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One-dimensional Variational Problems

An Introduction

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An Introduction

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PREFACE

This book provides an introduction to the one-dimensional variational calculus, with a special emphasis on *direct* methods. This topic, and in particular Tonelli's existence and regularity theory for solutions of one-dimensional variational problems, is somewhat neglected as authors usually treat minimum problems for multiple integrals, which lead to partial differential equations and are considerably more difficult to handle. One-dimensional problems are connected with ordinary differential equations and need much less technical prerequisites, but they exhibit the same kind of phenomena and surprises as variational problems for multiple integrals. Therefore our book might be welcomed by graduate students who want to get an idea of what the modern approach to variational problems is all about, without being bothered by too many technicalities, or by lecturers who want to use a modern text for a course on the calculus of variations. Except for results from the theory of measure and integration and from the theory of convex functions we develop all the tools needed in the text, including the basic results on one-dimensional Sobolev spaces, absolutely continuous functions, and functions of bounded variation. We concentrate our discussion on non-parametric problems, but the results so achieved can also be used to treat parametric variational problems without much more effort.

In the scholia of Chap. 6 we present some ramifications of the calculus of variations, and we point out various connections to pertinent problems for multiple variational integrals.

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INTRODUCTION

In the first 200 years of the history of the calculus of variations the prevalent approach was what we may call the *classical indirect approach*. This approach is based on the optimistic, but somewhat naïve, idea that every minimum problem has a solution. In order to determine this solution, one first looks for conditions which have to be satisfied by a minimizer; they are called *necessary conditions*. For example, if a differentiable functional $\mathcal{F} : \mathcal{C} \to \mathbf{R}$ is minimized at some inner point u of $\mathcal{C}$, the derivative $\mathcal{F}'(u)$ of $\mathcal{F}$ at u must vanish. If $\mathcal{F}$ is a *variational integral* defined on some class $\mathcal{C}$ of functions u, the necessary condition $\mathcal{F}'(u) = 0$ for a minimizer u usually leads to a differential equation, the so-called *Euler equation* which is to be satisfied by the function u.

An analysis of the necessary conditions often permits one to eliminate many candidates and eventually identifies a unique solution. For instance, if, by some luck, there is only one solution of the Euler equation satisfying all prescribed subsidiary conditions, then one is tempted to infer that this solution must also be a solution of the original minimum problem. This conclusion, drawn by masters like Gauss, Steiner, Lord Kelvin, Dirichlet, and Riemann, can be false since the minimum problem might have no solution. In other words, we have to prove the *existence of a minimizer* before we are allowed to conclude that a unique solution of the Euler equation satisfying all the required subsidiary conditions is a minimizer, or else we have to prove directly that the energy of our candidate is actually smaller than the energy of any other competing function.

The matter is even more complicated. First, it is neither evident nor always true that, for instance, a C^1-minimizer is necessarily of class C^2 and, therefore, a solution of the Euler equation. To ensure C^2-regularity of minimizers one has to prove *regularity theorems* that are usually based on ellipticity conditions. Secondly it is neither evident nor always true that a given Euler equation possesses a (classical) solution which fulfils the prescribed side conditions, and even if there is such a solution, it might not be the only one.

If there are several candidates for the position of a minimizer, then which one is a true (relative or absolute) minimizer? Jacobi's theory of conjugate points leads to sufficient conditions for an extremal to be a weak minimizer. Moreover, combining this theory with Weierstrass's field theory, we can even obtain sufficient conditions for an extremal to be a strong relative minimizer.

Contrary to this detour via solutions of Euler's equations, which becomes quite complicated and difficult for multidimensional variational problems, one may try to attack the minimum problem directly, by immediately proving the existence of a minimizer. This, in turn, would also give an existence theorem for solutions of Euler's equations satisfying prescribed restrictions, say, boundary conditions. This is the approach that one follows by using the so-called *direct methods of the calculus of variations*. It originated in the work of Gauss, Lord Kelvin, Dirichlet, and Riemann on boundary value problems for the potential equation $\Delta u = 0$. Following the example of Dirichlet,[1] Riemann applied a reasoning that he named *Dirichlet's principle: There exists a unique function which minimizes Dirichlet's integral* $\frac{1}{2}\int_\Omega |Du|^2\,dx$ *among all functions* $u \in C^1(\Omega)\cap C^0(\overline{\Omega})$ *which assume given values on the boundary* $\partial\Omega$; *moreover, this function is harmonic in* Ω.

After its refutation by Weierstrass,[2] Dirichlet's principle fell into disgrace. Yet the interest in the direct methods never quite faded and, about the turn of the century, Hilbert and Lebesgue exhibited several cases in which Dirichlet's principle could be rigorously stated. A few years later, Tonelli formulated his clean and modern direct approach to minimum problems which is based on the concept of *lower semicontinuity* of variational integrals.

More precisely, at the beginning of this century Hilbert succeeded in giving a direct proof of the existence of a shortest connection of two given points on a surface as well as of a minimizer for Dirichlet's integral in dimension 2 in the class of functions with prescribed smooth boundary values, thereby proving the validity of Dirichlet's principle in this case.

Analysing the notion of area of a surface, Lebesgue pointed out that the area functional is not continuous but merely semicontinuous with respect to uniform convergence of surfaces. To demonstrate this phenomenon by means of a one-dimensional analogue, he considered the zig-zag curves $c_k(t)$ pictured in Fig. 0.1. All these curves are of length $\sqrt{2}$, but their uniform limit $c(t) = (t, 0)$, $0 \le t \le 1$, is of length 1. Hence we have

$$1 = \int_0^1 |\dot{c}(t)|\,dt < \liminf_{k\to\infty} \int_0^1 |\dot{c}_k(t)|\,dt = \sqrt{2}.$$

The concept of semicontinuity of real-valued functions was introduced by Baire, and he noted that, on a compact set, a lower semicontinuous real function assumes its infimum.

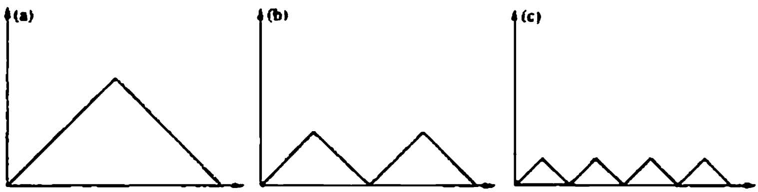

FIG. 0.1.

[1] Dirichlet never used this principle in one of his published papers, but his students at Göttingen have repeatedly stated that he frequently and without hesitation applied this reasoning in his lectures.

[2] It seems that Weierstrass already doubted Dirichlet's principle before 1860, but he published his objections only in 1870 in a lecture given to the Berlin Academy. Riemann, who died in 1864, knew of Weierstrass's criticism. He accepted it; nevertheless he was convinced that his work on *Abelian functions* would eventually be justified. See also $\boxed{1}$ of Section 3.2.

Tonelli realized that the Arzelà–Ascoli compactness theorem and Baire's semicontinuity concept can be transferred from real functions of one or several variables to variational integrals and that they are the perfect tools to prove the existence of minimizers of one-dimensional variational integrals by means of direct methods. Today we still use Tonelli's reasoning which proceeds as follows. In order to show that a functional $\mathcal{F}(u)$ defined on a class $\mathcal{C}$ has an absolute minimizer in $\mathcal{C}$, one first has to prove that the functional is bounded from below in $\mathcal{C}$, so that it has a finite infimum. Secondly, one tries to verify that the functional is sequentially lower semicontinuous with respect to some suitable kind of convergence for which, thirdly, the set turns out to be sequentially compact. Alternatively one can try to prove that $\mathcal{C}$ contains at least a converging minimizing sequence $\{u_k\}$ which tends to a limit u_0 belonging to $\mathcal{C}$. Then we obtain the inequalities

$$\inf_{\mathcal{C}} \mathcal{F} \leq \mathcal{F}(u_0) \leq \liminf_{k\to\infty} \mathcal{F}(u_k) \leq \inf_{\mathcal{C}} \mathcal{F} \tag{0.1}$$

which imply

$$\mathcal{F}(u_0) = \inf_{\mathcal{C}} \mathcal{F}. \tag{0.2}$$

i.e. u_0 is an absolute minimizer of $\mathcal{F}$ in $\mathcal{C}$.

Tonelli required the Lagrangian $F(x, u, p)$ of the variational integral

$$\mathcal{F}(u) := \int_a^b F(x, u(x), u'(x))\, dx, \quad [a, b] \subset \mathbf{R}, \tag{0.3}$$

to satisfy the following conditions:

$$F_{pp}(x, u, p) \geq 0 \tag{0.4}$$

$$F(x, u, p) \geq c_0|p|^m - c_1 \quad \text{for some constants } m > 1,\ c_0 > 0, \text{ and } c_1 \geq 0. \tag{0.5}$$

Under these assumptions he worked in the class of *absolutely continuous functions* on the interval $I = [a, b]$, equipped with the uniform convergence as the notion of convergence. He formalized, exploited, and popularized the idea of the direct methods in a series of papers and lectures during the first 30 years of this century, and in his monographs. The aim of our book is to present his ideas and some of his results for a large class of one-dimensional variational problems, together with some more recent developments of these ideas.

Tonelli's ideas are nowadays so much a part of our mathematical culture that many of them may even appear to be trivial, but they were by no means trivial at the beginning of this century. On the other hand, their application to minimum problems for variational integrals is not at all trivial and, owing to the enormous freedom we have in setting the stage, they might tempt us to investigate problems that, in principle, are far away from the problem we wanted to study initially. This will become apparent in the last chapter of this book when we deal with the applications of direct methods to specific variational problems.

Presently we will give the reader an idea of the difficulties one might face and of the questions one has to answer in order to use direct methods. To this end we now describe the key ideas of direct methods in a more detailed and formal way.

Roughly speaking, direct methods appear as a recipe of the form: try to apply the following extension of Weierstrass's theorem for real-valued functions of several real variables.

Theorem 0.1 *Let $\mathcal{F} : C \to \mathbb{R} \cup \{+\infty\}$ be a functional defined on a nonempty set C that is equipped with a notion of convergence for which C is sequentially compact and $\mathcal{F}$ is sequentially lower semicontinuous. Then there exists a minimizer of $\mathcal{F}$ in C.*

In order to apply this theorem to functionals of type (0.3) we proceed as follows:

(i) First we fix the class C of admissible or competing functions and a suitable notion of convergence on C which is denoted by $u_k \to u$.

(ii) Next we show that the functional $\mathcal{F}$ is well defined on C and bounded from below, so that $\inf_C \mathcal{F}$ is finite. This implies that we can find a minimizing sequence in C, i.e. a sequence of functions $u_k \in C$, $k = 1, 2, \ldots$, such that $\mathcal{F}(u_k) \to \inf_C \mathcal{F}$.

(iii) Then we prove that $\mathcal{F}$ is sequentially lower semicontinuous on C with respect to the convergence chosen in (i). That is, we have to verify that $u_k \to u$ implies

$$\mathcal{F}(u) \le \liminf_{k\to\infty} \mathcal{F}(u_k). \tag{0.6}$$

(iv) Finally we show sequential compactness of C with respect to the convergence of (i). In fact, it suffices to prove that any—or at least one—minimizing sequence contains a convergent subsequence with a limit in C.

Steps (i)–(iv) obviously allow us to apply the theorem above and to infer the existence of a minimizer u_0 of $\mathcal{F}$ in C. We then say that u_0 is a solution of the problem

$$\text{`}\mathcal{F} \to \min \ \text{in } C\text{'}.$$

The first question is how one should choose the set C of admissible functions and the notion of convergence within C. Clearly C should be *complete* with respect to this convergence so that converging sequences in C always possess a limit in C, and, secondly, this convergence notion should be rather weak in order to ensure that one can select from any minimizing sequence a convergent subsequence, or equivalently, to ensure sequential compactness of C. On the other hand, if we choose too weak a convergence, it might be hard or even impossible to verify that $\mathcal{F}(u)$ is lower semicontinuous with respect to this convergence (and that the limit of some 'convergent' sequence in C again belongs to C).

Actually the situation is usually even more complicated. The variational problems we are interested in come from physics or differential geometry and are formulated in a classical context. We are required to minimize $\mathcal{F}$ in a class of smooth functions satisfying suitable side conditions such as, for example, boundary conditions. To illustrate the situation we consider the following four *examples*.

$\boxed{1}$ Minimize the Dirichlet integral

$$\mathcal{D}(u) := \frac{1}{2}\int_0^1 |u'|^2\, dx$$

in the class $K_1 := \{u \in C^0(\overline{I}) \cap C^1(I)\colon u(0) = \alpha, u(1) = \beta\}$, α, β being prescribed real numbers and $I := (0, 1)$.

$\boxed{2}$ Minimize the length of the image or of the graph of mappings $u : [0, 1] \to \mathbf{R}$ with prescribed values at 0 and 1, i.e. minimize

$$\mathcal{L}(u) = \int_0^1 |u'|\, dx \quad \text{or} \quad \mathcal{A}(u) = \int_0^1 \sqrt{1 + (u')^2}\, dx$$

in K_1 or in the class $K_2 := \{u \in C^0(\overline{I}) \cap D^1(\overline{I})\colon u(0) = \alpha, u(1) = \beta\}$. Here $D^1(\overline{I})$ denotes the class of piecewise smooth functions $u : \overline{I} \to \mathbf{R}$.

$\boxed{3}$ Minimize the length of the image or of the graph of mappings of $[0, 1]$ onto $S^1 \subset \mathbf{R}^2$ whose values at 0 and 1 are the point $(1, 0)$ of S^1, i.e. minimize

$$\mathcal{L}(u) := \int_0^1 |u'|\, dx \quad \text{or} \quad \mathcal{A}(u) := \int_0^1 \sqrt{1 + (u')^2}\, dx$$

in the class $K_3 := \{u \in C^0(\overline{I}, \mathbf{R}^2) \cap D^1(\overline{I}, \mathbf{R}^2)\colon u(\overline{I}) = S^1, u(0) = u(1) = (1, 0)\}$ where $D^1(\overline{I}, \mathbf{R}^2)$ denotes the class of piecewise smooth functions $u : \overline{I} \to \mathbf{R}^2$.

$\boxed{4}$ Given a non-negative and continuous integrand $F(x, z, \xi)$, minimize

$$\mathcal{F}(u) := \int_0^1 F(x, u, u')\, dx$$

in the class

$$K_4 := \{u \in C^0(\overline{I}) \cap C^1(I)\colon u(0) = \alpha, u(1) = \beta\}$$

or in the class

$$K_5 := \{u \in C^1([0, 1])\colon u(0) = \alpha, u(1) = \beta\}.$$

As it cannot be expected that $\mathcal{F}$-equibounded sets in the admissible classes K_i above are sequentially compact with respect to the C^1-convergence, we will have to choose a weaker notion of convergence τ; but then we cannot expect the classes K_i to be complete with respect to this new convergence. If we insist on applying direct methods, we are therefore led to the task *to complete* the classes $K = K_i$ above with respect to the chosen notion of convergence τ, and consequently to work in classes $K_{(\tau)}$ of *generalized functions* which in principle are not differentiable in the classical sense and in many cases not even continuous. Having enlarged the class of competing functions from K to $K_{(\tau)}$, we are forced to *extend* our functional $\mathcal{F}$ to a new functional $\mathcal{F}_{(\tau)}$ defined on $K_{(\tau)}$. In general there are many different ways of extending $\mathcal{F}$; how should we proceed? Deferring this question for a while, let us emphasize that, applying direct methods, i.e. insisting on the fact that there must be a minimizer for any reasonable minimum problem, we are forced to work in classes of generalized functions and to accept minimizers which may not be smooth. This fact was pointed out by Hilbert in his celebrated lecture at the International Congress of Mathematicians, held in Paris in 1900. Hilbert's 20th problem, stated at this Congress, reads as follows: *Has not every regular variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also if need be that the notion of solution shall be suitably extended*?

Choosing a notion of convergence τ in such a way that $K_{(\tau)}$ is sequentially compact with respect to τ and $\mathcal{F}_{(\tau)}$ is sequentially lower semicontinuous with respect to τ, we can apply the theorem above with $C = K_{(\tau)}$ and $\mathcal{F} = \mathcal{F}_{(\tau)}$ and conclude that there exists a minimizer $u_{(\tau)}$ for '$\mathcal{F}_{(\tau)} \to$ min *in* $K_{(\tau)}$', i.e. for the problem

$$\min\{\mathcal{F}_{(\tau)}(u) : u \in K_{(\tau)}\}. \tag{0.7}$$

In principle the minimizer $u_{(\tau)}$ and the minimum problem in (0.7) depend on the notion of convergence τ. Only if τ is chosen as the *strongest* convergence for which $K_{(\tau)}$ is sequentially compact and $\mathcal{F}_{(\tau)}$ sequentially lower semicontinuous can the minimum problem (0.7) be considered as the reasonable *generalization* of the original problem and $u_{(\tau)}$ as a *generalized minimizer* of our original minimum problem. Otherwise, important properties of the competing functions in K_i might get lost in the closure procedure and problem (0.7) could turn out to be a substantially *different* minimum problem.

For instance, in example $\boxed{2}$ the sequence of piecewise smooth mappings

$$u_k(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2} - 1/k \\ k/2\left(x + 1/k - \frac{1}{2}\right) & \text{if } \frac{1}{2} - 1/k \le x \le \frac{1}{2} + 1/k \\ 1 & \text{if } \frac{1}{2} + 1/k \le x \le 1 \end{cases}$$

belongs to K_2 with $\alpha = 0$ and $\beta = 1$, and it is equibounded in 'energy', i.e.

$$\sup_k \mathcal{L}(u_k) < \infty \quad \text{or} \quad \sup_k \mathcal{A}(u_k) < \infty;$$

its limit is the function (see Fig. 0.2)

$$u(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Therefore, if we decide to work with the 'pointwise' convergence τ, the mapping $u(x)$ will belong to $K_{2(\tau)}$, and if, moreover, we decide to compute u' as the pointwise derivative of u which exists almost everywhere, we find that

$$\mathcal{L}(u) = 0,$$

while for obvious topological reasons any mapping v in K_2 at least covers the interval $[0, 1]$; hence $\mathcal{L}(v) \ge 1$.

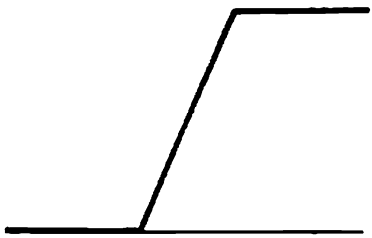

FIG. 0.2.

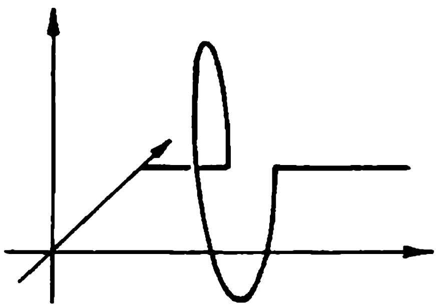

FIG. 0.3.

The situation is even more complicated in example $\boxed{3}$. The sequence

$$u_k(x) = \begin{cases} (1,0) & \text{if } 0 \le x \le \frac{1}{2} \\ \left(\cos 2\pi k\left(x - \frac{1}{2}\right), \sin 2\pi k(x - \frac{1}{2})\right) & \text{if } \frac{1}{2} \le x \le \frac{1}{2} + 1/k \\ (1,0) & \text{if } \frac{1}{2} + 1/k \le x \le 1 \end{cases}$$

is clearly seen to belong to K_3. From the graph of u_k (see Fig. 0.3) it is clear that no 'weak convergence' on K_3 will allow a prospective limit u to remember that u_k covers S^1. In fact, a reader with some experience in 'weak convergences' will notice that the sequence $\{u_k\}$ converges weakly to the constant function $u(x) \equiv (1,0)$. Thus, choosing any weak convergence on functions we are led to conclude that a generalized minimizer of the problem in example $\boxed{3}$ is the constant map $(1,0)$ with the energy $\mathcal{L}(u) = 0$. Obviously we cannot accept this function as a generalized solution of the minimum problem in example $\boxed{3}$ since mappings v in K_3 cover the sphere S^1 at least once and their 'energies' $\mathcal{L}(v)$ are at least 2π!

