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Group decision making with incomplete q-rung orthopair fuzzy preference relations

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ABSTRACT

In this paper, we propose a novel group decision making (GDM) method in incomplete q-rung orthopair fuzzy preference relations (q-ROFPRs) environments. We propose an additive consistency definition, which is characterized by a q-rung orthopair fuzzy priority vector. The property of the proposed additive consistency definition is offered and a model to obtain missing judgments in incomplete q-ROFPRs is proposed. We present an approach to adjust the inconsistency for q-ROFPRs, propose a model to obtain the priority vector, and propose a method to increase consensus degrees of q-ROFPRs. Finally, we present a GDM method in incomplete q-ROFPRs environments and use two illustrative examples and some comparisons to illustrate that our method outperforms the existing methods for GDM in incomplete q-ROFPRs environments. The proposed GDM method offers us a useful way for GDM in incomplete q-ROFPRs environments.

1. Introduction

The q-rung orthopair fuzzy sets (q-ROFSs) [41,43] are more general than intuitionistic fuzzy sets (IFSs) [1] and Pythagorean fuzzy sets (PFSs) [40,42]. The q-ROFSs have membership degrees and non-membership degrees, where the summation value of the qth power of the membership degree and the qth power of the non-membership degree is between 0 and 1. In this sense, q-ROFSs contain more information than IFSs and PFSs, which have attracted the attention of many researchers [14,30,31,32,44].

GDM has attracted extensive research interests in the past decades [2,6–10,13,15,16,19,21–23,27,28], whose aim is to find the optimal alternative(s) from a collection of alternatives according to the preferences of decision makers (DMs). Preference relations (PRs) are constructed by DMs to model decision making processes, where the consistency and the consensus are two important research issues of GDM [11,33,39,45,48]. At present, there are two classes of PRs: one is the basic PRs, including the multiplicative PRs [24,25], fuzzy PRs [20], linguistic PRs [4], with a certain degree to express the preference; the other is to extend the forms of these three basic PRs, including interval fuzzy PRs [5,18,38], interval multiplicative PRs [26], intuitionistic fuzzy PRs [3,29], intuitionistic multiplicative PRs [36], interval-valued intuitionistic fuzzy PRs [37], hesitant fuzzy PRs [50], hesitant multiplicative PRs [35], hesitant fuzzy linguistic PRs [49], hesitant multiplicative linguistic PRs [34], Pythagorean fuzzy PRs [17], and so on.

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As stated above, q-ROFSs are one of the most important concepts to accommodate more uncertainties than IFSs and PFSs to represent the fuzzy information. Due to its powerfulness in describing the ambiguity and the indeterminacy, Zhang et al. [46] presented the q-ROFPRs by applying q-ROFSs to PRs. The q-ROFPRs are suitable to represent preferred judgements and non-preferred judgements of DMs. At present, we find several studies about q-ROFPRs [12,46,47]. With the help of the optimization theory, two methods for inducing the priority vector from a single q-ROFPR and group q-ROFPRs are presented in [46]. Zhang et al. [47] presented a multiplicative consistency concept for q-ROFPRs and provided two models to derive the priority vector. Li et al. [12] presented a method by a new consistency measurement to forecast the missing values in an incomplete q-ROFPR.

However, the GDM methods presented in [3,29,38,46] have the following drawbacks:

(1) The GDM methods presented in [3,29,38,46] do not consider the consistency test and the consistency adjustment.
(2) In the GDM methods presented in [3,29,38,46], the weights of DMs are predefined, where there are no rules to describe how to set these weights. It should be noted that different DMs should have different weights in the group.
(3) The GDM methods presented in [3,29,38,46] have the drawback that they don't have the capability to deal with incomplete q-ROFPRs, where they cannot get the ranking orders of alternatives in incomplete q-ROFPRs environments.
(4) The GDM methods presented in [3,29,38,46] have the drawback that they don't consider the consensus reaching process of q-ROFPRs in GDM.

Therefore, we need to find a new GDM method in incomplete q-ROFPRs environments to overcome the drawbacks of the GDM methods presented in [3,29,38,46].

The main contributions of this paper are as follows:

(1) An additive consistency concept for q-ROFPRs is presented to ensure the ranking to be reasonable.
(2) A consistency index for assessing the consistency of q-ROFPRs is offered.
(3) When q-ROFPRs are inconsistent, an algorithm for repairing inconsistent judgments is proposed. Moreover, when q-ROFPRs are incomplete, a model for deriving unknown judgements is given.
(4) A method for obtaining the ranking values of alternatives from a q-ROFPR is proposed.
(5) A consensus index and a consensus reaching procedure are proposed to rationalize the decision making results because the DMs may be insufficient confidence in their judgements of q-ROFPRs.
(6) A GDM method with incomplete q-ROFPRs is proposed to guarantee reasonable ranking orders of alternatives with high consensus levels.
(7) Our method outperforms the methods provided in [3,29,38,46] for GDM in incomplete q-ROFPRs environments.

This paper is arranged as follows. Section 2 reviews some basic concepts of q-ROFSs [41,43]. In Section 3, we propose the definition of additive consistency for q-ROFPRs and discuss several attractive properties. We also propose the models for dealing with incomplete and inconsistent q-ROFPRs, where we propose a model to estimate missing values in an incomplete q-ROFPR and propose two models to form an acceptable consistent q-ROFPR from an inconsistent q-ROFPR. We also propose a method for acquiring the priority weights from a complete and acceptable consistent q-ROFPR. We also propose an algorithm to check and modify the preference values of a given q-ROFPR to reach an acceptable additive consistency. In Section 4, we perform the consensus analysis for q-ROFPRs. In Section 5, we consider two illustrative GDM examples. In Section 6, we present the conclusions.

2. Preliminaries

The definition of q-ROFSs [41,43], which generalize the concepts of intuitionistic fuzzy sets (IFSs) [1] and Pythagorean fuzzy sets (PFSs) [40,42], is reviewed as follows:

Definition 2.1 (*[41,43]*). A q-ROFS A in the universe of discourse Z is represented by $A = \{\langle z, \mu_A(z), v_A(z)\rangle | z \in Z\}$, where $\mu_A(z) \in [0,1]$ denotes the membership value of element z to the q-ROFS A, $v_A(z) \in [0,1]$ denotes the non-membership value of element z to the q-ROFS A, $z \in Z$, $0 \leqslant \mu_A^q(z) + v_A^q(z) \leqslant 1$ and $q \geqslant 1$. The grade of hesitancy of element z to the q-ROFS A is represented by $\pi_A(z) = \left(1 - \mu_A^q(z) - v_A^q(z)\right)^{1/q}$, where $z \in Z$ and $q \geqslant 1$.

Liu and Wang [14] called the pair $(\mu_A(z), v_A(z))$ a q-rung orthopair fuzzy number (q-ROFN). For convenience, we use $\alpha = (\mu_\alpha, v_\alpha)$ to represent a q-ROFN, which meets $\mu_\alpha \in [0,1]$, $v_\alpha \in [0,1]$, $0 \leqslant \mu_\alpha^q + v_\alpha^q \leqslant 1$ and $q \geqslant 1$.

Definition 2.2 (*[14]*). Let $\alpha = (\mu_\alpha, v_\alpha)$ be a q-ROFN. The score value $S(\alpha)$ of the q-ROFN $\alpha = (\mu_\alpha, v_\alpha)$ is defined as $S(\alpha) = \mu_\alpha^q - v_\alpha^q$, where $S(\alpha) \in [-1,1]$ and $q \geqslant 1$. The accuracy value $H(\alpha)$ of the q-ROFN $\alpha = (\mu_\alpha, v_\alpha)$ is defined as $H(\alpha) = \mu_\alpha^q + v_\alpha^q$, where $H(\alpha) \in [0,1]$ and $q \geqslant 1$.

Definition 2.3 (*[14]*). Let $\alpha_1 = \left(\mu_{\alpha_1}, \nu_{\alpha_1}\right)$ and $\alpha_2 = \left(\mu_{\alpha_2}, \nu_{\alpha_2}\right)$ be any two q-ROFNs, let $S(\alpha_1)$ and $S(\alpha_2)$ be the score values of the q-ROFNs α_1 and α_2, respectively, and let $H(\alpha_1)$ and $H(\alpha_2)$ be the accuracy values of the q-ROFNs α_1 and α_2, respectively. Then,

(1) If $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 > \alpha_2$.
(2) If $S(\alpha_1) = S(\alpha_2)$ and $H(\alpha_1) > H(\alpha_2)$, then $\alpha_1 > \alpha_2$.
(3) If $S(\alpha_1) = S(\alpha_2)$ and $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$.

To express the DMs' uncertain hesitancy by using q-ROFNs, Zhang *et al.* [46] constructed a q-ROFPR, shown as follows.

Definition 2.4 (*[46]*). A q-ROFPR on a set $X = \{x_1, x_2, \cdots, x_n\}$ of alternatives is represented by $R = (r_{ij})_{n\times n}$, where r_{ij} is a q-ROFN, $r_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$, $0 \leqslant \mu_{ij}^q + \nu_{ij}^q \leqslant 1$, $\mu_{ij} = \nu_{ji}$, $\nu_{ij} = \mu_{ji}$, $\mu_{ii} = \nu_{ii} = \sqrt[q]{0.5}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $q \geqslant 1$.

3. Additive consistency analysis for *q*-ROFPRs

In this section, we present the definition of additive consistency for q-ROFPRs and discuss several attractive properties. Then, we propose some models for dealing with incomplete and inconsistent q-ROFPRs, where we propose a model to estimate missing values in an incomplete q-ROFPR. Then, we propose two models to form an acceptable consistent q-ROFPR from an inconsistent q-ROFPR. Afterward, we propose a method for deriving the priorities from a complete and acceptable consistent q-ROFPR. Finally, we propose an algorithm to check and modify the preference values of a given q-ROFPR to reach an acceptable additive consistency.

3.1. Additive consistency for q-ROFPRs

Definition 3.1. A q-ROFPR $R = (r_{ij})_{n\times n}$ is additive consistent, where $r_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $q \geqslant 1$, if it satisfies

$$\mu_{ij}^q + \mu_{jk}^q + \mu_{ki}^q = \mu_{ik}^q + \mu_{kj}^q + \mu_{ji}^q, \tag{1}$$

for all $i, j, k = 1, 2, \cdots, n$ and $q \geqslant 1$.

Based on **Definition 2.4** and **Definition 3.1**, we have the following theorem.

Theorem 3.1. *For a q-ROFPR* $R = (r_{ij})_{n\times n}$, *where* $r_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ *and* $q \geqslant 1$, *the following statements are equivalent*:

(i) $\mu_{ij}^q + \mu_{jk}^q + \mu_{ki}^q = \mu_{ik}^q + \mu_{kj}^q + \mu_{ji}^q$, for all $i, j, k = 1, 2, \cdots, n$.
(ii) $\nu_{ij}^q + \nu_{jk}^q + \nu_{ki}^q = \nu_{ik}^q + \nu_{kj}^q + \nu_{ji}^q$, for all $i, j, k = 1, 2, \cdots, n$.
(iii) $\mu_{ij}^q + \mu_{jk}^q + \mu_{ki}^q = \nu_{ij}^q + \nu_{jk}^q + \nu_{ki}^q$, for all $i, j, k = 1, 2, \cdots, n$.
(iv) $\mu_{ij}^q + \mu_{jk}^q + \mu_{ki}^q = \mu_{ik}^q + \mu_{kj}^q + \mu_{ji}^q$, for all $i, j, k = 1, 2, \cdots, n$ with $i < j < k$.
(v) $\nu_{ij}^q + \nu_{jk}^q + \nu_{ki}^q = \nu_{ik}^q + \nu_{kj}^q + \nu_{ji}^q$, for all $i, j, k = 1, 2, \cdots, n$ with $i < j < k$.
(vi) $\mu_{ij}^q + \mu_{jk}^q + \mu_{ki}^q = \nu_{ij}^q + \nu_{jk}^q + \nu_{ki}^q$, for all $i, j, k = 1, 2, \cdots, n$ with $i < j < k$.
(vii) $\mu_{ij}^q + \mu_{jk}^q + \nu_{ik}^q = \nu_{ij}^q + \nu_{jk}^q + \mu_{ik}^q$, for all $i, j, k = 1, 2, \cdots, n$ with $i < j < k$.

Based on **Theorem 3.1**, we propose the definition of additive consistency of q-ROFPRs, shown as follows.

Definition 3.2. Let $R = (r_{ij})_{n\times n}$ be a q-ROFPR, where $r_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $q \geqslant 1$. The q-ROFPR R is additive consistent if

$$\mu_{ij}^q + \mu_{jk}^q + \nu_{ik}^q = \nu_{ij}^q + \nu_{jk}^q + \mu_{ik}^q, \tag{2}$$

for all $i, j, k = 1, 2, \cdots, n$ with $i < j < k$ and $q \geqslant 1$.

Considering the complete additive consistency is too strict, we propose the following definition of additive consistency index (ACI) for q-ROFPRs.

Definition 3.3. *Let $R = (r_{ij})_{n\times n}$ be a q-ROFPR, where $r_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $q \geqslant 1$. An additive consistency index $ACI(R)$ of the q-ROFPR R is defined as*

$$ACI(R) = 1 - \frac{2}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left| \mu_{ij}^q + \mu_{jk}^q + \nu_{ik}^q - \nu_{ij}^q - \nu_{jk}^q - \mu_{ik}^q \right|, \tag{3}$$

where $ACI(R) \in [0, 1]$.

Based on **Definition 3.3**, the q-ROFPR R is a q-ROFPR iff $ACI(R) = 1$. Following the proposed additive consistency index, we offer the following definition.

Definition 3.4. A q-ROFPR R is acceptable additive consistent if $ACI(R) \geqslant \eta$, where η is the given threshold and $\eta \in [0, 1]$.

