Information Sciences 561 (2021) 352–370

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Optimization-based group decision making using interval-valued intuitionistic fuzzy preference relations

Zhiming Zhang [a], Shyi-Ming Chen [b,*]

[a] Hebei Key Laboratory of Machine Learning and Computational Intelligence, College of Mathematics and Information Science, Hebei University, Baoding 071002, China
[b] Department of Computer Science and Information Engineering, National Taiwan University of Science and Technology, Taipei, Taiwan

ARTICLE INFO

Article history:
Received 3 October 2020
Received in revised form 10 December 2020
Accepted 18 December 2020
Available online 29 December 2020

Keywords:
GDM
IVIFPR
Consistency
Consensus
Optimization model

ABSTRACT

In this paper, we propose an optimization-based group decision making (GDM) method using interval-valued intuitionistic fuzzy preference relations (IVIFPRs). First, the concept of consistency of intuitionistic fuzzy preference relations (IFPRs) is provided. Moreover, the consistency index for IFPRs is presented. Subsequently, by splitting an IVIFPR into two IFPRs, an additive consistency is proposed for IVIFPRs. Afterward, a consensus index is presented for GDM. When the consistency and the consensus do not achieve the requirement, we propose several models to reach the requirement. Furthermore, individual IVIFPRs are integrated into a collective IVIFPR. After that, a procedure is offered to obtain the interval-valued intuitionistic fuzzy (IVIF) priority weights of the alternatives. Moreover, a new GDM method with IVIFPRs is offered. Finally, some application examples are offered. The proposed GDM method can conquer the shortcomings of the existing GDM methods. It offers us a useful way for GDM in the IVIF context.

1. Introduction

Group decision making (GDM) usually needs the decision makers (DMs) to rank the alternatives, where preference relations are good techniques to collect the wisdom of a group of DMs. Until now, many researchers have developed different kinds of preference relations [3,7,22,25,26,29,35,37]. For example, Xu [41] introduced intuitionistic fuzzy preference relations (IFPRs) based on intuitionistic fuzzy sets (IFSs) [1]. In recent years, some multiattribute decision making methods [4,8,9,20,24,27,49] have been proposed based on IFSs. In [27], Tang and Meng presented a GDM method based on linguistic intuitionistic fuzzy Hamacher aggregation operators. Up to now, many researchers [11,14,34,39,42,43,45,46] have studied IFPRs. For example, Wang [34] defined the concept of additive consistency for IFPRs. Based on which, Jin *et al.* [11] presented a novel multiplicative consistency. Liao and Xu [14] reviewed the definition of consistency in intuitionistic fuzzy environments. In practical applications, IFPRs provided by DMs are not always consistent or consensual. Xu *et al.* [46] improved the consistency and the consensus. In [2], Atanassov and Gargov proposed the theory of interval-valued intuitionistic fuzzy sets (IVIFSs). In recent years, some multiattribute decision making methods [19,21,48] have been proposed based on IVIFSs. Xu and Chen [44] presented the definition of interval-valued intuitionistic fuzzy preference relations (IVIFPRs), which can properly denote the uncertain preferred and the uncertain non-preferred judgements of DMs. In [17], Meng *et al.* presented

* Corresponding author.
E-mail address: smchen@mail.ntust.edu.tw (S.-M. Chen).

https://doi.org/10.1016/j.ins.2020.12.047

a GDM method based on interval-valued intuitionistic multiplicative preference relations. In [50], Zhang *et al.* presented a GDM method based on IVIFPRs for selecting cloud computing vendors for small and medium-sized enterprises. After the work of Xu and Chen [44], several scholars [15,33] presented some algorithms to repair the inconsistency of IVIFPRs. From the review of the researches for IFPRs and IVIFPRs [6,13,15,18,28,30–34,38,44], we can see that there are some drawbacks:

(1) Consistency plays a significant role in various kinds of preference relations. For different kinds of preference relations, a key point is to check and improve their consistency. In practical GDM with IVIFPRs, it is common that an IVIFPR is unacceptable consistent. In this case, we need to repair the IVIFPR until it reaches the consistency threshold. Besides the consistency, the consensus in a GDM process also plays a necessary part. The consensus reaching process can avoid the situation of non-consensus among DMs. However, the methods presented in [6,30,38,44] don't improve the consistency and the consensus. Consensus checking and reaching is overlooked in the methods presented in [15,31,33]. The method presented in [32] lacks the consideration of judging and improving the consistency.
(2) The methods presented in [15,18,28,32,33] may need several iteration times and they only improve one IVIFPR each time, which needs more computational efforts. Furthermore, the iterative algorithms presented in [15,18,28,32,33] cannot retain DMs' preferences as much as possible under the condition that the original IVIFPRs need to be adjusted into modified IVIFPRs having an acceptable consistency and/or an acceptable consensus.
(3) The methods presented in [6,13,34] ignored DMs' risk attitudes. Moreover, the methods presented in [6,13,34] derived alternatives' priority weights by constructing some programming models, which only consider DMs' satisfaction degrees and ignored their dissatisfaction degrees.

This paper further carries out the GDM with IVIFPRs and offers a new GDM method, where

(1) A definition of additive consistent of IFPRs is provided. Then, we propose a consistency index and a novel definition of acceptable consistency of IFPRs. Moreover, an acceptable additive consistency of IVIFPRs is presented and its basic properties are offered. The methods presented in [18,28] applied the complete consistency, whereas the proposed method utilizes the acceptable consistency. It should be noted that when we let the consistency threshold be equal to zero, we can derive decision-making methods with IVIFPRs following the complete consistency analysis.
(2) When the consistency and consensus of IVIFPRs are unacceptable, a model for reaching the requirements of the consistency and consensus is presented to ensure the minimum loss of the original DMs' information and the maximum consistency and consensus levels. Compared with the methods presented in [15,18,28,31–33] which offered iterative methods for improving the consistency or the consensus, the proposed consistency and consensus repairing method can improve several IVIFPRs simultaneously, where no iterative processes are required. Hence, our method can easily be performed and time-saving, which aims to preserve the DMs' most original information.
(3) The methods presented in [18,28] cannot ensure that the DMs' most original information is preserved in the process of improving the consistency and the consensus. The proposed method has the advantage that it can achieve three goals simultaneously, i.e., (1) reaching the acceptable consistency and consensus requirements, (2) retaining the largest amount of the original information, and (3) maximizing the consistency and the consensus levels of the modified IVIFPRs.
(4) For GDM with IVIFPRs, a model which involves the group consensus is established to calculate the DMs' weights.
(5) We derive the priority weights using a model which involves DMs' satisfaction and dissatisfaction simultaneously.
(6) A new GDM method with IVIFPRs is proposed. Some application examples are used to show the advantages of our method over the methods presented in [5,15,30,31,33,34,43]. The proposed GDM method can conquer the shortcomings of the existing GDM methods.

This paper is organized as follows. Section 2 reviews some basic concepts to help the readers to understand the contents of the following sections. Section 3 provides the consistency analyses of IFPRs and IVIFPRs. Section 4 constructs the models of reaching the additive consistency and consensus. Section 5 presents a novel GDM method using IVIFPRs. Section 6 offers some application examples. The conclusions are provided in Section 7.

2. Preliminaries

In this section, we present some related concepts. Let $\Lambda = \{1, 2, \cdots, m\}$ and let $\Gamma = \{1, 2, \cdots, n\}$.

2.1. IFPRs

Definition 2.1 *[1].* An IFS T in the universe of discourse course Y is represented by $T = \{\langle y, \xi_T(y), \eta_T(y)\rangle | y \in Y\}$, where $\xi_T : Y \to [0,1]$ and $\eta_T : Y \to [0,1]$ are the membership function and the non-membership function of the IFS T, respectively. For each $y \in Y$, $0 \leqslant \xi_T(y) + \eta_T(y) \leqslant 1$.

For convenience, $\beta = (\xi_\beta, \eta_\beta)$ is called the intuitionistic fuzzy value (IFV) [47], where $\xi_\beta \in [0,1]$, $\eta_\beta \in [0,1]$, and $\xi_\beta + \eta_\beta \leqslant 1$.

Definition 2.2 *[41].* An IFPR B on a finite set $R = \{r_1, r_2, \cdots, r_n\}$ of alternatives is represented as a matrix $B = (\beta_{ij})_{n\times n}$, where $\beta_{ij} = \left(\xi_{ij}, \eta_{ij}\right)$ is an IFV, which satisfies the following conditions:

$$\begin{cases} \xi_{ij}, \eta_{ij} \in [0,1], \ i,j \in \Gamma, \\ \xi_{ij} = \eta_{ji}, \ \eta_{ij} = \xi_{ji}, \ i,j \in \Gamma, \\ \xi_{ii} = \eta_{ii} = 0.5, \ i \in \Gamma, \\ \xi_{ij} + \eta_{ij} \leqslant 1, \ i,j \in \Gamma. \end{cases} \tag{1}$$

Definition 2.3 *[34].* Let $z = (z_1, z_2, \cdots, z_n)^T$ be an intuitionistic fuzzy (IF) priority weight vector, where $z_i = \left(z_i^{\xi}, z_i^{\eta}\right)$ is an IFV with $z_i^{\xi}, z_i^{\eta} \in [0,1]$ and $z_i^{\xi} + z_i^{\eta} \leqslant 1$. The vector z is called normalized iff

$$\sum_{j=1, j\neq i}^{n} z_j^{\xi} \leqslant z_i^{\eta}, \ z_i^{\xi} + n - 2 \geqslant \sum_{j=1, j\neq i}^{n} z_j^{\eta}, \ \forall i \in \Gamma \tag{2}$$

Definition 2.4 *[34].* An IFPR $B = (\beta_{ij})_{n\times n}$ is additive consistent, where $\beta_{ij} = \left(\xi_{ij}, \eta_{ij}\right)$ is an IFV, if

$$\xi_{ij} + \xi_{jk} + \xi_{ki} = \xi_{ik} + \xi_{kj} + \xi_{ji}, \ \forall i,j,k \in \Gamma \tag{3}$$

2.2. IVIFPRs

Definition 2.5 *[2].* An IVIFS $\widetilde{T}$ in Y is represented by $\widetilde{T} = \left\{\left\langle y, \widetilde{\xi}_{\widetilde{T}}(y), \widetilde{\eta}_{\widetilde{T}}(y)\right\rangle \middle| y \in Y\right\}$, where $\widetilde{\xi}_{\widetilde{T}}(y) \subseteq [0,1]$ and $\widetilde{\eta}_{\widetilde{T}}(y) \subseteq [0,1]$ are the interval-value membership degree and the interval-value non-membership degree, respectively, and $\sup\widetilde{\xi}_{\widetilde{T}}(y) + \sup\widetilde{\eta}_{\widetilde{T}}(y) \leqslant 1$.

For convenience, an interval-value intuitionistic fuzzy value (IVIFV) [44] is denoted by $\widetilde{\beta} = \left(\widetilde{\xi}_{\widetilde{\beta}}, \widetilde{\eta}_{\widetilde{\beta}}\right) = \left(\left[\underline{\xi}_{\widetilde{\beta}}, \bar{\xi}_{\widetilde{\beta}}\right], \left[\underline{\eta}_{\widetilde{\beta}}, \bar{\eta}_{\widetilde{\beta}}\right]\right)$, where $\underline{\xi}_{\widetilde{\beta}}, \bar{\xi}_{\widetilde{\beta}}, \underline{\eta}_{\widetilde{\beta}}, \bar{\eta}_{\widetilde{\beta}} \in [0,1]$ and $\bar{\xi}_{\widetilde{\beta}} + \bar{\eta}_{\widetilde{\beta}} \leqslant 1$.

Definition 2.6 *[44].* Let $\widetilde{\beta} = \left(\left[\underline{\xi}_{\widetilde{\beta}}, \bar{\xi}_{\widetilde{\beta}}\right], \left[\underline{\eta}_{\widetilde{\beta}}, \bar{\eta}_{\widetilde{\beta}}\right]\right)$ be an IVIFV. Then, $s\left(\widetilde{\beta}\right) = \frac{1}{2}\left(\underline{\xi}_{\widetilde{\beta}} - \underline{\eta}_{\widetilde{\beta}} + \bar{\xi}_{\widetilde{\beta}} - \bar{\eta}_{\widetilde{\beta}}\right)$ is called the score value of $\widetilde{\beta}$ and $\gamma\left(\widetilde{\beta}\right) = \frac{1}{2}\left(\underline{\xi}_{\widetilde{\beta}} + \underline{\eta}_{\widetilde{\beta}} + \bar{\xi}_{\widetilde{\beta}} + \bar{\eta}_{\widetilde{\beta}}\right)$ is called the accuracy value of $\widetilde{\beta}$. Let $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ be any two IVIFVs. Then,

(1) If $s\left(\widetilde{\beta}_1\right) < s\left(\widetilde{\beta}_2\right)$, then $\widetilde{\beta}_1 < \widetilde{\beta}_2$.
(2) If $s\left(\widetilde{\beta}_1\right) = s\left(\widetilde{\beta}_2\right)$, then
  (i) If $\gamma\left(\widetilde{\beta}_1\right) = \gamma\left(\widetilde{\beta}_2\right)$, then $\widetilde{\beta}_1 = \widetilde{\beta}_2$.
  (ii) If $\gamma\left(\widetilde{\beta}_1\right) < \gamma\left(\widetilde{\beta}_2\right)$, then $\widetilde{\beta}_1 < \widetilde{\beta}_2$.