These two examples show how difficult the choice of a suitable convergence τ and a reasonable extension $\mathcal{F}(\tau)$ of $\mathcal{F}$ might be.

Before we discuss a 'canonical' way to extend $\mathcal{F}$, we would like to make one more remark. In many cases, such as in examples $\boxed{1}$, $\boxed{2}$, $\boxed{4}$, there is a natural linear structure in the classes K where we want to minimize $\mathcal{F}$; the classes K are affine subspaces of

C^1 or $C^0 \cap D^1$. Therefore we might first 'close' or 'complete' the space of functions of class C^1 (or D^1) with finite energy to a space X and then recover the sets C by imposing the desired side conditions in a weak form. This procedure often works quite well, but it might also be a source of misunderstandings, since a correct interpretation of subsidiary conditions in the weak sense might not be easy. For instance, in example $\boxed{4}$ we shall see that minimization in K_4 or K_5 might lead to different results, but of course it is difficult to distinguish generalized functions belonging to $K_{4(\tau)}$ from those belonging to $K_{5(\tau)}$.

Let us now return to the extension $\mathcal{F}_{(\tau)}$ of $\mathcal{F}$. Clearly there are many ways of extending $\mathcal{F}$ from K to $K_{(\tau)}$. If $\mathcal{F}$ is already τ-sequentially lower semicontinuous in K (if not, the problem is even more complicated), we do not want to extend $\mathcal{F}$ arbitrarily as *any* τ-sequentially semicontinuous functional on $K_{(\tau)}$, but we want to find the *best extension*, i.e. the *largest lower semicontinuous extension of* $\mathcal{F}$ *on* $K_{(\tau)}$. This extension, already considered by Lebesgue for the area functional, is often called[3] *τ-relaxation* of $\mathcal{F}$; it is given by

$$\mathcal{F}_{(\tau)}(u) := \inf\{\liminf_{k\to\infty} \mathcal{F}(u_k) : u_k \in K \quad \tau\text{-converge to } u\}.$$

Note that, in principle, it is no longer clear whether $\mathcal{F}_{(\tau)}$ is a variational integral in the classical sense, and, even if it were a variational integral, we would not know how to compute its Lagrangian $F_{(\tau)}(x, u, \xi)$. Notice also that exact knowledge of the Lagrangian $F_{(\tau)}(x, u, \xi)$ is *essential* if we want to apply some *calculus*, for instance if we want to derive necessary conditions such as Euler's equations.

Suppose now that we are able to apply the direct methods of the calculus of variations to the minimum problem $\min\{\mathcal{F}(u) : u \in K\}$. This means that we have determined some τ-relaxation $\min\{\mathcal{F}_{(\tau)}(u) : u \in K_{(\tau)}\}$ of the original minimum problem '$\mathcal{F} \to \min$ *in* K' such that the theorem above is applicable to $C = K_{(\tau)}$ and $\mathcal{F} = \mathcal{F}_{(\tau)}$. Confronting the relaxed problem '$\mathcal{F}_{(\tau)} \to \min$ *in* $K_{(\tau)}$' with the original problem '$\mathcal{F} \to \min$ *in* K', we are led to the natural *regularity problem* for minimizers, a question that was also raised by Hilbert in 1900. Hilbert's 19th problem is the following question: *Are the solutions of regular problems in the calculus of variations always necessarily regular?*

This is an extremely delicate and complicated problem, at least for multidimensional integrals, and it has occupied a large part of the field of the calculus of variations during the last decades. Even for one-dimensional integrals it is not a completely trivial problem, and we shall discuss it only for a class of *regular integrals*; see Chap. 4. Of course a positive answer to the regularity problem also solves the initial question of whether by extending our given minimum problem we have chosen the best notion of convergence τ; in particular we see that the regularity problem is tightly connected with the existence problem.

For a long time it was thought that minimizers must always be regular, provided the Lagrangian satisfies some natural 'ellipticity and growth conditions'. This conjecture turned out to be false; in fact, even very reasonable problems may have singular

[3] We refer the interested reader to the book by Buttazzo [47], where relaxation is studied for several function spaces, and to the monograph by Giaquinta *et al.* [120].

minimizers, and their singularities might be both of mathematical and physical interest. Therefore, the regularity problem is nowadays replaced by a qualitative study of minimizers including their singularities.

Furthermore, the appearance of singularities shows that the choice of the convergence τ and of the τ-relaxed extension $\mathcal{F}_{(\tau)}$ are important questions. The study of singularities through relaxation becomes really relevant when one considers minimum problems for mappings between Riemannian manifolds, a subject that is not included in the present book, but since difficulties connected with the choices of τ and $\mathcal{F}_{(\tau)}$ already appear for one-dimensional problems, the reader should be alerted.

Our book treats one-dimensional variational problems; it is organized as follows. In Chap. 1 we present classical 'indirect methods' based on necessary and sufficient conditions for optimality. We briefly illustrate this method by some of the time-honoured examples of the calculus of variations. In Chap. 2 we introduce the framework of function spaces which is necessary to apply the direct methods of the calculus of variations, such as the classes of *absolutely continuous functions* and of *functions with bounded variation*. Chapter 3 is devoted to lower semicontinuous results, which imply, via direct methods, the existence of a solution of minimum problems; original proof of Tonelli's is presented, together with another, more recent and more general, proof. Moreover, lower semicontinuity in the space BV is also discussed. Chapter 4 deals with some regularity results for minimizers of one-dimensional variational problems; we present the case of *regular integrands* first studied by Hilbert and the remarkable *partial regularity theorem of Tonelli*. Finally, in the last chapter we discuss some applications and significant examples. For example, we treat boundary value problems as well as eigenvalue problems for linear differential operators of second order, and we discuss the existence of periodic solutions to Hamiltonian systems and to variational problems, including non-coercive problems. We also sketch an approach to existence results in optimal control theory. Finally we deal with the existence and regularity results for obstacle problems related to parametric and non-parametric variational problems. The parametric existence problem is treated by a method which goes back to an idea by Erdmann. This method yields a rather direct and brief approach to parametric problems.

Although our text mostly deals with minimizers of a given functional, we should like to emphasize that the study of critical points, i.e. of general solutions of the Euler equations, is equally of interest. However, the existence theory for critical points is more involved, as it usually also requires topological arguments, while the corresponding regularity theory in many respects resembles that of minimizers. Concerning the study of critical points, for instance, we refer the reader to the books by Ekeland [93], Mawhin–Willem [181], Rabinowitz [215], and Struwe [247].

1

CLASSICAL PROBLEMS AND INDIRECT METHODS

In this first chapter we describe the classical *indirect approach to solving variational problems*. The first step of this method consists in deriving *necessary conditions* for a function to be a minimizer. Among these, the principal condition is the vanishing of the first variation of the functional to be minimized at the prospective minimizer. As a consequence of this, any smooth candidate has to satisfy the Euler equation, which for one-dimensional variational problems is a quasilinear second-order differential equation. Following A. Kneser, a C^2-solution of the Euler equation is said to be an *extremal*. Now the second and more difficult step of the indirect method consists in deriving *sufficient* conditions for an extremal to be a minimizer. Traditionally this is achieved by Jacobi's theory of conjugate points and by Weierstrass's field theory. Here we content ourselves by describing *Carathéodory's royal road* to field theory and by proving that every sufficiently small piece of an extremal is a local minimizer provided that the Lagrangian is elliptic.

1.1 The Euler equation and other necessary conditions for optimality

In this section we derive necessary conditions which have to be satisfied by minimizers of variational integrals. The principal *necessary condition* is the vanishing of the first variation of variational integrals at their minimizers. For smooth minimizers this condition implies *the Euler equation* and, in the case of free boundary values, also the so-called *natural boundary condition*. Moreover, in many cases the *first inner variation* of a variational integral vanishes. This leads to *E. Noether's equation* or at least to its integral version, the so-called *Erdmann equation*.

We consider functionals $\mathcal{F}$ of the type

$$\mathcal{F}(u) = \int_I F(x, u(x), u'(x))\, dx \tag{1.1}$$

which will be called *variational integrals*; usually they are extended over a bounded interval $I = (a, b)$ in $\mathbb{R}$. Their integrand is the composition of a real-valued function $F(x, z, p)$ with the mapping $x \mapsto (x, u(x), u'(x))$ associated with any smooth function $u : \overline{I} \to \mathbb{R}^N$. The function $F(x, z, p)$ is usually called the *Lagrangian* of the variational integral $\mathcal{F}$ defined by (1.1). For the sake of simplicity we mostly assume that F is defined on $\overline{I} \times \mathbb{R}^N \times \mathbb{R}^N$ and that F is at least of class C^1. Then (1.1) is well defined for any $u \in C^1(\overline{I}, \mathbb{R}^N)$. Often we shall consider $\mathcal{F}$ only in some 'neighbourhood' of such a function u. Then it suffices to assume that $F \in C^1(\mathcal{U})$ where $\mathcal{U}$ denotes some open set in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ containing the 1-graph $\{(x, u(x), u'(x)) : x \in \overline{I}\}$; clearly, $\mathcal{F}(v)$ is defined for any $v \in C^1(\bar{I}, \mathbb{R}^N)$ satisfying $\|v - u\|_{C^1(I)} < \delta$ for some sufficiently small $\delta > 0$. It follows that the function

$$\Phi(\epsilon) := \mathcal{F}(u + \epsilon\varphi) \tag{1.2}$$

is defined for any choice of $\varphi \in C^1(\bar{I}, \mathbb{R}^N)$ and for $|\epsilon| < \epsilon_0$ where $\epsilon_0 := \delta/\|\varphi\|_{C^1(I)}$. Moreover, Φ is of class C^1 on $(-\epsilon_0, \epsilon_0)$ and we obtain

$$\Phi'(0) = \int_I \{F_z(x, u, u') \cdot \varphi + F_p(x, u, u') \cdot \varphi'\}\, dx. \tag{1.3}$$

We set

$$\delta\mathcal{F}(u, \varphi) := \Phi'(0) \tag{1.4}$$

and call $\delta\mathcal{F}(u, \varphi)$ *the first variation of $\mathcal{F}$ at u in the direction of φ*. We note that

$$\delta\mathcal{F}(u, \varphi) = \int_I \{F_z(x, u, u') \cdot \varphi + F_p(x, u, u') \cdot \varphi'\}\, dx \tag{1.5}$$

is a linear functional of $\varphi \in C^1(\bar{I}, \mathbb{R}^N)$.

Definition 1.1 *A function $u \in C^1(I, \mathbb{R}^N)$ satisfying*

$$\int_I \{F_z(x, u, u') \cdot \varphi + F_p(x, u, u') \cdot \varphi'\}\, dx = 0 \tag{1.6}$$

for all $\varphi \in C_c^\infty(I, \mathbb{R}^N)$ is said to be a weak extremal *of the functional $\mathcal{F}$.*

More precisely we should call u a weak C^1-extremal since later we shall also consider other kinds of weak extremals. If $u \in C^1(\bar{I}, \mathbb{R}^N)$ then (1.6) is equivalent to

$$\delta\mathcal{F}(u, \varphi) = 0 \quad \text{for all } \varphi \in C_c^\infty(I, \mathbb{R}^N).$$

In other words, weak $\mathcal{F}$-extremals of class $C^1(\bar{I}, \mathbb{R}^N)$ are *stationary points of $\mathcal{F}$ in all smooth directions $\varphi \in C_c^\infty(I, \mathbb{R}^N)$* in the sense that the directional derivative $\Phi'(0)$ vanishes, i.e. that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\{\mathcal{F}(u + \epsilon\varphi) - \mathcal{F}(u)\} = 0.$$

Obviously we have

Proposition 1.2 *If $u \in C^1(\bar{I}, \mathbb{R}^N)$ is a weak minimizer of $\mathcal{F}$*

$$\mathcal{F}(u) \le \mathcal{F}(u + \varphi) \tag{1.7}$$

for all $\varphi \in C_c^\infty(I, \mathbb{R}^N)$ with $\|\varphi\|_{C^1(I)} < \delta$, $0 < \delta \ll 1$, then u is a weak extremal of $\mathcal{F}$.

Remark 1 Note that extremals, i.e. solutions of the Euler equations, are not necessarily minimizers. For example, local minima and saddle points may also occur as solutions.

Now we want to show that weak extremals of class C^2 satisfy the Euler equation. For this purpose we need

Lemma 1.3 *The fundamental lemma Suppose that $f \in C^0(I)$ satisfies*

$$\int_I f(x)\eta(x)\,dx = 0 \quad \textit{for all } \eta \in C_c^\infty(I). \tag{1.8}$$

Then we have $f(x) = 0$ for all $x \in I$.

Proof Let χ_0 be the characteristic function of some interval $I_0 = (x_0 - \delta, x_0 + \delta) \subset\subset I$, $\delta > 0$. Since $C_c^\infty(I)$ is dense in $L^2(I)$ with respect to the L^2-norm, we infer from (1.8) that

$$\int_{I_0} f(x)\,dx = \int_I f(x)\chi_0(x)\,dx = 0, \tag{1.9}$$

and therefore

$$\frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta} f(x)\,dx = 0. \tag{1.10}$$

Letting $\delta \to +0$ we obtain $f(x_0) = 0$ for any $x_0 \in I$. □

We can strengthen the result of Lemma 1.3 in the following way.

Lemma 1.4 *If $f \in L^1(I)$ satisfies (1.8), then $f(x) = 0$ a.e. on I.*

Proof Choose I_0 and χ_0 as before, and consider the piecewise linear function $\eta_\epsilon \in C_c^0(I)$ defined by $\eta_\epsilon(x) := 1$ on I_0, $\eta_\epsilon(x) := 0$ for $|x - x_0| \geq \delta + \epsilon$, and linearly in $(x_0 - \delta - \epsilon, x_0 - \delta)$ and in $(x_0 + \delta, x_0 + \delta + \epsilon)$, $0 < \epsilon \ll 1$. Since $C_c^\infty(I)$ is dense in $C^0(\overline{I})$ with respect to the sup-norm on I, relation (1.8) implies

$$\int_I f(x)\eta_\epsilon(x)\,dx = 0 \quad \text{for any } \epsilon > 0.$$

If $\epsilon \to +0$ we infer that (1.9) and therefore also (1.10) hold true. Letting $\delta \to +0$ we obtain $f(x_0) = 0$ at all Lebesgue points $x_0 \in I$. □

Now we can derive the Euler equation.

Proposition 1.5 *Let $u : I \to \mathbb{R}^N$ be a weak extremal of $\mathcal{F}$, and assume that $u \in C^2(I, \mathbb{R}^N)$ and $F \in C^2(U)$ for some neighbourhood U of the 1-graph of u. Then u satisfies*

$$\frac{d}{dx}F_p(x, u(x), u'(x)) - F_z(x, u(x), u'(x)) = 0 \quad \textit{on } I. \tag{1.11}$$

Proof From (1.6) we infer by integration by parts that

$$\int_I \left\{ F_z(x,u,u') - \frac{d}{dx} F_p(x,u,u') \right\} \cdot \varphi(x)\,dx = 0$$

for all $\varphi \in C_c^\infty(I, \mathbb{R}^N)$. Choose $\varphi = (\varphi^1, \dots, \varphi^N)$ as $\varphi^k = 0$ if $k \neq i$, $\varphi^i = \eta \in C_c^\infty(I)$. Then we obtain

$$\int_I \left\{ F_{z^i}(x,u,u') - \frac{d}{dx} F_{p^i}(x,u,u') \right\} \eta(x)\,dx = 0$$

for all $\eta \in C_c^\infty(I)$, and Lemma 1.3 implies that

$$F_{z^i}(x,u(x),u'(x)) - \frac{d}{dx} F_{p^i}(x,u(x),u'(x)) = 0 \quad \text{on } I$$

for any $i \in \{1,2,\dots,N\}$. □

We call equation (1.11) the *Euler equation* associated with the Lagrangian F. Then Proposition 1.5 states that any weak C^2-extremal of a variational integral $\mathcal{F}$ with a C^2-Lagrangian F necessarily satisfies the Euler equation (1.11). Passing from the vectorial form (1.11) to its components, we arrive at the *system of Euler equations*

$$\frac{d}{dx} F_{p^i}(x,u(x),u'(x)) - F_{z^i}(x,u(x),u'(x)) = 0, \quad 1 \le i \le N, \tag{1.12}$$

which is to be satisfied by any C^2-extremal $u = (u^1, \dots, u^N)$. Clearly (1.12) is a system of N quasilinear equations of second order for the N functions $u^1, \dots, u^N$.

Let us consider three examples.

$\boxed{1}$ The Lagrangian $F(x,z,p) = \omega(x,z)\sqrt{1+p^2}$ with $\omega(x,z) > 0$ and $N = 1$ leads to the variational integral

$$\mathcal{F}(u) = \int_a^b \omega(x,u)\sqrt{1+u'^2}\,dx,$$

a kind of weighted length, with the corresponding Euler equation

$$\frac{d}{dx}\left[\omega(x,u)\frac{u'}{\sqrt{1+u'^2}}\right] - \omega_z(x,u)\sqrt{1+u'^2} = 0.$$

This can be written as

$$\kappa\omega\sqrt{1+u'^2} = \omega_z - u'\omega_x$$

where

$$\kappa = \frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right)$$

denotes the curvature of the graph of u.

$\boxed{2}$ The Lagrangian $F(x, z, p) = p^2 + q(x)z^2$, $N = 1$, defines the variational integral

$$\mathcal{F}(u) = \int_a^b [u'^2 + c(x)u^2]\,dx$$

with the Euler equation

$$-u'' + q(x)u = 0.$$

$\boxed{3}$ Let $V(x)$ be of class $C^1(\mathbb{R}^3)$ and m a positive constant, and consider the Lagrangian

$$F(x, v) = \frac{m}{2}|v|^2 + V(x)$$

for $N = 3$ where we have replaced z and p by x and v respectively, while the independent variable x will be denoted by t. We look for functions $x(t)$ which make the variational integral

$$\mathcal{F}(x) = \int_{t_1}^{t_2} \left[\frac{m}{2}|\dot{x}|^2 - V(x)\right] dt$$

stationary. In mechanics $\mathcal{F}(x)$ is the *action* of a motion $x = x(t)$, $t_1 \le t \le t_2$, of a point mass m in a *conservative force field* $-V_x$ with the *potential energy* V, and $T = \frac{1}{2}m\dot{x}^2$ is the *kinetic energy* of the motion $x(t)$. The Euler equation of the action $\mathcal{F}$ is equivalent to Newton's equation

$$m\ddot{x} = -V_x(x).$$

Combining Propositions 1.2 and 1.5, we obtain

Proposition 1.6 *Suppose that $\mathcal{F}(u) \le \mathcal{F}(v)$ for all $v \in C^1(\bar{I}, \mathbb{R}^N)$ satisfying $u = v$ on ∂I and $\|u - v\|_{C^1(I)} < \delta$ for some sufficiently small $\delta > 0$. Moreover, let $u \in C^2(I, \mathbb{R}^N) \cap C^1(\bar{I}, \mathbb{R}^N)$ and $F \in C^2$. Then u satisfies the Euler equation* (1.11).

Definition 1.7 *Any solution $u \in C^2(I, \mathbb{R}^N)$ of the Euler equation* (1.11) *is said to be an* extremal *of $\mathcal{F}$.*

In other words, weak minimizers of $\mathcal{F}$ in the sense of Proposition 1.6 are $\mathcal{F}$-extremals if they are of class C^2 and if the Lagrangian F is of class C^2. Note, however, that a (weak) minimizer of $\mathcal{F}$ in C^1 need not be of class C^2. Consider the following example:

$\boxed{4}$ $F(x, z, p) = z^2(2x - p)^2$, $N = 1$, and $I = (-1, 1)$; hence

$$\mathcal{F}(u) = \int_{-1}^{1} u^2(x)[2x - u'(x)]^2\,dx.$$

Then the function $u \in C^1(\bar{I})$ defined by $u(x) := 0$ on $[-1, 0]$, $u(x) := x^2$ on $[0, 1]$ is the unique minimizer of $\mathcal{F}$ among all $v \in C^1(\bar{I})$ satisfying $v(-1) = 0$, $v(1) = 1$; but $u \notin C^2(I)$.

