\{u^1 > k\right\}$, we notice that

$$Du + D\varphi = \begin{pmatrix} \left(1 - \eta^{p^*}\right) Du^1 + p^* \eta^{p^*-1}\left(k - u^1\right) D\eta \\ Du^2 \\ Du^3 \\ Du^4 \end{pmatrix} = \left(1 - \eta^{p^*}\right) Du + \eta^{p^*} A.$$

Since

$$\operatorname{adj}_2(Du + D\varphi) = \left(1 - \eta^{p^*}\right) \operatorname{adj}_2 Du + \eta^{p^*} \operatorname{adj}_2 A,$$

$$\operatorname{adj}_3(Du + D\varphi) = \left(1 - \eta^{p^*}\right) \operatorname{adj}_3 Du + \eta^{p^*} \operatorname{adj}_3 A$$

and

$$\det(Du + D\varphi) = \left(1 - \eta^{p^*}\right) \det Du + \eta^{p^*} \det A,$$

then thanks to the assumptions that $F_\alpha(x, \cdot)$, $G_\beta(x, \cdot)$, $H_\gamma(x, \cdot)$ and $I(x, \cdot)$ are convex, we get $f(x, \cdot)$ in (3.2) is polyconvex, thus almost everywhere in $\{\eta > 0\} \cap \{u^1 > k\}$,

$$\begin{aligned} f(x, Du + D\varphi) &= f(x, (1 - \eta^{p^*}) Du + \eta^{p^*} A) \\ &\le (1 - \eta^{p^*}) f(x, Du) + \eta^{p^*} f(x, A). \end{aligned} \tag{3.25}$$

By the minimality of u, $f(x, Du) \in L^1_{loc}(\Omega)$. Lemma 3.2 ensures that

$$\eta^{p^*} f(x, A) \in L^1\left(\left\{u^1 > k\right\} \cap \{\eta > 0\}\right),$$

therefore (3.24) and (3.25) imply $f(x, Du + D\varphi) \in L^1_{\mathrm{loc}}(\Omega)$, then

$$\int_{A^1_{k,R} \cap \{\eta > 0\}} f(x, Du) \mathrm{d}x \le \int_{A^1_{k,R} \cap \{\eta > 0\}} \left\{(1 - \eta^{p^*}) f(x, Du) + \eta^{p^*} f(x, A)\right\} \mathrm{d}x.$$

The inequality above implies

$$\int_{A^1_{k,R} \cap \{\eta > 0\}} \eta^{p^*} f(x, Du) \mathrm{d}x \le \int_{A^1_{k,R} \cap \{\eta > 0\}} \eta^{p^*} f(x, A) \mathrm{d}x. \tag{3.26}$$

Taking into account A in (3.11) and the particular structure of f in (3.2) we obtain

$$\begin{aligned}
F_2\big(x, A^2\big) &= F_2\big(x, (Du)^2\big),\\
F_3\big(x, A^3\big) &= F_3\big(x, (Du)^3\big),\\
F_4\big(x, A^4\big) &= F_4\big(x, (Du)^4\big)\\
G_4\big(x, \big(\operatorname{adj}_2 A\big)^4\big) &= G_4\big(x, \big(\operatorname{adj}_2 Du\big)^4\big),\\
G_5\big(x, \big(\operatorname{adj}_2 A\big)^5\big) &= G_5\big(x, \big(\operatorname{adj}_2 Du\big)^5\big),\\
G_6\big(x, \big(\operatorname{adj}_2 A\big)^6\big) &= G_6\big(x, \big(\operatorname{adj}_2 Du\big)^6\big),\\
H_1\big(x, \big(\operatorname{adj}_3 A\big)^1\big) &= H_1\big(x, \big(\operatorname{adj}_3 Du\big)^1\big),
\end{aligned}$$

then by (3.26) and the growth assumptions (3.5)-(3.8) one has

$$\begin{aligned}
&\int_{A^1_{k,R}\cap\{\eta>0\}} \eta^{p^*}\Big\{|Du^1|^p - 1\Big\}\mathrm{d}x\\
\le &\int_{A^1_{k,R}\cap\{\eta>0\}} \eta^{p^*}\Big\{|Du^1|^p + \sum_{\beta=1}^{3} |(\operatorname{adj}_2 Du)^\beta|^q + \sum_{\alpha=2}^{4} |(\operatorname{adj}_3 Du)^\alpha|^r - 1\Big\}\\
\le c &\int_{A^1_{k,R}\cap\{\eta>0\}} \eta^{p^*}\Big\{F_1\left(x, Du^1\right) + \sum_{\beta=1}^{3} G_\beta\Big(x, (\operatorname{adj}_2 Du)^\beta\Big)\\
&\qquad + \sum_{\gamma=2}^{4} H_\gamma\left(x, (\operatorname{adj}_3 Du)^\gamma\right) + I(x, \det Du)\Big\}\mathrm{d}x \qquad (3.27)\\
\le c &\int_{A^1_{k,R}\cap\{\eta>0\}} \eta^{p^*}\Big\{F_1\left(x, p^*\eta^{-1}\left(k - u^1\right) D\eta\right) + \sum_{\beta=1}^{3} G_\beta\big(x, (\operatorname{adj}_2 A)^\beta\,\big)\\
&\qquad + \sum_{\gamma=2}^{4} H_\gamma\left(x, (\operatorname{adj}_3 A)^\gamma\right) + I(x, \det A)\Big\}\mathrm{d}x,
\end{aligned}$$

where c is a constant depending upon c_1, c_2, c_3.

Our nearest goal is to estimate the right hand side integrals. By the assumptions (3.5) and (3.23), we use Young inequality in order to derive, for any $Q \in (p, p^*)$,

$$\begin{aligned}
&\int_{A^1_{k,R}\cap\{\eta>0\}} \eta^{p^*} F_1\left(x, p^*\eta^{-1}\left(k-u^1\right) D\eta\right) \mathrm{d}x \\
\leq& c\int_{A^1_{k,R}\cap\{\eta>0\}} \eta^{p^*}\left(|p^*\eta^{-1}\left(u^1-k\right) D\eta|^p + a(x)+1\right) \mathrm{d}x \\
\leq& c\int_{A^1_{k,R}\cap\{\eta>0\}} \left((p^*)^p\eta^{p^*-p}\left(\frac{u^1-k}{R-\rho}\right)^p + a(x)+1\right)\mathrm{d}x \\
\leq& c\int_{A^1_{k,R}} \left(\frac{u^1-k}{R-\rho}\right)^p \mathrm{d}x + c\int_{A^1_{k,R}} (a(x)+1)\mathrm{d}x \\
\leq& c\int_{A^1_{k,R}} \left(\frac{u^1-k}{R-\rho}\right)^Q \mathrm{d}x + c|A^1_{k,R}|^{1-\frac{1}{\sigma}} + c\int_{A^1_{k,R}} (a(x)+1)\mathrm{d}x,
\end{aligned} \tag{3.28}$$

here c is a constant depending on c_3, p, and we have used the facts $\eta^{p^*-p} \leq 1$ and

$$|A^1_{k,R}| = |A^1_{k,R}|^{\frac{1}{\sigma}}|A^1_{k,R}|^{1-\frac{1}{\sigma}} \leq |\Omega|^{\frac{1}{\sigma}}|A^1_{k,R}|^{1-\frac{1}{\sigma}}.$$