Definition 2.7 *[44].* Let $\widetilde{B} = \left(\widetilde{\beta}_{ij}\right)_{n\times n}$ be an IVIFPR on R, where $\widetilde{\beta}_{ij} = \left(\widetilde{\xi}_{ij}, \widetilde{\eta}_{ij}\right)$ is an IVIFV ($i,j \in \Gamma$), $\widetilde{\xi}_{ij}$ denotes an interval of degrees in $[0,1]$ that alternative r_i is better than alternative r_j, $\widetilde{\eta}_{ij}$ denotes an interval of degrees in $[0,1]$ that alternative r_i is worse than alternative r_j, which satisfy

$$\begin{cases} \widetilde{\xi}_{ij} = \left[\underline{\xi}_{ij}, \bar{\xi}_{ij}\right] \subseteq [0,1], \ i,j \in \Gamma, \\ \widetilde{\eta}_{ij} = \left[\underline{\eta}_{ij}, \bar{\eta}_{ij}\right] \subseteq [0,1], \ i,j \in \Gamma, \\ \widetilde{\xi}_{ij} = \widetilde{\eta}_{ji}, \ \widetilde{\eta}_{ij} = \widetilde{\xi}_{ji}, \ i,j \in \Gamma, \\ \widetilde{\xi}_{ii} = \widetilde{\eta}_{ii} = [0.5, 0.5], \ i \in \Gamma, \\ 0 \leqslant \bar{\xi}_{ij} + \bar{\eta}_{ij} \leqslant 1, \ i,j \in \Gamma. \end{cases} \tag{4}$$

3. Consistency analysis for IFPRs and IVIFPRs

In this section, we present the consistency analysis for IFPRs and IVIFPRs. For an IFPR $B = (\beta_{ij})_{n\times n}$, where $\beta_{ij} = \left(\xi_{ij}, \eta_{ij}\right)$ is an IFV, the following statements are equivalent:

(i) $\xi_{ij} + \xi_{jk} + \xi_{ki} = \xi_{ik} + \xi_{kj} + \xi_{ji}$, for all $i, j, k \in \Gamma$,
(ii) $\xi_{ij} + \xi_{jk} + \eta_{ik} = \eta_{ij} + \eta_{jk} + \xi_{ik}$, for all $i, j, k \in \Gamma$, where $i < j < k$.

The consistency for IFPRs is redefined as follows.

Definition 3.1. An IFPR $B = \left(\left(\xi_{ij}, \eta_{ij}\right)\right)_{n\times n}$ is called additive consistent if it satisfies

$$\xi_{ij} + \xi_{jk} + \eta_{ik} = \eta_{ij} + \eta_{jk} + \xi_{ik},\ \forall i, j, k \in \Gamma,\ i < j < k. \tag{5}$$

To quantify the consistency for an IFPR more accurately, we propose an additive consistency index based on Eq. (5), shown as follows.

Definition 3.2. The consistency index $f(B)$ of an IFPR $B = \left(\left(\xi_{ij}, \eta_{ij}\right)\right)_{n\times n}$ is defined as

$$f(B) = p \times \sum_{1\leqslant i<j<k\leqslant n} \left|\xi_{ij} + \xi_{jk} + \eta_{ik} - \eta_{ij} - \eta_{jk} - \xi_{ik}\right| \tag{6}$$

where $p = \frac{2}{n(n-1)(n-2)}$ and $0 \leqslant f(B) \leqslant 1$.

Definition 3.3. Let B be an IFPR. For a given threshold $\theta \in [0, 1]$, if $f(B) \leqslant \theta$, then B is said to be acceptable additive consistent.

Theorem 3.1. Let $B^h = \left(\beta^h_{ij}\right)_{n\times n}$ $(h = 1, 2, \cdots, m)$ be m IFPRs with $\beta^h_{ij} = \left(\xi^h_{ij}, \eta^h_{ij}\right)$ and let $B^c = \left(\beta^c_{ij}\right)_{n\times n}$ be their collective IFPR, where $\beta^c_{ij} = \left(\xi^c_{ij}, \eta^c_{ij}\right) = \left(\sum_{h=1}^m \lambda_h \xi^h_{ij}, \sum_{h=1}^m \lambda_h \eta^h_{ij}\right)$, $\sum_{h=1}^m \lambda_h = 1$ and $0 \leqslant \lambda_h \leqslant 1$. If all IFPRs have the additive consistency (acceptable additive consistency), then the collective IFPR has the additive consistency (acceptable additive consistency).

Proof. Because B^h is acceptable consistent, we have $f\left(B^h\right) \leqslant \theta$, where θ is a predefined consistency threshold. Let B^c be the collective IFPR obtained by the aggregation of these m IFPRs B^h $(h = 1, 2, \cdots, m)$. Then, we have

$$\begin{aligned}
f(B^c) &= p \times \sum_{1\leqslant i<j<k\leqslant n} \left|\xi^c_{ij} + \xi^c_{jk} + \eta^c_{ik} - \eta^c_{ij} - \eta^c_{jk} - \xi^c_{ik}\right| \\
&= p \times \sum_{1\leqslant i<j<k\leqslant n} \left|\sum_{h=1}^m \lambda_h \xi^h_{ij} + \sum_{h=1}^m \lambda_h \xi^h_{jk} + \sum_{h=1}^m \lambda_h \eta^h_{ik} - \sum_{h=1}^m \lambda_h \eta^h_{ij} - \sum_{h=1}^m \lambda_h \eta^h_{jk} - \sum_{h=1}^m \lambda_h \xi^h_{ik}\right| \\
&= p \times \sum_{1\leqslant i<j<k\leqslant n} \left|\sum_{h=1}^m \lambda_h \left(\xi^h_{ij} + \xi^h_{jk} + \eta^h_{ik} - \eta^h_{ij} - \eta^h_{jk} - \xi^h_{ik}\right)\right| \leqslant p \times \sum_{1\leqslant i<j<k\leqslant n} \sum_{h=1}^m \lambda_h \left|\xi^h_{ij} + \xi^h_{jk} + \eta^h_{ik} - \eta^h_{ij} - \eta^h_{jk} - \xi^h_{ik}\right)\right| \\
&= \sum_{h=1}^m \lambda_h f\left(B^h\right) \leqslant \sum_{h=1}^m \lambda_h \theta = \theta.
\end{aligned}$$

Thus, B^c is acceptable additive consistent. Moreover, let $\theta = 0$, we can conclude that B^c has additive consistency if all B^h $(h \in \Lambda)$ have additive consistency. The proof is finished. **Q. E. D.**

Theorem 3.2 *[34].* Let $B = (\beta_{ij})_{n\times n} = \left(\left(\xi_{ij}, \eta_{ij}\right)\right)_{n\times n}$ be an IFPR. If there exists a normalized IF priority weighting vector $z = (z_1, z_2, \cdots, z_n)^T$ with $z_i = \left(z^\xi_i, z^\eta_i\right)$, such that

$$\beta_{ij} = \left(\xi_{ij}, \eta_{ij}\right) = \begin{cases} (0.5, 0.5), & \text{if } i = j, \\ \left(0.5 z^\xi_i + 0.5 z^\eta_j, 0.5 z^\eta_i + 0.5 z^\xi_j\right), & \text{if } i \neq j, \end{cases} \tag{7}$$

then B is additive consistent. Since $\xi_{ij} = \eta_{ji}$ and $\eta_{ij} = \xi_{ji}$, Eq. (7) can easily be reduced into

$$\beta_{ij} = \left(\xi_{ij}, \eta_{ij}\right) = \begin{cases} (0.5, 0.5), & \text{if } i = j, \\ \left(0.5 z^\xi_i + 0.5 z^\eta_j, 0.5 z^\eta_i + 0.5 z^\xi_j\right), & \text{if } i < j. \end{cases} \tag{8}$$

3.1. Consistency analysis of IVIFPRs

For an IVIFPR $\widetilde{B}$, Wan *et al.* [33] defined its lower matrix $B^{-}=\left(\beta_{ij}^{-}\right)_{n\times n}$ and its upper matrix $B^{+}=\left(\beta_{ij}^{+}\right)_{n\times n}$, where

$$\beta_{ij}^{-}=\left(\xi_{ij}^{-},\eta_{ij}^{-}\right)=\begin{cases}\left(\underline{\xi}_{ij},\bar{\eta}_{ij}\right), & \text{if } i<j,\\ (0.5,0.5), & \text{if } i=j,\\ \left(\bar{\xi}_{ij},\underline{\eta}_{ij}\right), & \text{if } i>j,\end{cases} \tag{9}$$

$$\beta_{ij}^{+}=\left(\xi_{ij}^{+},\eta_{ij}^{+}\right)=\begin{cases}\left(\bar{\xi}_{ij},\underline{\eta}_{ij}\right), & \text{if } i<j,\\ (0.5,0.5), & \text{if } i=j,\\ \left(\underline{\xi}_{ij},\bar{\eta}_{ij}\right), & \text{if } i>j.\end{cases} \tag{10}$$

According to **Definition 2.2**, B^{-} and B^{+} are IFPRs [33]. If the lower matrix B^{-} and the upper matrix B^{+} of an IVIFPR $\widetilde{B}=\left(\widetilde{\beta}_{ij}\right)_{n\times n}=\left(\left(\widetilde{\xi}_{ij},\widetilde{\eta}_{ij}\right)\right)_{n\times n}$ are offered in advance, then

$$\widetilde{\beta}_{ij}=\left(\widetilde{\xi}_{ij},\widetilde{\eta}_{ij}\right)=\begin{cases}\left(\left[\xi_{ij}^{-},\xi_{ij}^{+}\right],\left[\eta_{ij}^{+},\eta_{ij}^{-}\right]\right), & \text{if } i<j,\\ ([0.5,0.5],[0.5,0.5]), & \text{if } i=j,\\ \left(\left[\xi_{ij}^{+},\xi_{ij}^{-}\right],\left[\eta_{ij}^{-},\eta_{ij}^{+}\right]\right), & \text{if } i>j.\end{cases} \tag{11}$$

For an IVIFPR $\widetilde{B}=\left(\widetilde{\beta}_{ij}\right)_{n\times n}=\left(\left(\widetilde{\xi}_{ij},\widetilde{\eta}_{ij}\right)\right)_{n\times n}$, its induced matrix $B(\lambda)=\left(\beta_{ij}(\lambda)\right)_{n\times n}$ is defined as follows:

$$\beta_{ij}(\lambda)=\left(\xi_{ij}(\lambda),\eta_{ij}(\lambda)\right)=\begin{cases}\left((1-\lambda)\underline{\xi}_{ij}+\lambda\bar{\xi}_{ij},(1-\lambda)\bar{\eta}_{ij}+\lambda\underline{\eta}_{ij}\right), & \text{if } i<j,\\ (0.5,0.5), & \text{if } i=j,\\ \left((1-\lambda)\bar{\xi}_{ij}+\lambda\underline{\xi}_{ij},(1-\lambda)\underline{\eta}_{ij}+\lambda\bar{\eta}_{ij}\right), & \text{if } i>j,\end{cases} \tag{12}$$

where $\lambda\in[0,1]$. Especially, it follows directly from Eq. (12) that $B(0)=B^{-}$ and $B(1)=B^{+}$. As B^{-} and B^{+} are IFPRs, it follows from **Definition 2.2** and Eq. (12) that $B(\lambda)$ is also an IFPR.

Definition 3.4. An IVIFPR $\widetilde{B}$ is additive consistent or acceptable additive consistent when $B(\lambda)$ is additive consistent or acceptable additive consistent for any $\lambda\in[0,1]$.

Theorem 3.3. An induced matrix $B(\lambda)$ is additive consistent or acceptable additive consistent for any $\lambda\in[0,1]$ iff both B^{-} and B^{+} are additive consistent or acceptable additive consistent.

Proof. Since $B^{-}=B(0)$ and $B^{+}=B(1)$, the necessity condition is obvious. We only have to prove the sufficiency. Note that $B(\lambda)$ is a combined IFPR composed of B^{-} and B^{+}. In virtue of **Theorem 3.1**, $B(\lambda)$ has the additive consistency or the acceptable additive consistency. This proof is completed. **Q. E. D.**

Theorem 3.4. is consistent or acceptable consistent if B^{-} and B^{+} are consistent or acceptable consistent.

Theorem 3.5. is additive consistent iff it satisfies

$$\underline{\xi}_{ij}+\underline{\xi}_{jk}+\bar{\eta}_{ik}=\bar{\eta}_{ij}+\bar{\eta}_{jk}+\underline{\xi}_{ik} \tag{13}$$

and

$$\bar{\xi}_{ij}+\bar{\xi}_{jk}+\underline{\eta}_{ik}=\underline{\eta}_{ij}+\underline{\eta}_{jk}+\bar{\xi}_{ik},\ \textit{for all}\ i<j<k \tag{14}$$

Theorem 3.6. is acceptable additive consistent iff it satisfies

$$\sum_{1\leqslant i<j<k\leqslant n}\left|\underline{\xi}_{ij}+\underline{\xi}_{jk}+\bar{\eta}_{ik}-\bar{\eta}_{ij}-\bar{\eta}_{jk}-\underline{\xi}_{ik}\right|\leqslant\frac{\theta}{p} \tag{15}$$

and

$$\sum_{1\leqslant i<j<k\leqslant n}\left|\bar{\xi}_{ij}+\bar{\xi}_{jk}+\underline{\eta}_{ik}-\underline{\eta}_{ij}-\underline{\eta}_{jk}-\bar{\xi}_{ik}\right|\leqslant\frac{\theta}{p} \tag{16}$$

where θ is a predefined consistency threshold.

Wan *et al.* [33] offered a formula for obtaining the IVIF priority vector $\tilde{z} = \left(\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_n\right)^T$ from an IVIFPR $\widetilde{B}$, where $\tilde{z}_i = \left(\left[\underline{z}_i^{\xi}, \bar{z}_i^{\xi}\right], \left[\underline{z}_i^{\eta}, \bar{z}_i^{\eta}\right]\right)$ $(i = 1, 2, \cdots, n)$. Let the IF priority vectors of B^- and B^+ be $z^- = \left(z_1^-, z_2^-, \cdots, z_n^-\right)^T$ and $z^+ = \left(z_1^+, z_2^+, \cdots, z_n^+\right)^T$, respectively, where $z_i^- = \left(z_i^{\xi-}, z_i^{\eta-}\right)$ and $z_i^+ = \left(z_i^{\xi+}, z_i^{\eta+}\right)$ $(i = 1, 2, \cdots, n)$. Then, it follows from Wan *et al.* [33] that

$$\begin{cases} \underline{z}_i^{\xi} = \min\{z_i^{\xi-}, z_i^{\xi+}\}, \\ \bar{z}_i^{\xi} = \max\{z_i^{\xi-}, z_i^{\xi+}\}, \\ \underline{z}_i^{\eta} = \min\{z_i^{\eta-}, z_i^{\eta+}\}, \\ \bar{z}_i^{\eta} = \max\{z_i^{\eta-}, z_i^{\eta+}\}. \end{cases} \tag{17}$$

4. Consistency and consensus-improving for IVIFPRs

This section studies GDM using IVIFPRs. First, a consensus index for IVIFPRs is proposed. Subsequently, several repairing models for IVIFPRs are proposed.