We shall treat the question of regularity for weak extremals and minimizers in Chap. 4.

Although a weak C^1-extremal u of $\mathcal{F}$ need not be of class C^2, it will nevertheless turn out that the Euler equation (1.11) holds true, even if $F_p(x, z, p)$ is merely assumed to be continuous. This will be proved by means of

Lemma 1.8 (DuBois-Reymond's lemma) *Suppose that $f \in L^1(I)$ satisfies*

$$\int_I f(x)\eta'(x)\,dx = 0 \quad \textit{for all } \eta \in C_c^\infty(I). \tag{1.13}$$

Then there is a constant $c \in \mathbb{R}$ such that $f(x) = c$ a.e. on I.

Proof Fix two Lebesgue points $x_0, \xi \in I$ of f and set $c := f(x_0)$. Suppose that $x_0 < \xi$ and $(x_0 - \epsilon, \xi + \epsilon) \subset\subset I$, $\epsilon > 0$. Then we choose a piecewise linear function $\zeta \in C_c^0(I)$ by setting $\zeta(x) := 1$ on $[x_0, \xi]$, $\zeta(x) := 0$ for $x \notin [x_0 - \epsilon, \xi + \epsilon]$, and $\zeta(x) := \epsilon^{-1}(x - x_0 + \epsilon)$ in $[x_0 - \epsilon, x_0]$, $\zeta(x) := \epsilon^{-1}(\xi - x + \epsilon)$ in $[\xi, \xi + \epsilon]$. By a straightforward approximation argument we infer from (1.13) that also

$$\int_I f(x)\zeta'(x)\,dx = 0$$

holds true, and this is equivalent to

$$\frac{1}{\epsilon}\int_{x_0-\epsilon}^{x_0} f(x)\,dx - \frac{1}{\epsilon}\int_{\xi}^{\xi+\epsilon} f(x)\,dx = 0.$$

Letting $\epsilon \to +0$ we arrive at $f(x_0) = f(\xi)$, i.e. $f(\xi) = c$ for any Lebesgue point $\xi > x_0$. If $\xi < x_0$ we reverse the roles of x_0 and ξ, and we obtain the same result. Thus we have $f(\xi) = c$ for any Lebesgue point ξ of f. □

Proposition 1.9 *Let $u \in C^1(\overline{I}, \mathbb{R}^N)$ be a weak extremal of $\mathcal{F}$, i.e.*

$$\int_I \{F_z(x, u, u') \cdot \varphi + F_p(x, u, u') \cdot \varphi'\}\,dx = 0$$

for all $\varphi \in C_c^\infty(I, \mathbb{R}^N)$. Then there is a constant vector $c \in \mathbb{R}^N$ such that

$$F_p(x, u(x), u'(x)) = c + \int_a^x F_z(t, u(t), u'(t))\,dt \tag{1.14}$$

for all $x \in (a, b) = I$.

Proof An integration by parts leads to

$$\int_a^b F_z(x, u, u') \cdot \varphi(x)\,dx = -\int_a^b \left(\int_a^x F_z(t, u(t), u'(t))\,dt\right) \cdot \varphi'(x)\,dx$$

for all $\varphi \in C_c^\infty(I)$, whence

$$\int_I \left[F_p(x, u(x), u'(x)) - \int_a^x F_z(t, u(t), u'(t))\,dt\right] \cdot \varphi'(x)\,dx = 0$$

for all $\varphi \in C_c^\infty(I)$. Then we obtain (1.14) on account of DuBois-Reymond's lemma.

□

Note that the function

$$\pi(x) := c + \int_a^x F_z(t, u(t), u'(t))\, dt \tag{1.15}$$

on the right-hand side of (1.14) is of class $C^1(\bar{I}, \mathbb{R}^N)$. Hence also $F_p(\cdot, u, u') \in C^1(\bar{I}, \mathbb{R}^N)$, and we obtain (1.11) by differentiating (1.14) with respect to x. Thus we have

Corollary 1.10 *Any weak C^1-extremal u of $\mathcal{F}$ satisfies*

$$\frac{d}{dx} F_p(x, u(x), u'(x)) - F_z(x, u(x), u'(x)) = 0.$$

Note, however, that we are not allowed to apply the chain rule to differentiate $F_p(\cdot, u, u')$ with respect to x. We denote equation (1.14) as *DuBois-Reymond's equation* or as *the Euler equation in integrated form.*

By essentially the same reasoning we obtain

Proposition 1.11 *If $u \in Lip(I, \mathbb{R}^N)$ is a weak Lipschitz-extremal of $\mathcal{F}$, i.e. if $\delta\mathcal{F}(u, \varphi) = 0$ for all $\varphi \in C_c^\infty(I, \mathbb{R}^N)$, then there exists a constant vector $c \in \mathbb{R}^N$ such that*

$$F_p(x, u(x), u'(x)) = c + \int_a^x F_z(t, u(t), u'(t))\, dt$$

holds true for almost all $x \in I = (a, b)$, and that

$$\frac{d}{dx} F_p(x, u(x), u'(x)) = F_z(x, u(x), u'(x)) \quad \text{a.e. on } I.$$

Now we want to derive another necessary condition, the so-called *natural boundary conditions*, which will be satisfied by minimizers with 'free boundary values'.

Let us introduce the *Euler operator* $L_F(u)$ by

$$L_F(u) := F_z(\cdot, u, u') - DF_p(\cdot, u, u'). \quad D = \frac{d}{dx}. \tag{1.16}$$

Consider $u \in C^2(\bar{I}, \mathbb{R}^N)$, $F \in C^2(U)$, and $(\alpha, \beta) \subset I$. Then we obtain for all $\varphi \in C^1(\bar{I}, \mathbb{R}^N)$ by means of an integration by parts that

$$\begin{aligned}
&\int_\alpha^\beta \{F_z(\cdot, u, u') \cdot \varphi + F_p(\cdot, u, u') \cdot \varphi'\}\, dx \\
&\quad = \int_\alpha^\beta L_F(u) \cdot \varphi\, dx + [F_p(x, u(x), u'(x)) \cdot \varphi(x)]_\alpha^\beta.
\end{aligned} \tag{1.17}$$

This identity leads to the following result:

Proposition 1.12 *Suppose that $u \in C^1(\bar{I}, \mathbb{R}^N)$, $F \in C^1(U)$, and*

$$\delta\mathcal{F}(u, \varphi) = 0 \quad \text{for all } \varphi \in C^1(\bar{I}, \mathbb{R}^N). \tag{1.18}$$

Then the function $F_p(x, u(x), u'(x))$ vanishes at the end points $x = a$ and $x = b$ of I.

Proof (i) Assume first that $u \in C^2(\bar{I}, \mathbb{R}^N)$ and $F \in C^2(U)$. Clearly $\delta\mathcal{F}(u, \varphi) = 0$ holds for all $\varphi \in C_c^\infty(I, \mathbb{R}^N)$, whence $L_F(u) = 0$ on account of Proposition 1.5. Then relations (1.17) and (1.18) imply

$$[F_p(x, u(x), u'(x)) \cdot \varphi(x)]_a^b = 0 \tag{1.19}$$

if we choose $\alpha = a$ and $\beta = b$. For any vector $\xi \in \mathbb{R}^N$ we can find a function $\varphi \in C^1(\bar{I}, \mathbb{R}^N)$ such that $\varphi(a) = 0$ and $\varphi(b) = \xi$. Then it follows from (1.19) that the function

$$\pi(x) := F_p(x, u(x), u'(x)), \quad x \in \bar{I}, \tag{1.20}$$

satisfies $\pi(b) \cdot \xi = 0$ for all $\xi \in \mathbb{R}^N$, whence $\pi(b) = 0$, and similarly we obtain $\pi(a) = 0$. Hence the claim is proved in the special case when $u \in C^2$ and $F \in C^2$.

(ii) Now we only assume that $u \in C^1(F, \mathbb{R}^N)$ and $F \in C^1(U)$. By Proposition 1.9 we obtain that the function $F_p(x, u(x), u'(x))$ is of class C^1, and so we deduce at once that (1.17) still holds. Then the previous argument applies again thus concluding the proof. □

To find solutions of the Euler equation $L_F(u) = 0$ it is useful to determine *first integrals* of this equation, i.e. functions which are constant along the 1-graph of any solution of the Euler equation. If $N = 1$ we can reduce $L_F(u) = 0$ to a scalar equation of first order. If F does not depend on x, i.e. if $F = F(z, p)$, an important first integral is given by the function

$$\Phi(z, p) := p \cdot F_p(z, p) - F(z, p). \tag{1.21}$$

In fact we have

Proposition 1.13 *If $F_x = 0$, then $\Phi(u(x), u'(x)) \equiv \text{const}$ on $\bar{I}$ for any solution $u \in C^2(I, \mathbb{R}^N)$ of $L_F(u) = 0$, i.e. for any extremal u of $\mathcal{F}$.*

Proof A straightforward computation yields the identity

$$\frac{d}{dx}\Phi(u, u') = u' \cdot L_F(u)$$

whence $d(\Phi(u, u'))/dx = 0$ if $L_F(u) = 0$. □

Remark 2 The *conservation law* '$\Phi(u, u') = \text{const}$' may have more solutions than the Euler equation $L_F(u) = 0$. For instance, if $u(x) \equiv \text{const} =: c$ and $F(c, 0) = -h$, then u is a solution of $\Phi(u, u') = h$, but it is a solution of $L_F(u) = 0$ only if $F_z(c, 0) = 0$. Thus one has to check that, by solving $\Phi(u, u') = h$, truly a solution of the Euler equation is picked up.

$\boxed{5}$ Let $F(x, v) = \frac{m}{2}|v|^2 - V(x)$ be the Lagrangian considered in $\boxed{3}$, and let

$$E(x, v) = v \cdot F_v(x, v) - F(x, v)$$

be the corresponding function (1.21). It is easily seen that

$$E(x, v) = \frac{m}{2}|v|^2 + V(x),$$

i.e. $E(x, \dot{x})$ is the *total energy* of a motion $x = x(t)$, and Proposition 1.13 states that

$$E(x(t), \dot{x}(t)) \equiv \text{const}$$

along any extremal of the action integral

$$\mathcal{F}(x) = \int_{t_1}^{t_2} \left[\frac{m}{2}|\dot{x}|^2 - V(x)\right] dt,$$

i.e. total energy is conserved along any solution $x(t)$ of Newton's equation

$$m\ddot{x} = -V_x(x).$$

In Section 1.3 we shall use the conservation law $\Phi(u, u') = h$ to determine the extremals of some classical minimum problems. Furthermore we can show that a rather large class of functions $u : [a, b] \to \mathbb{R}^N$ satisfies the conservation law

$$\Phi(u(x), u'(x)) = \text{const} \quad \text{on } [a, b]. \tag{1.22}$$

To this end we consider an arbitrary function $u \in C^1(\overline{I}, \mathbb{R}^N)$ and a mapping $(t, \epsilon) \mapsto x = \xi(t, \epsilon)$ of class C^1 on $\overline{I} \times (-\epsilon_0, \epsilon_0)$, $\epsilon_0 > 0$, such that, for any $\epsilon \in (-\epsilon_0, \epsilon_0) =: I_0$, the mapping $\xi(\cdot, \epsilon)$ is a C^1-diffeomorphism of $\overline{I}$ onto itself satisfying $\xi(a, \epsilon) = a$ and $\xi(b, \epsilon) = b$. Furthermore we assume that $\partial\xi/\partial\epsilon(\cdot, \epsilon) \in C^1(\overline{I})$ for $\epsilon \in I_0$ and $\xi(t, 0) = t$ for all $t \in \overline{I}$. We call $\{\xi(\cdot, \epsilon)\}_{\epsilon \in I_0}$ an *admissible parameter variation*, and the family $\{v(\cdot, \epsilon)\}_{\epsilon \in I_0}$ of functions

$$v(t, \epsilon) := u(\xi(t, \epsilon)), \quad (t, \epsilon) \in \overline{I} \times I_0, \tag{1.23}$$

is said to be an *admissible inner variation of* u. Note that $v(t, 0) = u(t)$ and that $v(t, \epsilon)$ has the same values as $u(t)$ at $t = a$ and $t = b$ respectively. We set

$$\lambda(t) := \frac{\partial\xi}{\partial\epsilon}(t, 0) \tag{1.24}$$

and

$$\Psi(\epsilon) := \int_a^b F(t, v(t, \epsilon), \dot{v}(t, \epsilon))\, dt = \mathcal{F}(v(\cdot, \epsilon)), \tag{1.25}$$

where $\dot{v}$ denotes the derivative with respect to t. If u is a minimizer in a class C of functions that is invariant with respect to admissible parameter variations (i.e. if $u \in C$, then also $v(\cdot, \epsilon) = u \circ \xi(\cdot, \epsilon) \in \mathcal{C}$ for any admissible parameter variation $\{\xi(\cdot, \epsilon)\}_{\epsilon \in I_0}$, then we have

$$\Psi(\epsilon) \geq \Psi(0) \quad \text{for } |\epsilon| < \epsilon_0$$

and therefore

$$\Psi'(0) = 0.$$

We set

$$\partial\mathcal{F}(u, \lambda) := \Psi'(0) \tag{1.26}$$

and call $\partial\mathcal{F}(u, \lambda)$ *the first inner variation of $\mathcal{F}$ at u in the direction of λ.*

Proposition 1.14 *For $u \in C^1(\overline{I}, \mathbf{R}^N)$ the inner variation of $\mathcal{F}$ in the direction of $\lambda = \partial\xi/\partial\epsilon(\cdot, 0)$ is given by*

$$\partial\mathcal{F}(u, \lambda) = \int_a^b \{[u' \cdot F_p(x, u, u') - F(x, u, u')]\lambda' - F_x(x, u, u')\lambda\}\, dx. \tag{1.27}$$

Proof From $t = \tau(\xi(t, \epsilon), \epsilon)$ we obtain by differentiation with respect to t and ϵ respectively that ($' = \partial/\partial x$, $\dot{} = \partial/\partial t$)

$$1 = \tau'(\xi(t, \epsilon), \epsilon)\dot{\xi}(t, \epsilon)$$

and

$$0 = \tau_\epsilon(\xi(t, \epsilon), \epsilon) + \tau'(\xi(t, \epsilon), \epsilon) \; \xi_\epsilon(t, \epsilon).$$

Furthermore we have $t = \xi(t, 0) = x$ and $\tau'(x, 0) = 1$, whence

$$\tau_\epsilon(x, 0) = -\lambda(x); \tag{1.28}$$

thus we have the Taylor expansion

$$\tau(x, \epsilon) = x - \epsilon\lambda(x) + \cdots \tag{1.29}$$

of $\tau(x, \epsilon)$ with respect to ϵ. Applying the coordinate transformation $x \mapsto t = \tau(x, \epsilon)$ to

$$\Psi(\epsilon) = \int_a^b F(t, u(\xi(t, \epsilon)), u'(\xi(t, \epsilon))\dot{\xi}(t, \epsilon))\, dt$$

we obtain

$$\Psi(\epsilon) = \int_a^b F\Big(\tau(x, \epsilon), u(x), u'(x)\, \frac{1}{\tau'(x, \epsilon)}\Big)\tau'(x, \epsilon)\, dx.$$

From

$$\frac{\partial}{\partial \epsilon}\tau'(x,\epsilon) = \frac{\partial}{\partial x}\frac{\partial}{\partial \epsilon}\tau(x,\epsilon)$$

we infer that

$$\frac{\partial}{\partial \epsilon}\tau'(x,\epsilon)\Big|_{\epsilon=0} = -\lambda'(x),$$

and consequently

$$\Psi'(0) = \int_a^b \{-\lambda F_x(x,u,u') + \lambda' u' \cdot F_p(x,u,u') + F(x,u,u')(-\lambda')\}\, dx.$$

□

Let $\lambda(x)$ be an arbitrary function of class $C_c^\infty(I)$, and set

$$\tau(x,\epsilon) := x - \epsilon\lambda(x)$$

for $x \in \bar{I}$ and $\epsilon \in (-\epsilon_0, \epsilon_0)$, $\epsilon_0 > 0$. Because of

$$\tau'(x,\epsilon) = 1 - \epsilon\lambda'(x)$$

we have $\tau'(x,\epsilon) > 0$ for $x \in \bar{I}$ and $|\epsilon| < \epsilon_0 \ll 1$, and $\lambda(a) = \lambda(b) = 0$. Therefore the mapping $x \mapsto t = \tau(x,\epsilon)$ yields a C^1-diffeomorphism of $[a,b]$ onto itself satisfying $\tau(a,\epsilon) = a$, $\tau(b,\epsilon) = b$ provided that $|\epsilon| < \epsilon_0 \ll 1$. It is easily seen that by $\xi(\cdot,\epsilon) := \tau^{-1}(\cdot,\epsilon)$, $|\epsilon| < \epsilon_0 \ll 1$, we define an admissible parameter variation satisfying (1.24). Hence we obtain the following result:

Proposition 1.15 *If*

$$\frac{\partial}{\partial \epsilon}\mathcal{F}(v(\cdot,\epsilon))\Big|_{\epsilon=0} = 0 \tag{1.30}$$

for any admissible inner variation $v(t,\epsilon) = u(\xi(t,\epsilon))$ of u, then we have

$$\partial\mathcal{F}(u,\lambda) = 0 \quad \textit{for all } \lambda \in C_c^\infty(I). \tag{1.31}$$

By applying DuBois-Reymond's lemma we infer analogously to Proposition 1.9 the following generalized version of the conservation law (1.22).

Proposition 1.16 *If (1.30) holds for any admissible inner variation $v(t,\epsilon) = u(\xi(t,\epsilon))$ of u, then there is a constant c such that Erdmann's equation*

$$\Phi(x,u(x),u'(x)) = c - \int_a^x F_x(t,u(t),u'(t))\, dt \quad \textit{for all } x \in \bar{I} \tag{1.32}$$

holds true with $\Phi(x, z, p) := F(x, z, p) - p \cdot F_p(x, z, p)$. *Moreover, since* $u \in C^1(\bar{I}, \mathbf{R}^N)$, *it follows that* $F_x(\cdot, u, u')$ *is of class* $C^0(\bar{I})$. *Then eqn* (1.32) *implies that* $\Phi(\cdot, u, u')$ *is of class* $C^1(\bar{I})$ *and differentiation of* (1.32) *yields Noether's equation*

$$\frac{d}{dx}\Phi(x, u(x), u'(x)) + F_x(x, u(x), u'(x)) = 0 \quad \text{in } \bar{I}. \tag{1.33}$$

In particular, if the Lagrangian F does not depend on x, i.e. if $F = F(z, p)$, *we obtain relation* (1.22).

Remark 3 Suppose that $F(z, p)$ is a Lagrangian of class C^1 on $\mathbf{R}^N \times \mathbf{R}^N$ which is positively homogeneous of second order with respect to p, i.e. $F(z, \lambda p) = \lambda^2 F(z, p)$ for any $\lambda > 0$. Then it follows that $F = p \cdot F_p - F = \Phi$. We conclude that

$$F(u(x), u'(x)) = \text{const} \quad \text{on } [a, b]$$

holds true if relation (1.30) is satisfied for any admissible inner variation $v(t, \epsilon) = u(\xi(t, \epsilon))$, $|\epsilon| < \epsilon_0$, of u. If, in addition, $F(z, p) > 0$ provided that $p \neq 0$, and if $u(a) \neq u(b)$, then we have

$$F(u(x), u'(x)) = h \quad \text{on } [a, b] \tag{1.34}$$

for some positive constant h, and in particular $u'(x) \neq 0$ on $[a, b]$.

Let us note that this reasoning extends even to functions u of class $H^{1,2}(I, \mathbf{R}^N)$ which satisfy (1.30) with respect to any admissible parameter variation. This observation will be used in Section 5.9 where we shall establish the existence of regular minimizers for parametric variational problems.

Usually it is a rather subtle problem to decide whether an extremal of $\mathcal{F}$ actually is a local or even a global minimizer of $\mathcal{F}$ in a given class of mappings $u \in C^1(\bar{I}, \mathbf{R}^N)$, say, in the class of mappings with prescribed boundary values. In the next section we describe a method that in principle can be used to decide the question of minimization, though it might be quite cumbersome to carry out the method for specific problems.