Recall $D\hat{u} = (0, Du_2, Du_3, Du_4)^t$. We use the growth condition in (3.6) for G_β and Lemma 3.1 (i) in order to derive

$$\begin{aligned}
&\eta^{p^*}\sum_{\beta=1}^{3} G_\beta\left(x, \left(\operatorname{adj}_2 A\right)^\beta\right) \\
\leq& c\eta^{p^*}\sum_{\beta=1}^{3}\left(\left|\left(\operatorname{adj}_2 A\right)^\beta\right|^q + b(x)+1\right) \\
\leq& c\eta^{p^*}\left(|p^*\eta^{-1}(u^1-k)D\eta|^q\left(\sum_{\alpha=2}^{4}|Du^\alpha|\right)^q + b(x)+1\right) \\
\leq& c\eta^{p^*}|p^*\eta^{-1}(u^1-k)D\eta|^q|D\hat{u}|^q + b(x)+1\big) \\
\leq& c\eta^{p^*-q}(p^*)^q\left(\frac{u^1-k}{R-\rho}\right)^q|D\hat{u}|^q + c\eta^{p^*}(b(x)+1),
\end{aligned}$$

where c is a constant depending on c_3, q. From (3.12) we know $q < \frac{pp^*}{p+p^*} < p < Q$, this allows us to use the Young inequality with exponents $\frac{Q}{q}$ and $\frac{Q}{Q-q}$ in order to derive, for almost every $x \in A^1_{k,R} \cap \{\eta > 0\}$,

$$\left(\frac{u^1-k}{R-\rho}\right)^q |D\hat{u}|^q \leq c\left(\left(\frac{u^1-k}{R-\rho}\right)^Q + |D\hat{u}|^{\frac{Qq}{Q-q}}\right).$$

We have thus derived that for any $Q \in (p, p^*)$,

$$\begin{aligned}&\int_{A^1_{k,R}\cap\{\eta>0\}} \eta^{p^*} \sum_{\beta=1}^{3} G_\beta\big(x, \big(\operatorname{adj}_2 A\big)^\beta\big)\mathrm{d}x \\ \le& c\int_{A^1_{k,R}} \Big\{\Big(\frac{u^1-k}{R-\rho}\Big)^Q + |D\hat{u}|^{\frac{Qq}{Q-q}} + b(x) + 1\Big\}\mathrm{d}x,\end{aligned} \tag{3.29}$$

with c a constant depending upon c_2, c_3, p, Q, q.

By the growth condition (3.7) for H_γ and Lemma 3.1 (ii) we get

$$\begin{aligned}&\eta^{p^*}\sum_{\gamma=2}^{4} H_\gamma\left(x, (\operatorname{adj}_3 A)^\gamma\right) \\ \le& c\eta^{p^*}\sum_{\gamma=2}^{4}\left(|(\operatorname{adj}_3 A)^\gamma|^r + c(x) + 1\right) \\ \le& c\eta^{p^*}\left(\left|p^*\eta^{-1}(u^1-k)|D\eta|\sum_{\beta=4}^{6}|(\operatorname{adj}_2 A)^\beta|\right|^r + c(x) + 1\right) \\ \le& c\eta^{p^*-r}(p^*)^r\Big(\frac{u^1-k}{R-\rho}\Big)^r\left(\sum_{\beta=4}^{6}|(\operatorname{adj}_2 D\hat{u})^\beta|\right)^r + c\eta^{p^*}(c(x)+1).\end{aligned}$$

From (3.12) we know that $r < \frac{qp^*}{q+p^*} < q < p < Q$, which allows us to use Young inequality with exponents $\frac{Q}{r}$ and $\frac{Q}{Q-r}$ to get almost everywhere in $A^1_{k,R}\cap\{\eta>0\}$,

$$\Big(\frac{u^1-k}{R-\rho}\Big)^r\left(\sum_{\beta=4}^{6}|(\operatorname{adj}_2 D\hat{u})^\beta|\right)^r \le c\left(\Big(\frac{u^1-k}{R-\rho}\Big)^Q + \left(\sum_{\beta=4}^{6}|(\operatorname{adj}_2 D\hat{u})^\beta|\right)^{\frac{Qr}{Q-r}}\right).$$

The above inequality, together with the fact $\eta^{p^*-r}\le 1$ implies

$$\begin{aligned}&\int_{A^1_{k,R}\cap\{\eta>0\}} \eta^{p^*}\sum_{\gamma=2}^{4} H_\gamma\left(x, (\operatorname{adj}_3 A)^\gamma\right)\mathrm{d}x \\ \le& c\int_{A^1_{k,R}}\left(\Big(\frac{u^1-k}{R-\rho}\Big)^Q + \left(\sum_{\beta=4}^{6}|(\operatorname{adj}_2 D\hat{u})^\beta|\right)^{\frac{Qr}{Q-r}} + c(x) + 1\right)\mathrm{d}x,\end{aligned} \tag{3.30}$$

with c a constant depending upon c_2, c_3, p, Q, r.

By (3.8) and Lemma 3.1 (iii),

$$\begin{aligned} & \eta^{p^*} I(x, \det A) \\ \le\ & c\eta^{p^*}(|\det A|^s + d(x) + 1) \\ \le\ & c\eta^{p^*}\big((|p^*\eta^{-1}(u^1 - k)|D\eta||(\mathrm{adj}_3 A)^1|)^s + d(x) + 1\big) \\ \le\ & c\eta^{p^*-s}(p^*)^s \left(\frac{u^1 - k}{R - \rho}\right)^s |(\mathrm{adj}_3 D\hat{u})^1|^s + c\eta^{p^*}(d(x) + 1). \end{aligned}$$

From (3.12) we know that $s < \frac{rp^*}{r+p^*} < r < p < Q$, which allows us to use Young inequality with exponents $\frac{Q}{s}$ and $\frac{Q}{Q-s}$ and get almost everywhere in $A^1_{k,R} \cap \{\eta > 0\}$,

$$\left(\frac{u^1 - k}{R - \rho}\right)^s |(\mathrm{adj}_3 Du)^1|^s \le c\left(\left(\frac{u^1 - k}{R - \rho}\right)^Q + |(\mathrm{adj}_3 D\hat{u})^1|^{\frac{Qs}{Q-s}}\right).$$

Therefore

$$\begin{aligned} & \int_{A^1_{k,R}\cap\{\eta>0\}} \eta^{p^*} I(x, \det A)\mathrm{d}x \\ \le & c\int_{A^1_{k,R}} \Big\{\Big(\frac{u^1 - k}{R - \rho}\Big)^Q + |(\mathrm{adj}_3 D\hat{u})^1|^{\frac{Qs}{Q-s}} + d(x) + 1\Big\}\mathrm{d}x, \end{aligned} \tag{3.31}$$

with c a constant depending upon c_2, c_3, p, Q, s.