4.1. A consensus index for IVIFPRs

In the following, we reach an acceptable consistency and consensus for IVIFPRs, where the deviation measure is an indispensable tool to measure the deviation between IVIFPRs.

Definition 4.1. For any two IVIFPRs $\widetilde{B}^h = \left(\tilde{\beta}_{ij}^h\right)_{n\times n} = \left(\left(\tilde{\xi}_{ij}^h, \tilde{\eta}_{ij}^h\right)\right)_{n\times n} (h = 1, 2)$ with $\tilde{\xi}_{ij}^h = \left[\underline{\xi}_{ij}^h, \bar{\xi}_{ij}^h\right]$ and $\tilde{\eta}_{ij}^h = \left[\underline{\eta}_{ij}^h, \bar{\eta}_{ij}^h\right]$, where $h = 1, 2$, the distance between them is defined as follows:

$$d\left(\widetilde{B}^1, \widetilde{B}^2\right) = \frac{1}{2n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left|\underline{\xi}_{ij}^1 - \underline{\xi}_{ij}^2\right| + \left|\bar{\xi}_{ij}^1 - \bar{\xi}_{ij}^2\right| + \left|\underline{\eta}_{ij}^1 - \underline{\eta}_{ij}^2\right| + \left|\bar{\eta}_{ij}^1 - \bar{\eta}_{ij}^2\right|\right) \tag{18}$$

Definition 4.2. The similarity degree between IVIFPRs $\widetilde{B}^1$ and $\widetilde{B}^2$ is defined as

$$S\left(\widetilde{B}^1, \widetilde{B}^2\right) = 1 - d\left(\widetilde{B}^1, \widetilde{B}^2\right) \tag{19}$$

Definition 4.3. The proximity degree $P\left(\widetilde{B}^h\right)$ of expert e_h $(h = 1, 2, \cdots, m)$ is defined as

$$P\left(\widetilde{B}^h\right) = \frac{1}{m-1} \sum_{t=1, t\neq h}^{m} S\left(\widetilde{B}^t, \widetilde{B}^h\right) \tag{20}$$

Definition 4.4. The consensus index which quantifies the consensus among $\widetilde{B}^1, \widetilde{B}^2, \cdots, \widetilde{B}^m$ of a group of experts is defined as

$$g\left(\widetilde{B}^1, \widetilde{B}^2, \cdots, \widetilde{B}^m\right) = \frac{1}{m} \sum_{h=1}^{m} P\left(\widetilde{B}^h\right) \tag{21}$$

After plugging Eqs. (18), (19) and (20) into Eq. (21), Eq. (21) is converted into:

$$\begin{aligned} g\left(\widetilde{B}^1, \widetilde{B}^2, \cdots, \widetilde{B}^m\right) &= \frac{2}{m(m-1)} \sum_{h=1}^{m-1} \sum_{t=h+1}^{m} S\left(\widetilde{B}^h, \widetilde{B}^t\right) \\ &= 1 - \sum_{h=1}^{m-1} \sum_{t=h+1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} q\left(\left|\underline{\xi}_{ij}^h - \underline{\xi}_{ij}^t\right| + \left|\bar{\xi}_{ij}^h - \bar{\xi}_{ij}^t\right| + \left|\underline{\eta}_{ij}^h - \underline{\eta}_{ij}^t\right| + \left|\bar{\eta}_{ij}^h - \bar{\eta}_{ij}^t\right|\right), \end{aligned} \tag{22}$$

$$q = \frac{1}{n(n-1)m(m-1)}$$

Definition 4.5. When $g\left(\widetilde{B}^1,\widetilde{B}^2,\cdots,\widetilde{B}^m\right)\geqslant\eta$ for a given threshold $\eta\in[0,1]$, the group of experts has an acceptable consensus.

4.2. Consistency and consensus-improving models

In this paper, when the consistency and the consensus of the initial IVIFPRs $\widetilde{B}^h$ $(h=1,2,\cdots,m)$ do not satisfy the given requirements, some optimization models for deriving the modified IVIFPRs are established to achieve several goals: (1) It ensures the modified IVIFPRs to possess acceptable consistency and consensus, (2) It minimizes the deviation between the revised IVIFPRs and the initial IVIFPRs, and (3) It minimizes the consistency index of the revised IVIFPRs and maximizes the consensus index of the revised IVIFPRs.

Let $\widetilde{B}'^h=\left(\widetilde{\beta}'^h_{ij}\right)_{n\times n}=\left(\left(\widetilde{\xi}'^h_{ij},\widetilde{\eta}'^h_{ij}\right)\right)_{n\times n}$ be the modified IVIFPR of the IVIFPR $\widetilde{B}^h=\left(\widetilde{\beta}^h_{ij}\right)_{n\times n}=\left(\left(\widetilde{\xi}^h_{ij},\widetilde{\eta}^h_{ij}\right)\right)_{n\times n}$, where $\widetilde{\xi}'^h_{ij}=\left[\underline{\xi}'^h_{ij},\overline{\xi}'^h_{ij}\right]$, $\widetilde{\eta}'^h_{ij}=\left[\underline{\eta}'^h_{ij},\overline{\eta}'^h_{ij}\right]$ and $h=1,2,\cdots,m$. Let $B'^{h-}=\left(\beta'^{h-}_{ij}\right)_{n\times n}$ and $B'^{h+}=\left(\beta'^{h+}_{ij}\right)_{n\times n}$ be the lower matrix and the upper matrix of the modified IVIFPR $\widetilde{B}'^h$, respectively, where $h=1,2,\cdots,m$. To achieve the aforesaid goals, the following model is established:

$$\min\ \iota=\frac{1}{m}\sum_{h=1}^{m}d\left(\widetilde{B}'^h,\widetilde{B}^h\right)$$
$$\text{s.t.}\begin{cases}f\left(B'^{h-}\right)\leqslant\theta,\ h\in\Lambda,\\ f\left(B'^{h+}\right)\leqslant\theta,\ h\in\Lambda,\\ g\left(\widetilde{B}'^1,\widetilde{B}'^2,\cdots,\widetilde{B}'^m\right)\geqslant\eta,\\ \widetilde{B}'^h\text{ is an IVIFPR},\ h\in\Lambda.\end{cases}\tag{M-1}$$

After inserting Eqs. (15), (16), (18) and (22) into the model (M−1), we have

$$\min\iota=\frac{1}{2mn(n-1)}\sum_{h=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\underline{\xi}'^h_{ij}-\underline{\xi}^h_{ij}\right|+\left|\overline{\xi}'^h_{ij}-\overline{\xi}^h_{ij}\right|+\left|\underline{\eta}'^h_{ij}-\underline{\eta}^h_{ij}\right|+\left|\overline{\eta}'^h_{ij}-\overline{\eta}^h_{ij}\right|\right)$$
$$\text{s.t.}\begin{cases}\sum\limits_{1\leqslant i<j<k\leqslant n}\left|\xi'^{h-}_{ij}+\xi'^{h-}_{jk}+\eta'^{h-}_{ik}-\eta'^{h-}_{ij}-\eta'^{h-}_{jk}-\xi'^{h-}_{ik}\right|\leqslant\frac{\theta}{p},\ h\in\Lambda,\\ \sum\limits_{1\leqslant i<j<k\leqslant n}\left|\xi'^{h+}_{ij}+\xi'^{h+}_{jk}+\eta'^{h+}_{ik}-\eta'^{h+}_{ij}-\eta'^{h+}_{jk}-\xi'^{h+}_{ik}\right|\leqslant\frac{\theta}{p},\ h\in\Lambda,\\ 1-\sum\limits_{h=1}^{m-1}\sum\limits_{t=h+1}^{m}\sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n}q\left(\left|\underline{\xi}'^h_{ij}-\underline{\xi}'^t_{ij}\right|+\left|\overline{\xi}'^h_{ij}-\overline{\xi}'^t_{ij}\right|+\left|\underline{\eta}'^h_{ij}-\underline{\eta}'^t_{ij}\right|+\left|\overline{\eta}'^h_{ij}-\overline{\eta}'^t_{ij}\right|\right)\geqslant\eta,\\ 0\leqslant\underline{\xi}'^h_{ij},\overline{\xi}'^h_{ij},\underline{\eta}'^h_{ij},\overline{\eta}'^h_{ij}\leqslant1,\ i,j\in\Gamma,\ h\in\Lambda,\\ \underline{\xi}'^h_{ii}=\overline{\xi}'^h_{ii}=\underline{\eta}'^h_{ii}=\overline{\eta}'^h_{ii}=0.5,\ i\in\Gamma,\ h\in\Lambda,\\ \underline{\xi}'^h_{ij}\leqslant\overline{\xi}'^h_{ij},\ i,j\in\Gamma,\ h\in\Lambda,\\ \underline{\eta}'^h_{ij}\leqslant\overline{\eta}'^h_{ij},\ i,j\in\Gamma,\ h\in\Lambda,\\ \overline{\xi}'^h_{ij}+\overline{\eta}'^h_{ij}\leqslant1,\ i,j\in\Gamma,\ h\in\Lambda,\\ \underline{\xi}'^h_{ij}=\underline{\eta}'^h_{ji},\ i,j\in\Gamma,\ h\in\Lambda,\\ \overline{\xi}'^h_{ij}=\overline{\eta}'^h_{ji},\ i,j\in\Gamma,\ h\in\Lambda.\end{cases}\tag{M-2}$$

To yield the modified IVIFPR $\widetilde{B}'^h$, the model (M−2) is rewritten as

$$\min \mathrm{i} = \frac{1}{2mn(n-1)} \sum_{h=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left| \underline{\xi}'^{h}_{ij} - \underline{\xi}^{h}_{ij} \right| + \left| \bar{\xi}'^{h}_{ij} - \bar{\xi}^{h}_{ij} \right| + \left| \underline{\eta}'^{h}_{ij} - \underline{\eta}^{h}_{ij} \right| + \left| \bar{\eta}'^{h}_{ij} - \bar{\eta}^{h}_{ij} \right| \right)$$

$$\text{s.t.} \begin{cases} \sum\limits_{1 \leqslant i<j<k \leqslant n} \left| \underline{\xi}'^{h}_{ij} + \underline{\xi}'^{h}_{jk} + \bar{\eta}'^{h}_{ik} - \bar{\eta}'^{h}_{ij} - \bar{\eta}'^{h}_{jk} - \underline{\xi}'^{h}_{ik} \right| \leqslant \frac{\theta}{p},\ h \in \Lambda, \\ \sum\limits_{1 \leqslant i<j<k \leqslant n} \left| \bar{\xi}'^{h}_{ij} + \bar{\xi}'^{h}_{jk} + \underline{\eta}'^{h}_{ik} - \underline{\eta}'^{h}_{ij} - \underline{\eta}'^{h}_{jk} - \bar{\xi}'^{h}_{ik} \right| \leqslant \frac{\theta}{p},\ h \in \Lambda, \\ 1 - \sum\limits_{h=1}^{m-1} \sum\limits_{t=h+1}^{m} \sum\limits_{i=1}^{n-1} \sum\limits_{j=i+1}^{n} q\left(\left| \underline{\xi}'^{h}_{ij} - \underline{\xi}'^{t}_{ij} \right| + \left| \bar{\xi}'^{h}_{ij} - \bar{\xi}'^{t}_{ij} \right| + \left| \underline{\eta}'^{h}_{ij} - \underline{\eta}'^{t}_{ij} \right| + \left| \bar{\eta}'^{h}_{ij} - \bar{\eta}^{t}_{ij} \right| \right) \geqslant \eta, \\ 0 \leqslant \underline{\xi}'^{h}_{ij}, \bar{\xi}'^{h}_{ij}, \underline{\eta}'^{h}_{ij}, \bar{\eta}'^{h}_{ij} \leqslant 1,\ i,j \in \Gamma,\ i<j, \\ h \in \Lambda, \\ \underline{\xi}'^{h}_{ij} \leqslant \bar{\xi}'^{h}_{ij},\ i,j \in \Gamma,\ i<j,\ h \in \Lambda, \\ \underline{\eta}'^{h}_{ij} \leqslant \bar{\eta}'^{h}_{ij},\ i,j \in \Gamma,\ i<j,\ h \in \Lambda, \\ \bar{\xi}'^{h}_{ij} + \bar{\eta}'^{h}_{ij} \leqslant 1,\ i,j \in \Gamma,\ i<j,\ h \in \Lambda. \end{cases} \tag{M-3}$$