There are several *other necessary conditions* to be satisfied by minimizers u of a variational integral $\mathcal{F}(u)$ which are treated in the classical texts. For instance, a minimizer must satisfy[4] the *necessary Legendre condition*

$$F_{p^i p^k}(x, u(x), u'(x))\xi^i \xi^k \geq 0 \tag{1.35}$$

for all $\xi \in \mathbf{R}^N$ and all $x \in \bar{I}$. In fact, if $u \in C^1(\bar{I}, \mathbf{R}^N)$ is a weak minimizer of $\mathcal{F}$ in the sense that $\mathcal{F}(u) \leq \mathcal{F}(u + \varphi)$ for all $\varphi \in C_c^\infty(I, \mathbf{R}^N)$ with $\|\varphi\|_{C^1(I)} < \delta$, $0 < \delta \ll 1$, and if F is of class C^2, then the function $\Phi(\epsilon) := \mathcal{F}(u + \epsilon\varphi)$ satisfies $\Phi(0) \leq \Phi(\epsilon)$

[4]Here and in the sequel we use the summation convention to sum from 1 to N with respect to repeated Latin indices.

for $|\epsilon| < \delta/\|\varphi\|_{C^1(I)}$ and $\varphi \in C_c^\infty(I, \mathbb{R}^N)$, whence $\Phi'(0) = 0$ and $\Phi''(0) \geq 0$. The first relation leads to (6) whereas the second yields

$$\delta^2\mathcal{F}(u, \varphi) \geq 0 \quad \text{for all } \varphi \in C_c^\infty(I, \mathbb{R}^N). \tag{1.36}$$

Here $\delta^2\mathcal{F}(u, \varphi) := \Phi''(0)$ is the so-called *second variation of $\mathcal{F}$ at u in the direction of φ*, and a straightforward computation shows that

$$\delta^2\mathcal{F}(u, \varphi) = 2\int_a^b Q(x, \varphi(x), \varphi'(x))\, dx \tag{1.37}$$

where we have set

$$2Q(x, \zeta, \pi) := a_{jk}(x)\pi^j\pi^k + 2b_{jk}(x)\zeta^j\pi^k + c_{jk}(x)\zeta^j\zeta^k \tag{1.38}$$

and

$$\begin{aligned} a_{jk}(x) &:= F_{p^j p^k}(e(x)), & b_{jk}(x) &:= F_{z^j p^k}(e(x)), \\ c_{jk}(x) &:= F_{z^j z^k}(e(x)), & e(x) &:= (x, u(x), u'(x)). \end{aligned} \tag{1.39}$$

Here ζ and π denote N-tuples $(\zeta^1, \ldots, \zeta^N)$ and $(\pi^1, \ldots, \pi^N)$. An appropriate approximation argument yields that (1.24) implies $\delta^2\mathcal{F}(u, \varphi) \geq 0$ for all Lipschitz-continuous functions $\varphi : \overline{I} \to \mathbb{R}^N$ with supp $\varphi \subset I$. In particular, if $[x_0 - \rho, x_0 + \rho] \subset I$, $\rho > 0$, and $\xi = (\xi^1, \ldots, \xi^N) \in \mathbb{R}^N$, we can choose $\varphi(x)$ as $\varphi(x) = 0$ in $[a, x_0 - \rho] \cup [x_0 + \rho, b]$, $\varphi(x) = \lambda\xi\rho^{-1}(x - x_0 + \rho)$ in $[x_0 - \rho, x_0]$, and $\varphi(x) = \lambda\xi\rho^{-1}(x_0 + \rho - x)$ in $[x_0, x_0 + \rho]$. Then we obtain

$$0 \leq \delta^2\mathcal{F}(u, \varphi) = \int_{x_0-\rho}^{x_0+\rho} a_{jk}(x)\xi^j\xi^k\, dx + o(\rho) \quad \text{as } \rho \to +0$$

and therefore

$$0 \leq \frac{1}{2\rho}\int_{x_0-\rho}^{x_0+\rho} a_{jk}(x)\xi^j\xi^k\, dx + o(1) \quad \text{as } \rho \to +0.$$

Letting $\rho \to +0$ we arrive at

$$a_{jk}(x_0)\xi^j\xi^k \geq 0$$

for any $x_0 \in I$ and all $\xi \in \mathbb{R}^N$, since the coefficients $a_{jk}(x)$ are continuous on $\overline{I}$, and the necessary Legendre condition (1.35) follows at once.

For a detailed discussion of necessary conditions we refer, for example, to the treatises of Bolza [36], Bliss [35], Carathéodory [57], Akhiezer [3], Gelfand–Fomin [112], or Giaquinta–Hildebrandt [113].

Finally we want to state the *Lagrange multiplier theorem* for variational problems with so-called *isoperimetric side conditions*. These are constraints of the form

$$\mathcal{G}(u) = \text{const} \tag{1.40}$$

where $\mathcal{G}(u)$ is a given functional of the form

$$\mathcal{G}(u) := \int_a^b G(x, u(x), u'(x))\,dt$$

with a Lagrangian $G(x, z, p)$ of class C^2 in a neighbourhood of 1-graph u. We are looking for functions $u : \overline{I} \to \mathbf{R}^N$, $I = (a, b)$, which minimize a given variational integral $\mathcal{F}(u)$ of kind (1.1) among all C^1-mappings $\overline{I} \to \mathbf{R}^N$ satisfying prescribed boundary conditions as well as the subsidiary condition (1.40).

Proposition 1.17 *Suppose that u is a weak minimizer of the variational integral $\mathcal{F}$ in the class $\mathcal{C}$ of all functions $v \in C^1(I, \mathbf{R}^N)$ satisfying the boundary conditions $v(a) = \alpha$, $v(b) = \beta$ and the constraint $\mathcal{G}(v) = c$ for some prescribed constant c. Assume also that $\delta\mathcal{G}(u, \varphi)$ does not vanish for all $\varphi \in C_c^\infty(I, \mathbf{R}^N)$. Then there is a real number λ such that*

$$\delta\mathcal{F}(u, \varphi) + \lambda\,\delta\mathcal{G}(u, \varphi) = 0 \quad \text{for all} \quad \varphi \in C_c^\infty(I, \mathbf{R}^N). \tag{1.41}$$

Moreover, if $u \in C^2(I, \mathbf{R}^N)$ then

$$\frac{d}{dx} H_p(x, u, u') - H_z(x, u, u') = 0 \tag{1.42}$$

where $H(x, z, p)$ denotes the Lagrangian $H := F + \lambda G$.

Proof By assumption there is a function $\psi \in C_c^\infty(I, \mathbf{R}^N)$ such that $\delta\mathcal{G}(u, \psi) = 1$. With this function and an arbitrarily chosen $\varphi \in C_c^\infty(I, \mathbf{R}^N)$ we define two functions $\Phi : Q \to \mathbf{R}$ and $\Psi : Q \to \mathbf{R}$ on $Q := \{(\epsilon, t) : |\epsilon| < \epsilon_0, \ |t| < t_0\}$, $0 < \epsilon_0$, $t_0 \ll 1$, by

$$\Phi(\epsilon, t) := \mathcal{F}(u + \epsilon\varphi + t\psi), \qquad \Psi(\epsilon, t) := \mathcal{G}(u + \epsilon\varphi + t\psi).$$

Since $\Psi_t(0, 0) = 1$ we can apply the implicit function theorem, and, for $|\epsilon_0| \ll 1$, we obtain a function $\tau \in C^1(-\epsilon_0, \epsilon_0)$ with $\tau(0) = 0$ such that $(\epsilon, \tau(\epsilon)) \in Q$ for $|\epsilon| < \epsilon_0$ and

$$\Psi(\epsilon, \tau(\epsilon)) = c \quad \text{for all } \epsilon \in (-\epsilon_0, \epsilon_0)$$

whence

$$\tau'(0) = -\Psi_\epsilon(0, 0).$$

Furthermore the functions u and $v = u + \epsilon\varphi + t\psi$ satisfy the same boundary conditions at $x = a$ and $x = b$, and their C^1-distance

$$\|v - u\|_{C^1(\bar{I})} \leq |\epsilon|\|\varphi\|_{C^1(\bar{I})} + |t|\|\psi\|_{C^1(\bar{I})}$$

tends to zero as $\epsilon \to 0$ and $t \to 0$. Thus we have

$$\Phi(\epsilon, \tau(\epsilon)) \geq \Phi(0, 0) \quad \text{for } |\epsilon| \ll 1,$$

and it follows that

$$\Phi_\epsilon(0, 0) + \Phi_t(0, 0)\tau'(0) = 0.$$

By introducing the Lagrange multiplier λ as

$$\lambda := -\Phi_t(0, 0) = -\delta\mathcal{G}(u, \psi),$$

which is independent of φ, we arrive at the equation

$$\Phi_\epsilon(0, 0) + \lambda\Psi_\epsilon(0, 0) = 0.$$

Thus we have proved that

$$\delta\mathcal{F}(u, \varphi) + \lambda\, \delta\mathcal{G}(u, \varphi) = 0 \quad \text{for all } \varphi \in C_c^\infty(I, \mathbb{R}^N),$$

which implies (1.42) on account of Proposition 1.5. □

$\boxed{6}$ Let

$$\mathcal{F}(u) = \int_a^b [u'^2 + q(x)u^2]\, dx$$

be the variational integral of $\boxed{2}$ which is to be minimized among all functions of class $C^1(\bar{I})$, $I = (a, b)$, satisfying $u(a) = 0$ and $u(b) = 0$ as well as the constraint

$$\int_a^b u^2\, dx = 1.$$

Then any minimizer u of this problem with $u \in C^2(I)$ satisfies

$$-u'' + q(x)u = \lambda u,$$
$$u(a) = 0, \qquad u(b) = 0,$$

for some suitable constant λ. Thus the Lagrange multiplier λ is an *eigenvalue* of the *Sturm–Liouville operator*

$$L = -\frac{d^2}{dx^2} + q(x)$$

for zero boundary conditions, and the minimizer u is an *eigenfunction* of L corresponding to λ, and it is easily seen that λ is the smallest eigenvalue of L for these boundary conditions.

1.2 Calibrators and sufficient conditions for minima

Now we want to describe an approach to sufficient conditions for optimality which was introduced by Carathéodory [57]. The basic idea of this method is due to Johann Bernoulli (cf. Carathéodory [58]).

We begin with a simple geometric observation.

Proposition 1.18 *Let $\mathcal{F} : C \to \mathbf{R}$ and $\mathcal{M} : C \to \mathbf{R}$ be two real-valued functionals on a set C, and suppose that for some element $u_0 \in C$ the following conditions are satisfied:*

(i) $\mathcal{F}(u_0) = \mathcal{M}(u_0)$, *and* $\mathcal{F}(u) \geq \mathcal{M}(u)$ *for all* $u \in C$;

(ii) $\mathcal{M}(u) \geq \mathcal{M}(u_0)$ *for all* $u \in C$.

Then we have

$$\mathcal{F}(u) \geq \mathcal{F}(u_0) \quad \textit{for all } u \in C.$$

Proof Since $\mathcal{F}(u_0) = \mathcal{M}(u_0)$ we can write

$$\begin{aligned}\mathcal{F}(u) - \mathcal{F}(u_0) &= \mathcal{F}(u) - \mathcal{M}(u_0)\\ &= [\mathcal{F}(u) - \mathcal{M}(u)] + [\mathcal{M}(u) - \mathcal{M}(u_0)].\end{aligned}$$

By (i) and (ii), both brackets are non-negative for any $u \in C$, whence

$$\mathcal{F}(u) - \mathcal{F}(u_0) \geq 0 \quad \text{for all } u \in C.$$

□

We call the functional $\mathcal{M}$ a *minimizing osculator for $\mathcal{F}$ at $u_0 \in C$*. Then we can state Proposition 1.18 as follows: *An element $u_0 \in C$ is a minimizer for the functional $\mathcal{F} : C \to \mathbf{R}$ if we can find a minimizing osculator for $\mathcal{F}$ at u_0.*

Let us now apply this geometric principle to variational integrals

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x))\, dx \tag{1.43}$$

with a Lagrangian $F(x, z, p)$ of class C^2 which is defined on all of $\mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N$. Set $I = (a, b)$. We consider $\mathcal{F}$ as a functional on the class $C^1(\overline{I}, \mathbf{R}^N)$, or on subclasses thereof, for instance

$$C(a, b) = \{u \in C^1(\overline{I}, \mathbf{R}^N) : u(a) = \alpha, u(b) = \beta\} \tag{1.44}$$

with fixed boundary values α and β. How can one find an osculating minimizer $\mathcal{M}$ for $\mathcal{F}$ at u_0 which is of the form

$$\mathcal{M}(u) = \int_a^b M(x, u(x), u'(x))\, dx \tag{1.45}$$

with a C^2-Lagrangian $M(x, z, p)$? The easiest way to satisfy conditions (i) and (ii) of Proposition 1.18 is to require that $\mathcal{M}(u)$ is constant on $\mathcal{C}$ and that the conditions

$$M(x, u_0(x), u_0'(x)) = F(x, u_0(x), u_0'(x)),$$
$$M(x, u(x), u'(x)) \leq F(x, u(x), u'(x))$$

are satisfied for all $x \in I$ and all $u \in \mathcal{C}$. This leads us to

Definition 1.19 *A* calibrator for the triple $\{F, u_0, \mathcal{C}\}$ with $u_0 \in \mathcal{C}$ *is a Lagrangian $M(x, z, p)$ satisfying the following two conditions:*

(i) *For all $x \in I$ and $u \in \mathcal{C}$ we have*

$$M(x, u(x), u'(x)) \leq F(x, u(x), u'(x))$$

where the equality sign holds if $u = u_0$.

(ii) *The associated integral*

$$\mathcal{M}(u) := \int_a^b M(x, u(x), u'(x))\, dx$$

is invariant on $\mathcal{C}$, i.e. $\mathcal{M}(u_1) = \mathcal{M}(u_2)$ for any $u_1, u_2 \in \mathcal{C}$.

On account of Proposition 1.18 we obtain the following result.

Proposition 1.20 *Suppose that M is a calibrator for the triple $\{F, u_0, \mathcal{C}\}$. Then $\mathcal{M}$ is a minimizing osculator for $\mathcal{F}$ at $u_0 \in \mathcal{C}$, whence $\mathcal{F}(u_0) \leq \mathcal{F}(u)$ for all $u \in \mathcal{C}$.*

Remark 1 Let M be a calibrator for $\{F, u_0, \mathcal{C}\}$, and suppose that, for any $u \in \mathcal{C}$, we have $u + \epsilon\varphi \in \mathcal{C}$ for any $\varphi \in C_c^\infty(I, \mathbb{R}^N)$ and all ϵ with $|\epsilon| \ll 1$. Then

$$\mathcal{M}(u + \epsilon\varphi) = \mathcal{M}(u) \quad \text{for } |\epsilon| \ll 1$$

whence

$$\delta\mathcal{M}(u, \varphi) = 0 \quad \text{for all } \varphi \in C_c^\infty(I, \mathbb{R}^N).$$

Then the fundamental lemma implies that

$$M_z(\cdot, u, u') - \frac{d}{dx} M_p(\cdot, u, u') = 0 \quad \text{on } I$$

for any $u \in \mathcal{C} \cap C^2(I, \mathbb{R}^N)$. Thus the calibrator $M(x, z, p)$ is a *null Lagrangian* on a suitable neighbourhood of 1-graph $u|_I$ in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$.

Hence we shall look for calibrators in the class of null Lagrangians $M(x, z, p)$. Such Lagrangians are given by

$$M(x, z, p) = S_x(x, z) + S_{z^i}(x, z)p^i \tag{1.46}$$

as one can easily check, where $S(x, z)$ is an arbitrary C^2-function, and

$$\mathcal{M}(u) = \int_a^b \frac{d}{dx} S(x, u(x))\, dx = S(b, u(b)) - S(a, u(a)) \tag{1.47}$$

for any $u \in C^1(\overline{I}, \mathbb{R}^N)$ such that the graph of u is contained in the domain of S.

Given an extremal $u \in C(a,b) \cap C^2(I,\mathbb{R}^N)$ for the functional $\mathcal{F}(v)$, when and how can we find a calibrator M for the triple $\{F, u, C_\delta(a,b)\}$ in order to guarantee that $\mathcal{F}(u) \leq \mathcal{F}(v)$ for all $v \in C_\delta(a,b) := C(a,b) \cap \{v:\ \sup_I |v(x) - u(x)| < \delta\}$ for some $\delta > 0$? To answer this question we assume that u is *embedded into a field of extremals of* F. By this we mean that there is a simply connected domain $\Gamma = \{(x,c): c \in I_0,\ x \in I(c)\}$ in $\mathbb{R} \times \mathbb{R}^N$, where I_0 is a non-empty parameter set in $\mathbb{R}^N$ and $I(c)$ is an interval on the real axis, and a C^1-diffeomorphism $f : \Gamma \to G$ of Γ onto a (simply connected) domain G in $\mathbb{R} \times \mathbb{R}^N$ such that f is of the form

$$f(x,c) = (x, \varphi(x,c)) \tag{1.48}$$

with $\varphi' = dy/dx \in C^1(\Gamma, \mathbb{R}^N)$ and satisfies

$$u(x) = \varphi(x, c_0) \quad \text{for all } x \in \bar{I} \tag{1.49}$$

and for some $c_0 \in I_0$, where $[a,b] \subset \operatorname{int} I(c_0)$. Furthermore it is assumed that, for any $c \in I_0$, the function $\varphi(\cdot, c)$ is an extremal of $\mathcal{F}$, i.e.

$$F_z(\cdot, \varphi(\cdot,c), \varphi'(\cdot,c)) - \frac{d}{dx} F_p(\cdot, \varphi(\cdot,c), \varphi'(\cdot,c)) = 0.$$

Let us introduce the *slope field* $\wp : G \to \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ of the extremal field f by

$$\wp(x,z) := (x, z, \mathcal{P}(x,z)) \quad \text{where } \mathcal{P} := \varphi' \circ f^{-1} \in C^1(G, \mathbb{R}^N). \tag{1.50}$$

One calls $\mathcal{P} : G \to \mathbb{R}^N$ the *slope function* of f on G. Then we have $\varphi' = \mathcal{P} \circ f$, i.e.

$$\varphi'(x,c) = \mathcal{P}(x, \varphi(x,c)) \quad \text{for all } (x,c) \in \Gamma \tag{1.51}$$

and in particular

$$u'(x) = \mathcal{P}(x, u(x)) \quad \text{for } a \leq x \leq b. \tag{1.52}$$

Definition 1.21 *A C^1-diffeomorphism $f : \Gamma \to G$ of a simply connected domain $\Gamma = \{(x,c): c \in I_0,\ x \in I(x)\}$, $G = f(\Gamma)$, of the form $f(x,c) = (x, \varphi(x,c))$ with $\varphi' \in C^1(\Gamma, \mathbb{R}^N)$ is said to be a* slope field *on G with the slope $\mathcal{P} := \varphi' \circ f^{-1} \in C^1(G, \mathbb{R}^N)$, and the mapping $\wp : (x,z) \mapsto (x, z, \mathcal{P}(x,z))$ is the slope field of f. We call f an* extremal field *on G if the functions $\varphi'(\cdot, c) : I(c) \to \mathbb{R}^N$ are extremals of $\mathcal{F}$. Furthermore f is said to be an* optimal field *on G if there exists a function $S(x,z)$ of class $C^2(G)$ such that the Lagrangian*

$$F^*(x,z,p) := F(x,z,p) - S_x(x,z) - S_z(x,z) \cdot p \tag{1.53}$$

satisfies

$$F^* \geq 0 \ \text{ on } G \times \mathbb{R}^N \quad \text{and} \quad F^* \circ \wp = 0 \ \text{ on } G. \tag{1.54}$$

We call S the eikonal *of the optimal field f. Finally f is called a* Weierstrass field *on G if it is an optimal field satisfying*

$$F^*(x,z,p) > 0 \ \text{ for } \ (x,z) \in G \ \text{ and } \ p \neq \mathcal{P}(x,z). \tag{1.55}$$

Proposition 1.22 *If there exists an optimal field f on G with the eikonal S, then we have*

$$\mathcal{F}(u) \geq S(b, u(b)) - S(a, u(a)) \quad \textit{for all } u \in C^1([a, b], \mathbf{R}^N) \textit{ with graph } u \subset G \tag{1.56}$$

and even

$$\mathcal{F}(u) = S(b, u(b)) - S(a, u(a)) \tag{1.57}$$

if u fits the field, i.e. if

$$u'(x) = \mathcal{P}(x, u(x)) \tag{1.58}$$

where $\mathcal{P}$ denotes the slope function of the field f. If f is a Weierstrass field on G, inequality (1.56) *can be strengthened to*

$$\mathcal{F}(u) > S(b, u(b)) - S(a, u(a)) \tag{1.59}$$

for all $u \in C^1([a, b], \mathbf{R}^N)$ with graph $u \subset G$ which do not fit the field, i.e. for which $u'(x_0) \neq \mathcal{P}(x_0, u(x_0))$ at some point $x_0 \in \bar{I}$. In particular we have

$$\mathcal{F}(u) \geq \mathcal{F}(u_0) \quad \textit{for all } u, u_0 \in C(a, b) \textit{ with graph } u \subset G \textit{ and graph } u_0 \subset G \tag{1.60}$$

provided that u_0 fits an optimal field on G, and we even have $\mathcal{F}(u) > \mathcal{F}(u_0)$ if u_0 fits an optimal field on G and $u \neq u_0$.