Substituting the estimates (3.28), (3.29), (3.30) and (3.31) into (3.27) we arrive at

$$\begin{aligned} & \int_{A^1_{k,\rho}} |Du^1|^p \mathrm{d}x \le \int_{A^1_{k,R}\cap\{\eta>0\}} \eta^{p^*} |Du^1|^p \mathrm{d}x \\ \le & c\int_{A^1_{k,R}} \left[\left(\frac{u^1 - k}{R - \rho}\right)^Q + |D\hat{u}|^{\frac{Qq}{Q-q}} + \left(\sum_{\beta=4}^{6} |(\mathrm{adj}_2 D\hat{u})^\beta|\right)^{\frac{Qr}{Q-r}} \right. \\ & \left. + |(\mathrm{adj}_3 D\hat{u})^1|^{\frac{Qs}{Q-s}} + \omega(x)\right] \mathrm{d}x + c|A^1_{k,R}|^{1-\frac{1}{\sigma}}, \end{aligned} \tag{3.32}$$

where c is a constant depending upon $c_1, c_2, c_3, p, q, r, s, Q$ and

$$\omega(x) = a(x) + b(x) + c(x) + d(x) + 1.$$

Considering the facts $Du \in L^p_{loc}(\Omega)$, $\mathrm{adj}_2 Du \in L^q_{loc}(\Omega, \mathbb{R}^{6\times 6})$ and $\mathrm{adj}_3 Du \in L^r_{loc}(\Omega, \mathbb{R}^{4\times 4})$ (which can be derived by using (3.5)-(3.7) and $f(x, Du) \in L^1_{loc}(\Omega)$), one can use Hölder inequality to derive

$$
\begin{aligned}
&\int_{A^1_{k,R}}\left[|D\hat{u}|^{\frac{Qq}{Q-q}}+\left(\sum_{\beta=4}^{6}|(\mathrm{adj}_2 D\hat{u})^\beta|\right)^{\frac{Qr}{Q-r}}+|(\mathrm{adj}_3 D\hat{u})^1|^{\frac{Qs}{Q-s}}+\omega(x)\right]\mathrm{d}x\\
\le&\left(\int_{B_{R_0}}(D\hat{u})^p\,\mathrm{d}x\right)^{\frac{qQ}{(Q-q)p}}\left|A^1_{k,R}\right|^{1-\frac{qQ}{p(Q-q)}}\\
&+\left(\int_{B_{R_0}}\left(\sum_{\beta=4}^{6}|(\mathrm{adj}_2 D\hat{u})^\beta|\right)^q\mathrm{d}x\right)^{\frac{Qr}{q(Q-r)}}\left|A^1_{k,R}\right|^{1-\frac{Qr}{q(Q-r)}} \qquad (3.33)\\
&+\left(\int_{B_{R_0}}\left(|(\mathrm{adj}_3 D\hat{u})^1|\right)^r\mathrm{d}x\right)^{\frac{Qs}{r(Q-s)}}\left|A^1_{k,R}\right|^{1-\frac{Qs}{r(Q-s)}}\\
&+\|\omega(x)\|_{L^\sigma(B_{R_0})}|A^1_{k,R}|^{1-\frac{1}{\sigma}}\\
\le& c\left(\left|A^1_{k,R}\right|^{1-\frac{qQ}{p(Q-q)}}+\left|A^1_{k,R}\right|^{1-\frac{Qr}{q(Q-r)}}+\left|A^1_{k,R}\right|^{1-\frac{Qs}{r(Q-s)}}+|A^1_{k,R}|^{1-\frac{1}{\sigma}}\right).
\end{aligned}
$$

We stress that, the above c is a constant not only depending on $c_1, c_2, c_3, p, q, r, s, Q$, but also on $\|D\hat{u}\|_{L^p(B_{R_0})}$ and $\|\omega\|_{L^\sigma(B_{R_0})}$, but independent of k. Let us take θ as in (3.22) and using the fact $\left|A^1_{k,R}\right| \le |B_{R_0}| < 1$, then (3.32) and (3.33) merge into

$$\int_{A^1_{k,\rho}}|Du^1|^p dx \le c\int_{A^1_{k,R}}\Big(\frac{u^1-k}{R-\rho}\Big)^Q dx + c|A^1_{k,R}|^\theta,$$

as desired. □

Lemma 3.4. *Let f be as in (3.2) satisfying the growth conditions (3.5)-(3.8). Let $u \in W^{1,1}_{loc}(\Omega,\mathbb{R}^4)$ be a local minimizer of $\mathcal{F}$. Let $1 < p \le 4$ and q, r, s satisfy the relations (3.12). Let $B_{R_0}(x_0) \Subset \Omega$ with $|B_{R_0}| < 1$, For $k \in \mathbb{R}$, denote*

$$B^1_{k,t} := \{x \in B_t(x_0) : u^1(x) < k\}, \quad 0 < t \le R_0. \qquad (3.34)$$

Then there exists $c > 0$, independent of k, such that for every $0 < \rho < R \le R_0$ and every $p < Q < p^*$,

$$\int_{B^1_{k,\rho}}|Du^1|^p dx \le \int_{B^1_{k,R}}\Big(\frac{k-u^1}{R-\rho}\Big)^Q dx + c|B^1_{k,R}|^\theta, \qquad (3.35)$$

where θ is defined by (3.22).

Proof. Denote $B^1_{k,t}$ as in (3.34). To prove that u^1 satisfies (3.35), we notice that $-u$ is a local minimizer of $\int_\Omega \tilde{f}(x, Dv(x))\mathrm{d}x$ where

$$\tilde{f}(\xi) := \sum_{\alpha=1}^{4}F_\alpha(x,-\xi^\alpha)+\sum_{\beta=1}^{6}G_\beta(x,(\mathrm{adj}_2\xi)^\beta)+\sum_{\gamma=1}^{4}H_\gamma(x,(-\mathrm{adj}_3\xi)^\gamma)+H(x,\det\xi).$$

If we denote

$$\tilde{F}_\alpha(x,\lambda) := F_\alpha(x,-\lambda), \quad \tilde{H}_\gamma(x,\lambda) = H_\gamma(x,-\lambda), \quad \lambda \in \mathbb{R}^4,$$

then the functions $\tilde{F}_\alpha$, $\tilde{H}_\gamma$ are convex and satisfy (3.5) and (3.7) respectively. The function $\tilde{f}$ satisfies the assumptions of Lemma 3.3, therefore we have obtained that $v^1 = -u^1$ satisfies (3.21), which is equivalent to u^1 satisfies (3.35). □

Remark 3.1. *The symmetric structure of the energy density $f(x,\xi)$ in (3.2) allows us to obtain analogous statements to Lemmas 3.3 and 3.4 also for u^2, u^3 and u^4.*

With the above lemmas in hands, we are now ready to prove Theorem 3.1.