Furthermore, in the model (M−3), let

$$\varepsilon^{h}_{ij} = \left(\underline{\xi}'^{h}_{ij} - \underline{\xi}^{h}_{ij} \right) \vee 0,\ \delta^{h}_{ij} = \left(\underline{\xi}^{h}_{ij} - \underline{\xi}'^{h}_{ij} \right) \vee 0$$

$$\phi^{h}_{ij} = \left(\bar{\xi}'^{h}_{ij} - \bar{\xi}^{h}_{ij} \right) \vee 0,\ \varphi^{h}_{ij} = \left(\bar{\xi}^{h}_{ij} - \bar{\xi}'^{h}_{ij} \right) \vee 0$$

$$\gamma^{h}_{ij} = \left(\underline{\eta}'^{h}_{ij} - \underline{\eta}^{h}_{ij} \right) \vee 0,\ \kappa^{h}_{ij} = \left(\underline{\eta}^{h}_{ij} - \underline{\eta}'^{h}_{ij} \right) \vee 0$$

$$\alpha^{h}_{ij} = \left(\bar{\eta}'^{h}_{ij} - \bar{\eta}^{h}_{ij} \right) \vee 0, o^{h}_{ij} = \left(\bar{\eta}^{h}_{ij} - \bar{\eta}'^{h}_{ij} \right) \vee 0$$

$$\pi^{h}_{ijk} = \left(\underline{\xi}'^{h}_{ij} + \underline{\xi}'^{h}_{jk} + \bar{\eta}'^{h}_{ik} - \bar{\eta}'^{h}_{ij} - \bar{\eta}'^{h}_{jk} - \underline{\xi}'^{h}_{ik} \right) \vee 0$$

$$\varpi^{h}_{ijk} = \left(\bar{\eta}'^{h}_{ij} + \bar{\eta}'^{h}_{jk} + \underline{\xi}'^{h}_{ik} - \underline{\xi}'^{h}_{ij} - \underline{\xi}'^{h}_{jk} - \bar{\eta}'^{h}_{ik} \right) \vee 0$$

$$\vartheta^{h}_{ijk} = \left(\bar{\xi}'^{h}_{ij} + \bar{\xi}'^{h}_{jk} + \underline{\eta}'^{h}_{ik} - \underline{\eta}'^{h}_{ij} - \underline{\eta}'^{h}_{jk} - \bar{\xi}'^{h}_{ik} \right) \vee 0$$

$$\varsigma^{h}_{ijk} = \left(\underline{\eta}'^{h}_{ij} + \underline{\eta}'^{h}_{jk} + \bar{\xi}'^{h}_{ik} - \bar{\xi}'^{h}_{ij} - \bar{\xi}'^{h}_{jk} - \underline{\eta}'^{h}_{ik} \right) \vee 0$$

$$\tau^{ht}_{ij} = \left(\underline{\xi}'^{h}_{ij} - \underline{\xi}'^{t}_{ij} \right) \vee 0, \upsilon^{ht}_{ij} = \left(\underline{\xi}'^{t}_{ij} - \underline{\xi}'^{h}_{ij} \right) \vee 0$$

$$\psi^{ht}_{ij} = \left(\bar{\xi}'^{h}_{ij} - \bar{\xi}'^{t}_{ij} \right) \vee 0,\ \zeta^{ht}_{ij} = \left(\bar{\xi}'^{t}_{ij} - \bar{\xi}'^{h}_{ij} \right) \vee 0$$

$$\partial^{ht}_{ij} = \left(\underline{\eta}'^{h}_{ij} - \underline{\eta}'^{t}_{ij} \right) \vee 0,\ \ell^{ht}_{ij} = \left(\underline{\eta}'^{t}_{ij} - \underline{\eta}'^{h}_{ij} \right) \vee 0$$

$$\gamma^{ht}_{ij} = \left(\bar{\eta}'^{h}_{ij} - \bar{\eta}'^{t}_{ij} \right) \vee 0,\ \nabla^{ht}_{ij} = \left(\bar{\eta}'^{t}_{ij} - \bar{\eta}'^{h}_{ij} \right) \vee 0$$

where the symbol "$\vee$" denotes the maximum operator. Therefore, the model (M−3) is transformed as

$$\min\ \imath = \frac{1}{2mn(n-1)}\sum_{h=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\varepsilon_{ij}^{h}+\delta_{ij}^{h}+\phi_{ij}^{h}+\varphi_{ij}^{h}+\gamma_{ij}^{h}+\kappa_{ij}^{h}+\alpha_{ij}^{h}+o_{ij}^{h}\right)$$

$$\text{s.t.}\begin{cases}
\underline{\xi}_{ij}'^{h}-\underline{\xi}_{ij}^{h}-\varepsilon_{ij}^{h}+\delta_{ij}^{h}=0,\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\bar{\xi}_{ij}'^{h}-\bar{\xi}_{ij}^{h}-\phi_{ij}^{h}+\varphi_{ij}^{h}=0,\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\underline{\eta}_{ij}'^{h}-\underline{\eta}_{ij}^{h}-\gamma_{ij}^{h}+\kappa_{ij}^{h}=0,\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\bar{\eta}_{ij}'^{h}-\bar{\eta}_{ij}^{h}-\alpha_{ij}^{h}+o_{ij}^{h}=0,\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\underline{\xi}_{ij}'^{h}+\underline{\xi}_{jk}'^{h}+\bar{\eta}_{ik}'^{h}-\bar{\eta}_{ij}'^{h}-\bar{\eta}_{jk}'^{h}-\underline{\xi}_{ik}'^{h}-\pi_{ijk}^{h}+\varpi_{ijk}^{h}=0,\ i,j,k\in\Gamma,\ i<j<k,\ h\in\Lambda,\\
\sum_{1\leqslant i<j<k\leqslant n}\left(\pi_{ijk}^{h}+\varpi_{ijk}^{h}\right)\leqslant\frac{2\theta}{p},\ h\in\Lambda,\\
\bar{\xi}_{ij}'^{h}+\bar{\xi}_{jk}'^{h}+\underline{\eta}_{ik}'^{h}-\underline{\eta}_{ij}'^{h}-\underline{\eta}_{jk}'^{h}-\bar{\xi}_{ik}'^{h}-\vartheta_{ijk}^{h}+\varsigma_{ijk}^{h}=0,\ i,j,k\in\Gamma,\ i<j<k,\ h\in\Lambda,\\
\sum_{1\leqslant i<j<k\leqslant n}\left(\vartheta_{ijk}^{h}+\varsigma_{ijk}^{h}\right)\leqslant\frac{2\theta}{p},\ h\in\Lambda,\\
\underline{\xi}_{ij}'^{h}-\underline{\xi}_{ij}'^{t}-\tau_{ij}^{ht}+\upsilon_{ij}^{ht}=0,\ i,j\in\Gamma,\ i<j,\ h,t\in\Lambda,\ h<t,\\
\bar{\xi}_{ij}'^{h}-\bar{\xi}_{ij}'^{t}-\psi_{ij}^{ht}+\zeta_{ij}^{ht}=0,\ i,j\in\Gamma,\ i<j,\ h,t\in\Lambda,\ h<t,\\
\underline{\eta}_{ij}'^{h}-\underline{\eta}_{ij}'^{t}-\partial_{ij}^{ht}+\ell_{ij}^{ht}=0,\ i,j\in\Gamma,\ i<j,\ h,t\in\Lambda,\ h<t,\\
\bar{\eta}_{ij}'^{h}-\bar{\eta}_{ij}'^{t}-\gamma_{ij}^{ht}+\nabla_{ij}^{ht}=0,\ i,j\in\Gamma,\ i<j,\ h,t\in\Lambda,\ h<t,\\
1-\sum_{h=1}^{m-1}\sum_{t=h+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}q\left(\tau_{ij}^{ht}+\upsilon_{ij}^{ht}+\psi_{ij}^{ht}+\zeta_{ij}^{ht}+\partial_{ij}^{ht}+\ell_{ij}^{ht}+\gamma_{ij}^{ht}+\nabla_{ij}^{ht}\right)\geqslant\eta,\\
0\leqslant\underline{\xi}_{ij}'^{h},\bar{\xi}_{ij}'^{h},\underline{\eta}_{ij}'^{h},\bar{\eta}_{ij}'^{h}\leqslant 1,\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\underline{\xi}_{ij}'^{h}\leqslant\bar{\xi}_{ij}'^{h},\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\underline{\eta}_{ij}'^{h}\leqslant\bar{\eta}_{ij}'^{h},\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\bar{\xi}_{ij}'^{h}+\bar{\eta}_{ij}'^{h}\leqslant 1,\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\varepsilon_{ij}^{h},\delta_{ij}^{h},\phi_{ij}^{h},\varphi_{ij}^{h},\gamma_{ij}^{h},\kappa_{ij}^{h},\alpha_{ij}^{h},o_{ij}^{h}\geqslant 0,\ i,j\in N,\ i<j,\ h\in\Lambda,\\
\pi_{ijk}^{h},\varpi_{ijk}^{h}\geqslant 0,\ i,j,k\in\Gamma,\ i<j<k,\ h\in\Lambda,\\
\tau_{ij}^{ht},\upsilon_{ij}^{ht},\psi_{ij}^{ht},\zeta_{ij}^{ht},\partial_{ij}^{ht},\ell_{ij}^{ht},\gamma_{ij}^{ht},\nabla_{ij}^{ht}\geqslant 0,\ i,j\in\Gamma,\ i<j,\ h,t\in\Lambda,\ h<t.
\end{cases}\tag{M-4}$$

Under the condition of keeping the minimum deviation between the original IVIFPRs and the modified IVIFPRs unchanged, we build a model through minimizing the consistency index and maximizing the consensus index, shown as follows:

$$
\begin{aligned}
&\min \sum_{1\leqslant i<j<k\leqslant n}\left(\pi_{ijk}^{h}+\varpi_{ijk}^{h}\right)\\
&\min \sum_{1\leqslant i<j<k\leqslant n}\left(\vartheta_{ijk}^{h}+\varsigma_{ijk}^{h}\right)\\
&\min \sum_{h=1}^{m-1}\sum_{t=h+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} q\left(\tau_{ij}^{ht}+\upsilon_{ij}^{ht}+\psi_{ij}^{ht}+\zeta_{ij}^{ht}+\partial_{ij}^{ht}+\ell_{ij}^{ht}+\gamma_{ij}^{ht}+\nabla_{ij}^{ht}\right)
\end{aligned}
$$

$$
s.t.\begin{cases}
\underline{\xi}_{ij}^{\prime h}-\underline{\xi}_{ij}^{h}-\varepsilon_{ij}^{h}+\delta_{ij}^{h}=0,\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\bar{\xi}_{ij}^{\prime h}-\bar{\xi}_{ij}^{h}-\phi_{ij}^{h}+\varphi_{ij}^{h}=0,\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\underline{\eta}_{ij}^{\prime h}-\underline{\eta}_{ij}^{h}-\gamma_{ij}^{h}+\kappa_{ij}^{h}=0,\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\bar{\eta}_{ij}^{\prime h}-\bar{\eta}_{ij}^{h}-\alpha_{ij}^{h}+o_{ij}^{h}=0,\ i,j\in\Gamma,\ i<j,\ h\in\Lambda,\\
\underline{\xi}_{ij}^{\prime h}+\underline{\xi}_{jk}^{\prime h}+\bar{\eta}_{ik}^{\prime h}-\bar{\eta}_{ij}^{\prime h}-\bar{\eta}_{jk}^{\prime h}-\underline{\xi}_{ik}^{\prime h}-\pi_{ijk}^{h}+\varpi_{ijk}^{h}=0,\ i,j,k\in\Gamma,\ i<j<k,\ h\in\Lambda,\\
\sum_{1\leqslant i<j<k\leqslant n}\left(\pi_{ijk}^{h}+\varpi_{ijk}^{h}\right)\leqslant\frac{2\theta}{p},\ h\in\Lambda,\\
\bar{\xi}_{ij}^{\prime h}+\bar{\xi}_{jk}^{\prime h}+\underline{\eta}_{ik}^{\prime h}-\underline{\eta}_{ij}^{\prime h}-\underline{\eta}_{jk}^{\prime h}-\bar{\xi}_{ik}^{\prime h}-\vartheta_{ijk}^{h}+\varsigma_{ijk}^{h}=0,\ i,j,k\in\Gamma,\ i<j<k,\ h\in\Lambda,\\
\sum_{1\leqslant i<j<k\leqslant n}\left(\vartheta_{ijk}^{h}+\varsigma_{ijk}^{h}\right)\leqslant\frac{2\theta}{p},\ h\in\Lambda,\\
\underline{\xi}_{ij}^{\prime h}-\underline{\xi}_{ij}^{\prime t}-\tau_{ij}^{ht}+\upsilon_{ij}^{ht}=0,\ i,j\in\Gamma,\ i<j,\ h,t\in\Lambda,\ h<t,\\
\bar{\xi}_{ij}^{\prime h}-\bar{\xi}_{ij}^{\prime t}-\psi_{ij}^{ht}+\zeta_{ij}^{ht}=0,\ i,j\in\Gamma,\ i<j,\ h,t\in\Lambda,\ h<t,\\
\underline{\eta}_{ij}^{\prime h}-\underline{\eta}_{ij}^{\prime t}-\partial_{ij}^{ht}+\ell_{ij}^{ht}=0,\ i,j\in\Gamma,\ i<j,\ h,t\in\Lambda,\ h<t,\\
\bar{\eta}_{ij}^{\prime h}-\bar{\eta}_{ij}^{\prime t}-\gamma_{ij}^{ht}+\nabla_{ij}^{ht}=0,i,j\in\Gamma,i<j,h,t\in\Lambda,h<t,\\
1-\sum_{h=1}^{m-1}\sum_{t=h+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} q\left(\tau_{ij}^{ht}+\upsilon_{ij}^{ht}+\psi_{ij}^{ht}+\zeta_{ij}^{ht}+\partial_{ij}^{ht}+\ell_{ij}^{ht}+\gamma_{ij}^{ht}+\nabla_{ij}^{ht}\right)\geqslant\eta,\\
0\leqslant\underline{\mu}_{ij}^{\prime h},\bar{\mu}_{ij}^{\prime h},\underline{\nu}_{ij}^{\prime h},\bar{\eta}_{ij}^{\prime h}\leqslant 1,i,j\in\Gamma,i<j,h\in\Lambda,\\
\underline{\xi}_{ij}^{\prime h}\leqslant\bar{\xi}_{ij}^{\prime h},i,j\in\Gamma,i<j,h\in\Lambda,\\
\underline{\eta}_{ij}^{\prime h}\leqslant\bar{\eta}_{ij}^{\prime h},i,j\in\Gamma,i<j,h\in\Lambda,\\
\bar{\xi}_{ij}^{\prime h}+\bar{\eta}_{ij}^{\prime h}\leqslant 1,i,j\in\Gamma,i<j,h\in\Lambda,\\
\frac{1}{2mn(n-1)}\sum_{h=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\varepsilon_{ij}^{h}+\delta_{ij}^{h}+\phi_{ij}^{h}+\varphi_{ij}^{h}+\gamma_{ij}^{h}+\kappa_{ij}^{h}+\alpha_{ij}^{h}+o_{ij}^{h}\right)=\iota^{*},\\
\varepsilon_{ij}^{h},\delta_{ij}^{h},\phi_{ij}^{h},\varphi_{ij}^{h},\gamma_{ij}^{h},\kappa_{ij}^{h},\alpha_{ij}^{h},o_{ij}^{h}\geqslant 0,i,j\in\Gamma,i<j,h\in\Lambda,\\
\pi_{ijk}^{h},\varpi_{ijk}^{h}\geqslant 0,\ i,j,k\in\Gamma,i<j<k,h\in\Lambda,\\
\tau_{ij}^{ht},\upsilon_{ij}^{ht},\psi_{ij}^{ht},\zeta_{ij}^{ht},\partial_{ij}^{ht},\ell_{ij}^{ht},\gamma_{ij}^{ht},\nabla_{ij}^{ht}\geqslant 0,\ i,j\in\Gamma,i<j,h,t\in\Lambda,h<t.
\end{cases}
\tag{M-5}
$$

where ι^{*} is the optimal solution derived from the model (M−4). Solving the model (M−5), a modified IVIFPR $\widetilde{B}^{\prime h}=\left(\widetilde{\beta}_{ij}^{\prime h}\right)_{n\times n}=\left(\left(\widetilde{\xi}_{ij}^{\prime h},\widetilde{\eta}_{ij}^{\prime h}\right)\right)_{n\times n}$ $(h=1,2,\cdots,m)$ is obtained as follows:

$$
\widetilde{\beta}_{ij}^{\prime h}=\left(\widetilde{\mu}_{ij}^{\prime h},\widetilde{\nu}_{ij}^{\prime h}\right)=\begin{cases}
\left(\left[\underline{\mu}_{ij}^{\prime h},\bar{\mu}_{ij}^{\prime h}\right],\left[\underline{\nu}_{ij}^{\prime h},\bar{\nu}_{ij}^{\prime h}\right]\right),\ i<j,\\
([0.5,0.5],[0.5,0.5]),\ i=j,\\
\left(\left[\underline{\nu}_{ji}^{\prime h},\bar{\nu}_{ji}^{\prime h}\right],\left[\underline{\mu}_{ji}^{\prime h},\bar{\mu}_{ji}^{\prime h}\right]\right),\ i>j.
\end{cases}
\tag{23}
$$