3.2. Models for dealing with incomplete and inconsistent q-ROFPRs

For a q-ROFPR $R = (r_{ij})_{n\times n}$, where $r_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $q \geqslant 1$, if some judgments in the q-ROFPR R is missing, then R is called an incomplete q-ROFPR. If there are some values in $[0, 1]$ for unknown judgments which let the incomplete q-ROFPR R be additive consistent, then we can derive Eq. (2). However, we cannot guarantee that Eq. (2) holds, i.e.,

$$\mu_{ij}^q + \mu_{jk}^q + \nu_{ik}^q \neq \nu_{ij}^q + \nu_{jk}^q + \mu_{ik}^q, \tag{4}$$

for some triples of (i, j, k) with $i < j < k$, where $q \geqslant 1$. Thus, we relax Eq. (2) by introducing positive deviation variables δ_{ijk}^+ and δ_{ijk}^-, where $\delta_{ijk}^+ = \frac{\left|\mu_{ij}^q+\mu_{jk}^q+\nu_{ik}^q-\nu_{ij}^q-\nu_{jk}^q-\mu_{ik}^q\right| + \left(\mu_{ij}^q+\mu_{jk}^q+\nu_{ik}^q-\nu_{ij}^q-\nu_{jk}^q-\mu_{ik}^q\right)}{2}$, $\delta_{ijk}^- = \frac{\left|\mu_{ij}^q+\mu_{jk}^q+\nu_{ik}^q-\nu_{ij}^q-\nu_{jk}^q-\mu_{ik}^q\right| - \left(\mu_{ij}^q+\mu_{jk}^q+\nu_{ik}^q-\nu_{ij}^q-\nu_{jk}^q-\mu_{ik}^q\right)}{2}$, and

$$\mu_{ij}^q + \mu_{jk}^q + \nu_{ik}^q - \nu_{ij}^q - \nu_{jk}^q - \mu_{ik}^q - \delta_{ijk}^+ + \delta_{ijk}^- = 0, \tag{5}$$

for each triple of (i, j, k) with $i < j < k$, where $\delta_{ijk}^+ \geqslant 0$, $\delta_{ijk}^- \geqslant 0$ and $q \geqslant 1$. Furthermore, we build the following model to determine unknown values in the q-ROFPR $R = (r_{ij})_{n\times n}$, where $r_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $q \geqslant 1$, based on Eq. (5):

$$\begin{aligned} & g = \min \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left(\delta_{ijk}^+ + \delta_{ijk}^-\right) \\ & s.t. \begin{cases} \mu_{ij}^q + \mu_{jk}^q + \nu_{ik}^q - \nu_{ij}^q - \nu_{jk}^q - \mu_{ik}^q - \delta_{ijk}^+ + \delta_{ijk}^- = 0, \ i, j, k = 1, 2, \cdots, n, \ i < j < k, \\ \delta_{ijk}^+, \delta_{ijk}^- \geqslant 0, \ i, j, k = 1, 2, \cdots, n, \ i < j < k, \\ \mu_{ij} \in [0, 1], \ \mu_{ij}^q + \nu_{ij}^q \leqslant 1, \text{ if } (i, j) \in Z_\mu \text{ and } (i, j) \notin Z_\nu, \\ \nu_{ij} \in [0, 1], \ \mu_{ij}^q + \nu_{ij}^q \leqslant 1, \text{ if } (i, j) \notin Z_\mu \text{ and } (i, j) \in Z_\nu, \\ \mu_{ij}, \nu_{ij} \in [0, 1], \ \mu_{ij}^q + \nu_{ij}^q \leqslant 1, \text{ if } (i, j) \in Z_\mu \text{ and } (i, j) \in Z_\nu, \end{cases} \end{aligned} \tag{M-1}$$

where $Z_\mu = \left\{(i, j) \middle| \mu_{ij} \text{ is missing}, i, j = 1, 2, \cdots, n, \text{ and } i < j\right\}$, $Z_\nu = \left\{(i, j) | \nu_{ij} \text{ is missing}, i, j = 1, 2, \cdots, n, \text{ and } i < j\right\}$ and $q \geqslant 1$. Solving the model (M-1) yields the optimal solutions, denoted by μ_{ij}^* ($(i, j) \in Z_\mu$) and ν_{ij}^* ($(i, j) \in Z_\nu$). Based on these optimal solutions, a complete q-ROFPR $R = (r_{ij})_{n\times n}$ is built, where

$$r_{ij} = \left(\mu_{ij}, \nu_{ij}\right) = \begin{cases} \left(\mu_{ij}, \nu_{ij}\right), \text{ if } (i, j) \notin Z_\mu \text{ and } (i, j) \notin Z_\nu, \\ \left(\mu_{ij}^*, \nu_{ij}\right), \text{ if } (i, j) \in Z_\mu \text{ and } (i, j) \notin Z_\nu, \\ \left(\mu_{ij}, \nu_{ij}^*\right), \text{ if } (i, j) \notin Z_\mu \text{ and } (i, j) \in Z_\nu, \\ \left(\mu_{ij}^*, \nu_{ij}^*\right), \text{ if } (i, j) \in Z_\mu \text{ and } (i, j) \in Z_\nu, \\ \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right), \text{ if } i = j, \\ \left(\nu_{ji}, \mu_{ji}\right), \text{ if } i > j, \end{cases} \tag{6}$$

$1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n$ and $q \geqslant 1$.

Let $R = (r_{ij})_{n\times n}$ be a complete q-ROFPR, where $r_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $q \geqslant 1$. When R is acceptable consistent, we obtain

$$\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left| \mu_{ij}^q + \mu_{jk}^q + \nu_{ik}^q - \nu_{ij}^q - \nu_{jk}^q - \mu_{ik}^q \right| \leqslant \frac{n(n-1)(n-2)(1-\eta)}{2}. \tag{7}$$

Notably, q-ROFPRs offered by DMs are generally unacceptable inconsistent, namely, Eq. (7) does not hold. In order to produce the ranking orders of alternatives reasonably, we must adjust original judgments of q-ROFPRs. We present a model to acquire an acceptable consistent q-ROFPR $R' = \left(r'_{ij}\right)_{n\times n}$, where $r'_{ij} = \left(\mu'_{ij}, v'_{ij}\right)$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, shown as follows:

$$f = \min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\mu_{ij} - \mu'_{ij}\right| + \left|v_{ij} - v'_{ij}\right|\right)$$
$$s.t.\begin{cases} \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\left(\mu'_{ij}\right)^q + \left(\mu'_{jk}\right)^q + \left(v'_{ik}\right)^q - \left(v'_{ij}\right)^q - \left(v'_{jk}\right)^q - \left(\mu'_{ik}\right)^q\right| \leqslant \frac{n(n-1)(n-2)(1-\eta)}{2}, \\ 0 \leqslant \mu'_{ij} \leqslant 1,\ i,j = 1,2,\cdots,n,\ i<j, \\ 0 \leqslant v'_{ij} \leqslant 1,\ i,j = 1,2,\cdots,n,\ i<j, \\ 0 \leqslant \left(\mu'_{ij}\right)^q + \left(v'_{ij}\right)^q \leqslant 1,\ i,j = 1,2,\cdots,n,\ i<j. \end{cases} \quad \text{(M-2)}$$

Let $a^+_{ij} = \frac{\left|\mu_{ij}-\mu'_{ij}\right| + \left(\mu_{ij}-\mu'_{ij}\right)}{2}$ and $a^-_{ij} = \frac{\left|\mu_{ij}-\mu'_{ij}\right| - \left(\mu_{ij}-\mu'_{ij}\right)}{2}$. Then, we have $\left|\mu_{ij} - \mu'_{ij}\right| = a^+_{ij} + a^-_{ij}$ and $\mu_{ij} - \mu'_{ij} = a^+_{ij} - a^-_{ij}$. Let $b^+_{ij} = \frac{\left|v_{ij}-v'_{ij}\right| + \left(v_{ij}-v'_{ij}\right)}{2}$ and $b^-_{ij} = \frac{\left|v_{ij}-v'_{ij}\right| - \left(v_{ij}-v'_{ij}\right)}{2}$. Then, we have $\left|v_{ij} - v'_{ij}\right| = b^+_{ij} + b^-_{ij}$ and $v_{ij} - v'_{ij} = b^+_{ij} - b^-_{ij}$. Let $c^+_{ijk} = \frac{\left|\left(\mu'_{ij}\right)^q + \left(\mu'_{jk}\right)^q + \left(v'_{ik}\right)^q - \left(v'_{ij}\right)^q - \left(v'_{jk}\right)^q - \left(\mu'_{ik}\right)^q\right| + \left(\left(\mu'_{ij}\right)^q + \left(\mu'_{jk}\right)^q + \left(v'_{ik}\right)^q - \left(v'_{ij}\right)^q - \left(v'_{jk}\right)^q - \left(\mu'_{ik}\right)^q\right)}{2}$ and let $c^-_{ijk} = \frac{\left|\left(\mu'_{ij}\right)^q + \left(\mu'_{jk}\right)^q + \left(v'_{ik}\right)^q - \left(v'_{ij}\right)^q - \left(v'_{jk}\right)^q - \left(\mu'_{ik}\right)^q\right| - \left(\left(\mu'_{ij}\right)^q + \left(\mu'_{jk}\right)^q + \left(v'_{ik}\right)^q - \left(v'_{ij}\right)^q - \left(v'_{jk}\right)^q - \left(\mu'_{ik}\right)^q\right)}{2}$. Then, we have $\left|\left(\mu'_{ij}\right)^q + \left(\mu'_{jk}\right)^q + \left(v'_{ik}\right)^q - \left(v'_{ij}\right)^q - \left(v'_{jk}\right)^q - \left(\mu'_{ik}\right)^q\right| = c^+_{ijk} + c^-_{ijk}$ and $\left(\mu'_{ij}\right)^q + \left(\mu'_{jk}\right)^q + \left(v'_{ik}\right)^q - \left(v'_{ij}\right)^q - \left(v'_{jk}\right)^q - \left(\mu'_{ik}\right)^q = c^+_{ijk} - c^-_{ijk}$. Therefore, the model (M-2) is transformed as

$$f = \min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(a^-_{ij} + a^+_{ij} + b^-_{ij} + b^+_{ij}\right)$$
$$s.t.\begin{cases} \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left(c^-_{ijk} + c^+_{ijk}\right)\frac{n(n-1)(n-2)(1-\eta)}{2}, \\ \left(\mu_{ij} + a^-_{ij} - a^+_{ij}\right)^q + \left(\mu_{jk} + a^-_{jk} - a^+_{jk}\right)^q + \left(v_{ik} + b^-_{ik} - b^+_{ik}\right)^q - \left(v_{ij} + b^-_{ij} - b^+_{ij}\right)^q - \left(v_{jk} + b^-_{jk} - b^+_{jk}\right)^q - \\ \left(\mu_{ik} + a^-_{ik} - a^+_{ik}\right)^q + c^-_{ijk} - c^+_{ijk} = 0,\ i,j,k = 1,2,\cdots,n,\ i<j<k, \\ a^-_{ij}, a^+_{ij}, b^-_{ij}, b^+_{ij} \geqslant 0,\ i,j = 1,2,\cdots,n,\ i<j, \\ c^-_{ijk}, c^+_{ijk} \geqslant 0,\ i,j,k = 1,2,\cdots,n,\ i<j<k, \\ 0 \leqslant \mu_{ij} + a^-_{ij} - a^+_{ij} \leqslant 1,\ i,j = 1,2,\cdots,n,\ i<j, \\ 0 \leqslant v_{ij} + b^-_{ij} - b^+_{ij} \leqslant 1,\ i,j = 1,2,\cdots,n,\ i<j, \\ 0 \leqslant \left(\mu_{ij} + a^-_{ij} - a^+_{ij}\right)^q + \left(v_{ij} + b^-_{ij} - b^+_{ij}\right)^q \leqslant 1,\ i,j = 1,2,\cdots,n,\ i<j. \end{cases} \quad \text{(M-3)}$$

After solving the model (M-3), we can derive the optimal solutions, denoted by $a^{*-}_{ij}, a^{*+}_{ij}, b^{*-}_{ij}, b^{*+}_{ij}$ $(i,j = 1,2,\cdots,n,\ i<j)$, and $c^{*-}_{ijk}, c^{*+}_{ijk}$ $(i,j,k = 1,2,\cdots,n,\ i<j<k)$. Then, we can gain a modified acceptable additive consistent q-ROFPR $R' = \left(r'_{ij}\right)_{n\times n}$, where

$$r'_{ij} = \left(\mu'_{ij}, v'_{ij}\right) = \begin{cases} \left(\mu_{ij} + a^{*-}_{ij} - a^{*+}_{ij}, v_{ij} + b^{*-}_{ij} - b^{*+}_{ij}\right), & \text{if } i<j, \\ \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right), & \text{if } i=j, \\ \left(v_{ji} + b^{*-}_{ji} - b^{*+}_{ji}, \mu_{ji} + a^{*-}_{ji} - a^{*+}_{ji}\right), & \text{if } i>j, \end{cases} \quad (8)$$

$1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n$ and $q \geqslant 1$.