Proof. Under the condition (3.9), one can choose $Q \in (p, p^*)$ sufficiently close to p^* such that

$$\frac{p}{p^*} < 1 - \frac{qQ}{p(Q-q)} < 1 - \frac{qp^*}{p(p^*-q)}, \tag{3.36}$$

$$\frac{p}{p^*} < 1 - \frac{rQ}{q(Q-r)} < 1 - \frac{rp^*}{q(p^*-r)} \tag{3.37}$$

$$\frac{p}{p^*} < 1 - \frac{sQ}{r(Q-s)} < 1 - \frac{sp^*}{r(p^*-s)}. \tag{3.38}$$

These inequalities can always be satisfied since the right hand side inequalities of the above three relations are all equivalent to $Q < p^*$. (3.36), (3.37), (3.38) together with $\sigma > \frac{4}{p}$ imply

$$\frac{p}{p^*} < 1 - \max\left\{\frac{qQ}{p(Q-q)}, \frac{rQ}{q(Q-r)}, \frac{sQ}{r(Q-s)}, \frac{1}{\sigma}\right\}.$$

We recall the definition of θ in (3.22) and we have

$$\theta > \frac{p}{p^*} = 1 - \frac{p}{4}.$$

The result (3.21) in Lemma 3.3 tells us that u^1 satisfies (2.1) and then belongs to GDG_p^+. Similarly, the result (3.35) in Lemma 3.4 tells us that u^1 satisfies (2.2) and so $u^1 \in GDG_p^-$. The result of Theorem 3.1 follows from Theorem 2.1. □

As a final remark of this section, we mention that we only considered just now integral functionals in 4-dimensional Euclidean spaces. The same idea can be used to deal with integral functionals

$$\mathcal{F}(u,\Omega) = \int_\Omega f(x, Du(x))dx$$

in n-dimensional Euclidean spaces, $n \geq 3$, with the integrand has splitting form

$$f(x,\xi) = \sum_{\alpha=1}^{n} F_\alpha(x,\xi^\alpha) + \sum_{\beta=1}^{n(n-1)/2} G_\beta(x, (\mathrm{adj}_2\xi)^\beta) + \cdots + H(x, \det \xi). \tag{3.39}$$

If one assume that there exist exponents

$$1 < p_1 \leq n, \ 1 < p_2, p_3, \cdots, p_{n-1}, \ 1 \leq p_n,$$

constants $c_1, c_3 > 0$, $c_2 \geq 0$, and nonnegative functions

$$a_i \in L^\sigma_{loc}(\Omega),\ \sigma > \frac{n}{p_1},\ i = 1, 2, \cdots, n,$$

such that

$$\begin{aligned}
&c_1|\lambda|^{p_1} - c_2 \leq F_\alpha(x,\lambda) \leq c_3(|\lambda|^{p_1}+1) + a_1(x),\ \forall \lambda \in \mathbb{R}^n,\ \alpha = 1,2,\cdots,n,\\
&c_1|\eta|^{p_2} - c_2 \leq G_\beta(x,\eta) \leq c_3(|\eta|^{p_2}+1) + a_2(x),\ \forall \eta \in \mathbb{R}^{n(n-1)/2},\ \beta = 1,2,\cdots,n(n-1)/2,\\
&\cdots\\
&0 \leq H(x,t) \leq c_3(|t|^{p_n}+1) + a_n(x),\ \forall t \in \mathbb{R},
\end{aligned} \tag{3.40}$$

then the following theorem is immediate.

Theorem 3.2. *Let f has splitting structure as in (3.39), and satisfy the growth conditions (3.40). Let $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ be a local minimizer of $\mathcal{F}$ in the sense that $f(x, Du(x)) \in L^1_{loc}(\Omega)$ and*

$$\mathcal{F}(u, \mathrm{supp}\varphi) \leq \mathcal{F}(u + \varphi, \mathrm{supp}\varphi).$$

Let

$$1 \leq p_n < p_{n-1} < \cdots < p_2 < p_1 \leq n.$$

Assume

$$\frac{p_1}{p_1^*} < 1 - \max\left\{\frac{p_2 p_1^*}{p_1(p_1^* - p_2)}, \frac{p_3 p_1^*}{p_2(p_1^* - p_3)}, \cdots, \frac{1}{\sigma}\right\},$$

where $p_1^* = \frac{np_1}{n-p_1}$, if $p_1 < n$, and, if $p_1 = n$, then p_1^* is any $\nu > n$.

Then all local minimizers $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ of $\mathcal{F}$ are locally bounded and locally Hölder continuous.

As a corollary of Theorem 3.2, we take $n = 3$, $p_1 = p$, $p_2 = q$ and $p_3 = r$, we then have

Corollary 3.1. *Let f be as in (3.10). We assume that there exist exponents $1 \leq r < q < p \leq 3$, constants $c_1, c_3 > 0$, $c_2 \geq 0$ and functions*

$$0 \leq a, b, c \in L^\sigma(\Omega),\ \sigma > \frac{3}{p},$$

such that

$$\begin{aligned}
&c_1|\lambda|^p - c_2 \leq F_\alpha(x,\lambda) \leq c_3(|\lambda|^p+1) + a(x),\ \forall \lambda \in \mathbb{R}^3,\ \alpha = 1,2,3,\\
&c_1|\lambda|^q - c_2 \leq G_\alpha(x,\lambda) \leq c_3(|\lambda|^q+1) + b(x),\ \forall \lambda \in \mathbb{R}^3,\ \alpha = 1,2,3,\\
&0 \leq H(x,t) \leq c_3(|t|^r+1),\ \forall t \in \mathbb{R}.
\end{aligned}$$

Assume

$$\frac{p}{p^*} < \min\left\{1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)}, 1 - \frac{1}{\sigma}\right\},$$

where $p^* = \frac{3p}{3-p}$, if $p < 3$, and, if $p = 3$, then p^* is any $\nu > 3$.

Then all the local minimizers $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^3)$ of $\mathcal{F}$ are locally bounded and locally Hölder continuous.

Let us compare the above corollary with Theorem 2.1 in [7]. We find that the assumptions are the same, but the result of Theorem 2.1 in [7] is locally bounded, while the results of the above lemma are, except for that u is locally bounded, u is locally Hölder continuous.

We remark that the proof's strategy of this section is the same of the paper [7]. We remark also that the paper [3] slightly improves [7].

Some noteworthy developments should be mentioned here, especially the recent papers [27–29] by T.Granucci (sometimes with M. Randolfi). In [27] the author derived everywhere Hölder continuity of vectorial local minimizers of integral functionals with rank one integrands of the form

$$\int_\Omega \sum_{\alpha=1}^{n} f_\alpha(x, u^\alpha, \nabla u^\alpha) + G(x, u, \nabla u) dx.$$

In [28] the author studied the regularity of the local minima of a class of vectorial functionals of the form

$$J(u, \Omega) = \int_\Omega \sum_{\alpha=1}^{n} |\nabla u^\alpha(x)|^p + G(x, u(x), \nabla u(x)) dx. \tag{3.41}$$

In [29] the authors established boundedness and Hölder continuity results for vectorial local minimizers of integral functionals of the form (3.41). It should be noted that, in these papers, the Hölder continuity results are obtained by proving that each component stays in a suitable De Giorgi class.

4 A degenerate linear elliptic equation.

In this section we shall give another application of Theorem 2.1 to regularity property of weak solutions of a linear elliptic equation with the form

$$-\text{div}(a(x)Du) = -\text{div}F, \quad \text{in } \Omega, \tag{4.1}$$

here Ω stands for an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, and

$$F \in (L^2_{loc}(\Omega))^n, \tag{4.2}$$

$$0 < a(x) \leq \bar{\beta} < +\infty, \quad \text{a.e. } \Omega. \tag{4.3}$$

Definition 4.1. *A function $u \in W^{1,2}_{loc}(\Omega)$ is called a solution to (4.1) if*

$$\int_\Omega a(x) Du D\varphi \mathrm{d}x = \int_\Omega F D\varphi \mathrm{d}x \tag{4.4}$$

for all $\varphi \in W^{1,2}(\Omega)$ with $supp\varphi \Subset \Omega$.