Proof Let us introduce the class C by

$$C := C(a, b) \cap \{u : \text{graph } u \subset G\}. \tag{1.61}$$

and suppose that $u_0 \in C$ fits an optimal field on G with the eikonal S. Define the null Lagrangian $M(x, z, p)$ on $G \times \mathbf{R}^N$ by

$$M(x, z, p) := S_x(x, z) + S_z(x, z) \cdot p. \tag{1.62}$$

Then the modified Lagrangian F^* defined by (1.53) can be written as $F^* = F - M$, and (1.54) implies that condition (i) of Definition 1.19 is satisfied. Thus the null Lagrangian M is a calibrator for the triple $\{F, u_0, C\}$, and by Proposition 1.20 we obtain the minimum property (1.60). Inspecting the proofs of Propositions 1.18 and 1.20, we readily verify the strengthened inequality $\mathcal{F}(u) > \mathcal{F}(u_0)$ if u_0 fits a Weierstrass field on G and $u \neq u_0$.

Let $\mathcal{M}(u) = \int_a^b M(x, u, u')\,dx$ be the variational integral associated with M. If u_0 fits the optimal field, then $\mathcal{F}(u_0) = \mathcal{M}(u_0)$, and we also have $\mathcal{M}(u) = \mathcal{M}(u_0)$ for all $u \in C$. By (1.47) it follows that

$$\mathcal{F}(u_0) = S(b, u(b)) - S(a, u(a)) \quad \text{for any } u \in C,$$

i.e. (1.57) holds true, and in conjunction with (1.60) we arrive at the desired inequality (1.56). Similarly we obtain inequality (1.59) for a Weierstrass field on G. $\square$

So far we have seen that the question as to whether a given extremal $u(x)$, $a \le x \le b$, of $\mathcal{F}$ is actually a local minimizer of $\mathcal{F}$ among all functions of class $C(a,b)$ is reduced to the problem of finding a calibrator M for the triple $\{F, u, C\}$ where $C = C(a,b) \cap \{v\colon \text{graph } v \subset G\}$ and G is a domain in $\mathbb{R} \times \mathbb{R}^N$ containing graph u, and this problem is reduced to the question of whether we can find an optimal field on a sufficiently small neighbourhood G of graph u in $\mathbb{R} \times \mathbb{R}^N$ such that u fits the field. When can we embed a given extremal u in an optimal field? To answer this question we first derive necessary conditions to be satisfied by an optimal field on G with the eikonal $S(x,z)$ and the slope field $\wp(x,z) = (x, z, \mathcal{P}(x,z))$. By (1.53) we have

$$F_p^*(x,z,p) = F_p(x,z,p) - S_z(x,z)$$

and (1.54) implies

$$F_p^* \circ \wp = 0$$

whence

$$S_z(x,z) = F_p(x,z,\mathcal{P}(x,z)). \tag{1.63}$$

Then it follows from equation $F_p^* \circ \wp = 0$ that

$$S_x(x,z) = F(x,z,\mathcal{P}(x,z)) - F_p(x,z,\mathcal{P}(x,z)) \cdot \mathcal{P}(x,z). \tag{1.64}$$

Thus eikonal S and slope field $\wp$ of an optimal field on $G \subset \mathbb{R} \times \mathbb{R}^N$ have to satisfy the *Carathéodory equations* (1.63), (1.64) which we can write as

$$S_z = F_p(\wp), \quad S_x = F(\wp) - F_p(\wp) \cdot \mathcal{P}. \tag{1.65}$$

This leads us to

Definition 1.23 *A field f on G with the slope field $\wp(x,z) = (x, z, \mathcal{P}(x,z))$ on G is said to be a* Mayer field *if there is a function $S \in C^2(G)$ such that the pair $\{S, \mathcal{P}\}$ satisfies the Carathéodory equations* (1.65). *The function S is called the eikonal of the Mayer field f.*

Let us introduce the so-called *Beltrami form* γ associated with F. This is a 1-form on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ defined by

$$\gamma := (F - p \cdot F_p)\, dx + F_{p^i} dz^i. \tag{1.66}$$

Denote by $\wp^*\gamma$ the pull-back of γ with respect to $\wp$, i.e.

$$\wp^*\gamma = (F(\wp) - \mathcal{P} \cdot F_p(\wp))\, dx + F_{p^i}(\wp) dz^i. \tag{1.67}$$

Then the Carathéodory equations (1.65) mean that

$$dS = \wp^*\gamma, \tag{1.68}$$

i.e. that the 1-form $\wp^*\gamma$ on G is exact. If the domain G is simply connected, eqn (1.68) is equivalent to

$$d(\wp^*\gamma) = 0, \tag{1.69}$$

i.e. *the field f is a Mayer field if the pull-back $\wp^*\gamma$ of the Beltrami form γ with the slope field $\wp$ of f is a closed 1-form on the simply connected domain G*. Moreover, we infer from (1.68) that on simply connected domains G the eikonals of optimal fields and of Mayer fields respectively are uniquely determined up to an additive constant.

Let us summarize the preceding results:

Proposition 1.24 *Let f be an optimal field on G with the eikonal S and the slope field $\wp$. Then f is a Mayer field on G with the eikonal S, i.e. $dS = \wp^*\gamma$.*

Definition 1.25 *Any field $f : \Gamma \to G$ on G with $f(x,c) = (x, \varphi(x,c))$, $(x,c) \in \Gamma$, is extended to the phase flow $e(x,c) := (x, \varphi(x,c), \varphi'(x,c))$ in the phase space $G \times \mathbb{R}^N$. Let us introduce the* canonical momenta *of f by*

$$\eta := F_p(e), \tag{1.70}$$

i.e.

$$\eta(x,c) = (\eta_1(x,c), \dots, \eta_N(x,c)) \quad \text{with } \eta_i(x,c) = F_{p^i}(e(c,x)).$$

The Lagrange brackets $[c^\alpha, c^\beta]$ *of the field f are defined by*

$$[c^\alpha, c^\beta] := \frac{\partial\eta}{\partial c^\alpha} \cdot \frac{\partial\varphi}{\partial c^\beta} - \frac{\partial\eta}{\partial c^\beta} \cdot \frac{\partial\varphi}{\partial c^\alpha}, \tag{1.71}$$

and $h(x,c) := (x, \varphi(x,c), \varphi'(x,c))$ is called the cophase flow in the cophase space $G \times \mathbb{R}^{N*}$. (Usually we identify $\mathbb{R}^N$ with $\mathbb{R}^{N*}$.)

Lemma 1.26 *If f is a field on G with the phase flow e, then*

$$d(e^*\gamma) = L_F(\varphi) \cdot \varphi_{c^\alpha}\, dc^\alpha \wedge dx + \sum_{(\alpha,\beta)} [c^\alpha, c^\beta] dc^\alpha \wedge dc^\beta \tag{1.72}$$

where the sum is to be extended over all pairs (α, β) ordered by $\alpha < \beta$.

Proof We have

$$\begin{aligned} d(e^*\gamma) &= d\{F(e)\,dx + F_{p^k}(e)[d\varphi^k - \varphi^{k'}\,dx]\} \\ &= F_{z^k}(e) d\varphi^k \wedge dx + F_{p^k}(e) d\varphi^{k'} \wedge dx - F_{p^k}(e) d\varphi^{k'} \wedge dx \\ &\quad + dF_{p^k}(e)\varphi^k_{c^\alpha} dc^\alpha. \end{aligned}$$

Since the second and the third term on the right-hand side cancel, we obtain

$$d(e^*\gamma) = \left[F_{z^k}(e) - \frac{\partial}{\partial x}(F_{p^k}(e))\right] \varphi^k_{c^\alpha} dc^\alpha \wedge dx + \frac{\partial\pi_k}{\partial c^\beta}\frac{\partial\varphi^k}{\partial c^\alpha} dc^\beta \wedge dc^\alpha.$$

Since the kth component of the covector $L_F(\varphi)$ is given by

$$L_F(\varphi)_k = F_{z^k}(e) - \frac{\partial}{\partial x}(F_{p^k}(e))$$

we obtain the identity (1.72). □

Proposition 1.27 *A field $f : \Gamma \to G$ with $f(x,c) = (x, \varphi(x,c))$ is a Mayer field on a simply connected domain G in $\mathbf{R} \times \mathbf{R}^N$ if and only if it is an extremal field with vanishing Lagrange brackets, i.e. if and only if*

$$L_F(\varphi) = 0 \ \textit{and} \ [c^\alpha, c^\beta] = 0 \quad \textit{for all } \alpha, \beta = 1, \ldots, n. \tag{1.73}$$

Proof The field f with the slope field $\wp$ is a Mayer field on G if and only if $d(\wp^*\gamma) = 0$. Since f is a diffeomorphism of Γ onto G, this equation is equivalent to $f^*d(\wp^*\gamma) = 0$. Furthermore we have $f^*\wp = e$ and therefore

$$f^*d(\wp^*\gamma) = d(f^*(\wp^*\gamma)) = d(e^*\gamma).$$

Thus f is a Mayer field on G if and only if

$$d(e^*\gamma) = 0. \tag{1.74}$$

On account of Lemma 1.26, this equation is equivalent to (1.73), and thus our assertion is proved. □

Now we turn to the converse question: *When is a Mayer field on G also an optimal field*? To decide this question we follow the example of Weierstrass and introduce the *excess function* $\mathcal{E}(x, z, q, p)$ on $G \times \mathbf{R}^N \times \mathbf{R}^N$ by

$$\mathcal{E}(x, z, q, p) := F(x, z, p) - F(x, z, q) - [p - q] \cdot F_p(x, z, q). \tag{1.75}$$

Clearly we have $\mathcal{E}(x, z, q, q) = 0$ for all $q \in \mathbf{R}^N$, and it follows that

$$\mathcal{E}(x, z, q, p) \geq 0 \quad \text{for all } (x, z) \in G \text{ and all } p, q \in \mathbf{R}^N \tag{1.76}$$

if and only if $F(x, z, p)$ is a convex function with respect to p for any $(x, z) \in G$. (We may write $\mathcal{E}_F$ instead of $\mathcal{E}$ to indicate the dependence on F.) Furthermore we have

$$\mathcal{E}(x, z, p, q) > 0 \quad \text{for all } (x, z) \in G \text{ and all } p, q \in \mathbf{R}^N \text{ with } p \neq q \tag{1.77}$$

if $F(x, z, p)$ is strictly convex with respect to p for any $(x, z) \in G$. Then we obtain in particular.

Proposition 1.28 *The Weierstrass excess function $\mathcal{E}$ of a Lagrangian F satisfies condition (1.77) if there is a constant $m > 0$ such that*

$$F_{p^ip^k}(x, z, p)\xi^i\xi^k \geq m|\xi|^2 \tag{1.78}$$

holds true for all $\xi = (\xi^1, \ldots, \xi^N) \in \mathbf{R}^N$ and for all $(x, z, p) \in G \times \mathbf{R}^N$.

Condition (1.78) is a *sufficient Legendre condition.*

Definition 1.29 *A C^2-Lagrangian $F(x, z, p)$ is said to be* elliptic *on $G \times \mathbf{R}^N$ if it satisfies the Legendre condition* (1.78).

Lemma 1.30 *If f is a Mayer field on G with the slope field $\wp(x,z) = (x, z, \mathcal{P}(x,z))$ on G and the eikonal $S(x,z)$, and if $M(x,z,p)$ denotes the null Lagrangian*

$$M(x,z,p) = S_x(x,z) + S_z(x,z) \cdot p,$$

then the modified Lagrangian $F^* := F - M$ can be written as

$$F^*(x,z,p) = \mathcal{E}_F(x,z,\mathcal{P}(x,z),p) \quad \text{for all } (x,z,p) \in G \times \mathbb{R}^N.$$

Proof By Definition 1.23 we have

$$S_x = F(\wp) - \mathcal{P} \cdot F_p(\wp), \qquad S_z = F_p \circ \wp.$$

Then M can be written as

$$\begin{aligned} M = S_x + p \cdot S_z &= (F(\wp) - \mathcal{P} \cdot F_p(\wp)) + p \cdot F_p(\wp) \\ &= F(\wp) + [p - \mathcal{P}] \cdot F_p(\wp). \end{aligned}$$

Therefore,

$$F^* = F - M = F - F(\wp) - [p - \mathcal{P}] \cdot F_p(\wp),$$

which means that

$$\begin{aligned} F^*(x,z,p) &= F(x,z,p) - F(x,z,\mathcal{P}(x,z)) \\ &\quad - [p - \mathcal{P}(x,z)] \cdot F_p(x,z,\mathcal{P}(x,z)) \\ &= \mathcal{E}_F(x,z,\mathcal{P}(x,z),p). \end{aligned} \tag{1.79}$$

□

By virtue of Definitions 1.21 and 1.29, Lemma 1.30, and Proposition 1.28 we obtain

Proposition 1.31 *A Mayer field on G is an optimal field if $F(x,z,p)$ is a convex function with respect to p for any $(x,z) \in G$, and it is a Weierstrass field on G if F satisfies the ellipticity condition* (1.78):

Combining Propositions 1.22 and 1.31 we obtain the following: *sufficient condition.*

Theorem 1.32 *Let $u(x)$, $a \le x \le b$, be an extremal of $\mathcal{F}(v) = \int_a^b F(x,v,v')\,dx$ which can be embedded into a Mayer field f on G, i.e. which fits the slope field of f. Then we have*

$$\mathcal{F}(u) \le \mathcal{F}(v) \text{ for all } v \in \mathcal{C} = \mathcal{C}(a,b) \cap \{v\colon \text{graph } v \subset G\},$$

where

$$\mathcal{C}(a,b) := \{v \in C^1([a,b], \mathbb{R}^N)\colon v(a) = u(a) = \alpha,\ v(b) = u(b) = \beta\},$$

provided that $F(x,z,\cdot)$ is convex for all $(x,z) \in G$, and we even have

$$\mathcal{F}(u) < \mathcal{F}(v) \quad \text{for all } v \in \mathcal{C} \text{ with } v \ne u$$

provided that the Lagrangian F is elliptic on $G \times \mathbb{R}^N$.

In view of this result there remains as a final problem the question of under which conditions we can embed a given extremal into a Mayer field. If $N = 1$, this question is fairly easy to answer since in this case there are no non-trivial Lagrange brackets. Consequently every extremal field is automatically a Mayer field. Therefore we just have to embed a given extremal into an extremal field. Locally this is a rather simple problem. In fact, let $u(x), a \le x \le b$, be an extremal for the Lagrangian F which is supposed to be of class C^3 and to satisfy $F_{pp} \ne 0$. Then the equation $L_F(u) = 0$ is equivalent to an equation in normal form, $u'' = \psi(x, u, u')$ with $\psi \in C^1$, to which we can apply the standard existence and uniqueness theorems for ordinary differential equations. Hence, for $c_0 := u(x_0)$ there exists a uniquely determined family of solutions $\varphi(\cdot, c)$ to the initial value problem

$$L_F(\varphi(\cdot, c)) = 0, \quad \varphi(x_0, c) = c, \quad \varphi'(x_0, c) = u'(x_0) \tag{1.80}$$

such that $\varphi(x, c)$ is defined on $\Gamma := (x_0 - \delta, x_0 + \delta) \times (c_0 - \delta, c_0 + \delta)$ provided that $0 < \delta \ll 1$. Moreover, $\varphi_c(x, c)$ exists and depends continuously on its variables. Then $\varphi(x_0, c) = c$ implies that $\varphi_c(x_0, c_0) = 1$, and by the implicit function theorem it follows that $f : \Gamma \to G$, defined by $f(x, c) := (x, \varphi(x, c))$ and $G := f(\Gamma)$, is a field on G, provided that $0 < \delta \ll 1$. Moreover, the uniqueness result implies $u(x) = \varphi(x, c_0)$ for $x_0 - \delta \le x \le x_0 + \delta$. Thus the extremal u is locally, i.e. in a neighbourhood of the point $(x_0, u(x_0))$, embedded in the extremal field f. Similarly we can argue for the boundary points $x_0 = a$ and $x_0 = b$. Hence, on account of Theorem 1.32, we obtain the following result for $N = 1$.

Theorem 1.33 *Let F be a Lagrangian of class C^3 and $u(x), a \le x \le b$, be an F-extremal satisfying the ellipticity condition*

$$F_{p^i p^k}(x, u(x), p)\xi^i \xi^k \ge m|\xi|^2 \tag{1.81}$$

for some $m > 0$ and all $x \in [a, b]$, $p \in \mathbb{R}^N$, and $\xi \in \mathbb{R}^N$. Then every sufficiently small piece of u is a local minimizer. Precisely speaking, this means: for $x_0 \in [a, b]$ there is an interval $I_0 = [a_0, b_0] \subset [a, b]$ with $x_0 \in (a_0, b_0)$ if $x_0 \in (a, b)$ and $x_0 = a_0$ or b_0 respectively if $x_0 = a$ or b, and some $\epsilon > 0$ such that

$$\int_{a_0}^{b_0} F(x, u, u')\,dx < \int_{a_0}^{b_0} F(x, v, v')\,dx$$

for all $v \in C^1([a_0, b_0], \mathbb{R}^N)$ satisfying

$$v(a_0) = u(a_0), \qquad v(b_0) = u(b_0),$$

and

$$0 < \sup_{x \in I_0} |u(x) - v(x)| < \epsilon.$$

In fact, Theorem 1.33 holds for every $N \geq 1$ and not only for $N = 1$. This can be proved as follows. Pick any point $x_0 \in (a, b)$ (the cases $x_0 = a$ and $x_0 = b$ can be handled in a similar way) and let x^* be a point in $\mathbb{R}$ with $x^* < x_0$ which is thought to be 'close to x_0'; set $z^* := u(x^*)$. Let $u(x), a \leq x \leq b$, be an F-extremal which in a neighbourhood of (x_0, z_0) with $z_0 := u(x_0)$ is to be embedded in a Mayer field. Condition (1.81) implies $\det F_{pp} \neq 0$ on $U \times \mathbb{R}^N$ where U is some neighbourhood of graph u in $\mathbb{R} \times \mathbb{R}^N$. Hence the Euler equation $L_F(\varphi) = 0$ is equivalent to some equation in normal form $\varphi'' = \psi(x, \varphi, \varphi')$ with $\psi \in C^1$. Hence we can solve the initial value problem

$$L_F(\varphi(\cdot, c)) = 0, \qquad \varphi(x^*, c) = z^*, \qquad \varphi'(x^*, c) = c \tag{1.82}$$

in a unique way by some function $\varphi(\cdot, c)$ such that $\varphi(x, c)$ is defined on $\Gamma_\delta^* := (x^* - \delta, x^* + \delta) \times (c^* - \delta, c^* + \delta)$ where $c^* := u'(x^*)$, and that $\varphi'(x, c)$ and $\varphi_c'(x, c)$ are continuous on Γ_δ^*. A Taylor expansion with respect to x at $x = x^*$ yields

$$\varphi(x, c) = z^* + \varphi'(x^*, c)(x - x^*) + \ldots$$

whence

$$\varphi_c(x, c) = \varphi_c'(x^*, c)(x - x^*) + \ldots$$

where $\ldots$ denotes higher-order terms in $(x - x^*)$. Then

$$\det \varphi_c(x, c) = (x - x^*)^N \det \varphi_c'(x^*, c) + \ldots,$$

and (1.82) implies

$$\det \varphi_c'(x^*, c) = 1 \quad \text{if } |c - c^*| < \delta.$$

Set $f(x, c) := (x, \varphi(x, c))$. Then $\det Df = \det \varphi_c$, and therefore

$$\det Df(x, c) = (x - x^*)^N + \ldots \quad \text{for } (x, c) \in \Gamma_\delta^*. \tag{1.83}$$

Note that $\varphi(x, c)$ and therefore $f(x, c)$ also depend on the parameter x^*. However, by (1.83) and a suitable continuity reasoning we can find positive numbers δ_0 and δ such that

$$\det Df(x, c^*) \neq 0 \quad \text{for all } x \in (x^*, x^* + \delta),\ c^* = u(x^*), \tag{1.84}$$

and any $x^* \in (x_0 - \delta_0, x_0)$.