On the premise of keeping the minimum distance between the initial q-ROFPR and the modified q-ROFPR, we hope that the number of adjusted elements in the adjusted q-ROFPR is as small as possible. Hence, we further propose a model to modify an unacceptable additive consistent q-ROFPR $R = (r_{ij})_{n\times n}$, where $r_{ij} = \left(\mu_{ij}, v_{ij}\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $q \geqslant 1$, shown as follows:

$$\varepsilon = \max\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\lambda_{ij}+\theta_{ij})$$
$$s.t.\begin{cases}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left(c_{ijk}^{-}+c_{ijk}^{+}\right)\leqslant\frac{n(n-1)(n-2)(1-\eta)}{2},\\ \left(\mu_{ij}+a_{ij}^{-}-a_{ij}^{+}\right)^{q}+\left(\mu_{jk}+a_{jk}^{-}-a_{jk}^{+}\right)^{q}+\left(v_{ik}+b_{ik}^{-}-b_{ik}^{+}\right)^{q}-\left(v_{ij}+b_{ij}^{-}-b_{ij}^{+}\right)^{q}-\left(v_{jk}+b_{jk}^{-}-b_{jk}^{+}\right)^{q}-\\ \left(\mu_{ik}+a_{ik}^{-}-a_{ik}^{+}\right)^{q}+c_{ijk}^{-}-c_{ijk}^{+}=0,\ i,j,k=1,2,\cdots,n,\ i<j<k,\\ \mu_{ij}+a_{ij}^{-}-a_{ij}^{+}=(1-\lambda_{ij})\left(\mu_{ij}+a_{ij}^{-}-a_{ij}^{+}\right)+\lambda_{ij}\mu_{ij},\ i,j=1,2,\cdots,n,\ i<j,\\ v_{ij}+b_{ij}^{-}-b_{ij}^{+}=(1-\theta_{ij})\left(v_{ij}+b_{ij}^{-}-b_{ij}^{+}\right)+\theta_{ij}v_{ij},\ i,j=1,2,\cdots,n,\ i<j,\\ \lambda_{ij},\theta_{ij}\in\{0,1\},\ i,j=1,2,\cdots,n,\ i<j,\\ f^{*}=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(a_{ij}^{-}+a_{ij}^{+}+b_{ij}^{-}+b_{ij}^{+}\right),\\ a_{ij}^{-},a_{ij}^{+},b_{ij}^{-},b_{ij}^{+}\geqslant 0,\ i,j=1,2,\cdots,n,\ i<j,\\ c_{ijk}^{-},c_{ijk}^{+}\geqslant 0,\ i,j,k=1,2,\cdots,n,\ i<j<k,\\ 0\leqslant\mu_{ij}+a_{ij}^{-}-a_{ij}^{+}\leqslant 1,\ i,j=1,2,\cdots,n,\ i<j,\\ 0\leqslant v_{ij}+b_{ij}^{-}-b_{ij}^{+}\leqslant 1,\ i,j=1,2,\cdots,n,\ i<j,\\ 0\leqslant\left(\mu_{ij}+a_{ij}^{-}-a_{ij}^{+}\right)^{q}+\left(v_{ij}+b_{ij}^{-}-b_{ij}^{+}\right)^{q}\leqslant 1,\ i,j=1,2,\cdots,n,\ i<j,\end{cases}\tag{M-4}$$

where $a_{ij}^{-},a_{ij}^{+},b_{ij}^{-},b_{ij}^{+}$ $(i,j=1,2,\cdots,n,\ i<j)$ and c_{ijk}^{-},c_{ijk}^{+} $(i,j,k=1,2,\cdots,n,\ i<j<k)$ are shown in the model (M-3), $f^{*}=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(a_{ij}^{*-}+a_{ij}^{*+}+b_{ij}^{*-}+b_{ij}^{*+}\right)$, and $a_{ij}^{*-},a_{ij}^{*+},b_{ij}^{*-},b_{ij}^{*+}$ are the optimal values obtained by the model (M-3).

In the model (M-4), λ_{ij} is a 0–1 variable, which is used to indicate whether the membership degree μ_{ij} in the original q-ROFPR $R=(r_{ij})_{n\times n}$ is adjusted or not, where $r_{ij}=\left(\mu_{ij},v_{ij}\right)$, $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant n$. If μ_{ij} is adjusted, then $\lambda_{ij}=0$. Otherwise, $\lambda_{ij}=1$. Similarly, θ_{ij} is a 0–1 variable, which is used to indicate whether the non-membership degree v_{ij} in the original q-ROFPR $R=(r_{ij})_{n\times n}$ is adjusted or not, where $r_{ij}=\left(\mu_{ij},v_{ij}\right)$, $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant n$. If v_{ij} is adjusted, then $\theta_{ij}=0$. Otherwise, $\theta_{ij}=1$. After solving the model (M-4), we can get the optimal solutions, denoted by $a_{ij}^{\circ-},a_{ij}^{\circ+},b_{ij}^{\circ-},b_{ij}^{\circ+}$ $(i,j=1,2,\cdots,n,\ i<j)$, $c_{ijk}^{\circ-},c_{ijk}^{\circ+}$ $(i,j,k=1,2,\cdots,n,\ i<j<k)$, and $\lambda_{ij}^{\circ},\theta_{ij}^{\circ}$ $(i,j=1,2,\cdots,n,\ i<j)$. Then, we can further acquire a revised acceptable additive consistent q-ROFPR $R'=\left(r'_{ij}\right)_{n\times n}$, where

$$r'_{ij}=\left(\mu'_{ij},v'_{ij}\right)=\begin{cases}\left(\mu_{ij}+a_{ij}^{\circ-}-a_{ij}^{\circ+},v_{ij}+b_{ij}^{\circ-}-b_{ij}^{\circ+}\right), & \text{if } i<j,\\ \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right), & \text{if } i=j,\\ \left(v_{ji}+b_{ji}^{\circ-}-b_{ji}^{\circ+},\mu_{ji}+a_{ji}^{\circ-}-a_{ji}^{\circ+}\right), & \text{if } i>j,\end{cases}\tag{9}$$

where $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n$ and $q\geqslant 1$.

3.3. Deriving the priority weights from a complete and acceptable consistent q-ROFPR

This subsection aims to acquire the priority weights from a complete and acceptable consistent q-ROFPR by a programming model.

Definition 3.5. Let $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the q-rung orthopair fuzzy priority weight vector of a q-ROFPR $R=(r_{ij})_{n\times n}$, where $r_{ij}=\left(\mu_{ij},v_{ij}\right)$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n$, $q\geqslant 1$, $\omega_i=\left(\mu_{\omega_i},v_{\omega_i}\right)$ is a q-ROFN with the conditions $\mu_{\omega_i},v_{\omega_i}\in[0,1]$ and $\mu_{\omega_i}^{q}+v_{\omega_i}^{q}\leqslant 1$. The q-rung orthopair fuzzy priority weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ is called normalized if it satisfies $\sum_{j=1,j\neq i}^{n}\mu_{\omega_j}^{q}\leqslant v_{\omega_i}^{q}$ and $\mu_{\omega_i}^{q}+n-2\geqslant\sum_{j=1,j\neq i}^{n}v_{\omega_j}^{q}$, for all $i=1,2,\cdots,n$.

According to the q-rung orthopair fuzzy priority weight vector ω, we build a matrix $P=(p_{ij})_{n\times n}$, where

$$p_{ij}=\left(\mu_{p_{ij}},v_{p_{ij}}\right)=\begin{cases}\left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right), & \text{if } i=j,\\ \left(\sqrt[q]{0.5\mu_{\omega_i}^{q}+0.5v_{\omega_j}^{q}},\sqrt[q]{0.5v_{\omega_i}^{q}+0.5\mu_{\omega_j}^{q}}\right), & \text{if } i\neq j.\end{cases}\tag{10}$$

Theorem 3.2. *The matrix $P=\left(p_{ij}\right)_{n\times n}$, where p_{ij} is defined in* Eq. (10)*, is an additive consistent q-ROFPR.*

Proof (*i*). By Eq. (10), we have

$$\begin{cases} 0 \leqslant \mu_{p_{ij}} = \sqrt[q]{0.5\mu_{\omega_i}^q + 0.5v_{\omega_j}^q} \leqslant 1, \\ 0 \leqslant v_{p_{ij}} = \sqrt[q]{0.5v_{\omega_i}^q + 0.5\mu_{\omega_j}^q} \leqslant 1, \\ \mu_{p_{ii}} = v_{p_{ii}} = \sqrt[q]{0.5}, \\ \mu_{p_{ij}} = \sqrt[q]{0.5\mu_{\omega_i}^q + 0.5v_{\omega_j}^q} = \sqrt[q]{0.5v_{\omega_j}^q + 0.5\mu_{\omega_i}^q} = v_{p_{ji}}, \\ v_{p_{ij}} = \sqrt[q]{0.5v_{\omega_i}^q + 0.5\mu_{\omega_j}^q} = \sqrt[q]{0.5\mu_{\omega_j}^q + 0.5v_{\omega_i}^q} = \mu_{p_{ji}}, \\ \mu_{p_{ij}}^q + v_{p_{ij}}^q = \left(\sqrt[q]{0.5\mu_{\omega_i}^q + 0.5v_{\omega_j}^q}\right)^q + \left(\sqrt[q]{0.5v_{\omega_i}^q + 0.5\mu_{\omega_j}^q}\right)^q \\ = 0.5\mu_{\omega_i}^q + 0.5v_{\omega_j}^q + 0.5v_{\omega_i}^q + 0.5\mu_{\omega_j}^q = 0.5(\mu_{\omega_i}^q + v_{\omega_i}^q) + 0.5\left(\mu_{\omega_j}^q + v_{\omega_j}^q\right) \leqslant 1. \end{cases}$$

Following **Definition 2.4**, we can see that P is a q-ROFPR.

(ii) For all $i,j,k = 1,2,\cdots,n$ with $i < j < k$, we derive

$$\begin{aligned} \mu_{p_{ij}}^q + \mu_{p_{jk}}^q + v_{p_{ik}}^q = & \left(\sqrt[q]{0.5\mu_{\omega_i}^q + 0.5v_{\omega_j}^q}\right)^q + \left(\sqrt[q]{0.5\mu_{\omega_j}^q + 0.5v_{\omega_k}^q}\right)^q + \left(\sqrt[q]{0.5v_{\omega_i}^q + 0.5\mu_{\omega_k}^q}\right)^q \\ = & \ 0.5\mu_{\omega_i}^q + 0.5v_{\omega_j}^q + 0.5\mu_{\omega_j}^q + 0.5v_{\omega_k}^q + 0.5v_{\omega_i}^q + 0.5\mu_{\omega_k}^q \\ = & \ 0.5v_{\omega_i}^q + 0.5\mu_{\omega_j}^q + 0.5v_{\omega_j}^q + 0.5\mu_{\omega_k}^q + 0.5\mu_{\omega_i}^q + 0.5v_{\omega_k}^q \\ = & \left(\sqrt[q]{0.5v_{\omega_i}^q + 0.5\mu_{\omega_j}^q}\right)^q + \left(\sqrt[q]{0.5v_{\omega_j}^q + 0.5\mu_{\omega_k}^q}\right)^q + \left(\sqrt[q]{0.5\mu_{\omega_i}^q + 0.5v_{\omega_k}^q}\right)^q \\ = & \ v_{p_{ij}}^q + v_{p_{jk}}^q + \mu_{p_{ik}}^q. \end{aligned}$$

Following **Definition 3.2**, P is additive consistent.

From (i) and (ii), P is additive consistent. **Q. E. D.**

Theorem 3.3. *Assume that* $R = (r_{ij})_{n\times n}$ *is a q-ROFPR, where* $r_{ij} = \left(\mu_{ij}, v_{ij}\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ *and* $q \geqslant 1$, *then R is additive consistent, if there exists a normalized q-rung orthopair fuzzy priority weight vector* $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ *with* $\omega_i = \left(\mu_{\omega_i}, v_{\omega_i}\right)$, *such that*

$$\left(\mu_{ij}, v_{ij}\right) = \begin{cases} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right), & \text{if } \ i = j, \\ \left(\sqrt[q]{0.5\mu_{\omega_i}^q + 0.5v_{\omega_j}^q}, \sqrt[q]{0.5v_{\omega_i}^q + 0.5\mu_{\omega_j}^q}\right), & \text{if } \ i \neq j. \end{cases} \tag{11}$$

Based on Eq. (11), if a q-ROFPR $R = (r_{ij})_{n\times n}$ is additive consistent, where $r_{ij} = \left(\mu_{ij}, v_{ij}\right)$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, we get:

$$\begin{cases} \mu_{ij} = \sqrt[q]{0.5\mu_{\omega_i}^q + 0.5v_{\omega_j}^q}, \\ v_{ij} = \sqrt[q]{0.5v_{\omega_i}^q + 0.5\mu_{\omega_j}^q}, \end{cases} \tag{12}$$

for all $i,j = 1,2,\cdots,n$ with $i < j$. Following Eq. (12), the smaller the value of $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\mu_{ij} - \sqrt[q]{0.5\mu_{\omega_i}^q + 0.5v_{\omega_j}^q}\right| + \left|v_{ij} - \sqrt[q]{0.5v_{\omega_i}^q + 0.5\mu_{\omega_j}^q}\right|\right)$, the higher the additive consistency level will be. Therefore, we offer the following model to derive a priority weight vector ω:

$$\begin{aligned} & h = \min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\mu_{ij} - \sqrt[q]{0.5\mu_{\omega_i}^q + 0.5v_{\omega_j}^q}\right| + \left|v_{ij} - \sqrt[q]{0.5v_{\omega_i}^q + 0.5\mu_{\omega_j}^q}\right|\right) \\ & s.t. \begin{cases} \mu_{\omega_i}, v_{\omega_i} \in [0,1], \ i = 1,2,\cdots,n, \\ \mu_{\omega_i}^q + v_{\omega_i}^q \leqslant 1, \ i = 1,2,\cdots,n, \\ \sum_{j=1,j\neq i}^{n}\mu_{\omega_j}^q \leqslant v_{\omega_i}^q, \ i = 1,2,\cdots,n, \\ \mu_{\omega_i}^q + n - 2 \geqslant \sum_{j=1,j\neq i}^{n} v_{\omega_j}^q, \ i = 1,2,\cdots,n. \end{cases} \end{aligned} \tag{M-5}$$

3.4. Decision making with a q-ROFPR

Associated with the above results, a procedure to check and modify the preference values of a given q-ROFPR $R = (r_{ij})_{n\times n}$ to reach an acceptable additive consistency is shown below:

Algorithm 1.