Remark 4.1. *Note that, the differential operator $-div(a(x)Du)$ is not coercive on $W^{1,2}_0(\Omega)$, even if it is well defined between $W^{1,2}_0(\Omega)$ and its dual. To see that it is sufficient to consider the sequence*

$$u_m(x) = |x|^{\frac{m(1-n)}{2(m+1)}} - 1, \quad m = 1, 2, \cdots,$$

and

$$a(x) = |x|$$

defined in $\Omega = B_1(0)$*. It satisfies*

$$\int_{B_1(0)} |Du_m|^2 \mathrm{d}x = \int_{B_1(0)} \frac{1}{|x|^{\frac{m(n+1)+2}{m+1}}} \mathrm{d}x = +\infty, \quad \textit{for } m \geq n-2,$$

thus

$$\|u_m\|_{W_0^{1,2}(\Omega)} = +\infty, \quad \textit{for } m \geq n-2, \tag{4.5}$$

and, at the same time

$$\int_{B_1(0)} a(x)|Du_m|^2 \mathrm{d}x = \int_{B_1(0)} \frac{1}{|x|^{\frac{nm+1}{m+1}}} \mathrm{d}x < +\infty, \quad \textit{for all } m = 1, 2, \cdots. \tag{4.6}$$

(4.5) together with (4.6) implies

$$\frac{1}{\|u_m\|_{W_0^{1,2}(\Omega)}} \int_{B_1(0)} a(x)|Du_m|^2 \mathrm{d}x = 0, \quad \textit{as } m \to +\infty.$$

For some developments related to elliptic partial differential equations, we refer to the classical books [35] by Ladyženskaya and Ural'tseva, [26] by Gilbarg and Trudinger, [32] by Heinonen, Kilpeläinen and Martio and [1] by Boccardo and Croce. In [33], the authors considered

$$-\mathrm{div}\mathcal{A}(x, Du(x)) = 0$$

with

$$\alpha(x)|\xi|^p \leq \mathcal{A}(x,\xi)\xi \leq \beta(x)|\xi|^p,$$

and obtained among other things that weak solutions of (4.1) are weakly monotone in the sense of [39] if $\beta \in L^\infty(\Omega)$ and $\alpha > 0$ a.e. Ω. Related results can be found in [36].

We remark that, under (4.2) and (4.3), the integrals in (4.4) are well-defined. We remark also that the main feature of (4.1) lies in the fact that the function $a(x)$ can be sufficiently close to 0, even if it is always positive. If $a(x)$ is uncontrollable near 0, then one can not expect any regularity property of solutions to (4.1), but for $a(x)$ satisfying

$$\frac{1}{a(x)} \in L^r_{loc}(\Omega), \quad r > n(n+1), \tag{4.7}$$

and $F(x)$ satisfying

$$F \in (L^m_{loc}(\Omega))^n, \ m > \frac{n(n+1)}{n-1}, \tag{4.8}$$

one can prove that any solution $u \in W^{1,2}_{loc}(\Omega)$ of (4.1) is locally bounded and locally Hölder continuous, as the following theorem shows.

Theorem 4.1. *Suppose (4.3), (4.7) and (4.8), then any weak solution* $u \in W^{1,2}_{loc}(\Omega)$ *of (4.1) is locally bounded and locally Hölder continuous.*

Proof. Let $B_{R_0}(x_0) \Subset \Omega$. Let s,t be such that $0 < s < t < R_0$. Consider a cut-off function $\eta \in C_0^\infty(B_t)$ satisfying the following assumptions

$$0 \leq \eta \leq 1, \ \eta \equiv 1 \text{ in } B_s(x_0) \ \text{ and } \ |D\eta| \leq \frac{2}{t-s}.$$

Let for $k \in \mathbb{R}$,

$$A_k = \{x \in \Omega : u(x) > k\} \quad \text{and} \quad A_{k,t} = A_k \cap B_t.$$

Take $\varphi = \eta^2(u-k)_+$ as a test function in the weak formulation (4.4). Note that

$$D\varphi = \big(\eta^2 Du + 2\eta D\eta(u-k)\big) \cdot 1_{A_{k,t}},$$

where $1_E(x)$ is the characteristic function of the set E, that is, $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ otherwise. Thus

$$\begin{aligned} &\int_{A_{k,t}} \eta^2 a(x)|Du|^2 \mathrm{d}x + 2\int_{A_{k,t}} \eta a(x) Du D\eta(u-k)\mathrm{d}x \\ =& \int_{A_{k,t}} \eta^2 F Du \mathrm{d}x + 2\int_{A_{k,t}} \eta F D\eta(u-k)\mathrm{d}x. \end{aligned}$$

(4.3) allows us to derive

$$\begin{aligned} &\int_{A_{k,t}} a(x)|\eta Du|^2 \mathrm{d}x \\ \le\ & 2\bar{\beta}\int_{A_{k,t}} \eta|Du||D\eta|(u-k)\mathrm{d}x + \int_{A_{k,t}} \eta^2|F||Du|\mathrm{d}x \\ &+2\int_{A_{k,t}} \eta|F||D\eta|(u-k)\mathrm{d}x. \end{aligned} \tag{4.9}$$

Let

$$\frac{2n}{n+1} < \delta < \frac{2n+1}{n+1} < 2$$

be fixed. Using Young inequality with exponents $\frac{2}{\delta}$ and $\frac{2}{2-\delta}$ and (4.9) one has

$$\begin{aligned} &\int_{A_{k,t}} |\eta Du|^\delta \mathrm{d}x \\ =& \int_{A_{k,t}} |\eta Du|^\delta a(x)^{\frac{\delta}{2}} \Big(\frac{1}{a(x)}\Big)^{\frac{\delta}{2}} \mathrm{d}x \\ \le\ & c\Big(\int_{A_{k,t}} a(x)|\eta Du|^2 \mathrm{d}x + \int_{A_{k,t}} \Big(\frac{1}{a(x)}\Big)^{\frac{\delta}{2-\delta}} \mathrm{d}x\Big) \\ \le\ & c\Big(\int_{A_{k,t}} \eta|Du||D\eta|(u-k)\mathrm{d}x + \int_{A_{k,t}} \eta^2|F||Du|\mathrm{d}x \\ &+ \int_{A_{k,t}} \eta|F||D\eta|(u-k)\mathrm{d}x + \int_{A_{k,t}} \Big(\frac{1}{\alpha(x)}\Big)^{\frac{\delta}{2-\delta}} \mathrm{d}x\Big) \\ :=\ & c(I_1 + I_2 + I_3 + I_4). \end{aligned} \tag{4.10}$$

Our next goal is to estimate each term in the right hand side of (4.10). We use Young inequality with exponent δ and δ' in order to estimate

$$\begin{aligned} I_1 =& \int_{A_{k,t}} \eta|Du||D\eta|(u-k)\mathrm{d}x \\ \le& \varepsilon \int_{A_{k,t}} |\eta Du|^{\delta}\mathrm{d}x + c(\varepsilon)\int_{A_{k,t}} (|D\eta|(u-k))^{\delta'}\mathrm{d}x \\ \le& \varepsilon \int_{A_{k,t}} |\eta Du|^{\delta}\mathrm{d}x + c(\varepsilon)\int_{A_{k,t}} \left(\frac{u-k}{t-s}\right)^{\delta'}\mathrm{d}x. \end{aligned} \tag{4.11}$$

We notice that

$$\frac{2n}{n+1} < \delta \Longleftrightarrow \delta' < \frac{2n}{n-1} \tag{4.12}$$

which together with fact $m > \frac{n(n+1)}{n-1}$ in (4.8) implies $\delta' < m$.