Choosing x^* sufficiently close to x_0 we then have $x^* < x_0 < x^* + \delta$, and if φ is the solution of (1.82) for this initial point x^* and f is the corresponding mapping $f(x, c) = (x, \varphi(x, c))$, we obtain

$$\det Df(x_0, c^*) \neq 0$$

where $c^* = u(x^*)$. Hence there is a neighbourhood Γ of the point (x_0, c^*) which is of the form $\Gamma = (x_0 - \rho, x_0 + \rho) \times (c^* - \rho, c^* + \rho)$, $\rho > 0$, such that $f\big|_\Gamma$ is a diffeomorphism of Γ onto $G := f(\Gamma)$, and $u(x) = \varphi(x, c^*)$ for $x^* - \delta < x < x^* + \delta$. Thus for $0 < \epsilon < \rho$, the piece $u(x)$, $x_0 - \epsilon \le x \le x_0 + \epsilon$, of the extremal u is embedded in the extremal field $f|_\Gamma$. We claim that $f|_\Gamma$ is a Mayer field. Since Γ is simply connected, we only have to prove that all Lagrange brackets $[c^\alpha, c^\beta]$ of f are identically zero. This can be seen as follows. Let $e(x, c) = (x, \varphi(x, c), \varphi'(x, c))$ be the phase flow of the map f, and set $\pi := \varphi'$ and $\bar{F}_p := F_p(e)$, $\bar{F}_z := F_z(e)$. Then

$$\frac{\partial}{\partial x}[c^\alpha, c^\beta] = \frac{\partial}{\partial c^\alpha}\frac{\partial}{\partial x}\bar{F}_p \cdot \frac{\partial \varphi}{\partial c^\beta} + \frac{\partial}{\partial c^\alpha}\bar{F}_p \cdot \frac{\partial \pi}{\partial c^\beta} - \frac{\partial}{\partial c^\beta}\frac{\partial}{\partial x}\bar{F}_p \cdot \frac{\partial \varphi}{\partial c^\alpha} - \frac{\partial}{\partial c^\beta}\bar{F}_p \cdot \frac{\partial \pi}{\partial c^\alpha}.$$

Since

$$\bar{F}_z = \frac{\partial}{\partial x}\bar{F}_p,$$

we obtain

$$\frac{\partial}{\partial x^\alpha}[c^\alpha, c^\beta] = \frac{\partial}{\partial c^\alpha}\bar{F}_z \cdot \frac{\partial \varphi}{\partial c^\beta} + \frac{\partial}{\partial c^\alpha}\bar{F}_p \cdot \frac{\partial \pi}{\partial c^\beta} - \frac{\partial}{\partial c^\beta}\bar{F}_z \cdot \frac{\partial \varphi}{\partial c^\alpha} - \frac{\partial}{\partial c^\beta}\bar{F}_z \cdot \frac{\partial \pi}{\partial c^\alpha}.$$

Now a straightforward computation shows that

$$\frac{\partial}{\partial x^\alpha}[c^\alpha, c^\beta] = 0.$$

Thus the Lagrange brackets depend only on c and not on x: hence they can be evaluated by setting $x = x^*$. However, by (1.82) we obtain $\varphi(x^*, c) = z^*$ and therefore $\varphi_{c^\alpha}(x^*, c) = 0$ for all $\alpha = 1, \dots, N$. Consequently we have

$$[c^\alpha, c^\beta](x^*, c) = 0,$$

and so all Lagrange brackets vanish on Γ^*_δ, in particular on Γ.

So far we have seen that in the elliptic situation (i.e. $F_{pp} > 0$) every sufficiently small piece of an extremal can be embedded in a Mayer field. Globally this is no longer true. Rather *a global extremal $u(x)$, $a \le x \le b$, can be embedded in a Mayer field if there is no pair x_1, x_2 of conjugate parameter values of u contained in the interval* $[a, b]$. This means that there is no non-trivial solution v of the Jacobi equation formed at u (i.e. no non-trivial 'Jacobi field v along u') vanishing at $x = x_1$ and $x = x_2$ where $a \le x_1 < x_2 \le b$. Here the *Jacobi equation at u* is the Euler–Lagrange equation of the quadratic functional

$$Q(v) := \frac{1}{2}\delta^2\mathcal{F}(u, v) \tag{1.85}$$

where

$$\delta^2\mathcal{F}(u, v) := \frac{d^2}{d\epsilon^2}\mathcal{F}(u + \epsilon v)\Big|_{\epsilon=0} \tag{1.86}$$

denotes the so-called second variation of $\mathcal{F}$ at u in the direction of v. Since $Q(v)$ is quadratic in v, the Jacobi equation is a linear equation of second order. The theory of such equations is well developed and can be found in many books. Concerning the theory of conjugate points we refer the reader, for example, to Carathéodory [57], Morse [194], Young [294], and Giaquinta–Hildebrandt [113]. In the last book, the global embedding problem is treated in Sections 2.1–2.4 of Chap. 6, volume I.

In the final part of this section we discuss the *Hamiltonian point of view*. We begin by introducing the *Legendre transformation* $\mathcal{L}_F$ generated by F. Here $F(x, z, p)$ is supposed to be a function of class C^s, $s \geq 2$, defined on $G \times \mathbb{R}^N$ where G denotes a domain in $\mathbb{R} \times \mathbb{R}^N$. We assume that $F_{pp}(x, z, p)$ is a positive definite matrix for all $(x, z) \in G$ and all $p \in \mathbb{R}^N$. Then the Legendre transformation is a two-step procedure. First we define the map $\mathcal{L}_F : (x, z, p) \mapsto \mathcal{L}_F(x, z, p)$ by

$$\mathcal{L}_F(x, z, p) := (x, z, y), \quad y := F_p(x, z, p). \tag{1.87}$$

Since $F_{pp} > 0$, the mapping $F_p(x, z, \cdot)$ is a C^1-diffeomorphism of $\mathbb{R}^N$ onto some domain $B^*(x, z) \subset \mathbb{R}^N$, for any $(x, z) \in G$. Therefore $\mathcal{L}_F$ is a C^1-diffeomorphism of $\Omega := G \times \mathbb{R}^N$ onto the domain $\Omega^* := \{(x, z, y) : (x, z) \in G, \ y \in B^*(x, z)\}$. Then the inverse $\mathcal{L}_F^{-1} : \Omega^* \to \Omega$ is of the form

$$\mathcal{L}_F^{-1}(x, z, y) = (x, z, \psi(x, z, y)) \tag{1.88}$$

with $\psi \in C^1(\Omega^*, \mathbb{R}^N)$.

In the second step we define the *Legendre transform* H of F as a function $H : \Omega^* \to \mathbb{R}$ which is given by

$$H := \{p \cdot F_p - F\} \circ \mathcal{L}_F^{-1}, \tag{1.89}$$

or equivalently by

$$\begin{aligned} H(x, z, y) &:= \psi(x, z, y) \cdot F_p(x, z, \psi(x, z, y)) - F(x, z, \psi(x, z, y)) \\ &= \psi(x, z, y) \cdot y - F(x, z, \psi(x, z, y)). \end{aligned}$$

The function $H(x, z, y)$ is also called the *Hamiltonian associated with the Lagrangian* $F(x, z, p)$. From (1.89) it follows that only $H \in C^{s-1}$ if $F \in C^s$. Now we shall show that H is even of class C^s, i.e. H is as smooth as F. In fact, (1.89) implies

$$\begin{aligned} &H_x \, dx + H_{z^k} dz^k + H_{y_k} dy_k \\ &\quad = y_k d\psi^k + \psi^k dy_k - F_x \, dx - F_{z^k} dz^k - F_{p^k} d\psi^k \end{aligned}$$

where (x, z, y) is the argument of $H_x, H_{z^k}, H_{y_k}, \psi^k$, and $(x, z, \psi(x, z, y))$ is the argument of F_x, F_{z^k}, F_{p^k}. Since $y_k = F_{p^k}(x, z, \psi(x, z, y))$, it follows that

$$H_x \, dx + H_{z^k} dz^k + H_{y_k} dy_k = -F_x \, dx - F_{z^k} dz^k + \psi^k dy_k$$

whence

$$\psi^k(x, z, y) = H_{y_k}(x, z, y) \tag{1.90}$$

and

$$
\begin{aligned}
H_x(x,z,y) &= -F_x(x,z,\psi(x,z,y)) \\
H_{z^k}(x,z,y) &= -F_{z^k}(x,z,\psi(x,z,y)).
\end{aligned}
\tag{1.91}
$$

Since F_x, F_z, and ψ are of class C^{s-1} we infer that H_x, H_z, H_y are of class C^{s-1}, whence H is of class C^s. Moreover, equations (1.88) and (1.90) imply

$$
\mathcal{L}_F^{-1}(x,z,y) = (x,z,H_y(x,z,y)), \tag{1.92}
$$

i.e.

$$
\mathcal{L}_F^{-1} = \mathcal{L}_H. \tag{1.93}
$$

Thus the Legendre transformation is an involution. We can express the previous formulae by the following table:

$$
\begin{gathered}
F(x,z,p) + H(x,z,y) = y \cdot p \\
y = F_p(x,z,p), \quad p = H_y(x,z,y) \\
F_x(x,z,p) + H_x(x,z,y) = 0, \quad F_z(x,z,p) + H_z(x,z,y) = 0.
\end{gathered}
\tag{1.94}
$$

Here (x,z,p) and (x,z,y) are corresponding triples in Ω and Ω^* respectively, i.e. $(x,z,y) = \mathcal{L}_F(x,z,p)$ or $(x,z,p) = \mathcal{L}_H(x,z,y)$.

Consider now an F-extremal $u \in C^2([a,b], \mathbf{R}^N)$ whose graph is contained in G, i.e. 1-graph $u \subset \Omega$, and set $\pi(x) := u'(x)$. Then the prolongation $e(x) = (x, u(x), \pi(x))$ of u into Ω satisfies the Euler equations

$$
\frac{du}{dx} = \pi, \qquad \frac{d}{dx} F_p(e) = F_z(e). \tag{1.95}
$$

Let us map e by applying the Legendre transformation $\mathcal{L}_F$:

$$
h := \mathcal{L}_F \circ e = \mathcal{L}_F(e). \tag{1.96}
$$

Then we have

$$
h(x) = (x, u(x), \eta(x)) \quad \text{with } \eta(x) = F_p(x, u(x), \pi(x))
$$

and it follows that

$$
e = \mathcal{L}_H \circ h = \mathcal{L}_H(h), \tag{1.97}
$$

i.e.

$$
e(x) = (x, u(x), \pi(x)) \quad \text{with } \pi(x) = H_y(x, u(x), \eta(x)).
$$

Thus (1.95) is transformed into

$$
u' = H_y(x,u,\eta), \qquad \eta' = -H_z(x,u,\eta). \tag{1.98}
$$

Conversely, if h is a solution of (1.98) then e is a solution of (1.95). The system (1.98) is called a *Hamiltonian system*. We can formulate this result as

Proposition 1.34 *If $F_{pp}(x, z, p) > 0$ on $\Omega = G \times \mathbb{R}^N$ then the Legendre transformation $\mathcal{L}_F$ is invertible, and we can define the associated Hamiltonian H by* (1.89). *Therefore the Euler equation* (1.95) *and the Hamiltonian system* (1.98) *are equivalent in the sense that if e is a solution of* (1.95), *then $h = \mathcal{L}_F(e)$ is a solution of* (1.98), *and, vice versa, if h satisfies* (1.98), *then $e = \mathcal{L}_H(h)$ fulfils* (1.95).

Suppose now that $f : \Gamma \to G$ is a Mayer field on G with the slope field $\wp(x, z) = (x, z, \mathcal{P}(x, z))$ and the eikonal S. Then we have the Carathéodory equations

$$S_x = F(\wp) - \mathcal{P} \cdot F_p(\wp), \quad S_z = F_p(\wp). \tag{1.99}$$

Let us introduce the *dual slope field* $\psi(x, z) = (x, z, \Psi(x, z))$ by

$$\psi := \mathcal{L}_F \circ \wp = \mathcal{L}_F(\wp); \tag{1.100}$$

that is,

$$\Psi(x, z) = F_p(x, z, \mathcal{P}(x, z)) \tag{1.101}$$

and

$$\mathcal{P}(x, z) = H_y(x, z, \Psi(x, z)). \tag{1.102}$$

Then the Carathéodory equations are equivalent to

$$S_x(x, z) = -H(x, z, \Psi(x, z)), \qquad S_z(x, z) = \Psi(x, z), \tag{1.103}$$

and these two equations lead to the so-called *Hamilton–Jacobi equation*

$$S_x + H(x, z, S_z) = 0 \tag{1.104}$$

for the eikonal $S(x, z)$. This is one scalar partial differential equation of first order for S which turns out to be equivalent to the system (1.99) for $\{S, \mathcal{P}\}$. In fact, if S is a C^2-solution of the Hamilton–Jacobi equation (1.104) on G, and if we set $\Psi := S_z$, then $\{S, \Psi\}$ satisfy (1.103). Introducing $\wp(x, z) = (x, z, \mathcal{P}(x, z))$ by $\mathcal{P}(x, y) = H_y(x, z, \Psi(x, z)) = H_y(x, z, S_z(x, z))$ it follows that $\{S, \mathcal{P}\}$ satisfy the Carathéodory equations (1.99). Thus we have found

Proposition 1.35 *The Hamilton–Jacobi equation* (1.104) *and the Carathéodory equations* (1.99) *are equivalent relations. In particular, if $S(x, z)$ is a solution of* (1.104) *on the domain G, then $\wp(x, z) = (x, z, \mathcal{P}(x, z))$ and*

$$\mathcal{P}(x, z) = H_y(x, z, S_z(x, z)) \tag{1.105}$$

is the slope field of some Mayer field $f(x, c) = (x, \varphi(x, c))$ on G which can be obtained from $\wp$ 'by integration', i.e. by solving the first-order equation

$$\varphi' = \mathcal{P}(\cdot, \varphi). \tag{1.106}$$

Thus we have found another method for constructing Mayer fields. This method can also be used to embed a given F-extremal in a Mayer field.

1.3 Some classical problems

In this section we want to present a few of the classical variational problems which were discussed before or briefly after the discovery of the infinitesimal calculus.[5]

$\boxed{1}$ *Fermat's principle and the laws of geometrical optics*

In 1662 Fermat derived the refraction law of geometrical optics by means of his celebrated principle, according to which *nature always acts in the shortest way* (*la nature agit toujours par les voies les plus courtes*). In the context of geometrical optics this principle means that light moves from one point to another in the quickest possible way.

To obtain the refraction law from Fermat's principle, we consider two media, separated by a plane, and two points A and B as in Fig. 1.1, and assume that light moves in a medium with a speed inversely proportional to the *optical density* of the medium. Then, denoting the optical densities of the two media by n_1 and n_2, Fermat's principle requires that the light path φ_{opt} joining A with B will be the one for which the time $T(\varphi_{opt})$ used by the light to proceed from A to B is minimal. It is clear that the curve φ_{opt} lies in the plane passing through A and B and orthogonal to the plane separating the two media. Let us introduce Cartesian coordinates x, y in this plane, and assume that the separation

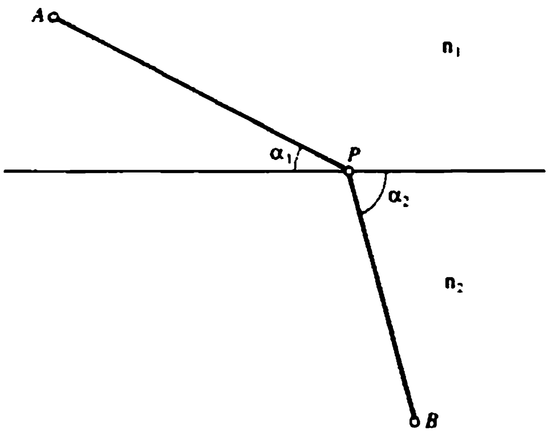

FIG. 1.1.

[5] In H.H. Goldstine [124], the interested reader can find a rich source of detailed historical information. We also refer to the historical remarks contained in the scholia of the treatise on the calculus of variations by Giaquinta–Hildebrandt [113].

line of the two media is the axis $y = 0$. If we set for every $z = (x, y) \in \mathbb{R}^2$

$$n(z) = \begin{cases} n_1 & \text{if } y \geq 0, \\ n_2 & \text{if } y < 0, \end{cases}$$

we easily obtain that for every plane curve $\varphi : [0, 1] \to \mathbb{R}^2$ the time $T(\varphi)$ is given by

$$T(\varphi) \doteq \int_0^1 n(\varphi)|\varphi'|\,dt.$$

Therefore, the minimum problem we are dealing with is

$$\min\left\{\int_0^1 n(\varphi)|\varphi'|\,dt : \varphi(0) = A, \varphi(1) = B\right\}.$$

The solution turns out to be the piecewise linear curve of Fig. 1.1, where

$$\frac{n_1}{n_2} = \frac{\cos\alpha_2}{\cos\alpha_1}, \tag{1.107}$$

which is the usual refraction law of geometrical optics.

Note that this problem can be reduced to a minimization problem for a function of one real variable. Indeed, in each of the two homogeneous media, light rays are straight lines since the paths of shortest time are just the paths of shortest length. Thus a path of shortest time joining point A in medium 1 with point B in medium 2 has to be a piecewise linear curve consisting of two straight segments AP and PB, where $P = (x, 0)$ is a point on the x-axis. The time $T(x)$ needed by the light to propagate along this curve will then be

$$T(x) = n_1|A - P| + n_2|P - B|.$$

The problem of shortest time is therefore reduced to a minimum problem for the function $T(x)$ as x varies in $\mathbb{R}$. If $A = (a, b)$ and $B = (\alpha, \beta)$ with $a < \alpha$ and $\beta < 0 < b$, the time $T(x)$ is given by

$$T(x) = n_1\left[(a - x)^2 + b^2\right]^{1/2} + n_2\left[(\alpha - x)^2 + \beta^2\right]^{1/2}.$$

Then the vanishing of the first derivative $T'(x)$ gives the equation

$$n_1 \frac{x - a}{[(x - a)^2 + b^2]^{1/2}} = n_2 \frac{\alpha - x}{[(x - \alpha)^2 + \beta^2]^{1/2}}$$

which is just the refraction law (1.107).

$\boxed{2}$ ***The Newton problem of optimal aerodynamic profile***

In 1685 Newton treated the question of determining the shape of a rotational symmetric body of least resistance moving in a fluid, thereby solving the first problem of the calculus of variations that cannot be reduced to a minimum problem for a function of a finite number of variables. In his *Principia* he wrote:

If in a rare medium, consisting of equal particles freely disposed at equal distance from each other, a globe and a cylinder described on equal diameter move with equal velocities in the direction of the axis of the cylinder, (then) the resistance of the globe will be half as great as that of the cylinder. ... I reckon that this proposition will be not without application in the building of ships.