Step 1: If the given q-ROFPR R is complete, then go to **Step 2**. Otherwise, by inserting R into the model (M-1), its optimal solutions are obtained and substituted into Eq. (6). Then, a complete q-ROFPR is derived and it is still denoted by $R = (r_{ij})_{n\times n}$, where $r_{ij} = \left(\mu_{ij}, v_{ij}\right)$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$.
Step 2: Compute the consistency index $ACI(R)$ of the q-ROFPR R via Eq. (3).
Step 3: Examine the acceptable additive consistency of R by **Definition 3.4**. If the q-ROFPR R is acceptable additive consistency, then go to **Step 4**. Otherwise, plug the q-ROFPR R into the model (M-3) and solve it to derive its optimal objective function value f^*. Then, plug R and f^* into the model (M-4) and solve it to derive its optimal solutions. Furthermore, insert these optimal solutions into Eq. (9) and obtain the adjusted q-ROFPR with the acceptable additive consistency, which is still denoted as $R = (r_{ij})_{n\times n}$, where $r_{ij} = \left(\mu_{ij}, v_{ij}\right)$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$.
Step 4: Substitute the q-ROFPR R into the model (M-5) and solve it to obtain the optimal q-rung orthopair fuzzy priority vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$.
Step 5: Calculate the score value $S(\omega_i)$ and the accuracy value $H(\omega_i)$ of ω_i $(i = 1, 2, \cdots, n)$ by **Definition 2.2**. Based on **Definition 2.3**, derive the ranking order of the alternatives x_1, x_2,..., and x_n based on the score value $S(\omega_i)$ and the accuracy value $H(\omega_i)$ of ω_i $(i = 1, 2, \cdots, n)$.

We provide an example to display the proposed **Algorithm 1** for managing an incomplete q-ROFPR.

Example 3.1. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of alternatives and let $R = (r_{ij})_{4\times 4}$ be an incomplete q-ROFPR on X, where $q = 3$, shown as follows:

$$R = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.3, 0.7) & (0.9, 0.6) & (0.6, 0.5) \\ (0.7, 0.3) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (-, -) & (0.5, 0.9) \\ (0.6, 0.9) & (-, -) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6, -) \\ (0.5, 0.6) & (0.9, 0.5) & (-, 0.6) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix}.$$

[Step 1] Substituting $R = (r_{ij})_{4\times 4}$ into the model (M-1), we can get the optimal solutions μ^*_{23}, v^*_{23} and v^*_{34}, where $\mu^*_{23} = 0.7841$, $v^*_{23} = 0.8031$ and $v^*_{34} = 0.9221$. After substituting the obtained three optimal solutions $\mu^*_{23} = 0.7841$, $v^*_{23} = 0.8031$ and $v^*_{34} = 0.9221$ into Eq. (6), we obtain a complete q-ROFPR $R = (r_{ij})_{4\times 4}$, shown as follows:

$$R = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.3, 0.7) & (0.9, 0.6) & (0.6, 0.5) \\ (0.7, 0.3) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.7841, 0.8031) & (0.5, 0.9) \\ (0.6, 0.9) & (0.8031, 0.7841) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6, 0.9221) \\ (0.5, 0.6) & (0.9, 0.5) & (0.9221, 0.6) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix},$$

where $q = 3$.

[Step 2] Based on Eq. (3), we obtain $ACI(R) = 0.8315$.
[Step 3] Let $\eta = 0.9$. Because $ACI(R) < \eta$, R is unacceptable additive consistent. Plug R into the model (M-3) and solve it, we can derive its optimal objective function value $f^* = 0.3665$. Then, plug R and $f^* = 0.3665$ into the model (M-4) and solve it, we can derive its optimal solutions, shown as follows:

$$a^{\circ-}_{12} = 0,\ a^{\circ-}_{13} = 0,\ a^{\circ-}_{14} = 0,\ a^{\circ-}_{23} = 0.0259,\ a^{\circ-}_{24} = 0, a^{\circ-}_{34} = 0, a^{\circ+}_{12} = 0, a^{\circ+}_{13} = 0, a^{\circ+}_{14} = 0.0144, a^{\circ+}_{23} = 0, a^{\circ+}_{24} = 0, a^{\circ+}_{34} = 0,$$
$$b^{\circ-}_{12} = 0, b^{\circ-}_{13} = 0, b^{\circ-}_{14} = 0, b^{\circ-}_{23} = 0, b^{\circ-}_{24} = 0, b^{\circ-}_{34} = 0, b^{\circ+}_{12} = 0.1388, b^{\circ+}_{13} = 0, b^{\circ+}_{14} = 0, b^{\circ+}_{23} = 0.0263, b^{\circ+}_{24} = 0.1066,$$
$$b^{\circ+}_{34} = 0.0545,$$
$$c^{\circ-}_{123} = 0.6, c^{\circ-}_{124} = 0.6, c^{\circ-}_{134} = 0, c^{\circ-}_{234} = 0, c^{\circ+}_{123} = 0, c^{\circ+}_{124} = 0, c^{\circ+}_{134} = 0, c^{\circ+}_{234} = 0, \lambda^{\circ}_{12} = 1,$$
$$\lambda^{\circ}_{13} = 1, \lambda^{\circ}_{14} = 0, \lambda^{\circ}_{23} = 0, \lambda^{\circ}_{24} = 1, \lambda^{\circ}_{34} = 1, \theta^{\circ}_{12} = 0,\ \theta^{\circ}_{13} = 1,\ \theta^{\circ}_{14} = 1,\ \theta^{\circ}_{23} = 0,\ \theta^{\circ}_{24} = 0,\ \theta^{\circ}_{34} = 0.$$

Furthermore, after inserting the above optimal solutions into Eq. (9), we can obtain the adjusted q-ROFPR $R = (r_{ij})_{4\times 4}$ with the acceptable additive consistency, where

$$R = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.3, 0.5612) & (0.9, 0.6) & (0.5856, 0.5) \\ (0.5612, 0.3) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.81, 0.7767) & (0.5, 0.7934) \\ (0.6, 0.9) & (0.7767, 0.81) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6, 0.8676) \\ (0.5, 0.5856) & (0.7934, 0.5) & (0.8676, 0.6) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix}.$$

[Step 4] After inserting the adjusted q-ROFPR $R = (r_{ij})_{4\times4}$ into the model (M-5) and solving it, we can obtain the priority vector ω, shown as follows:

$$\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T = ((0.539, 0.7037), (0.1706, 0.9486), (0.437, 0.9714), (0.5256, 0.6258))^T.$$

[Step 5] By **Definition 2.2**, we get $S(\omega_1) = -0.1919$, $S(\omega_2) = -0.8487$, $S(\omega_3) = -0.8331$ and $S(\omega_4) = -0.0998$. Because $S(\omega_4) > S(\omega_1) > S(\omega_3) > S(\omega_2)$, where $S(\omega_1) = -0.1919$, $S(\omega_2) = -0.8487$, $S(\omega_3) = -0.8331$ and $S(\omega_4) = -0.0998$, the ranking order of the alternatives x_1, x_2, x_3 and x_4 is: $x_4 \succ x_1 \succ x_3 \succ x_2$.

Li *et al.*'s method [12] only estimates the q-ROFN in which both the membership degree and the non-membership degree are unknown simultaneously, but it cannot estimate the q-ROFN in which only one of membership degree and non-membership degree is unknown. Therefore, Li *et al.*'s method [12] has the drawback that it cannot deal with the incomplete q-ROFPR $R = (r_{ij})_{4\times4}$ shown in **Example 3.1**.

In order to facilitate the comparison, let us utilize the presented estimation method to deal with the incomplete q-ROFPR $R = (r_{ij})_{5\times5}$ shown in **Example 7** of [12], shown as follows:

$$R = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.8, 0.3) & (0.7, 0.4) & (0.8, 0.3) & (0.5, 0.8) \\ (0.3, 0.8) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (-, -) & (0.6, 0.5) & (0.5, 0.6) \\ (0.4, 0.7) & (-, -) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (-, -) & (0.4, 0.7) \\ (0.3, 0.8) & (0.5, 0.6) & (-, -) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (-, -) \\ (0.8, 0.5) & (0.6, 0.5) & (0.7, 0.4) & (-, -) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix},$$

where $q = 3$. Based on the model (M-1), we acquire a complete q-ROFPR R, shown as follows:

$$R = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.8, 0.3) & (0.7, 0.4) & (0.8, 0.3) & (0.5, 0.8) \\ (0.3, 0.8) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.1697, 0.4931) & (0.6, 0.5) & (0.5, 0.6) \\ (0.4, 0.7) & (0.4931, 0.1697) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.5906, 0.0337) & (0.4, 0.7) \\ (0.3, 0.8) & (0.5, 0.6) & (0.0337, 0.5906) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.0281, 0.7857) \\ (0.8, 0.5) & (0.6, 0.5) & (0.7, 0.4) & (0.7857, 0.0281) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix},$$

where $q = 3$. Based on Eq. (3), we get $ACI(R) = 0.9219$.

If we use Eq. (22) presented in [12] to estimate the unknown values in the incomplete q-ROFPR R, then we gain a complete q-ROFPR $\dot{R}$, shown as follows:

$$\dot{R} = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.8, 0.3) & (0.7, 0.4) & (0.8, 0.3) & (0.5, 0.8) \\ (0.3, 0.8) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6050, 0.5207) & (0.6, 0.5) & (0.5, 0.6) \\ (0.4, 0.7) & (0.5207, 0.6050) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6869, 0.5657) & (0.4, 0.7) \\ (0.3, 0.8) & (0.5, 0.6) & (0.5657, 0.6869) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.4664, 0.4401) \\ (0.5, 0.8) & (0.6, 0.5) & (0.7, 0.4) & (0.4401, 0.4664) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix},$$

Based on Eq. (3), we get $ACI\left(\dot{R}\right) = 0.8876$.

If we use Eq. (23) presented in [12] to estimate the unknown values in the incomplete q-ROFPR R, then we gain a complete q-ROFPR $\ddot{R}$, shown as follows:

$$\ddot{R} = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.8, 0.3) & (0.7, 0.4) & (0.8, 0.3) & (0.5, 0.8) \\ (0.3, 0.8) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.5207, 0.6207) & (0.6, 0.5) & (0.5, 0.6) \\ (0.4, 0.7) & (0.6207, 0.5207) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.5657, 0.5853) & (0.4, 0.7) \\ (0.3, 0.8) & (0.5, 0.6) & (0.5853, 0.5657) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.4401, 0.7252) \\ (0.5, 0.8) & (0.6, 0.5) & (0.7, 0.4) & (0.7252, 0.4401) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix},$$

where $q = 3$. Based on Eq. (3), we get $ACI\left(\ddot{R}\right) = 0.9062$.

Clearly, the complete q-ROFPR R gained via our estimation method has a higher consistency degree (i.e., 0.9219) than the consistency degrees (i.e., 0.8876 and 0.9062) of the two complete q-ROFPRs obtained by Li *et al.*'s estimation method [12], respectively. Moreover, Li *et al.*'s estimation method [12] only can estimate the q-ROFNs in which both the membership degrees and the non-memberships are unknown simultaneously, whereas the proposed estimation method can estimate the q-ROFNs in which membership degrees or non-membership degrees or both membership degrees and non-membership degrees are not given by the DM. Thus, our estimation method is more practical to be used than Li *et al.*'s estimation method [12].

4. GDM with incomplete *q*-ROFPRs

In this section, we present a GDM method in incomplete q-ROFPRs environments. Let $X = \{x_1, x_2, \cdots, x_n\}$ be set of alternatives, and let $E = \{e_1, e_2, \cdots, e_m\}$ be a set of DMs. Assume that $R^s = \left(r_{ij}^s\right)_{n \times n}$ is the q-ROFPR offered by DM e_s, where $r_{ij}^s = \left(\mu_{ij}^s, \nu_{ij}^s\right)$ is the q-ROFN for $s = 1, 2, \cdots, m$, $i, j = 1, 2, \cdots, n$ and $q \geqslant 1$.

4.1. The consensus analysis

For the individual q-ROFPRs $R^s = \left(r_{ij}^s\right)_{n \times n}$, where $r_{ij}^s = \left(\mu_{ij}^s, \nu_{ij}^s\right)$, $s = 1, 2, \cdots, m$, $i, j = 1, 2, \cdots, n$ and $q \geqslant 1$, let $w = (w_1, w_2, \cdots, w_m)^T$ be their weight vector, where w_s denotes the weight of DM e_s, $w_s \in [0, 1]$, $s = 1, 2, \cdots, m$ and $\sum_{s=1}^m w_s = 1$. Then, the collective q-ROFPR $R^c = \left(r_{ij}^c\right)_{n \times n}$ is defined as follows:

$$r_{ij}^c = \left(\mu_{ij}^c, \nu_{ij}^c\right) = \left(\left(\sum_{s=1}^m w_s \left(\mu_{ij}^s\right)^q\right)^{1/q}, \left(\sum_{s=1}^m w_s \left(\nu_{ij}^s\right)^q\right)^{1/q}\right), \tag{13}$$

for all $i, j = 1, 2, \cdots, n$ and $q \geqslant 1$.