We use Young inequality again with exponents δ and δ' and Hölder inequality with exponents $\frac{m}{\delta'}$ and $\frac{m}{m-\delta'}$ to derive

$$\begin{aligned} I_2 =& \int_{A_{k,t}} \eta^2|F||Du|\mathrm{d}x \\ \le& \varepsilon \int_{A_{k,t}} |\eta Du|^{\delta}\mathrm{d}x + c(\varepsilon)\int_{A_{k,t}} (\eta|F|)^{\delta'}\mathrm{d}x \\ \le& \varepsilon \int_{A_{k,t}} |\eta Du|^{\delta}\mathrm{d}x + c(\varepsilon)\int_{A_{k,t}} |F|^{\delta'}\mathrm{d}x \\ \le& \varepsilon \int_{A_{k,t}} |\eta Du|^{\delta}\mathrm{d}x + c(\varepsilon)\|F\|^{\delta'}_{L^m(B_{R_0})}|A_{k,t}|^{1-\frac{\delta'}{m}}. \end{aligned} \tag{4.13}$$

Similarly, we use Young inequality again with exponents δ and δ' and Hölder inequality with exponents $\frac{m}{\delta}$ and $\frac{m}{m-\delta}$ (note that $\delta < 2 \le n < m$),

$$\begin{aligned} I_3 =& \int_{A_{k,t}} \eta|F||D\eta|(u-k)\mathrm{d}x \\ \le& c\int_{A_{k,t}} (|D\eta|(u-k))^{\delta'}\mathrm{d}x + c\int_{A_{k,t}} (|F|\eta)^{\delta}\mathrm{d}x \\ \le& c\int_{A_{k,t}} \left(\frac{u-k}{t-s}\right)^{\delta'}\mathrm{d}x + c\int_{A_{k,t}} |F|^{\delta}\mathrm{d}x \\ \le& c\int_{A_{k,t}} \left(\frac{u-k}{t-s}\right)^{\delta'}\mathrm{d}x + c\|F\|^{\delta}_{L^m(B_{R_0})}|A_{k,t}|^{1-\frac{\delta}{m}}. \end{aligned} \tag{4.14}$$

The assumption $\delta < \frac{2n+1}{n+1}$ together with $r > n(n+1)$ and $n \ge 2$ implies $r(2-\delta) > \delta$.

Using Hölder inequality with exponents $\frac{r(2-\delta)}{\delta}$ and $\frac{r(2-\delta)}{r(2-\delta)-\delta}$,

$$\begin{aligned} I_4 =& \int_{A_{k,t}} \Big(\frac{1}{\alpha(x)}\Big)^{\frac{\delta}{2-\delta}} \mathrm{d}x \\ \le & c\Big(\int_{A_{k,t}} \Big(\frac{1}{\alpha(x)}\Big)^{r} \mathrm{d}x\Big)^{\frac{\delta}{(2-\delta)r}} \Big(\int_{A_{k,t}} 1 \mathrm{d}x\Big)^{1-\frac{\delta}{(2-\delta)r}} \\ \le & c\Big\|\frac{1}{\alpha(x)}\Big\|^{\frac{\delta}{2-\delta}}_{L^r(B_{R_0})} |A_{k,t}|^{1-\frac{\delta}{(2-\delta)r}}. \end{aligned} \tag{4.15}$$

Substituting (4.11), (4.13), (4.14) and (4.15) into (4.10), one has

$$\begin{aligned} & \int_{A_{k,t}} |\eta Du|^{\delta} \mathrm{d}x \\ \le & 2\varepsilon \int_{A_{k,t}} |\eta Du|^{\delta} \mathrm{d}x + c\int_{A_{k,t}} \Big(\frac{u-k}{t-s}\Big)^{\delta'} \mathrm{d}x + c\|F\|^{\delta'}_{L^m(B_{R_0})} |A_{k,t}|^{1-\frac{\delta'}{m}} \\ & + c\|F\|^{\delta}_{L^m(B_{R_0})} |A_{k,t}|^{1-\frac{\delta}{m}} + \Big\|\frac{1}{\alpha(x)}\Big\|^{\frac{\delta}{2-\delta}}_{L^r(B_{R_0})} |A_{k,t}|^{1-\frac{\delta}{(2-\delta)r}}. \end{aligned}$$

Since $\delta < 2 < \delta'$, then $1-\frac{\delta'}{m} < 1-\frac{\delta}{m}$, thus

$$|A_k|^{1-\frac{\delta}{m}} = |A_k|^{\frac{\delta'-\delta}{m}} |A_k|^{1-\frac{\delta'}{m}} \le |\Omega|^{\frac{\delta'-\delta}{m}} |A_k|^{1-\frac{\delta'}{m}}.$$

We use this fact, and take $\varepsilon = \frac{1}{4}$, then

$$\begin{aligned} & \int_{A_{k,s}} |Du|^{\delta} \mathrm{d}x \le \int_{A_{k,t}} |\eta Du|^{\delta} \mathrm{d}x \\ \le & c\left(\int_{A_{k,t}} \Big(\frac{u-k}{t-s}\Big)^{\delta'} \mathrm{d}x + |A_{k,t}|^{1-\frac{\delta'}{m}} + |A_{k,t}|^{1-\frac{\delta}{m}} + |A_{k,t}|^{1-\frac{\delta}{(2-\delta)r}}\right) \\ \le & c\left(\int_{A_{k,t}} \Big(\frac{u-k}{t-s}\Big)^{\delta'} \mathrm{d}x + |A_{k,t}|^{1-\frac{\delta'}{m}} + |A_{k,t}|^{1-\frac{\delta}{(2-\delta)r}}\right). \end{aligned}$$

Let $\tilde{\theta} = \min\left\{1-\frac{\delta'}{m}, 1-\frac{\delta}{(2-\delta)r}\right\}$. (4.7) and (4.8) together with the condition on δ in (4.12) ensure $\tilde{\theta} > 1-\frac{\delta}{n}$. The condition $\delta > \frac{2n}{n+1}$ implies $Q = \delta' < \delta^*$. The above inequality has the form

$$\int_{A_{k,s}} |Du|^{\delta} \mathrm{d}x \le c\left[\int_{A_{k,t}} \left(\frac{u-k}{t-s}\right)^{Q} + |A_{k,s}|^{1-\frac{\delta}{n}+\varepsilon}\right]$$

with $s < t$, $\delta < Q < \delta^*$ and $\varepsilon > 0$, thus $u \in GDG^{+}_{\delta}$.