4.3. Determining the DMs' weights

Considering all modified IVIFPRs obtained by the model (M−5), let $\widetilde{B}^{c}=\left(\widetilde{\beta}_{ij}^{c}\right)_{n\times n}$ be their collective IVIFPR [30,31], where

$$
\widetilde{B}_{ij}^{c}=\left(\widetilde{\xi}_{ij}^{c},\widetilde{\eta}_{ij}^{c}\right)=\left(\left[\underline{\xi}_{ij}^{c},\bar{\xi}_{ij}^{c}\right],\left[\underline{\eta}_{ij}^{c},\bar{\eta}_{ij}^{c}\right]\right)=\left(\left[\sum_{h=1}^{m}w_h\underline{\xi}_{ij}^{\prime h},\sum_{h=1}^{m}w_h\bar{\xi}_{ij}^{\prime h}\right],\left[\sum_{h=1}^{m}w_h\underline{\eta}_{ij}^{\prime h},\sum_{h=1}^{m}w_h\bar{\eta}_{ij}^{\prime h}\right]\right),
\tag{24}
$$

where $w=(w_1,w_2,\cdots,w_m)^T$ is the weighting vector of the DMs.

Theorem 4.1. If all the modified IVIFPRs have acceptable additive consistency, then their collective IVIFPR has acceptable additive consistency.

Proof. To prove the acceptable consistency of $\widetilde{B}^c$, we need to prove its lower matrix $B^{c-} = \left(\left(\xi_{ij}^{c-}, \eta_{ij}^{c-}\right)\right)_{n\times n}$ and its upper matrix $B^{c+} = \left(\left(\xi_{ij}^{c+}, \eta_{ij}^{c+}\right)\right)_{n\times n}$ are acceptable consistent. Following Eqs. (9), (10) and (24), we can get

$$\left(\xi_{ij}^{c-}, \eta_{ij}^{c-}\right) = \begin{cases} \left(\underline{\xi}_{ij}^{c}, \bar{\eta}_{ij}^{c}\right), \text{ if } i<j \\ (0.5, 0.5), \text{ if } i=j \\ \left(\bar{\xi}_{ij}^{c}, \underline{\eta}_{ij}^{c}\right), \text{ if } i>j \end{cases} = \begin{cases} \left(\sum_{h=1}^{m} w_h \underline{\xi}_{ij}^{\prime h}, \sum_{h=1}^{m} w_h \bar{\eta}_{ij}^{\prime h}\right), \text{ if } i<j, \\ (0.5, 0.5), \text{ if } i=j, \\ \left(\sum_{h=1}^{m} w_h \bar{\xi}_{ij}^{\prime h}, \sum_{h=1}^{m} w_h \underline{\eta}_{ij}^{\prime h}\right), \text{ if } i>j, \end{cases}$$

$$\left(\xi_{ij}^{c+}, \eta_{ij}^{c+}\right) = \begin{cases} \left(\bar{\xi}_{ij}^{c}, \underline{\eta}_{ij}^{c}\right), \text{ if } i<j \\ (0.5, 0.5), \text{ if } i=j \\ \left(\underline{\xi}_{ij}^{c}, \bar{\eta}_{ij}^{c}\right), \text{ if } i>j \end{cases} = \begin{cases} \left(\sum_{h=1}^{m} w_h \bar{\xi}_{ij}^{\prime h}, \sum_{h=1}^{m} w_h \underline{\eta}_{ij}^{\prime h}\right), \text{ if } i<j, \\ (0.5, 0.5), \text{ if } i=j, \\ \left(\sum_{h=1}^{m} w_h \underline{\xi}_{ij}^{\prime h}, \sum_{h=1}^{m} w_h \bar{\eta}_{ij}^{\prime h}\right), \text{ if } i>j. \end{cases}$$

Because $B^{\prime h-}$ and $B^{\prime h+}$ are acceptable consistent, B^{c-} and B^{c+} are acceptable consistent. The proof is completed. **Q. E. D.**

When calculating the collective IVIFPR, we need to calculate the DMs' weighting vector. From Eq. (24), we can conclude that the group $\left\{\widetilde{B}^{\prime 1}, \widetilde{B}^{\prime 2}, \cdots, \widetilde{B}^{\prime m}\right\}$ of the modified IVIFPRs is full consensus iff the following equations hold true:

$$\begin{cases} \underline{\xi}_{ij}^{\prime h} = \sum_{z=1}^{m} w_z \underline{\xi}_{ij}^{\prime z}, \; i,j \in \Gamma, \; h \in \Lambda, \\ \bar{\xi}_{ij}^{\prime h} = \sum_{z=1}^{m} w_z \bar{\xi}_{ij}^{\prime z}, \; i,j \in \Gamma, \; h \in \Lambda, \\ \underline{\eta}_{ij}^{\prime h} = \sum_{z=1}^{m} w_z \underline{\eta}_{ij}^{\prime z}, \; i,j \in \Gamma, \; h \in \Lambda, \\ \bar{\eta}_{ij}^{\prime h} = \sum_{z=1}^{m} w_z \bar{\eta}_{ij}^{\prime z}, \; i,j \in \Gamma, \; h \in \Lambda. \end{cases} \tag{25}$$

Eq. (25) shows that the smaller the value of the following equation is, the higher the consensus level will be:

$$\sum_{h=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\underline{\xi}_{ij}^{\prime h} - \sum_{z=1}^{m} w_z \underline{\xi}_{ij}^{\prime z}\right| + \left|\bar{\xi}_{ij}^{\prime h} - \sum_{z=1}^{m} w_z \bar{\xi}_{ij}^{\prime z}\right| + \left|\underline{\eta}_{ij}^{\prime h} - \sum_{z=1}^{m} w_z \underline{\eta}_{ij}^{\prime z}\right| + \left|\bar{\eta}_{ij}^{\prime h} - \sum_{z=1}^{m} w_z \bar{\eta}_{ij}^{\prime z}\right|\right) \tag{26}$$

Therefore, the following model to obtain the DMs' weights is built:

$$\min Q = \sum_{h=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\delta_{ij}^{h+} + \delta_{ij}^{h-} + \chi_{ij}^{h+} + \chi_{ij}^{h-} + \sigma_{ij}^{h+} + \sigma_{ij}^{h-} + \zeta_{ij}^{h+} + \zeta_{ij}^{h-}\right)$$

$$\text{s.t.}\begin{cases} \underline{\xi}_{ij}^{\prime h} - \sum_{z=1}^{m} w_z \underline{\xi}_{ij}^{\prime z} - \delta_{ij}^{h+} + \delta_{ij}^{h-} = 0, \; i,j \in \Gamma, \; h \in \Lambda, \\ \bar{\xi}_{ij}^{\prime h} - \sum_{z=1}^{m} w_z \bar{\xi}_{ij}^{\prime z} - \chi_{ij}^{h+} + \chi_{ij}^{h-} = 0, \; i,j \in \Gamma, \; h \in \Lambda, \\ \underline{\eta}_{ij}^{\prime h} - \sum_{z=1}^{m} w_z \underline{\eta}_{ij}^{\prime z} - \sigma_{ij}^{h+} + \sigma_{ij}^{h-} = 0, \; i,j \in \Gamma, \; h \in \Lambda, \\ \bar{\eta}_{ij}^{\prime h} - \sum_{z=1}^{m} w_z \bar{\eta}_{ij}^{\prime z} - \zeta_{ij}^{h+} + \zeta_{ij}^{h-} = 0, \; i,j \in \Gamma, \; h \in \Lambda, \\ \delta_{ij}^{h+}, \delta_{ij}^{h-}, \chi_{ij}^{h+}, \chi_{ij}^{h-}, \sigma_{ij}^{h+}, \sigma_{ij}^{h-}, \zeta_{ij}^{h+}, \zeta_{ij}^{h-} \geqslant 0, \\ i,j \in \Gamma, \; h \in \Lambda, \\ 0 \leqslant w_h \leqslant 1, \; h \in \Lambda, \\ \sum_{h=1}^{m} w_h = 1. \end{cases} \tag{M-6}$$

5. Deriving the IVIF priority weights

In case of uncertainty, preference values are always imprecise to some extent. It is argued that IVIF priority weights are more flexible than exact priority weights. This section focuses on obtaining IVIF priority weights from the collective IVIFPR via programming models. Afterward, a new GDM method is proposed.

5.1. A model to gain the IVIF priority weights

Let $\widetilde{B}^h$ be an IVIFPR, where $h = 1, 2, \cdots, m$. Let $\widetilde{B}'^h$ be the modified IVIFPR with an acceptable consistency and an acceptable consensus obtained by the model (M−5). Eq. (24) is applied to obtain the modified collective IVIFPR $\widetilde{B}^c = \left(\widetilde{\beta}_{ij}^c\right)_{n\times n} = \left(\left(\widetilde{\xi}_{ij}^c, \widetilde{\eta}_{ij}^c\right)\right)_{n\times n}$ from $\widetilde{B}'^h$ ($h = 1, 2, \cdots .m$), where $\widetilde{\xi}_{ij}^c = \left[\underline{\xi}_{ij}^c, \bar{\xi}_{ij}^c\right]$ and $\widetilde{\eta}_{ij}^c = \left[\underline{\eta}_{ij}^c, \bar{\eta}_{ij}^c\right]$. Let the lower matrix and the upper matrix of $\widetilde{B}^c$ be $B^{c-} = \left(\beta_{ij}^{c-}\right)_{n\times n}$ and $B^{c+} = \left(\beta_{ij}^{c+}\right)_{n\times n}$, respectively, where $\beta_{ij}^{c-} = \left(\xi_{ij}^{c-}, \eta_{ij}^{c-}\right)$ and $\beta_{ij}^{c+} = \left(\xi_{ij}^{c+}, \eta_{ij}^{c+}\right)$. Let the IF priority vectors of B^{c-} and B^{c+} be $z^- = \left(z_1^-, z_2^-, \cdots, z_n^-\right)^T$ and $z^+ = \left(z_1^+, z_2^+, \cdots, z_n^+\right)^T$, respectively, where $z_i^- = (z_i^{\xi-}, z_i^{\eta-})$ and $z_i^+ = (z_i^{\xi+}, z_i^{\eta+})$ are IFVs. Based on **Theorem 3.5** and Eq. (8), we can see that $\widetilde{B}^c$ is consistent if and only if

$$\begin{cases} \xi_{ij}^{c-} = 0.5z_i^{\xi-} + 0.5z_j^{\eta-}, \ i,j \in \Gamma, \ i < j, \\ \eta_{ij}^{c-} = 0.5z_i^{\eta-} + 0.5z_j^{\xi-}, \ i,j \in \Gamma, \ i < j, \\ \xi_{ij}^{c+} = 0.5z_i^{\xi+} + 0.5z_j^{\eta+}, \ i,j \in \Gamma, \ i < j, \\ \eta_{ij}^{c+} = 0.5z_i^{\eta+} + 0.5z_j^{\xi+}, \ i,j \in \Gamma, \ i < j. \end{cases} \tag{27}$$

The above restrictions could be relaxed when $\widetilde{B}^c$ is inconsistent to some extent. For convenience, the following notations are introduced, where the symbol "$\vee$" denotes the maximum operator:

$$\begin{cases} \Delta_{ij}^1 = \left(\xi_{ij}^{c-} - 0.5z_i^{\xi-} - 0.5z_j^{\eta-}\right) \vee 0, \ i,j \in \Gamma, \ i < j, \\ \Delta_{ij}^2 = \left(0.5z_i^{\xi-} + 0.5z_j^{\eta-} - \xi_{ij}^{c-}\right) \vee 0, \ i,j \in \Gamma, \ i < j, \\ \Delta_{ij}^3 = \left(\eta_{ij}^{c-} - 0.5z_i^{\eta-} - 0.5z_j^{\xi-}\right) \vee 0, \ i,j \in \Gamma, \ i < j, \\ \Delta_{ij}^4 = \left(0.5z_i^{\eta-} + 0.5z_j^{\xi-} - \eta_{ij}^{c-}\right) \vee 0, \ i,j \in \Gamma, \ i < j, \\ \Delta_{ij}^5 = \left(\xi_{ij}^{c+} - 0.5z_i^{\xi+} - 0.5z_j^{\eta+}\right) \vee 0, \ i,j \in \Gamma, \ i < j, \\ \Delta_{ij}^6 = \left(0.5z_i^{\xi+} + 0.5z_j^{\eta+} - \xi_{ij}^{c+}\right) \vee 0, \ i,j \in \Gamma, \ i < j, \\ \Delta_{ij}^7 = \left(\eta_{ij}^{c+} - 0.5z_i^{\eta+} - 0.5z_j^{\xi+}\right) \vee 0, \ i,j \in \Gamma, \ i < j, \\ \Delta_{ij}^8 = \left(0.5z_i^{\eta+} + 0.5z_j^{\xi+} - \eta_{ij}^{c+}\right) \vee 0, \ i,j \in \Gamma, \ i < j. \end{cases} \tag{28}$$