Note that $0 \leqslant \mu_{ij}^s, \nu_{ij}^s \leqslant 1$, $\mu_{ii}^s = \nu_{ii}^s = \sqrt[q]{0.5}$, $\mu_{ij}^s = \nu_{ji}^s$, $\nu_{ij}^s = \mu_{ji}^s$, and $\left(\mu_{ij}^s\right)^q + \left(\nu_{ij}^s\right)^q \leqslant 1$, for each pair of (i, j). Thus, we obtain

$$\begin{cases} 0 \leqslant \mu_{ij}^c = \left(\sum_{s=1}^m w_s \left(\mu_{ij}^s\right)^q\right)^{1/q} \leqslant \left(\sum_{s=1}^m w_s 1^q\right)^{1/q} = 1, \\ 0 \leqslant \nu_{ij}^c = \left(\sum_{s=1}^m w_s \left(\nu_{ij}^s\right)^q\right)^{1/q} \leqslant \left(\sum_{s=1}^m w_s 1^q\right)^{1/q} = 1, \\ \mu_{ii}^c = \left(\sum_{s=1}^m w_s (\mu_{ii}^s)^q\right)^{1/q} = \left(\sum_{s=1}^m w_s \left(\sqrt[q]{0.5}\right)^q\right)^{1/q} = \sqrt[q]{0.5}, \\ \nu_{ij}^c = \left(\sum_{s=1}^m w_s \left(\nu_{ij}^s\right)^q\right)^{1/q} = \left(\sum_{s=1}^m w_s \left(\sqrt[q]{0.5}\right)^q\right)^{1/q} = \sqrt[q]{0.5}, \\ \mu_{ij}^c = \left(\sum_{s=1}^m w_s \left(\mu_{ij}^s\right)^q\right)^{1/q} = \left(\sum_{s=1}^m w_s \left(\nu_{ji}^s\right)^q\right)^{1/q} = \nu_{ji}^c, \\ \nu_{ij}^c = \left(\sum_{s=1}^m w_s \left(\nu_{ij}^s\right)^q\right)^{1/q} = \left(\sum_{s=1}^m w_s \left(\mu_{ji}^s\right)^q\right)^{1/q} = \mu_{ji}^c, \\ \left(\mu_{ij}^c\right)^q + \left(\nu_{ij}^c\right)^q = \left(\left(\sum_{s=1}^m w_s \left(\mu_{ij}^s\right)^q\right)^{1/q}\right)^q + \left(\left(\sum_{s=1}^m w_s \left(\nu_{ij}^s\right)^q\right)^{1/q}\right)^q \\ = \sum_{s=1}^m w_s \left(\mu_{ij}^s\right)^q + \sum_{s=1}^m w_s \left(\nu_{ij}^s\right)^q = \sum_{s=1}^m w_s \left(\left(\mu_{ij}^s\right)^q + \left(\nu_{ij}^s\right)^q\right) \leqslant \sum_{s=1}^m w_s = 1. \end{cases}$$

Thus, R^c is a q-ROFPR.

Theorem 4.1. *Let R^s be a q-ROFPR, where $s = 1, 2, \cdots, m$. Then, their collective q-ROFPR R^c introduced in Eq. (13) is acceptable additive consistent if all q-ROFPRs R^s, where $s = 1, 2, \cdots, m$, are acceptable additive consistent, namely, $ACI(R^s) \geqslant \eta$ for all $s = 1, 2, \cdots, m$, with η being the given additive consistency threshold.*

Proof. As per Eq. (13) and the acceptable consistency of R^s, we have

$$\begin{aligned}
ACI(R^c) =\ & 1 - \tfrac{2}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left| \left(\mu_{ij}^c\right)^q + \left(\mu_{jk}^c\right)^q + (\nu_{ik}^c)^q - \left(\nu_{ij}^c\right)^q - \left(\nu_{jk}^c\right)^q - (\mu_{ik}^c)^q \right| \\
=\ & 1 - \tfrac{2}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left| \begin{array}{l} \left(\left(\sum_{s=1}^m w_s \left(\mu_{ij}^s\right)^q\right)^{1/q}\right)^q + \left(\left(\sum_{s=1}^m w_s \left(\mu_{jk}^s\right)^q\right)^{1/q}\right)^q + \left(\left(\sum_{s=1}^m w_s (\nu_{ik}^s)^q\right)^{1/q}\right)^q - \\ \left(\left(\sum_{s=1}^m w_s \left(\nu_{ij}^s\right)^q\right)^{1/q}\right)^q - \left(\left(\sum_{s=1}^m w_s \left(\nu_{jk}^s\right)^q\right)^{1/q}\right)^q - \left(\left(\sum_{s=1}^m w_s (\mu_{ik}^s)^q\right)^{1/q}\right)^q \end{array} \right| \\
=\ & 1 - \tfrac{2}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left| \begin{array}{l} \sum_{s=1}^m \left(w_s \left(\mu_{ij}^s\right)^q\right) + \sum_{s=1}^m \left(w_s \left(\mu_{jk}^s\right)^q\right) + \sum_{s=1}^m \left(w_s (\nu_{ik}^s)^q\right) - \\ \sum_{s=1}^m \left(w_s \left(\nu_{ij}^s\right)^q\right) - \sum_{s=1}^m \left(w_s \left(\nu_{jk}^s\right)^q\right) - \sum_{s=1}^m \left(w_s (\mu_{ik}^s)^q\right) \end{array} \right| \\
=\ & 1 - \tfrac{2}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left| \sum_{s=1}^m \left(w_s \left(\left(\mu_{ij}^s\right)^q + \left(\mu_{jk}^s\right)^q + (\nu_{ik}^s)^q - \left(\nu_{ij}^s\right)^q - \left(\nu_{jk}^s\right)^q - (\mu_{ik}^s)^q\right)\right) \right| \\
\geqslant\ & 1 - \tfrac{2}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left(w_s \sum_{s=1}^m \left| \left(\mu_{ij}^s\right)^q + \left(\mu_{jk}^s\right)^q + (\nu_{ik}^s)^q - \left(\nu_{ij}^s\right)^q - \left(\nu_{jk}^s\right)^q - (\mu_{ik}^s)^q \right| \right) \\
=\ & \sum_{s=1}^m \left(w_s \left(1 - \tfrac{2}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left| \left(\mu_{ij}^s\right)^q + \left(\mu_{jk}^s\right)^q + (\nu_{ik}^s)^q - \left(\nu_{ij}^s\right)^q - \left(\nu_{jk}^s\right)^q - (\mu_{ik}^s)^q \right| \right) \right) \\
=\ & \sum_{s=1}^m (w_s ACI(R^s)) \geqslant \sum_{s=1}^m (w_s \eta) = \eta.
\end{aligned}$$

Following **Definition 3.4**, it is concluded that R^c is acceptable additive consistent. **Q. E. D.**

According to the individual q-ROFPR and the collective q-ROFPR, the consensus index is proposed as follows.

Definition 4.1. Let $R^s = \left(r_{ij}^s\right)_{n\times n} (s = 1, 2, \cdots, m)$ be m q-ROFPRs, where $r_{ij}^s = \left(\mu_{ij}^s, \nu_{ij}^s\right)$, and let $w = (w_1, w_2, \cdots, w_m)^T$ be their weight vector. Let $R^c = \left(r_{ij}^c\right)_{n\times n}$ be the collective q-ROFPR of the q-ROFPRs R^1, R^2, …, and R^m, where $r_{ij}^c = \left(\mu_{ij}^c, \nu_{ij}^c\right)$. The consensus index $GCI(R^s)$ of the q-ROFPR R^s is of defined as follows:

$$GCI(R^s) = 1 - \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left|\mu_{ij}^s - \mu_{ij}^c\right| + \left|\nu_{ij}^s - \nu_{ij}^c\right| \right), \tag{14}$$

where $s = 1, 2, \cdots, m$.

Because $0 \leqslant \left|\mu_{ij}^s - \mu_{ij}^c\right|, \left|\nu_{ij}^s - \nu_{ij}^c\right| \leqslant 1$, we obtain $0 \leqslant GCI(R^s) \leqslant 1$ for each $s = 1, 2, \cdots, m$. Furthermore, $GCI(R^s) = 1$ if and only if $R^s = R^t$ for all $s, t = 1, 2, \cdots, m$.

Let τ be the given consensus threshold. To rank alternatives from acceptable additive consistent q-ROFPRs R'^s $(s = 1, 2, \cdots, m)$ with the given consensus level, we offer the following model:

$$p = \min \sum_{s=1}^m \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\phi_{ij}^{s-} + \phi_{ij}^{s+} + \varphi_{ij}^{s-} + \varphi_{ij}^{s+}\right)$$

$$\begin{cases}
GCI(R'^s) \geqslant \tau, s = 1, 2, \cdots, m, \\
ACI(R'^s) \geqslant \eta, s = 1, 2, \cdots, m, \\
R'^s = \left(r'^s_{ij}\right)_{n\times n}, s = 1, 2, \cdots, m, \\
r'^s_{ij} = \left(\mu'^s_{ij}, \nu'^s_{ij}\right) = \left(\mu_{ij}^s + \phi_{ij}^{s-} - \phi_{ij}^{s+}, \nu_{ij}^s + \varphi_{ij}^{s-} - \varphi_{ij}^{s+}\right),\ i,j = 1, 2, \cdots, n,\ i < j,\ s = 1, 2, \cdots, m, \\
R'^c = \left(r'^c_{ij}\right)_{n\times n}, \\
r'^c_{ij} = \left(\mu'^c_{ij}, \nu'^c_{ij}\right) = \left(\left(\sum_{s=1}^m w_s \left(\mu'^s_{ij}\right)^q\right)^{1/q}, \left(\sum_{s=1}^m w_s \left(\nu'^s_{ij}\right)^q\right)^{1/q}\right), i,j = 1, 2, \cdots, n, i < j, \\
0 \leqslant \mu_{ij}^s + \phi_{ij}^{s-} - \phi_{ij}^{s+} \leqslant 1, i,j = 1, 2, \cdots, n, i < j, s = 1, 2, \cdots, m, \\
0 \leqslant \nu_{ij}^s + \varphi_{ij}^{s-} - \varphi_{ij}^{s+} \leqslant 1, i,j = 1, 2, \cdots, n, i < j, s = 1, 2, \cdots, m, \\
0 \leqslant \left(\mu_{ij}^s + \phi_{ij}^{s-} - \phi_{ij}^{s+}\right)^q + \left(\nu_{ij}^s + \varphi_{ij}^{s-} - \varphi_{ij}^{s+}\right)^q \leqslant 1, i,j = 1, 2, \cdots, n,\ i < j, s = 1, 2, \cdots, m, \\
\phi_{ij}^{s-}, \phi_{ij}^{s+}, \varphi_{ij}^{s-}, \varphi_{ij}^{s+} \geqslant 0, i,j = 1, 2, \cdots, n, i < j, s = 1, 2, \cdots, m,
\end{cases} \tag{M-6}$$

where $\phi_{ij}^{s+}=\frac{\left|\mu_{ij}^{s}-\mu_{ij}^{s}\right|+\left(\mu_{ij}^{s}-\mu_{ij}^{s}\right)}{2}$, $\phi_{ij}^{s-}=\frac{\left|\mu_{ij}^{s}-\mu_{ij}^{s}\right|-\left(\mu_{ij}^{s}-\mu_{ij}^{s}\right)}{2}$, $\varphi_{ij}^{s+}=\frac{\left|v_{ij}^{s}-v_{ij}^{s}\right|+\left(v_{ij}^{s}-v_{ij}^{s}\right)}{2}$, $\varphi_{ij}^{s-}=\frac{\left|v_{ij}^{s}-v_{ij}^{s}\right|-\left(v_{ij}^{s}-v_{ij}^{s}\right)}{2}$, $\left|\mu_{ij}^{s}-\mu_{ij}^{s}\right|=\phi_{ij}^{s+}+\phi_{ij}^{s-}$, $\mu_{ij}^{s}-\mu_{ij}^{s}=\phi_{ij}^{s+}-\phi_{ij}^{s-}$, $\left|v_{ij}^{s}-v_{ij}^{s}\right|=\varphi_{ij}^{s+}+\varphi_{ij}^{s-}$, $v_{ij}^{s}-v_{ij}^{s}=\varphi_{ij}^{s+}-\varphi_{ij}^{s-}$, $i,j=1,2,\cdots,n,\ i<j,\ s=1,2,\cdots,m$, and $q\geqslant 1$.

To solve model (M-6), we convert it into the following model by means of Eqs. (3) and (14):

$$p=\min\sum_{s=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\phi_{ij}^{s-}+\phi_{ij}^{s+}+\varphi_{ij}^{s-}+\varphi_{ij}^{s+}\right)$$
$$\begin{cases}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\mu_{ij}^{s}-\mu_{ij}^{c}\right|+\left|v_{ij}^{s}-v_{ij}^{c}\right|\right)\leqslant n(n-1)(1-\tau),s=1,2,\cdots,m,\\ \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\left(\mu_{ij}^{s}\right)^{q}+\left(\mu_{jk}^{s}\right)^{q}+\left(v_{ik}^{s}\right)^{q}-\left(v_{ij}^{s}\right)^{q}-\left(v_{jk}^{s}\right)^{q}-\left(\mu_{ik}^{s}\right)^{q}\right|\leqslant\frac{n(n-1)(n-2)(1-\eta)}{2},s=1,2,\cdots,m,\\ R^{s}=\left(r_{ij}^{s}\right)_{n\times n},s=1,2,\cdots,m,\\ r_{ij}^{s}=\left(\mu_{ij}^{s},v_{ij}^{s}\right)=\left(\mu_{ij}^{s}+\phi_{ij}^{s-}-\phi_{ij}^{s+},v_{ij}^{s}+\varphi_{ij}^{s-}-\varphi_{ij}^{s+}\right),i,j=1,2,\cdots,n,i<j,s=1,2,\cdots,m,\\ R^{c}=\left(r_{ij}^{c}\right)_{n\times n},\\ r_{ij}^{c}=\left(\mu_{ij}^{c},v_{ij}^{c}\right)=\left(\left(\sum_{s=1}^{m}w_{s}\left(\mu_{ij}^{s}\right)^{q}\right)^{1/q},\left(\sum_{s=1}^{m}w_{s}\left(v_{ij}^{s}\right)^{q}\right)^{1/q}\right),i,j=1,2,\cdots,n,i<j,\\ 0\leqslant\mu_{ij}^{s}+\phi_{ij}^{s-}-\phi_{ij}^{s+}\leqslant 1,i,j=1,2,\cdots,n,i<j,s=1,2,\cdots,m,\\ 0\leqslant v_{ij}^{s}+\varphi_{ij}^{s-}-\varphi_{ij}^{s+}\leqslant 1,i,j=1,2,\cdots,n,i<j,s=1,2,\cdots,m,\\ 0\leqslant\left(\mu_{ij}^{s}+\phi_{ij}^{s-}-\phi_{ij}^{s+}\right)^{q}+\left(v_{ij}^{s}+\varphi_{ij}^{s-}-\varphi_{ij}^{s+}\right)^{q}\leqslant 1,i,j=1,2,\cdots,n,i<j,s=1,2,\cdots,m,\\ \phi_{ij}^{s-},\phi_{ij}^{s+},\varphi_{ij}^{s-},\varphi_{ij}^{s+}\geqslant 0,i,j=1,2,\cdots,n,i<j,s=1,2,\cdots,m,\end{cases}\tag{M-7}$$

where the first constraint is the consensus requirement, the second constraint is the acceptable additive consistency condition, and the remaining constraints ensure the modified q-ROFPRs to be still q-ROFPRs.