In order to understand Newton's model, we assume that the fluid is so rarefied that it can be thought to consist of many small particles which are uniformly distributed and independent of each other. Then the only interaction between the body and the particles is caused by shocks, if the tangential friction is neglected.

Suppose now the body to be radially symmetric; then we can describe its profile as graph of a function $u(r)$. A computation shows that each shock between the body and a particle with mass m and speed v slows the body down by a quantity of

$$m v \cos^2 \vartheta = m v \; \frac{1}{1 + |u'(r)|^2}$$

where the meaning of the angle ϑ is indicated in Fig. 1.2.

Therefore, the total resistance of the profile given by the function u is proportional to the integral

$$\mathcal{F}(u) = \int_0^R \frac{r}{1 + |u'(r)|^2} \, dr, \tag{1.108}$$

R being the radius of the maximal cross-section. For instance, we obtain from (1.108) that the resistance of a sphere is half the resistance of a cylinder of equal diameter, as shown by Newton. One sees immediately that, without assuming a bound on the maximal height of the body, the infimum of the functional $\mathcal{F}$ is zero. In fact, the sequence of functions

$$u_h(r) = h\left(1 - \frac{r}{R}\right)$$

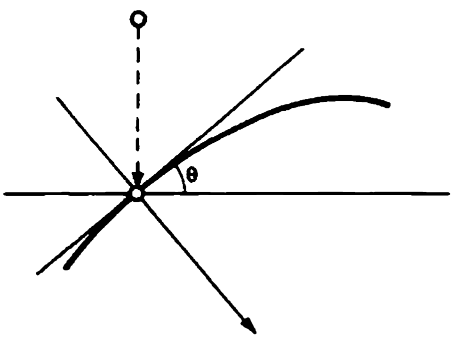

FIG. 1.2.

gives

$$\lim_{h\to\infty} \mathcal{F}(u_h) = 0.$$

while it is clear that $\mathcal{F}(u)$ differs from zero for any function u. Furthermore, even a bound on u of the form $0 \le u \le M$ is not sufficient to give the existence of a minimizer. This can be seen from the sequence of functions

$$u_h(r) = M \sin^2(hr),$$

which satisfy $0 \le u_h \le M$, but still give

$$\lim_{h\to\infty} \mathcal{F}(u_h) = 0.$$

Assuming that the function u is decreasing we obtain the problem to determine:

$$\min\{\mathcal{F}(u) : u(0) = M, u(R) = 0, u' \le 0\}. \tag{1.109}$$

By considering the function $v(s) = u^{-1}(M - s)$, problem (1.109) is often written in the form

$$\min\left\{\int_0^M \frac{vv'^3}{1+|v'|^2} ds + \frac{|v(0)|^2}{2} : v(0) \ge 0, v(M) = R, v' \ge 0\right\}.$$

By computing the first variation of the functional which appears in this minimization problem we obtain the Euler equation

$$\begin{cases} G(v') &= (vG_p(v'))' \quad \text{if } v' > 0 \\ G_p(v'(0)) &= 1 \quad\quad\quad\;\; \text{if } v(0) > 0 \end{cases}$$

where G denotes the function $G(p) = p^3/(1 + p^2)$. From the Euler equation we obtain DuBois-Reymond's equation

$$v'vG_p(v') - vG(v') = c$$

together with the condition $v'(0) = 1$. By a straightforward computation we then infer that

$$v\frac{v'^3}{(1+v'^2)^2} = \frac{c}{2}.$$

In order to integrate this equation it is convenient to use the parameter $z = v'$ which yields the two equations

$$v(z) = \frac{c}{2}\frac{(1+z^2)^2}{z^3},$$
$$\frac{ds}{dz} = \frac{1}{v'(s)}\frac{dv}{dz} = \frac{c}{2}\left(-\frac{2}{z^3} + \frac{1}{z} - \frac{3}{z^5}\right).$$

Therefore we obtain the solution in the parametric form

$$s(z) = \frac{c}{2}\Big(\frac{1}{z^2} + \log z + \frac{3}{4z^4} + A\Big),$$
$$v(z) = \frac{c}{2}\,\frac{(1+z^2)^2}{z^3},$$

where the constants A and c have to be determined through the boundary conditions $s(z)\big|_{z=1} = 0$ and $v(s)\big|_{s=M} = R$. If we want to express the solution by using the previous variables r and u, it is convenient to consider the function

$$f(t) = \frac{t}{(1+t^2)^2}\Big(-\frac{7}{4} + \frac{3}{4}t^4 + t^2 - \log t\Big)$$

defined for $t \geq 1$; it is strictly increasing and satisfies

$$\lim_{t\to\infty} f(t) = +\infty.$$

Set

$$T = f^{-1}\Big(\frac{M}{R}\Big), \quad r_0 = \frac{4RT}{(1+T^2)^2}.$$

Then we obtain

$$u(r) = M \quad \text{for all } r \in [0, r_0],$$

and for $r \geq r_0$ we can write the curve $u = u(r)$ in parametric form as

$$r(t) = \frac{r_0}{4t}(1+t^2)^2,$$
$$u(t) = M - \frac{r_0}{4t} f(t)(1+t^2)^2,$$

with $t \in [1, T]$. It is interesting to note that $du/dr \leq -1$ and that on the class

$$\Big\{u = M \quad \text{on } [0, r_0], \quad u(R) = 0, \quad u' \leq -\frac{\sqrt{3}}{3}\Big\}$$

the functional $\mathcal{F}$ turns out to be convex. For instance, by taking $M = R = 1$ we get the optimal profile of Fig. 1.3 where $r_0 \sim 0.35$, and the normalized resistance

$$C_0 = \frac{2}{R^2}\int_0^R \frac{r}{1+|u'(r)|^2}\, dr$$

is about $C_0 \sim 0.37$.

The literature on the Newton problem is rather extensive; we refer to Buttazzo and Kawohl [50] for a survey of the problem. Some interesting applications to engineering are given in the book by Miele [185].

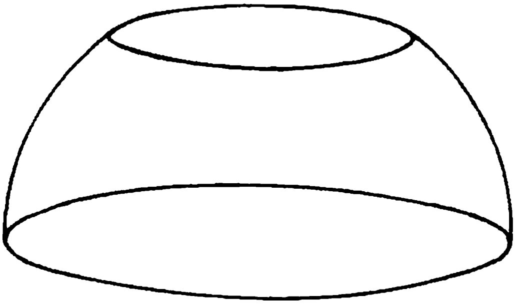

FIG. 1.3.

It is interesting to note that in general, even for a circular maximal cross-section, the optimal profile is not radially symmetric (see Brock *et al.* [45]) if for general cross-sections $\Omega \subset \mathbf{R}^2$ and functions $u : \Omega \to \mathbb{R}$ the 'Newtonian resistance' is chosen as (see Buttazzo–Kawohl [50])

$$\mathcal{F}(u) = \int_\Omega \frac{1}{1 + |\operatorname{grad} u|^2}\, dx.$$

However, in the radially symmetric situation, the optimal profile always has a flat region with radius r_0 on its top, and the following asymptotic estimates hold as M/R tends to $+\infty$:

$$\frac{r_0}{R} \approx \frac{27}{16}\left(\frac{M}{R}\right)^{-3}, \quad C_0 \approx \frac{27}{32}\left(\frac{M}{R}\right)^{-2}.$$

$\boxed{3}$ *The brachistochrone*

In 1638 Galileo formulated the problem of finding a curve, connecting two given points A and B, on which a point mass moves without friction under the influence of gravity in the least possible time from the initial point A to the end point B below A. Galileo erroneously stated that the optimal curve is a circular path. The correct solution of the so-called brachistochrone problem was found by Johann Bernoulli in 1697. If in a Cartesian coordinate system with gravity acting in direction of the negative y-axis, $A = (x_1, y_1)$ denotes the initial point and $B = (x_2, y_2)$, with $x_1 < x_2, y_2 < y_1$, the end point, and if $u : [x_1, x_2] \to \mathbf{R}$ is a function such that $u(x_1) = y_1, u(x_2) = y_2$, and $u(x) < y_1$ for $x_1 < x \leq x_2$, the time needed by the point mass to slide from A to B along the graph of u, starting at A with zero velocity, is given by the quantity

$$\mathcal{F}(u) = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + |u'(x)|^2}{y_1 - u(x)}}\, dx$$

where g denotes the acceleration due to gravity. In order to compute the solution of the problem

$$\min\{\mathcal{F}(u) \colon u(x_1) = y_1, u(x_2) = y_2\}$$

we use DuBois-Reymond's equation

$$\frac{u'^2}{(1+u'^2)^{1/2}(y_1 - u)^{1/2}} - \frac{(1+u'^2)^{1/2}}{(y_1 - u)^{1/2}} = c.$$

which, after some calculations, reduces to

$$c^2(1+u'^2)(y_1 - u) = 1.$$

It is convenient to express its solution in parametric form; since it is evident that $u \le y_1$, we can write $u(t) = y_1 - k(1 - \cos t)$ where k denotes a suitable positive constant. The above equation then becomes

$$c^2\left(1 + \frac{k^2 \sin^2 t}{\dot{x}^2(t)}\right) k(1 - \cos t) = 1,$$

and with the choice $kc^2 = 1/2$ we obtain

$$\dot{x}(t) = k(1 - \cos t).$$

Therefore the solution turns out to be a cycloid, which in parametric form can be given as

$$\begin{aligned} x(t) &= x_1 + k(t - \sin t), \\ u(t) &= y_1 - k(1 - \cos t), \quad t \in [0, T], \end{aligned}$$

where the constants k and T are determined by the conditions

$$\begin{aligned} x(T) &= x_2 \\ u(T) &= y_2. \end{aligned}$$

Figure 1.4 shows the brachistochrone, the curve of quickest descent, with $A = (0, 1)$ and $B = B_1, B_2, B_3$, where $B_1 = (\pi/4, \eta)$, $B_2 = (\pi/2, 0)$, $B_3 = (3\pi/4, \eta)$, $2\eta = 1 + \cos T$, and $T = \pi/2 + \sin T$, $0 < T < \pi$.

It is interesting to note that, setting

$$y_1 - u(x) = \frac{v^2(x)}{2},$$

the functional $\mathcal{F}$ becomes

$$\frac{1}{\sqrt{g}} \int_{x_1}^{x_2} \sqrt{\frac{1}{v^2} + |v'|^2}\, dx.$$

In this way the brachistochrone problem reduces to a convex minimization.

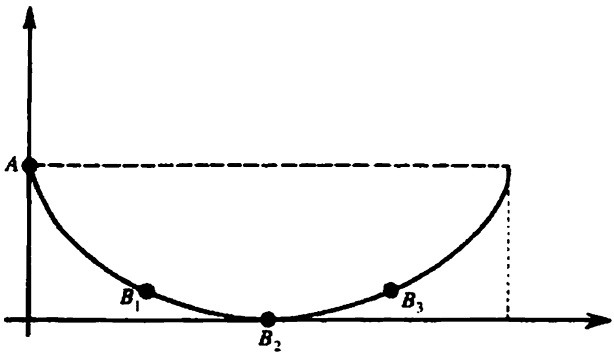

FIG. 1.4.

$\boxed{4}$ *The heavy-chain problem*

Consider the following problem which was proposed by Galileo in 1638. Find the shape of a very thin, heavy, inextensible cable (chain), suspended at its extrema. The solution was found independently by Jacob and Johann Bernoulli, Huygens, and Leibniz between 1690 and 1692.

Choose a Cartesian system of coordinates x, y in a vertical plane, such as in the previous example, and let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ with $x_1 < x_2$ be the two extrema of the chain. Suppose also that the chain is geometrically described by the graph of a function $z = u(x)$, $x_1 \le x \le x_2$. Then the potential energy of the entire chain is proportional to the quantity

$$\mathcal{F}(u) = \int_{x_1}^{x_2} u\sqrt{1 + |u'|^2}\, dx$$

while the assumption of inextensibility leads to the equality

$$\int_{x_1}^{x_2} \sqrt{1 + |u'|^2}\, dx = L$$

as a constraint, L being the total length of the chain. The shape of the heavy chain in equilibrium is then described by the minimizer u of the potential energy $\mathcal{F}(u)$ under the subsidiary conditions

$$\int_{x_1}^{x_2} \sqrt{1 + |u'|^2}\, dx = L, \quad u(x_1) = y_1, \; u(x_2) = y_2.$$

As in the case of constrained minimum problems for functions on a finite dimensional space, the first order necessary conditions of optimality involve a Lagrange multiplier λ, and DuBois-Reymond's equation then becomes

$$u\frac{u'^2}{(1+u'^2)^{1/2}} - u(1+u'^2)^{1/2} + \lambda\left(\frac{u'^2}{(1+u'^2)^{1/2}} - (1+u'^2)^{1/2}\right) = c$$

with a suitable constant c. After some simplifications we obtain the equation

$$c^2(1+u'^2) = (u+\lambda)^2$$

which can easily be integrated. The solution $u(x)$ of the heavy-chain problem turns out to be of the form

$$u(x) = \frac{1}{\alpha}[\cosh(\alpha x + \beta) + \gamma]$$

where the constants α, β, γ are determined by the conditions

$$u(x_1) = y_1, \quad u(x_2) = y_2, \quad \int_{x_1}^{x_2} \sqrt{1 + |u'(x)|^2}\, dx = L.$$

Figure 1.5 shows the shape of a heavy chain suspended at points $A = (-1, 1)$ and $B = (1, 1)$.

$\boxed{5}$ *Radially symmetric minimal surfaces*
Consider two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in an x, y-plane with $x_1 < x_2$ and $y_1, y_2 > 0$. We want to find a function $u : [x_1, x_2] \to \mathbb{R}$ with $u(x_1) = y_1, u(x_2) = y_2$, whose graph, after rotation about the x-axis, gives the surface of least area. Since the area of the rotated surface is

$$\mathcal{F}(u) = 2\pi \int_{x_1}^{x_2} |u(x)| \sqrt{1 + |u'(x)|^2}\, dx, \tag{1.110}$$

we have the problem to find the solution of

$$\min \left\{ \int_{x_1}^{x_2} |u| \sqrt{1 + |u'|^2}\, dx : u(x_1) = y_1,\ u(x_2) = y_2, u > 0 \right\}. \tag{1.111}$$

Besides the connected surfaces obtained by rotating the graphs of functions $u(x)$, $x_1 \le x \le x_2$, we may also consider a degenerate surface consisting of two disks of radii y_1 and

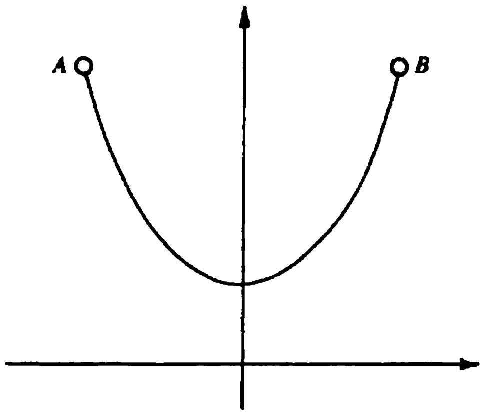

FIG. 1.5.

y_2 respectively, which lie in parallel planes perpendicular to the x-axis and are centred at the points $P_1 = (x_1, 0)$ and $P_2 = (x_2, 0)$. This surface has area $\pi(y_1^2 + y_2^2)$, which must be compared with the area $\mathcal{F}(u)$ of minimal graphs u, where $\mathcal{F}(u)$ is given by (1.110). Note that the degenerate surface consisting of the two disks furnishes a smaller value of area than any surface obtained by rotating the graph of a function $u : [x_1, x_2] \to \mathbb{R}$ with $u \geq 0$, connecting A and B, provided that $u(x)$ assumes the value zero.

DuBois-Reymond's equation is

$$u \frac{u'^2}{(1+u'^2)^{1/2}} - u(1+u'^2)^{1/2} = c$$

which can be transformed into the equation

$$u^2 = c^2(1+u'^2),$$

and this equation can be easily integrated. Therefore, if $x_2 - x_1$ is sufficiently small with respect to y_1 and y_2, the solution of problem (1.111) is given by

$$u(x) = \frac{1}{a}\cosh(ax+b)$$

where the constants a, b are determined by the boundary conditions $u(x_1) = y_1$ and $u(x_2) = y_2$. If, however, $x_2 - x_1$ is sufficiently large, problem (1.111) will not have a solution. For some 'intermediate' size of $x_2 - x_1$, problem (1.111) has a relative minimizer, but its area is larger than that of the two disks.

Let us restrict the analysis to the case when $y_1 > y_2$ and the admissible functions u are taken to be decreasing; then it is convenient to invert the axes and to consider the problem in terms of the function $v = u^{-1}$. Under this proviso the problem is to determine the Cartesian graph of minimal area spanned by a circle of radius y_1 at level x_1 and a circle of radius y_2 at level x_2, i.e. we have to find the solution of the convex minimum problem

$$\min\left\{\int_{y_2}^{y_1} r\sqrt{1+|v'(r)|^2}\,dr : v(y_1) = x_1,\ v(y_2) = x_2\right\}. \tag{1.112}$$

If $x_2 - x_1$ is sufficiently small with respect to y_1 and y_2, then there is a solution of (1.112) which is given by

$$v(r) = \beta + \alpha\log(r + \sqrt{r^2 - \alpha^2})$$

where the constants α and β are determined by means of the boundary conditions $v(y_1) = x_1$ and $v(y_2) = x_2$. On the other hand, if $x_2 - x_1$ is large, then problem (1.112) does not have a solution in the class of smooth functions; in fact, the optimal v is given by

$$v(r) = \begin{cases} x_1 - y_2\log\left(\dfrac{r+\sqrt{r^2-y_2^2}}{y_1+\sqrt{y_1^2-y_2^2}}\right) & \text{if } r > y_2, \\ x_2 & \text{if } r = y_2, \end{cases}$$

see Fig. 1.6. In other words the minimal Cartesian surface spanned by the two circles has a vertical region when $x_2 - x_1$ is large relative to $y_1 - y_2$.

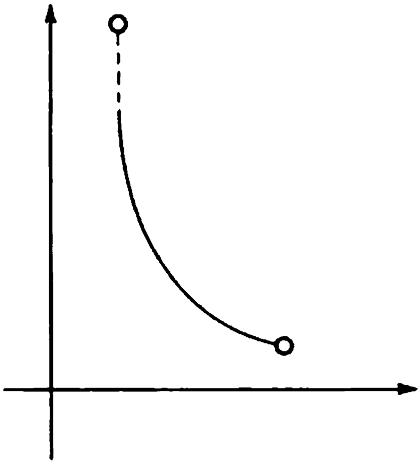

FIG. 1.6.

$\boxed{6}$ *Elastic strings and beams*

A. Consider an *elastic string* which, in its rest position, is described by the segment $[-1, 1]$ on the horizontal axis. If we load the string, and denote by $u(x)$ its vertical displacement, then, according to the simplest model of linear elasticity, the elastic energy of the string is given by

$$k \int_{-1}^{1} |u'(x)|^2 \, dx,$$

where k is a positive constant depending on the material of which the string is made. Assuming that the string is fixed at its boundary points and that its load is uniformly distributed, the shape of the string will be obtained by minimizing the total energy of the system. Denoting by g a positive constant which gives the uniform load distribution, we have to minimize the energy

$$\int_{-1}^{1} [k|u'|^2 + gu] \, dx \tag{1.113}$$

under the constraints $u(-1) = u(1) = 0$ at the boundary points $x = \pm 1$. This leads us to the boundary value problem

$$-2ku'' + g = 0, \quad u(-1) = 0, \quad u(1) = 0, \tag{1.114}$$

where the differential equation is the Euler equation of the potential energy (1.113). We infer immediately from (1.114) that the shape of the uniformly loaded string is given by (see Fig. 1.7a)

$$u(x) = \frac{g}{4k}(x^2 - 1).$$

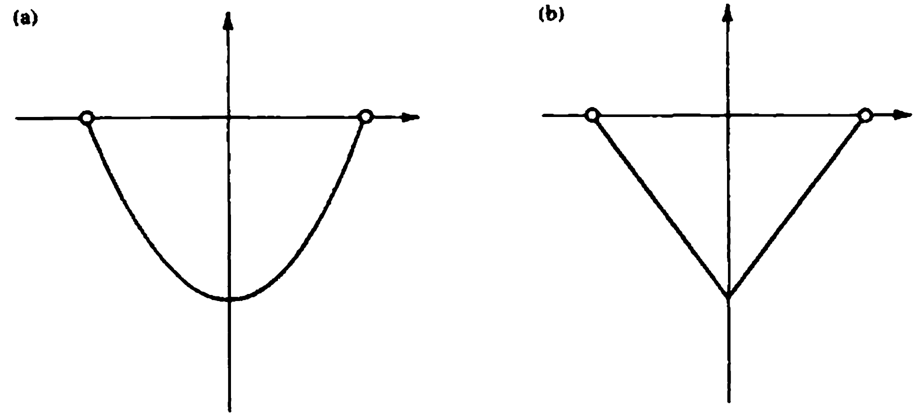

FIG. 1.7.