In order to prove that $u \in GDG^{-}_{\delta}$ it suffices that $\tilde{u} = -u \in GDG^{+}_{\delta}$. We note that $\tilde{u}$ satisfies

$$-\mathrm{div}(a(x)D\tilde{u}) = -\mathrm{div}\tilde{F},$$

with $\tilde{F} = -F$ and

$$\tilde{F} \in (L^m_{loc}(\Omega))^n.$$

Reasoning as above, one can derive that $-u \in GDG_\delta^+$, which together with $u \in GDG_\delta^+$ implies $u \in GDG_\delta$. Theorem 2.1 gives the result. □

5 A nonlinear elliptic equation.

This section gives an application of Theorem 2.1 to regularity property of weak solutions of nonlinear elliptic equations of the form

$$-\mathrm{div}\mathcal{A}(x, u(x), Du(x)) = f(x), \quad \text{in } \Omega, \tag{5.1}$$

here Ω stands for an open bounded subset of $\mathbb{R}^n$, $n \geq 2$. We assume that there exists positive constants α, β, and

$$1 < p \leq n, \ \frac{np}{n+1} < \bar{p} \leq p,$$

such that for all $\xi \in \mathbb{R}^n$,

$$\mathcal{A}(x, s, \xi)\xi \geq \alpha|\xi|^{\bar{p}} \tag{5.2}$$

and

$$|\mathcal{A}(x, s, \xi)| \leq \beta|\xi|^{p-1}. \tag{5.3}$$

As far as the function f in (5.1) is concerned, we assume

$$f \in L^m_{loc}(\Omega), \ m > \frac{n}{p-1}. \tag{5.4}$$

Definition 5.1. *A function $u \in W^{1,p}_{loc}(\Omega)$ is said to be a (weak) solution to (5.1) if*

$$\int_\Omega \mathcal{A}(x, u, Du)D\varphi \mathrm{d}x = \int_\Omega f\varphi \mathrm{d}x \tag{5.5}$$

for all $\varphi \in W^{1,p}(\Omega)$ with compact support.

We note that in (5.3), the growth of $|\mathcal{A}(x, s, \xi)|$ is controlled by $|\xi|^{p-1}$, while in (5.2), $\mathcal{A}(x, s, \xi)$ is coercive with $|\xi|^{\bar{p}}$, where $\bar{p}$ may be smaller than p but greater than $\frac{np}{n+1}$. If $\bar{p} = p$ then we are in the natural growth conditions. But if $\bar{p} < p$ then we are in the sense of general growth conditions or nonstandard conditions. For some results related to integral functionals with nonstandard growth conditions, we refer to Marcellini [40–46], Bogelein-Duzaar-Marcellini [2] and Esposito-Leonetti-Mingione [12] and the references therein.

We prove the following

Theorem 5.1. *Assume (5.2), (5.3) and (5.4). Then all weak solutions $u \in W^{1,p}_{loc}(\Omega)$ to (5.1) are locally bounded and locally Hölder continuous.*

Proof. For $B_{R_1} \Subset \Omega$, $0 < s < t \leq R_1 \leq 1$, let us take $\eta \in C_0^\infty(B_t)$ as follows

$$0 \leq \eta \leq 1, \ \eta \equiv 1 \text{ in } B_s \text{ and } |D\eta| \leq \frac{2}{t-s}.$$

If we take

$$\varphi = \eta(u-k)_+ \in W_0^{1,p}(B_t),$$

then

$$D\varphi = [D\eta(u-k) + \eta Du]\, 1_{A_{k,t}},$$

where $A_{k,t} = \{u > k\} \cap B_t$. We use the above φ as a test function in the weak formulation (5.5) and we have

$$\int_{A_{k,t}} \mathcal{A}(x,u,Du)\,[D\eta(u-k) + \eta Du]\,\mathrm{d}x = \int_{A_{k,t}} f\eta(u-k)\mathrm{d}x.$$

We use the above inequality, (5.2), (5.3) and we derive

$$\begin{aligned} & \int_{A_{k,s}} |Du|^{\bar p}\mathrm{d}x \le \int_{A_{k,t}} \eta|Du|^{\bar p}\mathrm{d}x \\ \le\ & \int_{A_{k,t}} \mathcal{A}(x,u,Du)\eta Du\mathrm{d}x \\ =\ & \int_{A_{k,t}} \mathcal{A}(x,u,Du)D\eta(u-k)\mathrm{d}x + \int_{A_{k,t}} f\eta(u-k)\mathrm{d}x \\ :=\ & I_1 + I_2. \end{aligned} \tag{5.6}$$

Using (5.3) and Young inequality with exponents $\frac{\bar p}{p-1}$ and $\frac{\bar p}{\bar p - p + 1}$, $|I_1|$ can be estimated as

$$\begin{aligned} |I_1| &\le \beta \int_{A_{k,t}} |Du|^{p-1}|D\eta|(u-k)\mathrm{d}x \\ &\le \varepsilon \int_{A_{k,t}} |Du|^{\bar p}\mathrm{d}x + c(\varepsilon)\int_{A_{k,t}} \left(\frac{u-k}{t-s}\right)^{\frac{\bar p}{\bar p - p + 1}} \mathrm{d}x. \end{aligned} \tag{5.7}$$

$|I_2|$ can be estimated by using Young inequality with exponents $\frac{\bar p}{p-1}$ and $\frac{\bar p}{\bar p - p + 1}$ again as

$$\begin{aligned} |I_2| &\le \int_{A_{k,t}} |f|\eta(u-k)\mathrm{d}x \\ &\le c\left(\int_{A_{k,t}} |f|^{\frac{\bar p}{p-1}}\mathrm{d}x + \int_{A_{k,t}} (u-k)^{\frac{\bar p}{\bar p - p + 1}}\,\mathrm{d}x\right) \\ &\le c\int_{A_{k,t}} |f|^{\frac{\bar p}{p-1}}\mathrm{d}x + c\int_{A_{k,t}} \left(\frac{u-k}{t-s}\right)^{\frac{\bar p}{\bar p - p + 1}} \mathrm{d}x, \end{aligned} \tag{5.8}$$

where we have used the fact $s < t < R_1 \le 1$, which implies $1 \le \frac{1}{t-s}$.

Substituting (5.7) and (5.8) into (5.6) one has

$$\begin{aligned} & \int_{A_{k,s}} |Du|^{\bar p}\mathrm{d}x \\ \le\ & \varepsilon \int_{A_{k,t}} |Du|^{\bar p}\mathrm{d}x + (c(\varepsilon)+1)\int_{A_{k,t}} \left(\frac{u-k}{t-s}\right)^{\frac{\bar p}{\bar p - p + 1}} \mathrm{d}x + c\int_{A_{k,t}} |f|^{\frac{\bar p}{p-1}}\mathrm{d}x. \end{aligned} \tag{5.9}$$

Now we want to eliminate the first term in the right hand side including $|Du|^{\bar p}$. We use a very useful lemma for real functions which can be found, for example, in [25] Lemma 6.1 or [23] Lemma 3.1.