After solving the model (M-7), we can acquire the optimal solutions, denoted by $\phi_{ij}^{*s-},\phi_{ij}^{*s+},\varphi_{ij}^{*s-},\varphi_{ij}^{*s+}$. Then, we can obtain the adjusted acceptable additive consistent q-ROFPRs $R^{s}=\left(r_{ij}^{s}\right)_{n\times n}$ $(s=1,2,\cdots,m)$ with the given consensus level, where

$$r_{ij}^{s}=\left(\mu_{ij}^{s},v_{ij}^{s}\right)=\begin{cases}\left(\mu_{ij}^{s}+\phi_{ij}^{*s-}-\phi_{ij}^{*s+},v_{ij}^{s}+\varphi_{ij}^{*s-}-\varphi_{ij}^{*s+}\right),\text{ if } i<j,\\ \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right),\text{ if } i=j,\\ \left(v_{ji}^{s}+\varphi_{ji}^{*s-}-\varphi_{ji}^{*s+},\mu_{ji}^{s}+\phi_{ji}^{*s-}-\phi_{ji}^{*s+}\right),\text{ if } i>j,\end{cases}\tag{15}$$

where $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n$ and $q\geqslant 1$.

We further offer the following model to adjust individual q-ROFPRs:

$$\vartheta=\max\sum_{s=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\lambda_{ij}^{s}+\theta_{ij}^{s}\right)$$
$$\begin{cases}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\mu_{ij}^{s}-\mu_{ij}^{c}\right|+\left|v_{ij}^{s}-v_{ij}^{c}\right|\right)\leqslant n(n-1)(1-\tau),s=1,2,\cdots,m,\\ \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\left(\mu_{ij}^{s}\right)^{q}+\left(\mu_{jk}^{s}\right)^{q}+\left(v_{ik}^{s}\right)^{q}-\left(v_{ij}^{s}\right)^{q}-\left(v_{jk}^{s}\right)^{q}-\left(\mu_{ik}^{s}\right)^{q}\right|\leqslant\frac{n(n-1)(n-2)(1-\eta)}{2},s=1,2,\cdots,m,\\ R^{s}=\left(r_{ij}^{s}\right)_{n\times n},s=1,2,\cdots,m,\\ r_{ij}^{s}=\left(\mu_{ij}^{s},v_{ij}^{s}\right)=\left(\mu_{ij}^{s}+\phi_{ij}^{s-}-\phi_{ij}^{s+},v_{ij}^{s}+\varphi_{ij}^{s-}-\varphi_{ij}^{s+}\right),i,j=1,2,\cdots,n,i<j,s=1,2,\cdots,m,\\ R^{c}=\left(r_{ij}^{c}\right)_{n\times n},\\ r_{ij}^{c}=\left(\mu_{ij}^{c},v_{ij}^{c}\right)=\left(\left(\sum_{s=1}^{m}w_{s}\left(\mu_{ij}^{s}\right)^{q}\right)^{1/q},\left(\sum_{s=1}^{m}w_{s}\left(v_{ij}^{s}\right)^{q}\right)^{1/q}\right),i,j=1,2,\cdots,n,i<j,\\ \mu_{ij}^{s}+\phi_{ij}^{s-}-\phi_{ij}^{s+}=\left(1-\lambda_{ij}^{s}\right)\left(\mu_{ij}^{s}+\phi_{ij}^{s-}-\phi_{ij}^{s+}\right)+\lambda_{ij}^{s}\mu_{ij}^{s},i,j=1,2,\cdots,n,i<j,s=1,2,\cdots,m,\\ v_{ij}^{s}+\varphi_{ij}^{s-}-\varphi_{ij}^{s+}=\left(1-\theta_{ij}^{s}\right)\left(v_{ij}^{s}+\varphi_{ij}^{s-}-\varphi_{ij}^{s+}\right)+\theta_{ij}^{s}v_{ij}^{s},i,j=1,2,\cdots,n,i<j,s=1,2,\cdots,m,\\ \lambda_{ij}^{s},\theta_{ij}^{s}\in\{0,1\},i,j=1,2,\cdots,n,i<j,s=1,2,\cdots,m,\\ p^{*}=\sum_{s=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\phi_{ij}^{s-}+\phi_{ij}^{s+}+\varphi_{ij}^{s-}+\varphi_{ij}^{s+}\right),\\ 0\leqslant\mu_{ij}^{s}+\phi_{ij}^{s-}-\phi_{ij}^{s+}\leqslant 1,i,j=1,2,\cdots,n,i<j,\ s=1,2,\cdots,m,\\ 0\leqslant v_{ij}^{s}+\varphi_{ij}^{s-}-\varphi_{ij}^{s+}\leqslant 1,i,j=1,2,\cdots,n,i<j,\ s=1,2,\cdots,m,\\ 0\leqslant\left(\mu_{ij}^{s}+\phi_{ij}^{s-}-\phi_{ij}^{s+}\right)^{q}+\left(v_{ij}^{s}+\varphi_{ij}^{s-}-\varphi_{ij}^{s+}\right)^{q}\leqslant 1,\ i,j=1,2,\cdots,n,i<j,\ s=1,2,\cdots,m,\\ \phi_{ij}^{s-},\phi_{ij}^{s+},\varphi_{ij}^{s-},\varphi_{ij}^{s+}\geqslant 0,i,j=1,2,\cdots,n,i<j,\ s=1,2,\cdots,m,\end{cases}\tag{M-8}$$

where $\phi_{ij}^{s-},\phi_{ij}^{s+},\varphi_{ij}^{s-},\varphi_{ij}^{s+}$ are shown in model (M-7) and p^{*} is the optimal objective value of model (M-7).

Solving the model (M-8) derives its optimal solutions, denoted by $\phi_{ij}^{\circ s-}, \phi_{ij}^{\circ s+}, \varphi_{ij}^{\circ s-}, \varphi_{ij}^{\circ s+}$. Then, we can obtain the adjusted acceptable additive consistent q-ROFPR $R^{s} = \left(r_{ij}^{s}\right)_{n\times n}$ with the given consensus level, where

$$r_{ij}^{s} = \left(\mu_{ij}^{s}, \nu_{ij}^{s}\right) = \begin{cases} \left(\mu_{ij}^{s} + \phi_{ij}^{\circ s-} - \phi_{ij}^{\circ s+}, \nu_{ij}^{s} + \varphi_{ij}^{\circ s-} - \varphi_{ij}^{\circ s+}\right), & \text{if } i < j, \\ \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right), & \text{if } i = j, \\ \left(\nu_{ji}^{s} + \varphi_{ji}^{\circ s-} - \varphi_{ji}^{\circ s+}, \mu_{ji}^{s} + \phi_{ji}^{\circ s-} - \phi_{ji}^{\circ s+}\right), & \text{if } i > j, \end{cases} \tag{16}$$

where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $s = 1, 2, \cdots, m$ and $q \geqslant 1$.

4.2. An algorithm to GDM with q-ROFPRs

Using the above analysis, this part proposes an algorithm, called **Algorithm 2**, to GDM with q-ROFPRs, shown as follows:

Algorithm 2.

Step 1: Let $R^{s} = \left(r_{ij}^{s}\right)_{n\times n}$ be the individual q-ROFPR offered by the DM e_s, where $s = 1, 2, \cdots, m$. If all q-ROFPRs R^{s} ($s = 1, 2, \cdots, m$) are complete, then go to **Step 2**. If R^{s} is incomplete, then plug R^{s} into the model (M-1) and solve this model. Insert the obtained optimal solutions into Eq. (6) and get a complete q-ROFPR, which is still denoted by $R^{s} = \left(r_{ij}^{s}\right)_{n\times n}$. Repeat this process until all q-ROFPRs R^{s} ($s = 1, 2, \cdots, m$) are complete.

Step 2: Let η be the given threshold of acceptable additive consistency. For each complete q-ROFPR $R^{s} = \left(r_{ij}^{s}\right)_{n\times n}$, where $s = 1, 2, \cdots, m$, Eq. (3) is used to calculate its consistency index. If all of them are acceptable additive consistent, skip to **Step 3**. Otherwise, plug R^{s} with unacceptable additive consistency into the model (M-3) and solve this model to derive its optimal objective function value. Then, plug R^{s} and this optimal objective function value into the model (M-4) and solve this model to derive its optimal solutions. Furthermore, insert these optimal solutions into Eq. (9) and obtain the adjusted q-ROFPR with the acceptable additive consistency, which is still denoted as R^{s}.

Step 3: The following equation is presented to acquire the DMs' weights:

$$w_{e_s} = \frac{1/\left(\sum_{t=1,t\neq s}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\mu_{ij}^{s} - \mu_{ij}^{t}\right| + \left|\nu_{ij}^{s} - \nu_{ij}^{t}\right|\right)\right)}{\sum_{s=1}^{m}\left(1/\left(\sum_{t=1,t\neq s}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\mu_{ij}^{s} - \mu_{ij}^{t}\right| + \left|\nu_{ij}^{s} - \nu_{ij}^{t}\right|\right)\right)\right)}, \tag{17}$$

where $s = 1, 2, \cdots, m$. Based on Eq. (13), calculate the collective q-ROFPR R^{c}.

Step 4: Let τ be the predetermined consensus threshold. If $\min_{1\leqslant l\leqslant m}\left\{GCI\left(R^{l}\right)\right\} \geqslant \tau$, skip to **Step 5**. Otherwise, substitute R^{s} ($s = 1, 2, \cdots, m$) into the model (M-7) and solve this model to derive its optimal objective function value. Then, insert R^{s} ($s = 1, 2, \cdots, m$) and this optimal objective function value into the model (M-8) and solve this model to derive its optimal solutions. Moreover, insert these optimal solutions into Eq. (16) and obtain the adjusted q-ROFPRs with the acceptable additive consistency and acceptable consensus, which is still denoted by R^{s} ($s = 1, 2, \cdots, m$).

Step 5: Compute the collective q-ROFPR R^{c} by aggregating q-ROFPRs R^{s} ($s = 1, 2, \cdots, m$) via Eq. (13). Plugging R^{c} into model (M-5), a q-rung orthopair fuzzy priority vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{T}$ can be derived from R^{c}.

Step 6: Definition 2.2 is applied to get the scores and accuracies of ω_i, based on which, **Definition 2.3** is utilized to rank ω_i. Based on the score value of ω_i ($i = 1, 2, \cdots, n$), we obtain the ranking order of alternatives x_i ($i = 1, 2, \cdots, n$). The greater ω_i is, the more important the alternative x_i ($i = 1, 2, \cdots, n$).

5. Illustrative examples and comparison analysis

Example 5.1. Assume that there are four alternatives x_1, x_2, x_3 and x_4 and assume that four DMs e_1, e_2, e_3 and e_4 are invited to form a group to provide their evaluations with respect to the alternatives x_1, x_2, x_3 and x_4 by performing pairwise comparisons between any two alternatives. The DMs e_1, e_2, e_3 and e_4 compare each pair of alternatives and give their judgments with incomplete q-ROFPRs R^1, R^2, R^3 and R^4, respectively, where $R^{s} = \left(r_{ij}^{s}\right)_{4\times 4}$ ($s = 1, 2, 3, 4$), shown as follows:

$$R^1 = \begin{pmatrix} (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (-,-) & (0.8, 0.6) & (0.7, 0.9) \\ (-,-) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (-, 0.4) & (0.3, 0.5) \\ (0.6, 0.8) & (0.4, -) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.6, -) \\ (0.9, 0.7) & (0.5, 0.3) & (-, 0.6) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) \end{pmatrix},$$

$$R^2 = \begin{pmatrix} (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.5, 0.9) & (-,-) & (0.7, 0.2) \\ (0.9, 0.5) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.7, 0.1) & (0.8, 0.1) \\ (-,-) & (0.1, 0.7) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (-,-) \\ (0.2, 0.7) & (0.1, 0.8) & (-,-) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) \end{pmatrix},$$

$$R^3 = \begin{pmatrix} (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.6, 0.8) & (-,-) & (0.7, 0.6) \\ (0.8, 0.6) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (-, 0.3) & (0.5, 0.9) \\ (-,-) & (0.3, -) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.8, 0.7) \\ (0.6, 0.7) & (0.9, 0.5) & (0.7, 0.8) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) \end{pmatrix},$$

$$R^4 = \begin{pmatrix} (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (-,-) & (0.6, -) & (0.7, 0.8) \\ (-,-) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.7, 0.1) & (0.6, 0.9) \\ (-, 0.6) & (0.1, 0.7) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (-,-) \\ (0.8, 0.7) & (0.9, 0.6) & (-,-) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) \end{pmatrix},$$

where the symbol "–" denotes an unknown value *Let q = 4*. **[Step 1]** Plugging R^s $(s = 1, 2, 3, 4)$ into the model (M-1) and solving it, the optimal solutions are obtained as follows:

$\mu_{12}^{*1} = 0.7517$, $v_{12}^{*1} = 0.9084$, $\mu_{23}^{*1} = 0.9038$, $v_{34}^{*1} = 0.9532$, $\mu_{13}^{*2} = 0.9532$, $v_{13}^{*2} = 0.826$, $\mu_{34}^{*2} = 0.866$, $v_{34}^{*2} = 0.7917$, $\mu_{13}^{*3} = 0.8282$, $v_{13}^{*3} = 0.853$, $\mu_{23}^{*3} = 0.6918$, $\mu_{12}^{*4} = 0.9076$, $v_{12}^{*4} = 0.753$, $v_{13}^{*4} = 0$, $\mu_{34}^{*4} = 0.6804$, $v_{34}^{*4} = 0.9415$.