Based on Eqs. (27) and (28), we build the following model to get the priority weights:

$$\min \ \varepsilon = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\Delta_{ij}^1 + \Delta_{ij}^2 + \Delta_{ij}^3 + \Delta_{ij}^4 + \Delta_{ij}^5 + \Delta_{ij}^6 + \Delta_{ij}^7 + \Delta_{ij}^8\right)$$

$$\text{s.t.} \begin{cases} \xi_{ij}^{c-} - 0.5z_i^{\xi-} - 0.5z_j^{\eta-} - \Delta_{ij}^1 + \Delta_{ij}^2 = 0, \ i,j \in \Gamma, \ i < j, \\ \eta_{ij}^{c-} - 0.5z_i^{\eta-} - 0.5z_j^{\xi-} - \Delta_{ij}^3 + \Delta_{ij}^4 = 0, \ i,j \in \Gamma, \ i < j, \\ \xi_{ij}^{c+} - 0.5z_i^{\xi+} - 0.5z_j^{\eta+} - \Delta_{ij}^5 + \Delta_{ij}^6 = 0, \ i,j \in \Gamma, \ i < j, \\ \eta_{ij}^{c+} - 0.5z_i^{\eta+} - 0.5z_j^{\xi+} - \Delta_{ij}^7 + \Delta_{ij}^8 = 0, \ i,j \in \Gamma, \ i < j, \\ 0 \leqslant z_i^{\xi-}, z_i^{\eta-}, z_i^{\xi+}, z_i^{\eta+} \leqslant 1, \ i \in \Gamma, \\ z_i^{\xi-} + z_i^{\eta-} \leqslant 1, \ z_i^{\xi+} + z_i^{\eta+} \leqslant 1, \ i \in \Gamma, \\ \Delta_{ij}^1, \Delta_{ij}^2, \Delta_{ij}^3, \Delta_{ij}^4, \Delta_{ij}^5, \Delta_{ij}^6, \Delta_{ij}^7, \Delta_{ij}^8 \geqslant 0, \ i,j \in \Gamma, \ i < j, \\ \sum_{j=1, j\neq i}^{n} z_j^{\xi-} \leqslant z_i^{\eta-}, \ z_i^{\xi-} + n - 2 \geqslant \sum_{j=1, j\neq i}^{n} z_j^{\eta-}, \ i \in \Gamma, \\ \sum_{j=1, j\neq i}^{n} z_j^{\xi+} \leqslant z_i^{\eta+}, \ z_i^{\xi+} + n - 2 \geqslant \sum_{j=1, j\neq i}^{n} z_j^{\eta+}, \ i \in \Gamma, \\ \max\{z_i^{\xi-}, z_i^{\xi+}\} + \max\{z_i^{\eta-}, z_i^{\eta+}\} \leqslant 1, \ i \in \Gamma. \end{cases} \tag{M-7}$$

In the model (M−7), the purpose from the first constraint to the sixth constraint is to calculate the priority weighting vectors. The seventh constraint is the requirement of the deviation variables. The eighth constraint to the tenth constraint are the normalization constraints on the intuitionistic fuzzy weighting vectors z^- and z^+. Let $\max\{z_i^{\xi-}, z_i^{\xi+}\} = s_i$ and let $\max\{z_i^{\eta-}, z_i^{\eta+}\} = t_i$, where $i \in \Gamma$. The model (M−7) is transformed into:

$$\min\ \varepsilon = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\Delta_{ij}^1+\Delta_{ij}^2+\Delta_{ij}^3+\Delta_{ij}^4+\Delta_{ij}^5+\Delta_{ij}^6+\Delta_{ij}^7+\Delta_{ij}^8\right)$$
$$\text{s.t.}\begin{cases} \xi_{ij}^{c-} - 0.5z_i^{\xi-} - 0.5z_j^{\eta-} - \Delta_{ij}^1 + \Delta_{ij}^2 = 0,\ i,j\in\Gamma,\ i<j, \\ \eta_{ij}^{c-} - 0.5z_i^{\eta-} - 0.5z_j^{\xi-} - \Delta_{ij}^3 + \Delta_{ij}^4 = 0,\ i,j\in\Gamma,\ i<j, \\ \xi_{ij}^{c+} - 0.5z_i^{\xi+} - 0.5z_j^{\eta+} - \Delta_{ij}^5 + \Delta_{ij}^6 = 0,\ i,j\in\Gamma,\ i<j, \\ \eta_{ij}^{c+} - 0.5z_i^{\eta+} - 0.5z_j^{\xi+} - \Delta_{ij}^7 + \Delta_{ij}^8 = 0,\ i,j\in\Gamma,\ i<j, \\ \Delta_{ij}^1,\Delta_{ij}^2,\Delta_{ij}^3,\Delta_{ij}^4,\Delta_{ij}^5,\Delta_{ij}^6,\Delta_{ij}^7,\Delta_{ij}^8 \geqslant 0,\ i,j\in\Gamma,\ i<j, \\ 0 \leqslant z_i^{\xi-}, z_i^{\eta-}, z_i^{\xi+}, z_i^{\eta+} \leqslant 1,\ i\in\Gamma; \\ z_i^{\xi-} + z_i^{\eta-} \leqslant 1,\ z_i^{\xi+} + z_i^{\eta+} \leqslant 1,\ i\in\Gamma; \\ \sum_{j=1,j\neq i}^{n} z_j^{\xi-} \leqslant z_i^{\eta-},\ z_i^{\xi-} + n - 2 \geqslant \sum_{j=1,j\neq i}^{n} z_j^{\eta-},\ i\in\Gamma, \\ \sum_{j=1,j\neq i}^{n} z_j^{\xi+} \leqslant z_i^{\eta+},\ z_i^{\xi+} + n - 2 \geqslant \sum_{j=1,j\neq i}^{n} z_j^{\eta+},\ i\in\Gamma, \\ s_i + t_i \leqslant 1,\ i\in\Gamma, \\ z_i^{\xi-}, z_i^{\xi+} \leqslant s_i,\ i\in\Gamma, \\ z_i^{\eta-}, z_i^{\eta+} \leqslant t_i,\ i\in\Gamma, \\ s_i, t_i \geqslant 0,\ i\in\Gamma. \end{cases} \tag{M-8}$$

5.2. A new GDM method

Following the previous discussions, we present the following GDM method with IVIFPRs:

Step 1: Let $\widetilde{B}^h$ be the IVIFPR given by DM e_h.

Step 2: Through Eqs. (9) and (10), gain B^{h-} and B^{h+} , respectively.

Step 3: Utilize Eq. (6) to compute $f\left(B^{h-}\right)$ and $f\left(B^{h+}\right)$, where $h = 1, 2, \cdots, m$. Employ Eq. (22) to quantify the consensus index.

Step 4: When all IVIFPRs are acceptable additive consistent and have acceptable consensus, let $\widetilde{B}'^h = \widetilde{B}^h$ $(h = 1, 2, \cdots, m)$ and go to **Step 5**. Otherwise, the models (M−4) and (M−5) are applied to adjust individual IVIFPRs to possess the acceptable consistency and consensus. Let $\widetilde{B}'^h$ $(h = 1, 2, \cdots, m)$ be the modified IVIFPRs.

Step 5: Based on the model (M−6), obtain the DMs' weighing vector w.

Step 6: Using Eq. (24) to get the collective IVIFPR $\widetilde{B}^c$.

Step 7: Using the model (M−8) to gain the IF priority weighting vectors z^- and z^+.

Step 8: Using Eq. (17), obtain the IVIF priority weights $\widetilde{z}_i = \left(\left[\underline{z}_i^{\xi}, \overline{z}_i^{\xi}\right], \left[\underline{z}_i^{\eta}, \overline{z}_i^{\eta}\right]\right)$ $(i = 1, 2, \cdots, n)$ of the alternatives.

Step 9: As per **Definition 2.6**, calculate $s\left(\widetilde{z}_i\right)$ and $\gamma\left(\widetilde{z}_i\right)$, respectively, and rank the alternatives r_i $(i = 1, 2, \cdots, n)$ following these score values and accuracy values.

6. Application example and discussions

6.1. Case study

Example 6.1. A group of three experts denoted as $E = \{e_1, e_2, e_3\}$ are gathered to select the best destination for a summer vacation. After a pre-evaluation, four feasible alternatives are put forward for further consideration, i.e., r_1: Istanbul, Turkey; r_2: Barcelona, Spain; r_3: Beijing, China; r_4: Rome, Italy. The experts e_1, e_2 and e_3 express their individual preferences of these four alternatives in the form of IVIFPRs $\widetilde{B}^1$, $\widetilde{B}^2$ and $\widetilde{B}^3$, respectively, shown as follows:

$$\widetilde{B}^1 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.1,0.2],[0.6,0.7]) & ([0.2,0.3],[0.5,0.6]) & ([0.3,0.5],[0.3,0.5]) \\ ([0.6,0.7],[0.1,0.2]) & ([0.5,0.5],[0.5,0.5]) & ([0.1,0.3],[0.5,0.6]) & ([0.5,0.7],[0.1,0.3]) \\ ([0.5,0.6],[0.2,0.3]) & ([0.5,0.6],[0.1,0.3]) & ([0.5,0.5],[0.5,0.5]) & ([0.2,0.3],[0.6,0.7]) \\ ([0.3,0.5],[0.3,0.5]) & ([0.1,0.3],[0.5,0.7]) & ([0.6,0.7],[0.2,0.3]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

$$\widetilde{B}^2 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.5,0.6],[0.2,0.3]) & ([0.3,0.5],[0.4,0.5]) & ([0.6,0.7],[0.1,0.3]) \\ ([0.2,0.3],[0.5,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.5,0.7],[0.1,0.2]) & ([0.1,0.4],[0.3,0.5]) \\ ([0.4,0.5],[0.3,0.5]) & ([0.1,0.2],[0.5,0.7]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.2,0.3]) \\ ([0.1,0.3],[0.6,0.7]) & ([0.3,0.5],[0.1,0.4]) & ([0.2,0.3],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

$$\widetilde{B}^3 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.5,0.7],[0.1,0.2]) & ([0.4,0.6],[0.3,0.4]) & ([0.1,0.4],[0.5,0.6]) \\ ([0.1,0.2],[0.5,0.7]) & ([0.5,0.5],[0.5,0.5]) & ([0.6,0.8],[0.1,0.2]) & ([0.6,0.7],[0.1,0.3]) \\ ([0.3,0.4],[0.4,0.6]) & ([0.1,0.2],[0.6,0.8]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.5],[0.2,0.3]) \\ ([0.5,0.6],[0.1,0.4]) & ([0.1,0.3],[0.6,0.7]) & ([0.2,0.3],[0.3,0.5]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

These four destinations can be ranked via our method, shown as follows:

Step 1: Three IVIFPRs $\widetilde{B}^1$, $\widetilde{B}^2$ and $\widetilde{B}^3$ are provided above.

Step 2: Based on Eqs. (9) and (10), B^{1-}, B^{1+}, B^{2-}, B^{2+}, B^{3-} and B^{3+} are obtained as follows:

$$B^{1-} = \begin{bmatrix} (0.5,0.5) & (0.1,0.7) & (0.2,0.6) & (0.3,0.5) \\ (0.7,0.1) & (0.5,0.5) & (0.1,0.6) & (0.5,0.3) \\ (0.6,0.2) & (0.6,0.1) & (0.5,0.5) & (0.2,0.7) \\ (0.5,0.3) & (0.3,0.5) & (0.7,0.2) & (0.5,0.5) \end{bmatrix}$$

$$B^{1+} = \begin{bmatrix} (0.5,0.5) & (0.2,0.6) & (0.3,0.5) & (0.5,0.3) \\ (0.6,0.2) & (0.5,0.5) & (0.3,0.5) & (0.7,0.1) \\ (0.5,0.3) & (0.5,0.3) & (0.5,0.5) & (0.3,0.6) \\ (0.3,0.5) & (0.1,0.7) & (0.6,0.3) & (0.5,0.5) \end{bmatrix}$$

$$B^{2-} = \begin{bmatrix} (0.5,0.5) & (0.5,0.3) & (0.3,0.5) & (0.6,0.3) \\ (0.3,0.5) & (0.5,0.5) & (0.5,0.2) & (0.1,0.5) \\ (0.5,0.3) & (0.2,0.5) & (0.5,0.5) & (0.4,0.3) \\ (0.3,0.6) & (0.5,0.1) & (0.3,0.4) & (0.5,0.5) \end{bmatrix}$$

$$B^{2+} = \begin{bmatrix} (0.5,0.5) & (0.6,0.2) & (0.5,0.4) & (0.7,0.1) \\ (0.2,0.6) & (0.5,0.5) & (0.7,0.1) & (0.4,0.3) \\ (0.4,0.5) & (0.1,0.7) & (0.5,0.5) & (0.5,0.2) \\ (0.1,0.7) & (0.3,0.4) & (0.2,0.5) & (0.5,0.5) \end{bmatrix}$$

$$B^{3-} = \begin{bmatrix} (0.5,0.5) & (0.5,0.2) & (0.4,0.4) & (0.1,0.6) \\ (0.2,0.5) & (0.5,0.5) & (0.6,0.2) & (0.6,0.3) \\ (0.4,0.4) & (0.2,0.6) & (0.5,0.5) & (0.3,0.3) \\ (0.6,0.1) & (0.3,0.6) & (0.3,0.3) & (0.5,0.5) \end{bmatrix}$$

$$B^{3+} = \begin{bmatrix} (0.5,0.5) & (0.7,0.1) & (0.6,0.3) & (0.4,0.5) \\ (0.1,0.7) & (0.5,0.5) & (0.8,0.1) & (0.7,0.1) \\ (0.3,0.6) & (0.1,0.8) & (0.5,0.5) & (0.5,0.2) \\ (0.5,0.4) & (0.1,0.7) & (0.2,0.5) & (0.5,0.5) \end{bmatrix}$$

Step 3: Based on Eqs. (6) and (22), we can obtain $f\left(B^{1-}\right) = 0.2333$, $f\left(B^{1+}\right) = 0.1833$, $f\left(B^{2-}\right) = 0.2$, $f\left(B^{2+}\right) = 0.1667$, $f\left(B^{3-}\right) = 0.2$, $f\left(B^{3+}\right) = 0.2833$ and $g\left(\widetilde{B}^1, \widetilde{B}^2, \widetilde{B}^3\right) = 0.7639$.