Lemma 5.1. *Let $f(\tau)$ be a non-negative bounded function defined for $0 \le R_0 \le \tau \le R_1$. Suppose that for $R_0 \le s < t \le R_1$ we have*

$$f(s) \le A(t-s)^{-\alpha} + B + \varepsilon f(t),$$

where $A, B, \alpha, \varepsilon$ are non-negative constants, and $\varepsilon < 1$. Then there exists a constant c, depending only on α and ε, such that for every ρ, R, $R_0 \le \rho < R \le R_1$ we have

$$f(\rho) \le c[A(R-\rho)^{-\alpha} + B].$$

We let ρ, R fixed with $\rho < R \le R_1$. Thus, from (5.9) we deduce for every s, t such that $\rho \le s < t \le R$, it results

$$\begin{aligned} &\int_{A_{k,s}} |Du|^{\bar p} \mathrm{d}x \\ \le \quad & \varepsilon \int_{A_{k,t}} |Du|^{\bar p} \mathrm{d}x + (c(\varepsilon)+1) \int_{A_{k,R}} \left(\frac{u-k}{t-s} \right)^{\frac{\bar p}{\bar p - p + 1}} \mathrm{d}x + \int_{A_{k,R}} |f|^{\frac{\bar p}{p-1}} \mathrm{d}x. \end{aligned} \tag{5.10}$$

Applying Lemma 5.1 in (5.10) we conclude that

$$\int_{A_{k,\rho}} |Du|^{\bar p} \mathrm{d}x \le c \int_{A_{k,R}} \left(\frac{u-k}{R-\rho} \right)^{\frac{\bar p}{\bar p - p + 1}} \mathrm{d}x + \int_{A_{k,R}} |f|^{\frac{\bar p}{p-1}} \mathrm{d}x. \tag{5.11}$$

Since $\frac{np}{n+1} < \bar p$, then $Q = \frac{\bar p}{\bar p - p + 1} < \bar p^*$. The condition $m > \frac{n}{p-1}$ in (5.4) and $\bar p \le n$ imply $\frac{m(p-1)}{\bar p} > 1$. Hölder inequality with exponents $\frac{m(p-1)}{\bar p}$ and $\frac{m(p-1)}{m(p-1)-\bar p}$ yields

$$\int_{A_{k,R}} |f|^{\frac{\bar p}{p-1}} \mathrm{d}x \le \left(\int_\Omega |f|^m \mathrm{d}x \right)^{\frac{\bar p}{m(p-1)}} |A_{k,R}|^{1-\frac{\bar p}{m(p-1)}}.$$

We use the condition (5.4) for f again and we derive $1 - \frac{\bar p}{m(p-1)} > 1 - \frac{\bar p}{n}$. Therefore (5.11) has the form

$$\int_{A_{k,\rho}} |Du|^{\bar p} \mathrm{d}x \le c \int_{A_{k,t}} \left(\frac{u-k}{R-\rho} \right)^{Q} \mathrm{d}x + c_* |A_{k,R}|^{1-\frac{\bar p}{n}+\varepsilon}, \tag{5.12}$$

with $Q = \frac{\bar p}{\bar p - p + 1} < \bar p^*$ and $\varepsilon = \frac{\bar p}{n} - \frac{\bar p}{m(p-1)} > 0$. (5.12) tells us that $u \in GDG^+_{\bar p}$. Similarly, one can derive that $u \in GDG^-_{\bar p}$. The result of Theorem 5.1 follows from Theorem 2.1. □

Final remarks:

As a first remark, we note that the local boundedness in this section has been studied in a more general framework in [8].

As a second remark, we notice from Sections 3,4,5 that Theorem 2.1 can be used to deal with regularity properties with non-standard growth conditions, while the classical De Giorgi class can be used to deal with standard growth conditions.

As a third remark, we note in Definition 2.1 that, a function $u \in W^{1,p}_{loc}(\Omega)$ belongs to the generalized De Giorgi class $GDG^+_p(\Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$ if it satisfies (2.1). For a vector valued function $u = (u^1, \cdots, u^N) \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$, $N \ge 1$, one can give a similar definition as follows.

Definition 5.2. *We say that* $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$*,* $N \geq 1$*, belongs to the generalized De Giorgi class* $GDG^+_p(N, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$*,* $1 < p \leq n$*,* $p \leq Q < p^*$*,* y *and* $\varepsilon > 0$*,* y_* *and* $\kappa_0 \geq 0$*, if*

$$\sum_{\alpha=1}^{N} \int_{A^\alpha_{k,\sigma\rho}} |Du^\alpha|^2 \, \mathrm{d}x \leq y \sum_{\alpha=1}^{N} \int_{A^\alpha_{k,\rho}} \left(\frac{u^\alpha - k}{(1-\sigma)\rho} \right)^Q \mathrm{d}x + y_* \sum_{\alpha=1}^{N} |A^\alpha_{k,\rho}|^{1-\frac{p}{n}+\varepsilon},$$

for all $k \geq \kappa_0$*,* $\sigma \in (0,1)$*, and all pairs of concentric cubes* $Q_{\sigma\rho}(x_0) \subset Q_\rho(x_0) \subset \Omega$ *centered at* x_0*, where*

$$A^\alpha_{k,\rho} = \{x \in \Omega : u^\alpha > k\} \cap Q_\rho.$$

We can define similarly $GDG^-_p(N, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$ to be the class of functions u such that $-u \in GDG^+_p(N, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$. More explicitly, they are the vectors in $W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ such that for all $k \leq -\kappa_0$, all $\sigma \in (0,1)$, and all pairs of concentric cubes $Q_{\sigma\rho} \subset Q_\rho \subset \Omega$,

$$\sum_{\alpha=1}^{N} \int_{B^\alpha_{k,\sigma\rho}} |Du^\alpha|^2 \, \mathrm{d}x \leq y \sum_{\alpha=1}^{N} \int_{B^\alpha_{k,\rho}} \left(\frac{k - u^\alpha}{(1-\sigma)\rho} \right)^Q \mathrm{d}x + y_* \sum_{\alpha=1}^{N} |B^\alpha_{k,\rho}|^{1-\frac{p}{n}+\varepsilon},$$

where

$$B^\alpha_{k,\rho} = \{x \in \Omega : u^\alpha < k\} \cap Q_\rho.$$

We indicate by $GDG_p(N, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$ the class of functions belonging both to $GDG^+_p\ (N, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$ and $GDG^-_p(N, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$:

$$\begin{aligned} & GDG_p(N, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0) \\ = \; & GDG^+_p(N, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0) \cap GDG^-_p(N, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0). \end{aligned}$$

It is clear that

$$GDG_p(1, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0) = GDG_p(\Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$$

and

$$GDG_p(1, \Omega, p, p, y, y_*, \varepsilon, \kappa_0) = DG_p(\Omega, p, y, y_*, \varepsilon, \kappa_0).$$

We ask the following question: Is the following statement correct ?

Let $u \in GDG_p(N, \Omega, p, Q, y, y_*, \varepsilon, \kappa_0)$ *for* $1 < p \leq n$*,* $N \geq 1$ *and some* $Q \in [p, p^*)$*, then* u *is locally bounded and locally Hölder continuous in* Ω*.*

We leave it for a future work.

Declaration: We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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