Substituting these optimal solutions into Eq. (6), complete q-ROFPRs $R^s = \left(r_{ij}^s\right)_{4\times 4}$ $(s = 1, 2, 3, 4)$ are obtained as follows:

$$R^1 = \begin{pmatrix} (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.7517, 0.9084) & (0.8, 0.6) & (0.7, 0.9) \\ (0.9084, 0.7517) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.9038, 0.4) & (0.3, 0.5) \\ (0.6, 0.8) & (0.4, 0.9038) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.6, 0.9532) \\ (0.9, 0.7) & (0.5, 0.3) & (0.9532, 0.6) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) \end{pmatrix},$$

$$R^2 = \begin{pmatrix} (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.5, 0.9) & (0.855, 0.826) & (0.7, 0.2) \\ (0.9, 0.5) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.7, 0.1) & (0.8, 0.1) \\ (0.826, 0.855) & (0.1, 0.7) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) & (0.866, 0.7917) \\ (0.2, 0.7) & (0.1, 0.8) & (0.7917, 0.866) & (\sqrt[q]{0.5}, \sqrt[q]{0.5}) \end{pmatrix},$$

$$R^3 = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6, 0.8) & (0.8282, 0.853) & (0.7, 0.6) \\ (0.8, 0.6) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6918, 0.3) & (0.5, 0.9) \\ (0.853, 0.8282) & (0.3, 0.6918) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.8, 0.7) \\ (0.6, 0.7) & (0.9, 0.5) & (0.7, 0.8) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix},$$

$$R^4 = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.9076, 0.753) & (0.6, 0) & (0.7, 0.8) \\ (0.753, 0.9076) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.7, 0.1) & (0.6, 0.9) \\ (0, 0.6) & (0.1, 0.7) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6804, 0.9415) \\ (0.8, 0.7) & (0.9, 0.6) & (0.9415, 0.6804) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix}.$$

[Step 2] Based on Eq. (3), we have $ACI\left(R^1\right) = 1$, $ACI\left(R^2\right) = 0.9296$, $ACI\left(R^3\right) = 0.836$, and $ACI\left(R^4\right) = 0.9221$. Let $\eta = 0.9$. All of q-ROFPRs are acceptable additive consistent except for the q-ROFPR R^3. Plug R^3 into the model (M-3) and solve this model to derive its optimal objective function value $f^* = 0.1778$. Then, plug R^3 and $f^* = 0.1778$ into the model (M-4) and solve this model to derive its optimal solutions, which are as follows:

$a^{\circ-}_{12} = 0, a^{\circ-}_{13} = 0, a^{\circ-}_{14} = 0, a^{\circ-}_{23} = 0, a^{\circ-}_{24} = 0, a^{\circ-}_{34} = 0, a^{\circ+}_{12} = 0, a^{\circ+}_{13} = 0, a^{\circ+}_{14} = 0, a^{\circ+}_{23} = 0, a^{\circ+}_{24} = 0, a^{\circ+}_{34} = 0,$

$b^{\circ-}_{12} = 0, b^{\circ-}_{13} = 0, b^{\circ-}_{14} = 0, b^{\circ-}_{23} = 0, b^{\circ-}_{24} = 0, b^{\circ-}_{34} = 0, b^{\circ+}_{12} = 0, b^{\circ+}_{13} = 0, b^{\circ+}_{14} = 0, b^{\circ+}_{23} = 0, b^{\circ+}_{24} = 0.1778, b^{\circ+}_{34} = 0, c^{\circ-}_{123} = 0,$

$c^{\circ-}_{124} = 0.6, c^{\circ-}_{134} = 0, c^{\circ-}_{234} = 0, c^{\circ+}_{123} = 0, c^{\circ+}_{124} = 0, c^{\circ+}_{134} = 0, c^{\circ+}_{234} = 0.6,$

$\lambda^{\circ}_{12} = 1, \lambda^{\circ}_{13} = 1, \lambda^{\circ}_{14} = 1, \lambda^{\circ}_{24} = 1, \ \lambda^{\circ}_{34} = 1$

$\theta^{\circ}_{12} = 1, \ \theta^{\circ}_{13} = 1, \ \theta^{\circ}_{14} = 1, \ \theta^{\circ}_{23} = 1, \ \theta^{\circ}_{24} = 0, \ \theta^{\circ}_{24} = 0, \ \theta^{\circ}_{34} = 1$

Furthermore, insert these optimal solutions into Eq. (9) and obtain the adjusted q-ROFPR R^3 with the acceptable additive consistency, where

$$R^3 = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6, 0.8) & (0.8282, 0.853) & (0.7, 0.6) \\ (0.8, 0.6) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6918, 0.3) & (0.5, 0.7222) \\ (0.853, 0.8282) & (0.3, 0.6918) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.8, 0.7) \\ (0.6, 0.7) & (0.7222, 0.5) & (0.7, 0.8) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix}.$$

[Step 3] Based on Eq. (17), the weight vector of the DMs is $w = (0.2529, 0.2241, 0.2982, 0.2248)^T$. According to Eq. (13), the collective q-ROFPR R^c is obtained as follows:

$$R^c = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.7312, 0.8466) & (0.7917, 0.737) & (0.7, 0.7382) \\ (0.8466, 0.7312) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.7663, 0.3074) & (0.6135, 0.7031) \\ (0.737, 0.7917) & (0.3074, 0.7663) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.7574, 0.8592) \\ (0.7382, 0.7) & (0.7031, 0.6135) & (0.8592, 0.7574) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix}.$$

[Step 4] Based on Eq. (14), we have $GCI\left(R^1\right) = 0.8844$, $GCI\left(R^2\right) = 0.8154$, $GCI\left(R^3\right) = 0.9263$, and $GCI\left(R^4\right) = 0.8414$. Let $\tau = 0.9$. Substitute R^s $(s = 1, 2, 3, 4)$ into the model (M-7) and solve this model to derive its optimal objective function value $p^* = 1.7285$. Then, insert R^s $(s = 1, 2, 3, 4)$ and $p^* = 1.7285$ into the model (M-8) and solve this model to derive its optimal solutions, which are as follows:

$\phi_{12}^{\circ 1-}=0,\ \phi_{13}^{\circ 1-}=0,\ \phi_{14}^{\circ 1-}=0,\ \phi_{23}^{\circ 1-}=0,\ \phi_{24}^{\circ 1-}=0,\ \phi_{34}^{\circ 1-}=0,\ \phi_{12}^{\circ 1+}=0,\ \phi_{13}^{\circ 2+}=0.01,\phi_{13}^{\circ 1+}=0,\ \phi_{14}^{\circ 1+}=0,\ \phi_{23}^{\circ 1+}=0,$
$\phi_{24}^{\circ 1+}=0,\ \phi_{34}^{\circ 1+}=0,\ \varphi_{12}^{\circ 1-}=0,\varphi_{13}^{\circ 1-}=0,\varphi_{14}^{\circ 1-}=0,\varphi_{23}^{\circ 1-}=0,\ \varphi_{24}^{\circ 1-}=0,\varphi_{34}^{\circ 1-}=0,\varphi_{12}^{\circ 1+}=0,\varphi_{13}^{\circ 1+}=0,\varphi_{14}^{\circ 1+}=0,$
$\varphi_{23}^{\circ 1+}=0.042,\varphi_{24}^{\circ 1+}=0,\varphi_{34}^{\circ 1+}=0,\phi_{12}^{\circ 2-}=0.2386,\phi_{13}^{\circ 2-}=0,\ \phi_{14}^{\circ 2-}=0,\ \phi_{23}^{\circ 2-}=0,$
$\phi_{24}^{\circ 2-}=0,\ \phi_{34}^{\circ 2-}=0,\ \phi_{12}^{\circ 2+}=0,\ \phi_{14}^{\circ 2+}=0,\ \phi_{23}^{\circ 2+}=0,\ \phi_{24}^{\circ 2+}=0.043,\phi_{34}^{\circ 2+}=0.1524,\varphi_{12}^{\circ 2-}=0,\ \varphi_{13}^{\circ 2-}=0,$
$\varphi_{14}^{\circ 2-}=0.0074,\varphi_{23}^{\circ 2-}=0.0135,\varphi_{24}^{\circ 2-}=0.5611,\varphi_{34}^{\circ 2-}=0,\ \varphi_{12}^{\circ 2+}=0.0001,\varphi_{13}^{\circ 2+}=0.132,\varphi_{14}^{\circ 2+}=0,\ \varphi_{23}^{\circ 2+}=0,$
$\varphi_{24}^{\circ 2+}=0,\ \varphi_{34}^{\circ 2+}=0,\ \phi_{12}^{\circ 3-}=0,\phi_{13}^{\circ 3-}=0,\phi_{14}^{\circ 3-}=0,\phi_{23}^{\circ 3-}=0,\phi_{24}^{\circ 3-}=0,\phi_{34}^{\circ 3-}=0,\phi_{12}^{\circ 3+}=0,$
$\phi_{13}^{\circ 3+}=0,\phi_{14}^{\circ 3+}=0,\phi_{23}^{\circ 3+}=0,\phi_{24}^{\circ 3+}=0,\phi_{34}^{\circ 3+}=0,\varphi_{12}^{\circ 3-}=0,\varphi_{13}^{\circ 3-}=0,\ \varphi_{14}^{\circ 3-}=0,\varphi_{23}^{\circ 3-}=0,\varphi_{24}^{\circ 3-}=0,\varphi_{34}^{\circ 3-}=0,$
$\varphi_{12}^{\circ 3+}=0,\varphi_{13}^{\circ 3+}=0,\varphi_{14}^{\circ 3+}=0,\varphi_{23}^{\circ 3+}=0,\varphi_{24}^{\circ 3+}=0,\varphi_{34}^{\circ 3+}=0,\phi_{12}^{\circ 4-}=0,\ \phi_{13}^{\circ 4-}=0.0968,\phi_{14}^{\circ 4-}=0,\ \phi_{23}^{\circ 4-}=0,$
$\phi_{24}^{\circ 4-}=0.0038,\phi_{34}^{\circ 4-}=0,\phi_{12}^{\circ 4+}=0,\ \phi_{13}^{\circ 4+}=0,\ \phi_{14}^{\circ 4+}=0,\ \phi_{23}^{\circ 4+}=0,\ \phi_{24}^{\circ 4+}=0,\ \phi_{34}^{\circ 4+}=0,$
$\varphi_{12}^{\circ 4-}=0,\ \varphi_{13}^{\circ 4-}=0.2089,\varphi_{14}^{\circ 4-}=0,\ \varphi_{23}^{\circ 4-}=0,\ \varphi_{24}^{\circ 4-}=0,\ \varphi_{34}^{\circ 4-}=0,\ \varphi_{12}^{\circ 4+}=0,\ \varphi_{13}^{\circ 4+}=0,\ \varphi_{14}^{\circ 4+}=0.0095,$
$\varphi_{23}^{\circ 4+}=0,\ \varphi_{24}^{\circ 4+}=0.2094,\varphi_{34}^{\circ 4+}=0.$

Moreover, insert the above solutions into Eq. (16) and obtain four adjusted q-ROFPRs with acceptable additive consistency and consensus, which are still denoted by R^s $(s=1,2,3,4)$, where

$$R^1=\begin{pmatrix} \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.7517,0.9084) & (0.8,0.6) & (0.7,0.9) \\ (0.9084,0.7517) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.9038,0.358) & (0.3,0.5) \\ (0.6,0.8) & (0.358,0.9038) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.6,0.9532) \\ (0.9,0.7) & (0.5,0.3) & (0.9532,0.6) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) \end{pmatrix},$$

$$R^2=\begin{pmatrix} \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.7385,0.8999) & (0.845,0.694) & (0.7,0.2074) \\ (0.8999,0.7385) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.7,0.1135) & (0.757,0.6611) \\ (0.694,0.845) & (0.1135,0.7) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.7136,0.7917) \\ (0.2074,0.7) & (0.6611,0.757) & (0.7917,0.7136) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) \end{pmatrix},$$

$$R^3=\begin{pmatrix} \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.6,0.8) & (0.8282,0.853) & (0.7,0.6) \\ (0.8,0.6) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.6918,0.3) & (0.5,0.7222) \\ (0.853,0.8282) & (0.3,0.6918) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.8,0.7) \\ (0.6,0.7) & (0.7222,0.5) & (0.7,0.8) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) \end{pmatrix},$$

$$R^4=\begin{pmatrix} \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.9076,0.753) & (0.6968,0.2089) & (0.7,0.7905) \\ (0.753,0.9076) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.7,0.1) & (0.6038,0.6906) \\ (0.2089,0.6968) & (0.1,0.7) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.6804,0.9415) \\ (0.7905,0.7) & (0.6906,0.6038) & (0.9415,0.6804) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) \end{pmatrix}.$$

Based on Eq. (17), the weight vector of the DMs is $w=(0.2351,0.2578,0.2622,0.2449)^T$. Furthermore, according to Eq. (13), the collective q-ROFPR R^c is obtained as follows:

$$R^c=\begin{pmatrix} \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.7702,0.8472) & (0.7996,0.6922) & (0.7,0.7302) \\ (0.8472,0.7702) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.7623,0.2789) & (0.6067,0.6611) \\ (0.6922,0.7996) & (0.2789,0.7623) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.7121,0.8615) \\ (0.7302,0.7) & (0.6611,0.6067) & (0.8615,0.7121) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) \end{pmatrix}.$$

[Step 5] By substituting R^c into model (M-5) and solving it, four priority weights of alternatives are $\omega_1 = (0.6356, 0.8232)$, $\omega_2 = (0, 0.7226)$, $\omega_3 = (0, 0.9065)$, and $\omega_4 = (0.5752, 0.7504)$.
[Step 6] Following **Definition 2.2**, we derive $S(\omega_1) = -0.2960$, $S(\omega_2) = -0.2726$, $S(\omega_3) = -0.6754$, and $S(\omega_4) = -0.2076$. Since $S(\omega_4) > S(\omega_2) > S(\omega_1) > S(\omega_3)$, the ranking order is $x_4 \succ x_2 \succ x_1 \succ x_3$, where the best alternative is x_4.