Step 4: Let $\theta = 0.1$ and $\eta = 0.9$ be the threshold of the acceptable additive consistency and the threshold of the acceptable consensus, respectively. One can check that all of IVIFPRs are unacceptable additive consistency and unacceptable consensus. According to the model (M−5), three modified IVIFPRs are derived as follows:

$$\widetilde{B}'^{1} = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.2,0.2],[0.6,0.6]) & ([0.3,0.35],[0.5,0.6]) & ([0.3,0.5],[0.3,0.5]) \\ ([0.6,0.6],[0.2,0.2]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.45],[0.2,0.2]) & ([0.5,0.7],[0.1,0.3]) \\ ([0.5,0.6],[0.3,0.35]) & ([0.2,0.2],[0.3,0.45]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.5],[0.2,0.3]) \\ ([0.3,0.5],[0.3,0.5]) & ([0.1,0.3],[0.5,0.7]) & ([0.2,0.3],[0.3,0.5]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

$$\widetilde{B}'^{2} = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.5,0.6],[0.3,0.4]) & ([0.3,0.5],[0.4,0.5]) & ([0.3,0.5],[0.1,0.3]) \\ ([0.2,0.3],[0.5,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.5,0.5],[0.2,0.2]) & ([0.4,0.4],[0.3,0.5]) \\ ([0.4,0.5],[0.3,0.5]) & ([0.2,0.2],[0.5,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.2,0.3]) \\ ([0.1,0.3],[0.3,0.5]) & ([0.3,0.5],[0.4,0.4]) & ([0.2,0.3],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

$$\widetilde{B}'^{3} = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.5,0.6],[0.3,0.4]) & ([0.4,0.6],[0.3,0.4]) & ([0.3,0.5],[0.2,0.5]) \\ ([0.3,0.4],[0.5,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.5,0.5],[0.2,0.2]) & ([0.6,0.7],[0.1,0.3]) \\ ([0.3,0.4],[0.4,0.6]) & ([0.2,0.2],[0.5,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.5],[0.2,0.3]) \\ ([0.2,0.5],[0.3,0.5]) & ([0.1,0.3],[0.6,0.7]) & ([0.2,0.3],[0.3,0.5]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

Step 5: Based on the model (M−6), the DMs' weighting vector is $w = (0,0,0.1)^T$.

Step 6: According to the modified IVIFPRs $\widetilde{B}'^h$ $(h = 1,2,3)$ and their weighting vector w, the collective IVIFPR $\widetilde{B}^c$ is obtained via Eq. (24), shown as follows:

$$\widetilde{B}^{c} = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.5,0.6],[0.3,0.4]) & ([0.4,0.6],[0.3,0.4]) & ([0.3,0.5],[0.2,0.5]) \\ ([0.3,0.4],[0.5,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.5,0.5],[0.2,0.2]) & ([0.6,0.7],[0.1,0.3]) \\ ([0.3,0.4],[0.4,0.6]) & ([0.2,0.2],[0.5,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.5],[0.2,0.3]) \\ ([0.2,0.5],[0.3,0.5]) & ([0.1,0.3],[0.6,0.7]) & ([0.2,0.3],[0.3,0.5]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

Step 7: With respect to $\widetilde{B}^c$, IF priority weights z^- and z^+ are derived via the model (8), shown as follows:

$$z^- = ((0.18,0.4),(0.38,0.38),(0.02,0.62),(0,0.58))^T$$

$$z^+ = ((0.4667,0.5333),(0.2667,0.6),(0.1333,0.7333),(0.062,0.8667))^T$$

Step 8: Applying Eq. (17), IVIF priority weights are determined, shown as follows:

$$\tilde{z}_1 = ([0.18,0.4667],[0.4,0.5333]),\ \tilde{z}_2 = ([0.2667,0.38],[\ 0.38,0.6])$$

$$\tilde{z}_3 = ([0.02,0.1333],[0.62,0.7333]),\ \tilde{z}_4 = ([0,0],[0.58,0.8667])$$

Step 9: Following **Definition 2.6**, we obtain $s\left(\tilde{z}_1\right) = -0.1433$, $s\left(\tilde{z}_2\right) = -0.1667$, $s\left(\tilde{z}_3\right) = -0.6000$ and $s\left(\tilde{z}_4\right) = -0.7233$, which shows that $\tilde{z}_1 \succ \tilde{z}_2 \succ \tilde{z}_3 \succ \tilde{z}_4$. Thus, we obtain $r_1 \succ r_2 \succ r_3 \succ r_4$.

6.2. Some comparative analysis

Next, we compare our method with several existing methods [5,15,30,31,33,34,43].

(1) A comparison with the methods presented in [31,34,43]: Consider a GDM problem which aims at choosing a network system from four feasible alternatives for a new multi-function building (Note: Please see page 1006 of [31]). We take $\theta = 0.1$ and $\eta = 0.9$. When the proposed method is applied to handle this example shown in [31], the ranking results of several methods are shown in Table 1. From Table 1, we can see that:

(i) Wan *et al.*'s method [31] and Wang's method [34] do not check and improve the consensus for IVIFPRs. For example, let $\widetilde{\boldsymbol{R}}^k$ $(k = 1,2,3)$ be three IVIFPRs shown in page 1007 of [31]. According to Eq. (22) of this paper, we can get $g\left(\widetilde{\boldsymbol{R}}^1, \widetilde{\boldsymbol{R}}^2, \widetilde{\boldsymbol{R}}^3\right) = 0.8528 < \eta = 0.9$, which indicates that the consensus among the three IVIFPRs shown in page 1007 of [31] are unacceptable. By contrast, a model is built in the proposed method to modify IVIFPRs without an acceptable consistency and an acceptable consensus.

(ii) The priority weights gained by Wan *et al.*'s method [31] take the form of IFVs, which seems to be unreasonable due to the fact that the elements in IVIFPRs are IVIFVs. By contrast, the priority weights obtained by the proposed method take the form of IVIFVs, which can well reflect the uncertainty of the DMs' preferences.

(iii) Wan *et al.*'s method [31] and Wang's method [34] applied the complete additive consistency, whereas the proposed method utilizes the acceptable additive consistency. It should be noted that when we let $\theta = 0$, we derived decision-making methods with IVIFPRs following the complete consistency and consensus analysis.

Table 1
A comparison between the methods presented in [31,34,43] and the proposed method (All collected from [31]).

Methods	Ranking values		Ranking orders
Wan *et al.*'s method [31]	$L(w_1)=0.2068$, $L(w_2)=0.3877$, $L(w_3)=0.3309$, $L(w_4)=0.1289$		$r_2 \succ r_3 \succ r_1 \succ r_4$
Xu's method [43]	$w=(\frac{1}{3},\frac{1}{3},\frac{1}{3})^T$	$c(\tilde{r}_1)=0.5134$, $c(\tilde{r}_2)=0.5673$, $c(\tilde{r}_3)=0.4800$, $c(\tilde{r}_4)=0.4401$	$r_2 \succ r_1 \succ r_3 \succ r_4$
	$w=(0.3,0.2,0.5)^T$	$c(\tilde{r}_1)=0.5262$, $c(\tilde{r}_2)=0.5580$, $c(\tilde{r}_3)=0.4919$, $c(\tilde{r}_4)=0.4246$	$r_2 \succ r_1 \succ r_3 \succ r_4$
	$w=(0.3145,0.2601,0.4254)^T$	$c(\tilde{r}_1)=0.5214$, $c(\tilde{r}_2)=0.5571$, $c(\tilde{r}_3)=0.4915$, $c(\tilde{r}_4)=0.4309$	$r_2 \succ r_1 \succ r_3 \succ r_4$
Wang's method [34]	$w=(\frac{1}{3},\frac{1}{3},\frac{1}{3})^T$	$S(\tilde{\omega}_1)=-0.5686$, $S(\tilde{\omega}_2)=-0.2710$, $S(\tilde{\omega}_3)=-0.4819$, $S(\tilde{\omega}_4)=-0.6525$	$r_2 \succ r_3 \succ r_1 \succ r_4$
	$w=(0.3,0.2,0.5)^T$	$S(\tilde{\omega}_1)=-0.5759$, $S(\tilde{\omega}_2)=-0.2778$, $S(\tilde{\omega}_3)=-0.4845$, $S(\tilde{\omega}_4)=-0.6502$	$r_2 \succ r_3 \succ r_1 \succ r_4$
	$w=(0.3145,0.2601,0.4254)^T$	$S(\tilde{\omega}_1)=-0.5726$, $S(\tilde{\omega}_2)=-0.2746$, $S(\tilde{\omega}_3)=-0.4833$, $S(\tilde{\omega}_4)=-0.6512$	$r_2 \succ r_3 \succ r_1 \succ r_4$
The proposed method	$s(\tilde{z}_1)=-0.4167$, $s(\tilde{z}_2)=-0.0834$, $s(\tilde{z}_3)=-0.4833$, $s(\tilde{z}_4)=-0.7166$		$r_2 \succ r_1 \succ r_3 \succ r_4$

(iv) Xu's method [43] ignored the consistency analysis that is a very important topic to obtain reasonable ranking orders of alternatives. Our method adjusts the consistency of IVIFPRs to get logical ranking orders of alternatives.

(2) A comparison with the methods presented in [5,15,30]**:** Consider the decision making problem about ERP system selection adopted from [30]. The proposed method is employed to rank the alternatives of this problem. We take $\theta=0.1$ and $\eta=0.9$. The ranking results of alternatives for different methods are shown in Table 2. From Table 2, we can see that:

(i) The lack of the consistency might lead to an inaccurate ending. Let $\widetilde{\boldsymbol{R}}^k$ $(k=1,2,3)$ be three IVIFPRs shown in page 579 of [30], let $\widetilde{\boldsymbol{R}}$ be the collective IVIFPR derived by Eq. (18) of [30], and let $\boldsymbol{R}_\mu$ and $\boldsymbol{R}_\nu$ be two special IFPRs extracted from $\widetilde{\boldsymbol{R}}$ based on Eqs. (9) and (10) of [30], respectively. By applying Eq. (6) of this paper, we can get $f(\boldsymbol{R}_\mu)=0.0648 < \theta=0.1$ and $f(\boldsymbol{R}_\nu)=0.1179 > \theta=0.1$, which means that $\boldsymbol{R}_\nu$ is an unacceptable additive consistency. That is, Wan *et al.*'s method [30] has the drawback that it does not develop some repairing methods for improving the consistency index of IVIFPRs. By applying Eq. (22) of this paper, we can get $g\left(\widetilde{\boldsymbol{R}}^1,\widetilde{\boldsymbol{R}}^2,\widetilde{\boldsymbol{R}}^3\right)=0.7347<\eta=0.9$, which can find that the consensus among the three DMs obtained by Wan *et al.*'s method [30] is unacceptable. Wan *et al.*'s method [30] disregards the consensus. Our method can simultaneously ensure three goals: (1) consistency and consensus are simultaneously reached, (2) the smallest information loss is guaranteed, and (3) the maximum consistency and consensus levels are guaranteed.

(ii) Wan *et al.*'s method [30] derived alternatives' priority weights by constructing several programming models, which considers the DMs' satisfaction but ignores the DMs' dissatisfaction. The proposed method has the advantage that it derives alternatives' priority weights by building a programming model which involves the DMs' satisfaction and dissatisfaction simultaneously to contain much valuable information and reducing the information loss.

Table 2
A comparison between the methods presented in [5,15,30] and the proposed method (All collected from [30]).

Methods	Ranking values		Ranking orders
Wan *et al.*'s method [30]	$\psi=0$	$RD_1=0.8343$, $RD_2=0.9284$, $RD_3=0.0765$, $RD_4=0.0906$	$r_2 \succ r_1 \succ r_4 \succ r_3$
	$\psi=0.5$	$RD_1=0.9441$, $RD_2=0.7244$, $RD_3=0.1112$, $RD_4=0.0555$	$r_1 \succ r_2 \succ r_3 \succ r_4$
	$\psi=1$	$RD_1=0.7511$, $RD_2=0.9403$, $RD_3=0.0970$, $RD_4=0.0697$	$r_2 \succ r_1 \succ r_3 \succ r_4$
Chu *et al.*'s method [5]	$r_4 \succ r_2 \succ r_1 \succ r_3$		
Liao *et al.*'s method [15]	$r_2 \succ r_1 \succ r_3 \succ r_4$		
The proposed method	$s(\tilde{z}_1)=-0.2804$, $s(\tilde{z}_2)=-0.0750$, $s(\tilde{z}_3)=-0.6597$, $s(\tilde{z}_4)=-0.6822$		$r_2 \succ r_1 \succ r_3 \succ r_4$

Table 3
A comparison between the methods presented in [15,33] and the proposed method (All collected from [33]).

Methods	Ranking values		Ranking orders
Wan *et al.*'s method [33]	$\xi = 0$	$q \in [0, 0.70)$	$r_1 \succ r_2 \succ r_3 \succ r_4$
		$q = 0.70$	$r_1 \sim r_2 \succ r_3 \succ r_4$
		$q \in (0.70, 1]$	$r_2 \succ r_1 \succ r_3 \succ r_4$
	$\xi = 1$	$q \in [0, 1]$	$r_1 \succ r_2 \succ r_3 \succ r_4$
	$\xi = 0.5$	$q \in [0, 1]$	$r_1 \succ r_2 \succ r_3 \succ r_4$
Liao *et al.*'s method [15]	$s(\tilde{r}_1) = 0.7154$, $s(\tilde{r}_2) = 0.8973$, $s(\tilde{r}_3) = -0.3024$,$s(\tilde{r}_4) = -0.8667$		$r_2 \succ r_1 \succ r_3 \succ r_4$
The proposed method	$s(\tilde{z}_1) = 0.1413$, $s(\tilde{z}_2) = -0.1413$, $s(\tilde{z}_3) = -0.6863$,$s(\tilde{z}_4) = -1.0000$		$r_1 \succ r_2 \succ r_3 \succ r_4$

Table 4
A comparison between the proposed method and the existing methods [5,15,30,31,33,34,43].