The methods presented in [3,29,38,46] have the drawbacks that they cannot deal with incomplete q-ROFPRs for GDM, whereas the proposed **Algorithm 2** can deal with incomplete q-ROFPRs for GDM, as shown in **Example 5.1**. From Table 1 that shows the rankings obtained by different methods, our **Algorithm 2** outperforms the methods offered in [3,29,38,46] for GDM with incomplete q-ROFPRs due to the fact that they have the drawbacks that they cannot get the ranking order, whereas our **Algorithm 2** can get the ranking order "$x_4 \succ x_2 \succ x_1 \succ x_3$" of the alternatives in this situation.

Example 5.2 (*[46]*). Assume that there are four alternatives x_1, x_2, x_3 and x_4 and assume that four DMs e_1, e_2, e_3 and e_4 are invited to form a group to provide their evaluations with respect to the alternatives x_1, x_2, x_3 and x_4 by performing pairwise comparisons between any two alternatives. The DMs e_1, e_2, e_3 and e_4 compare each pair of alternatives and give their judgments with complete q-ROFPRs R^1, R^2, R^3 and R^4, respectively, where $R^s = \left(r_{ij}^s\right)_{4\times 4}$ $(s = 1, 2, 3, 4)$, shown as follows:

$$R^1 = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.4, 0.7) & (0.8, 0.6) & (0.4, 0.8) \\ (0.7, 0.4) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.8, 0.3) & (0.8, 0.9) \\ (0.6, 0.8) & (0.3, 0.8) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.3, 0.9) \\ (0.8, 0.4) & (0.9, 0.8) & (0.9, 0.3) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix},$$

$$R^2 = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.4, 0.5) & (0.7, 0.4) & (0.3, 0.7) \\ (0.5, 0.4) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.8, 0.4) & (0.6, 0.9) \\ (0.4, 0.7) & (0.4, 0.8) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.2, 0.9) \\ (0.7, 0.3) & (0.9, 0.6) & (0.9, 0.2) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix},$$

$$R^3 = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.5, 0.6) & (0.6, 0.5) & (0.2, 0.6) \\ (0.6, 0.5) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.5, 0.3) & (0.5, 0.8) \\ (0.5, 0.6) & (0.3, 0.5) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.3, 0.8) \\ (0.6, 0.2) & (0.8, 0.5) & (0.8, 0.3) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix},$$

$$R^4 = \begin{pmatrix} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.5, 0.6) & (0.7, 0.5) & (0.3, 0.6) \\ (0.6, 0.5) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.6, 0.3) & (0.7, 0.9) \\ (0.5, 0.7) & (0.3, 0.6) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) & (0.2, 0.7) \\ (0.6, 0.3) & (0.9, 0.7) & (0.7, 0.2) & \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right) \end{pmatrix}.$$

Let $\eta = 0.95$, let $\tau = 0.95$ and let $q = 5$. According to the proposed **Algorithm 2**, we get the adjusted q-ROFPRs, shown as follows:

Table 1
The ranking orders obtained by five methods for Example 5.1 with incomplete q-ROFPRs.

Methods	Ranking orders of alternatives
Gong et al.'s method [3]	Cannot obtain the ranking order
Wang's method [29]	Cannot obtain the ranking order
Xu and Chen's method [38]	Cannot obtain the ranking order
Zhang et al.'s method [46]	Cannot obtain the ranking order
The proposed Algorithm 2	$x_4 \succ x_2 \succ x_1 \succ x_3$

$$R^{\cdot 1}=\begin{pmatrix} \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.4,0.6576) & (0.708,0.5529) & (0.3109,0.7985) \\ (0.6576,0.4) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.8,0.3) & (0.7062,0.9) \\ (0.5529,0.708) & (0.3,0.8) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.3,0.9) \\ (0.7985,0.3109) & (0.9,0.7062) & (0.9,0.3) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) \end{pmatrix},$$

$$R^{\cdot 2}=\begin{pmatrix} \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.4,0.5) & (0.7,0.4) & (0.3,0.7) \\ (0.5,0.4) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.7623,0.3813) & (0.6,0.8531) \\ (0.4,0.7) & (0.3813,0.7623) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.2,0.9) \\ (0.7,0.3) & (0.8531,0.6) & (0.9,0.2) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) \end{pmatrix},$$

$$R^{\cdot 3}=\begin{pmatrix} \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.5,0.6) & (0.6926,0.5) & (0.2889,0.6) \\ (0.6,0.5) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.5133,0.3) & (0.6775,0.8007) \\ (0.5,0.6926) & (0.3,0.5133) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.3,0.8) \\ (0.6,0.2889) & (0.8007,0.6775) & (0.8,0.3) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) \end{pmatrix},$$

$$R^{\cdot 4}=\begin{pmatrix} \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.5,0.6) & (0.7,0.5) & (0.3,0.6) \\ (0.6,0.5) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.6,0.3) & (0.7,0.8533) \\ (0.5,0.7) & (0.3,0.6) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.2,0.7) \\ (0.6,0.3) & (0.8533,0.7) & (0.7,0.2) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) \end{pmatrix}.$$

Following Eq. (17), the weight vector of DMs e_1, e_2, e_3 and e_4, where $w=(0.2329,0.2262,0.2638,0.2770)^T$, is gained. Furthermore, according to Eq. (13), we obtain the collective q-ROFPR R^c, shown as follows:

$$R^{c}=\begin{pmatrix} \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.4644,0.6) & (0.7,0.5) & (0.3,0.6894) \\ (0.6,0.4644) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.6978,0.3264) & (0.6775,0.8531) \\ (0.5,0.7) & (0.3264,0.6978) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) & (0.2674,0.8343) \\ (0.6894,0.3) & (0.8531,0.6775) & (0.8343,0.2674) & \left(\sqrt[q]{0.5},\sqrt[q]{0.5}\right) \end{pmatrix}.$$

By substituting R^c into the model (M-5), we get $\omega_1=(0.0075,0.6849)$, $\omega_2=(0.3446,0.5335)$, $\omega_3=(0,0.8041)$ and $\omega_4=(0.5335,0.3446)$. Using **Definition 2.2**, we derive $S(\omega_1)=-0.2200$, $S(\omega_2)=-0.0669$, $S(\omega_3)=-0.4180$ and $S(\omega_4)=0.0669$. Because $S(\omega_4)>S(\omega_2)>S(\omega_1)>S(\omega_3)$, where $S(\omega_1)=-0.2200$, $S(\omega_2)=-0.0669$, $S(\omega_3)=-0.4180$ and $S(\omega_4)=0.0669$, the ranking order of the alternatives x_1, x_2, x_3 and x_4 is: $x_4\succ x_2\succ x_1\succ x_3$.

Zhang *et al.* [46] used the model (M-5) and Eq. (14) in [46] to yield the priority vector of alternatives x_1, x_2, x_3 and x_4, shown as follows:

$$\begin{aligned}\omega=&\ (\omega_1,\omega_2,\omega_3,\omega_4)^T\\ =&\ \left(\left(0,\left(1-0.4436^q\right)^{1/q}\right),\left(0.1366,\left(1-0.5803^q\right)^{1/q}\right),\left(0,\left(1-0.0358^q\right)^{1/q}\right),\left(0.4197,\left(1-0.8039^q\right)^{1/q}\right)\right)^T.\end{aligned}$$

Let $q=5$. Based on Eq. (2) in [46], Zhang *et al.* [46] calculated the scores $S(\omega_1)$, $S(\omega_2)$, $S(\omega_3)$ and $S(\omega_4)$ of the q-ROF priority weights ω_1, ω_2, ω_3 and ω_4, respectively, where $S(\omega_1)=-0.9828$, $S(\omega_2)=-0.9341$, $S(\omega_3)=-1$, and $S(\omega_4)=-0.6512$. Because $S(\omega_4)>S(\omega_2)>S(\omega_1)>S(\omega_3)$, the ranking order of the alternatives x_1, x_2, x_3 and x_4 obtained by the method presented in [46] is $x_4\succ x_2\succ x_1\succ x_3$.

Table 2 shows the ranking orders produced via five approaches for dealing with **Example 5.2** with complete q-ROFPRs. From Table 2, we can see that the methods presented in [3,29,38,46] and the proposed **Algorithm 2** get the same ranking order "$x_4\succ x_2\succ x_1\succ x_3$" of the alternatives for GDM with complete q-ROFPRs shown in **Example 5.2**.

In summary, according to Tables 1 and 2, our **Algorithm 2** outperforms the methods offered in [3,29,38,46] for GDM. Table 3 shows a comparative analysis for different methods.

Table 2
The ranking orders produced via five methods for Example 5.2 with complete q-ROFPRs (All collected from [46]).

Methods	Ranking orders of alternatives
Gong *et al.*'s method [3]	$x_4 \succ x_2 \succ x_1 \succ x_3$
Wang's method [29]	$x_4 \succ x_2 \succ x_1 \succ x_3$
Xu and Chen's method [38]	$x_4 \succ x_2 \succ x_1 \succ x_3$
Zhang *et al.*'s method [46]	$x_4 \succ x_2 \succ x_1 \succ x_3$
The proposed Algorithm 2	$x_4 \succ x_2 \succ x_1 \succ x_3$

Table 3
A comparative analysis of the characteristics for different methods.

Methods	PRs	DMs' weights	Consistency detection and improving	Consensus detection and improving
Gong et al.'s method [3]	Complete intuitionistic fuzzy PRs	Predefined	No consider	No consider
Wang's method [29]	Complete intuitionistic fuzzy PRs	Predefined	No consider	No consider
Xu and Chen's method [38]	Complete interval fuzzy PRs	Predefined	No consider	No consider
Zhang et al.'s method [46]	Complete q-ROFPRs	Predefined	No consider	No consider
The proposed Algorithm 2	Incomplete q-ROFPRs	Determined by Eq. (17)	Consider	Consider

The proposed **Algorithm 2** for GDM has more advantages than the methods presented in [3,29,38,46], described as follows:

(1) In the proposed **Algorithm 2** for GDM, we consider that some unknown elements exist in incomplete q-ROFPRs. Our proposed estimation model can estimate unknown elements in incomplete q-ROFPRs in a reasonable way. However, the methods presented in [3,29,38,46] for GDM only can deal with complete PRs. Hence, our proposed method outperforms the methods presented in [3,29,38,46] for dealing with GDM problems due to the fact that our method for GDM can deal with incomplete q-ROFPRs, whereas the methods presented in [3,29,38,46] for GDM cannot deal with incomplete q-ROFPRs.

(2) In the GDM process of the methods presented in [3,29,38,46], the weights of DMs are predefined. In the GDM process of the proposed **Algorithm 2**, it uses the formula shown in Eq. (17) to produce the DMs' weights in which the group consensus level is taken into account. Hence, our **Algorithm 2** is more rational than the methods presented in [3,29,38,46] for GDM.

(3) The methods presented in [3,29,38,46] only give the definitions of consistency, but they don't identify the consistency level and don't provide a consistency improvement approach for GDM. In the proposed **Algorithm 2** for GDM, it provides a consistency adjustment mechanism, which not only contains a consistency check process, but also contains a consistency modification approach. Thus, the proposed **Algorithm 2** is more comprehensive in practical applications than the methods presented in [3,29,38,46].

(4) In the method presented in [3,29,38,46], the acceptable consensus of PRs is not checked and achieved. By contrast, the proposed **Algorithm 2** for GDM considers the consistency and the consensus simultaneously, which ensures the minimum total adjustment as well as the minimum number of adjustments in q-ROFPRs, and guarantees reliable and effective results. For example, based on Eq. (14), we can get the consensus indices $GCI\left(R^1\right)$, $GCI\left(R^2\right)$, $GCI\left(R^3\right)$ and $GCI\left(R^4\right)$ of the original q-ROFPRs R^1, R^2, R^3 and R^4 used in [46], respectively, where $GCI\left(R^1\right) = 0.9274$, $GCI\left(R^2\right) = 0.9405$, $GCI\left(R^3\right) = 0.9185$ and $GCI\left(R^4\right) = 0.9517$. Based on Eq. (14), we can get the consensus indices $GCI\left(R'^1\right)$, $GCI\left(R'^2\right)$, $GCI\left(R'^3\right)$ and $GCI\left(R'^4\right)$ of the modified q-ROFPRs R'^1, R'^2, R'^3 and R'^4 in **Example 5.2**, respectively, where $GCI\left(R'^1\right) = 0.9500$, $GCI\left(R'^2\right) = 0.9500$, $GCI\left(R'^3\right) = 0.9605$ and $GCI\left(R'^4\right) = 0.9605$. The results clearly show that the modified q-ROFPRs gained via our Algorithm 2 have better consensus levels than the ones of the original q-ROFPRs used in [46].

6. Conclusions

In this paper, we have presented a new GDM method in incomplete q-ROFPRs environments. We have proposed an additive consistency definition, which is characterized by a q-rung orthopair fuzzy priority vector. We have presented the property of the proposed additive consistency definition and have proposed a model to obtain missing judgments in incomplete

q-ROFPRs. We have presented a procedure to adjust the inconsistency of q-ROFPRs, have provided a model to gain the priorities, and have proposed a method to increase consensus degrees of q-ROFPRs. We have used two illustrative examples to illustrate that the proposed GDM method can overcome the drawbacks of the existing GDM methods. Our GDM method offers a very useful way for GDM in incomplete q-ROFPRs environments. In this paper, the thresholds of the consistency and the consensus are directly given in advance, which are subjectively defined. In the future, we will propose a reasonable method to assist DMs to determine the objective thresholds of the consistency and the consensus. In addition, this paper only considers the traditional GDM, which only involves several DMs. However, with the development of the Internet and the arrival of the era of big data, more and more DMs can participate in decision-making processes. Therefore, large-scale group decision making (LSGDM) involving a large amount of DMs becomes a very important research topic. In the future, we also will propose some LSGDM methods based on incomplete q-ROFPRs.

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