Methods	Preference relations	Checkingconsistency	Repairing inconsistency	Checking consensus	Improving consensus	Needing iterations	The forms of the obtained priority weights	Deriving the DMs' weights
Wang's method [34]	IFPRs	No	No	No	No	No	IFVs	No
Wan *et al.*'s method [33]	IVIFPRs	Yes	Yes	No	No	Yes	IVIFVs	Yes
Liao *et al.*'s method [15]	IVIFPRs	Yes	Yes	No	No	Yes	IVIFVs	No
Wan *et al.*'s method [30]	IVIFPRs	No	No	No	No	No	IVIFVs	Yes
Wan *et al.*'s method [31]	IVIFPRs	Yes	Yes	No	No	No	IFVs	Yes
Xu's method [43]	IFPRs	No	No	Yes	Yes	Yes	IFVs	No
Chu *et al.*'s method [5]	IFPRs	Yes	Yes	Yes	Yes	Yes	IFVs	No
The proposed method	IVIFPRs	Yes	Yes	Yes	Yes	No	IVIFVs	Yes

(iii) In [30], the priority vector is not obtained from the acceptable consistent IVIFPRs. However, the proposed method has the advantage that it calculates the IVIF priorities based on the acceptable consistent IVIFPRs with an acceptable consensus, which ensure that the ranking orders are reasonable.

(iv) The methods presented in [5,15] offered iterative methods for improving the multiplicative consistency, which needs more computational efforts to reach the consistency requirement. Our method improves the consistency and consensus of IVIFPRs without the iterative process.

(v) The methods presented in [15,30] disregarded the consensus analysis, such that the ranking cannot reflect the agreement degree among the opinions of the experts. The proposed method has the advantage that it considers the consensus reaching processes for IVIFPRs.

(3) A comparison with the methods presented in [15,33]: In [33], Wan *et al.* presented an example for virtual enterprise partner selection. We take $\theta = 0.1$ and $\eta = 0.9$ as the thresholds of acceptable consistency and consensus. The ranking results for different methods are displayed in Table 3. From Table 3, we can see that:

(i) The methods presented in [15,33] offered iterative methods for improving the multiplicative consistency, which needs more computational efforts to achieve the consistency requirement. Our method improves the consistency without the iterative process.

(ii) The methods presented in [15,33] only focus on improving the consistency level and cannot ensure that the DMs' most original information is preserved. The proposed method has the advantage that it can achieve three goals simultaneously, i.e., (1) reaching the acceptable consistency and consensus requirements, (2) retaining the largest amount of the original information, and (3) maximizing the consistency and consensus levels of the modified IVIFPRs.

(iii) The methods presented in [15,33] only restricted to the consistency analysis for IVIFPRs and ignored the consensus analysis for IVIFPRs. The consistency and consensus of IVIFPRs are simultaneously improved in our method.

(iv) The method presented in [15] does not consider how to derive the DMs' weights. By contrast, our method considers this issue and applies an optimization model to calculate the DMs' weights.

In summary, a comparison between our proposed approach and the existing approaches [5,15,30,31,33,34,43] is shown in Table 4.

7. Conclusions

In this paper, we have proposed an optimization-based GDM method using IVIFPRs. An additive consistency concept of IFPRs is offered. By using this concept, we defined the consistency index of IFPRs for the consistency checking and presented a novel concept of acceptable additive consistency for IFPRs. Then, through splitting an IVIFPR into two IFPRs, the concept of acceptable additive consistency is proposed in accordance with that of these two IFPRs. A consensus index is defined for

GDM by means of distance measures. Considering the case where the complete consensus may be too strict, a concept of acceptable consensus is further proposed to determine whether a sufficient consensus level has been reached. When the consistency and the consensus are unacceptable, several models are proposed to achieve acceptable additive consistency and consensus. To achieve the highest possible group consensus, a model to get the experts' weights is proposed, where the collective IVIFPR is obtained using the DMs' weights and the individual IVIFPRs. A procedure is presented to produce the priority weights. A GDM approach with IVIFPRs is presented. The proposed GDM method can conquer the shortcomings of the existing GDM methods. It offers us a useful way for GDM in the IVIF context. In the future, we will develop new GDM approaches using other kinds of preference relations [10,16,36,40]. Moreover, in recent years, some GDM methods [12,21,23] have been proposed based on various kinds of fuzzy sets. It is also worth of future research to propose new GDM methods based on the GDM methods presented in [12,21,23].

References

[1] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1) (1986) 87–96.
[2] K. Atanassov, G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets Syst. 31 (3) (1989) 343–349.
[3] J.J. Buckley, Fuzzy hierarchical analysis, Fuzzy Sets Syst. 17 (3) (1985) 233–247.
[4] S.M. Chen, C.H. Chang, Fuzzy multiattribute decision making based on transformation techniques of intuitionistic fuzzy values and intuitionistic fuzzy geometric averaging operators, Inf. Sci. 352 (2016) 133–149.
[5] J. Chu, X. Liu, Y. Wang, K.S. Chin, A group decision making model considering both the additive consistency and group consensus of intuitionistic fuzzy preference relations, Comput. Ind. Eng. 101 (2016) 227–242.
[6] J. Chu, X. Liu, L. Wang, Y. Wang, A group decision making approach based on newly defined additively consistent interval-valued intuitionistic preference relations, Int. J. Fuzzy Syst. 20 (3) (2018) 1027–1046.
[7] Y. Dong, J. Xu, H. Li, On consistency measures of linguistic preference relations, Eur. J. Oper. Res. 189 (2) (2008) 430–444.
[8] H. Garg, G. Kaur, Novel distance measures for cubic intuitionistic fuzzy sets and their applications to pattern recognitions and medical diagnosis, Granular Computing 5 (2) (2020) 169–184.
[9] H. Garg, K. Kumar, Improved possibility degree method for ranking intuitionistic fuzzy numbers and their application in multiattribute decision making, Granular Computing 4 (2) (2019) 237–247.
[10] M.J. Huang, F. Liu, Y.N. Peng, Q. Yu, Goal programming models for incomplete interval additive reciprocal preference relations with permutations, Granular Computing 5 (3) (2020) 373–386.
[11] F. Jin, Z. Ni, H. Chen, Y. Li, Approaches to group decision making with intuitionistic fuzzy preference relations based on multiplicative consistency, Knowl.-Based Syst. 97 (2016) 48–59.
[12] M.S.A. Khan, S. Abdullah, A. Ali, F. Amin, Pythagorean fuzzy prioritized aggregation operators and their application to multi-attribute group decision making, Granular Computing 4 (2) (2019) 249–263.
[13] H. Liao, Z. Xu, Priorities of intuitionistic fuzzy preference relation based on multiplicative consistency, IEEE Trans. Fuzzy Syst. 22 (2014) 1669–1681.
[14] H. Liao, Z. Xu, Consistency and consensus of intuitionistic fuzzy preference relations in group decision making, imprecision and uncertainty in information representation and processing, Studies in Fuzziness and Soft Computing 332 (2016) 189–206.
[15] H. Liao, Z. Xu, M. Xia, Multiplicative consistency of interval-valued intuitionistic fuzzy preference relation, J. Intell. Fuzzy Syst. 27 (6) (2014) 2969–2985.
[16] P. Mandal, A.S. Ranadive, Multi-granulation interval-valued fuzzy probabilistic rough sets and their corresponding three-way decisions based on interval-valued fuzzy preference relations, Granular Computing 4 (1) (2019) 89–108.
[17] F. Meng, J. Tang, F.J. Cabrerizo, E. Herrera-Viedma, A rational and consensual method for group decision making with interval-valued intuitionistic multiplicative preference relations, Engineering Applications of Artificial Intelligence 90 (2020) 103514 (https://doi.org/10.1016/j.engappai.2020.103514).
[18] F. Meng, J. Tang, P. Wang, X. Chen, A programming-based algorithm for interval-valued intuitionistic fuzzy group decision making, Knowl.-Based Syst. 144 (2018) 122–143.
[19] A.R. Mishra, A. Chandel, D. Motwani, Extended MABAC method based on divergence measures for multi-criteria assessment of programming language with interval-valued intuitionistic fuzzy sets, Granular Computing 5 (1) (2020) 97–117.
[20] A.R. Mishra, R.K. Singh, D. Motwani, Multi-criteria assessment of cellular mobile telephone service providers using intuitionistic fuzzy WASPAS method with similarity measures, Granular Computing 4 (3) (2019) 511–529.
[21] Z. Mu, S. Zeng, Q. Liu, Some interval-valued intuitionistic fuzzy Zhenyuan aggregation operators and their application to multi-attribute decision making, Int. J. Uncertainty Fuzziness Knowledge Based Syst. 26 (4) (2018) 633–653.
[22] S. Orlovsky, Decision-making with a fuzzy preference relation, Fuzzy Sets Syst. 1 (3) (1978) 155–167.
[23] K. Rahman, S. Abdullah, Generalized interval-valued Pythagorean fuzzy aggregation operators and their application to group decision making, Granular Computing 4 (1) (2019) 15–25.
[24] P. Rani, D. Jain, D.S. Hooda, Extension of intuitionistic fuzzy TODIM technique for multi-criteria decision making method based on Shapley weighted divergence measure, Granular Computing 4 (3) (2019) 407–420.
[25] T.L. Saaty, The Analytical Hierarchy Process, McGraw-Hill, New York, 1980.
[26] T.L. Saaty, L.G. Vargas, Uncertainty and rank order in the analytic hierarchy process, Eur. J. Oper. Res. 32 (1) (1987) 107–117.
[27] J. Tang, F. Meng, Linguistic intuitionistic fuzzy Hamacher aggregation operators and their application to group decision making, Granular Computing 4 (1) (2019) 109–124.
[28] J. Tang, F. Meng, Y. Zhang, Decision making with interval-valued intuitionistic fuzzy preference relations based on additive consistency analysis, Inf. Sci. 467 (2018) 115–134.
[29] P.J.M. van Laarhoven, W. Pedrycz, A fuzzy extension of Saaty's priority theory, Fuzzy Sets Syst. 11 (1–3) (1983) 229–241.
[30] S. Wan, F. Wang, J. Dong, Additive consistent interval-valued Atanassov intuitionistic fuzzy preference relation and likelihood comparison algorithm based group decision making, Eur. J. Oper. Res. 263 (2) (2017) 571–582.
[31] S. Wan, F. Wang, J. Dong, A three-phase method for group decision making with interval-valued intuitionistic fuzzy preference relations, IEEE Trans. Fuzzy Syst. 26 (2) (2018) 998–1010.
[32] S. Wan, F. Wang, J. Dong, A group decision-making method considering both the group consensus and multiplicative consistency of interval-valued intuitionistic fuzzy preference relations, Inf. Sci. 466 (2018) 109–128.
[33] S.P. Wan, G.L. Xu, J.Y. Dong, A novel method for group decision making with interval-valued Atanassov intuitionistic fuzzy preference relations, Inf. Sci. 372 (2016) 53–71.
[34] Z.J. Wang, Derivation of intuitionistic fuzzy weights based on intuitionistic fuzzy preference relations, Appl. Math. Model. 37 (9) (2013) 6377–6388.
[35] Z.J. Wang, A goal-programming-based heuristic approach to deriving fuzzy weights in analytic form from triangular fuzzy preference relations, IEEE Trans. Fuzzy Syst. 27 (2) (2019) 234–248.
[36] Z.J. Wang, J. Lin, And-like-uninorm based consistency analysis and optimized fuzzy weight closed-form solution of triangular fuzzy additive preference relations, Inf. Sci. 516 (2020) 429–452.

[37] Z.J. Wang, X. Tong, Consistency analysis and group decision making based on triangular fuzzy additive reciprocal preference relations, Inf. Sci. 361–362 (2016) 29–47.
[38] J. Wu, F. Chiclana, Non-dominance and attitudinal prioritisation methods for intuitionistic and interval-valued intuitionistic fuzzy preference relations, Expert Syst. Appl. 39 (18) (2012) 13409–134165.
[39] J. Wu, F. Chiclana, Multiplicative consistency of intuitionistic reciprocal preference relations and its application to missing values estimation and consensus building, Knowl.-Based Syst. 71 (2014) 187–200.
[40] H. Wu, P. Ren, Z. Xu, Addressing site selection for earthquake shelters with hesitant multiplicative linguistic preference relation, Inf. Sci. 516 (2020) 370–387.
[41] Z. Xu, Intuitionistic preference relations and their application in group decision making, Inf. Sci. 177 (2007) 2363–2379.
[42] Z. Xu, An error-analysis-based method for the priority of intuitionistic preference relation in decision making, Knowl.-Based Syst. 33 (3) (2012) 173–179.
[43] Z. Xu, Compatibility analysis of intuitionistic fuzzy preference relations in group decision making, Group Decis. Negot. 22 (2013) 463–482.
[44] Z. Xu, J. Chen, An approach to group decision making based on interval-valued intuitionistic judgment matrices, System Engineering and Theory Practice 27 (4) (2007) 126–133.
[45] Z. Xu, H. Liao, A survey of approaches to decision making with intuitionistic fuzzy preference relations, Knowl.-Based Syst. 80 (2015) 131–142.
[46] G. Xu, S. Wan, F. Wang, J. Dong, Y. Zeng, Mathematical programming methods for consistency and consensus in group decision making with intuitionistic fuzzy preference relations, Knowl.-Based Syst. 98 (2016) 30–43.
[47] Z. Xu, R.R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, Int. J. Gen Syst 35 (2006) 417–433.
[48] S. Zeng, S.M. Chen, K.Y. Fan, Interval-valued intuitionistic fuzzy multiple attribute decision making based on nonlinear programming methodology and TOPSIS method, Inf. Sci. 506 (2020) 424–442.
[49] S. Zeng, S.M. Chen, L.W. Kuo, Multiattribute decision making based on novel score function of intuitionistic fuzzy values and modified VIKOR method, Inf. Sci. 488 (2019) 76–92.
[50] S. Zhang, J. Tang, F. Meng, R. Yuan, A group decision making method with interval-valued intuitionistic fuzzy preference relations and its application in the selection of cloud computing vendors for SMEs, Informatica (2020), https://doi.org/10.15388/20-INFOR416.