If the load is concentrated at one point, say the origin $x = 0$, the minimum problem becomes

$$\min\left\{\int_{-1}^{1} k|u'|^2\,dx + gu(0) : u(-1) = u(1) = 0\right\}.$$

In this case the shape of the string is given by (see Fig. 1.7b)

$$u(x) = \frac{g}{4k}\big(|x| - 1\big).$$

B. An analogous problem is provided if, instead of an elastic string, we have a very thin elastic *beam*, whose elastic energy in the simplest linear elasticity model is given by

$$H\int_{-1}^{1} |u''(x)|^2\,dx$$

where, as before, $u(x)$ denotes the vertical deflection of the beam at x, and H is a positive constant depending on the material of the beam. Assume that the beam is clamped at its boundary points and that the load g is uniformly distributed; then the shape $u(x)$, $x - 1 \leq x \leq 1$, of the beam will be obtained by solving the following minimum problem:

$$\min\left\{\int_{-1}^{1}[H|u''|^2 + gu]\,dx : u(-1) = u(1) = 0,\right.$$
$$\left. u'(-1) = u'(1) = 0\right\}. \tag{1.115}$$

The Euler equation corresponding to (1.115) is

$$2Hu^{(4)} + g = 0.$$

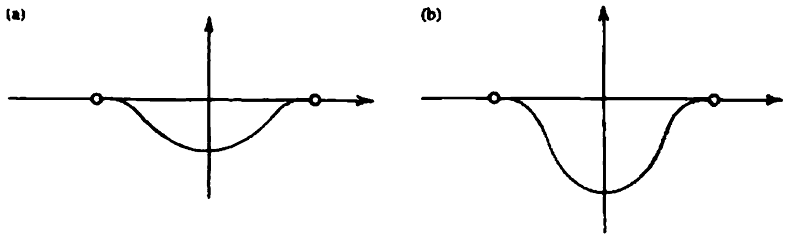

FIG. 1.8.

Therefore we have to solve the boundary problem

$$2Hu^{(4)} + g = 0, \quad u(-1) = u(1) = 0, \quad u'(-1) = u'(1) = 0,$$

whose solution is given by (see Fig. 1.8a)

$$u(x) = -\frac{g}{48H}(x^2 - 1)^2,$$

whereas the analogous problem with a load concentrated at the origin has the solution (see Fig. 1.8b)

$$u(x) = -\frac{g}{48H}\left(2|x|^3 - 3x^2 + 1\right).$$

C. The two-dimensional problem analogous to the problem of the elastic string is the equilibrium problem for *elastic membrane* which, in its rest position, is described by an open subset Ω of $\mathbb{R}^2$, and whose elastic energy is given by

$$k\int_\Omega |Du(x)|^2\,dx. \tag{1.116}$$

We consider here the case when Ω is the unit disk and the load g is uniformly distributed, so that, in polar coordinates, the problem becomes to determine

$$\min\left\{\int_0^1 [k|u'|^2 + gu]r\,dr : u(1) = 0\right\}, \tag{1.117}$$

once the membrane is assumed to be fixed at its boundary. In fact solutions must be radially symmetric. The boundary value problem for the corresponding Euler equation becomes

$$-2k(ru')' + gr = 0, \quad u(1) = 0,$$

and it is solved by all functions of the form

$$u(r) = \frac{g}{8k}(r^2 - 1) + c\log r$$

where c is an arbitrary real constant. The only solution of finite energy is the one with $c = 0$, so that the shape of a uniformly loaded circular membrane is given by the function

$$u(r) = \frac{g}{8k}(r^2 - 1).$$

For a membrane with the load concentrated at some point, the problem has no solution. In fact,

$$\inf\left\{\int_0^1 k|u'|^2 r\,dr + gu(0) : u(1) = 0\right\} = -\infty.$$

Indeed, if we take the functions

$$u_\epsilon(r) = \frac{g}{2k}\log\left(\frac{r+\epsilon}{1+\epsilon}\right)$$

the condition $u_\epsilon(1) = 0$ is fulfilled, and their total energies are

$$\begin{aligned} k\int_0^1 |u_\epsilon'|^2 r\,dr + gu_\epsilon(0) &= \frac{g^2}{4k}\left[\int_0^1 \frac{r}{(r+\epsilon)^2}dr + 2\log\left(\frac{\epsilon}{1+\epsilon}\right)\right] \\ &= \frac{g^2}{4k}\left[\log\left(\frac{\epsilon}{1+\epsilon}\right) - \frac{1}{1+\epsilon}\right]. \end{aligned}$$

This expression tends to $-\infty$ as $\epsilon \to 0$.

D. The two-dimensional problem analogous to that of the elastic beam is the equilibrium problem for an *elastic plate*, whose elastic energy is given by

$$k\int_\Omega |\Delta u|^2\,dx \tag{1.118}$$

where Δ is the Laplace operator. If we take as Ω the unit disk and assume that the load g is uniformly distributed, the equilibrium problem for the plate clamped at its boundary becomes, in polar coordinates, the minimization problem

$$\min\left\{\int_0^1 \left[k|u''|^2 + k\frac{|u'|^2}{r^2} + gu\right] r\,dr : u(1) = u'(1) = 0\right\}. \tag{1.119}$$

By solving the Euler equation

$$ru^{(4)} + 2u''' - \frac{1}{r}u'' + \frac{1}{r^2}u' + \frac{g}{2k}r = 0$$

we obtain the solution (see Fig. 1.9a)

$$u(r) = -\frac{g}{128k}(r^2 - 1)^2.$$

The similar problem for the clamped circular plate with a load concentrated at the origin

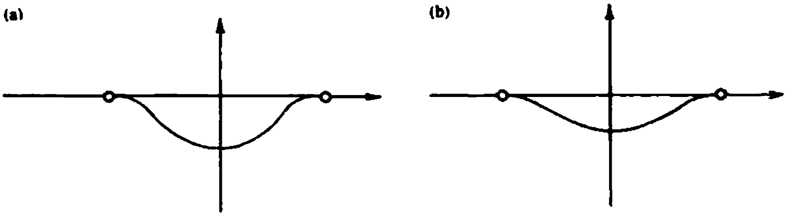

FIG. 1.9.

reduces, in polar coordinates, to the minimization problem

$$\min\left\{\int_0^1 kr\left[\left|u''\right|^2 + \frac{\left|u'\right|^2}{r^2}\right]dr + gu(0) : u(1) = u'(1) = 0\right\}$$

and has the solution (see Fig. 1.9b)

$$u(r) = -\frac{g}{16k}(1 - r^2 + 2r^2 \log r).$$

2

ABSOLUTELY CONTINUOUS FUNCTIONS AND SOBOLEV SPACES

In this chapter we shall introduce and discuss the class of *absolutely continuous functions*. There are essentially two ways of introducing these functions, namely the classical one going back to Vitali and Tonelli, and the more functional–analytic way in terms of the so-called *Sobolev spaces*. A detailed description of these spaces in any dimension can be found in several books; see for instance Adams [1], Maz'ya [182], Ziemer [295]. Here we confine ourselves to discuss Sobolev spaces of functions defined on intervals of the real line. We begin by introducing Sobolev spaces $H^{1,p}(a,b)$, $p \geq 1$, on intervals (a,b) of $\mathbf{R}$, and then we shall relate them to the class of absolutely continuous functions introduced by Vitali. The last section of the chapter will be devoted to the stu dy of the larger class of functions with bounded variation, introduced by Jordan.

2.1 Sobolev spaces in dimension 1

To motivate the use of Sobolev spaces we shall first discuss a simple example. Suppose that we want to apply the direct methods to the problem of minimizing the one-dimensional Dirichlet integral in $(0,1)$

$$\mathcal{D}(u) = \int_0^1 |u'(x)|^2\,dx$$

in the class

$$K(\alpha,\beta) := \{u \in C^1([0,1]) \colon u(0) = \alpha,\ u(1) = \beta\}, \quad \alpha,\beta \in \mathbf{R}.$$

Clearly, we have

$$0 \leq \inf_{K(\alpha,\beta)} \mathcal{D}(u) < +\infty.$$

If $\{u_k\}$ is a minimizing sequence, i.e. if $u_k \in K(\alpha,\beta)$ and

$$\mathcal{D}(u_k) \to \inf_{K(\alpha,\beta)} \mathcal{D},$$

we can assume that

$$\int_0^1 |u_k'|^2\,dx \leq \inf_{K(\alpha,\beta)} \mathcal{D} + 1$$

holds for all $k \in \mathbb{N}$. Also, for all $x, y \in [0, 1]$ we have

$$u_k(x) - u_k(y) = \int_y^x u_k'(t)\, dt.$$

Applying Hölder's inequality we deduce that

$$|u_k(x) - u_k(y)| \le \left(\int_0^1 |u_k'|^2\, dx\right)^{1/2} |x - y|^{1/2} \tag{2.1}$$

and

$$|u_k(x)| \le |u_k(0)| + \left(\int_0^1 |u_k'|^2\, dt\right)^{1/2};$$

thus $\{u_k\}$ is a bounded sequence in $C^{0,1/2}([0, 1])$. Consequently, the functions u_k are *equibounded* and *equicontinuous*. By the Arzelà–Ascoli theorem we then infer, passing to a subsequence, that

$$u_k \to u \quad \text{uniformly in } [0, 1].$$

Moreover, since the L^2-norms of the derivatives u_k' are equibounded and $L^2(0, 1)$ is a reflexive space it follows that *u_k' converge weakly in $L^2(0, 1)$ to some function $v \in L^2(0, 1)$*. We recall that the last claim means that

$$\int_0^1 u_k'\varphi\, dx \to \int_0^1 v\varphi\, dx$$

for all $\varphi \in L^2(0, 1)$ or, equivalently, for all $\varphi \in C_c^\infty(0, 1)$. In particular, if φ is smooth and has compact support in $(0, 1)$, we find from

$$\int_0^1 u_k'\varphi\, dx = -\int_0^1 u_k\varphi'\, dx$$

that

$$\int_0^1 u\varphi'\, dx = -\int_0^1 v\varphi\, dx \quad \text{for all } \varphi \in C_c^\infty(0, 1). \tag{2.2}$$

Equality (2.2) may be interpreted as saying that v is the *weak derivative* of u (the reader who knows something about distributions will recognize that v is the *distributional derivative of u*), and we shall write u' instead of v, although u is in principle only Hölder-continuous and not necessarily differentiable.

The previous discussion shows that every sequence in $K(\alpha, \beta)$ with equibounded Dirichlet integral will have a subsequence converging uniformly to some function u, and such that the sequence of the classical derivatives u_k' converges weakly to an L^2-function denoted by u' which can be interpreted as the weak derivative of u. It is therefore natural to consider on $K(\alpha, \beta)$ the following notion of convergence: *A sequence of functions $u_k \in K(\alpha, \beta)$ with equibounded Dirichlet's integral τ-converges to u if and only if*

$$\begin{cases} u_k \to u & \textit{uniformly} \\ u'_k \to u' & \textit{weakly in } L^2(0, 1) \end{cases} \tag{2.3}$$

where u' denotes the weak derivative of u, i.e. the function v in $L^2(0, 1)$ satisfying (2.2).

It is easily seen that $K(\alpha, \beta)$ is *not* complete with respect to the convergence τ; we shall therefore complete it by defining $K_{(\tau)}$ as *the abstract sequential closure of $K(\alpha, \beta)$ with respect to the convergence τ introduced in* (2.3), i.e. as the *smallest sequentially closed class* (with respect to the convergence τ) *which contains $K(\alpha, \beta)$*.

For any $u \in K_{(\tau)}(\alpha, \beta)$, with weak derivative u', we now define its Dirichlet integral simply as the Lebesgue integral of the L^1-function $|u'|^2$

$$\mathcal{D}_{(\tau)}(u) := \int_0^1 |u'|^2 \, dx.$$

Since for any two functions $w, z \in L^2(0, 1)$ we have

$$|w|^2 - |z|^2 \geq 2z(w - z)$$

we infer that

$$\liminf_{k \to \infty} \int_0^1 |u'_k|^2 \, dx - \int_0^1 |u'|^2 \geq \liminf_{k \to \infty} \int 2u'(u'_k - u') \, dx = 0.$$

Therefore, $\mathcal{D}_{(\tau)}$ is sequentially lower semicontinuous on $K_{(\tau)}$ with respect to the convergence (2.3).

Applying the direct methods presented in Section 2.3 to $C = K_{(\tau)}(\alpha, \beta)$ and $\mathcal{F} = \mathcal{D}_{(\tau)}$ we find a minimizer $u_0 \in C$ of $\mathcal{F}$ in C. But several questions still remain to be answered:

(i) Is $\mathcal{D}_{(\tau)}$ on $K_\tau(\alpha, \beta)$ the best extension of $\mathcal{D}$ on $K(\alpha, \beta)$, i.e. is $\mathcal{D}_{(\tau)}$ the maximal sequential lower semicontinuous extension of $\mathcal{D}$?

(ii) Is τ the strongest convergence we can use, compatible with the sequential compactness of minimizing sequences and the sequential lower semicontinuity of $\mathcal{D}$ or $\mathcal{D}_{(\tau)}$?

(iii) Can the minimizer $u_0 \in K_{(\tau)}(\alpha, \beta)$ be interpreted as the 'most appropriate' generalized solution of our original problem?

(iv) What is the class $K_{(\tau)}(\alpha. \beta)$?

Concerning questions (ii) and (iii) only a regularity theorem can give a positive answer; we shall see in Chap. 5 that, in fact, this is the case. (Actually in this specific example this can easily be proved.) We shall now try to answer questions (i) and (iv).

Consider the space of smooth functions in (0, 1) with finite L^2-norm and Dirichlet integral, equipped with the norm

$$\|u\|_{H^{1,2}(0,1)} := \left(\int_0^1 |u|^2\, dx + \int_0^1 |u'|^2\, dx \right)^{1/2} \tag{2.4}$$

induced by the scalar product

$$(u, v)_{H^{1,2}(0,1)} := \int_0^1 (uv + u'v')\, dx, \tag{2.5}$$

and denote by $H^{1,2}(0, 1)$ its *completion*. $H^{1,2}(0, 1)$ is a Hilbert space with scalar product (2.5) and norm (2.4),

$$\|u\|_{H^{1,2}(0,1)} = (u, u)^{1/2}_{H^{1,2}};$$

moreover, (2.1) also yields

$$H^{1,2}(0, 1) \subset C^{0,1/2}(0, 1).$$

In particular the boundary values of functions u in $H^{1,2}(0, 1)$ are well defined. We shall also see that

$$H^{1,2}(0, 1) = \{u \in C^{0,1/2}(0, 1) : u' \in L^2(0, 1)\}$$

or

$$H^{1,2}(0, 1) = \{u \in L^2(0, 1) : u' \in L^2(0, 1)\},$$

or in words: that $H^{1,2}(0, 1)$ agrees with the space of square integrable or Hölder-continuous functions with exponent 1/2 having weak derivatives in $L^2(0, 1)$. Finally it is readily seen that the convergence in (2.3) is nothing more than the *sequential weak convergence* in the Hilbert space $H^{1,2}(0, 1)$.

Since in a Hilbert space weakly sequentially closed sets and weakly closed sets coincide, and, moreover, convex weakly closed sets and convex strongly closed sets coincide too, we deduce at once that

(i_1) *$K_{(\tau)}(\alpha, \beta)$ is the strong closure of $K(\alpha, \beta)$ in* $H^{1,2}(0, 1)$, which can be proved to coincide with $\{u \in H^{1,2}(0, 1) : u(0) = \alpha, u(1) = \beta\}$;

(i_2) *$\mathcal{D}_{(\tau)}(u)$ is the maximal sequential lower semicontinuous extension of $\mathcal{D}$ to $K_{(\tau)}(\alpha, \beta)$.*

In fact, for every $u \in K_{(\tau)}(\alpha, \beta)$ there exists a sequence $\{u_k\}$ in $K(\alpha, \beta)$ such that

$$\mathcal{D}(u_k) \to \mathcal{D}_{(\tau)}(u).$$

In conclusion we can say that the problem

$$\min\left\{\int_0^1 |u'|^2\,dx : u \in H^{1,2}(0,1),\ u(0)=\alpha,\ u(1)=\beta\right\} \qquad (\mathcal{P})$$

can reasonably be considered as the generalized reformulation of the original problem

$$\min\{\mathcal{D}(u) : u \in K(\alpha,\beta)\}, \qquad (\mathcal{P}^r)$$

and its minimizers u_0 can be viewed as the generalized minimizers of $(\mathcal{P})$.

We should mention that this is not quite Tonelli's original approach but only an a posteriori reformulation of it; later we shall compare it with Tonelli's approach. Here we wanted only to motivate the introduction of Sobolev spaces by means of a simple example. A more geometric motivation of the Sobolev space $H^{1,2}$ strongly related to the Dirichlet integral will be discussed later. We now introduce the Sobolev spaces $H^{1,p}$ for every real number $p \geq 1$.

Let I be an open interval in $\mathbb{R}$ and let p be any real number with $p \geq 1$. We denote by X the linear subspace of $C^1(I)$ of functions for which

$$\|u\|_{H^{1,p}(I)} := \left(\int_I (|u|^p + |u'|^p)\,dx\right)^{1/p} \qquad (2.6)$$

is finite. Clearly X contains $C_c^1(I)$, as well as $C^1(\bar{I})$ if I is bounded; moreover, $\|\cdot\|_{H^{1,p}(I)}$ is a norm on X. One easily sees that X is not complete with respect to this norm.

Definition 2.1 *The completion of X with respect to the norm in* (2.6) *of X is denoted by $H^{1,p}(I)$ and referred to as a* Sobolev space. *The closure of $C_c^1(I)$ in $H^{1,p}(I)$ is denoted by $H_0^{1,p}(I)$.*

By definition Sobolev spaces are Banach spaces with the norm (2.6). Their elements are equivalence classes of Cauchy sequences in $C^1(I)$ and can be identified with elements of $L^p(I)$ since, by the definition of $H^{1,p}(I)$, the identity map defines an embedding of $H^{1,p}(I)$ into $L^p(I)$. So, as is customary, we shall refer to elements of $H^{1,p}(I)$ as *functions*. For example, we shall say that $u \in H^{1,p}(I)$ is a continuous function if, in the equivalence class of u, there is a continuous function, or, equivalently, if any representative of u in the equivalence class becomes a continuous function after an appropriate redefinition on a set of measure zero.

Clearly $H^{1,2}(I)$ is a Hilbert space since the norm (2.6) is induced in this case by the scalar product

$$(u,v)_{H^{1,2}(I)} := \int_I uv\,dx + \int_I u'v'\,dx, \qquad u,v \in C^1(I).$$

In the sequel we shall simply write $H^1(I)$, $H_0^1(I)$ for $H^{1,2}(I)$ and $H_0^{1,2}(I)$ respectively and, if no confusion may arise, $\|\cdot\|_{1,p,I}$ or simply $\|\cdot\|_{1,p}$ for $\|\cdot\|_{H^{1,p}(I)}$ and $(\cdot,\cdot)_{1,2}$ or simply $(\cdot,\cdot)$ for $(\cdot,\cdot)_{H^{1,2}(I)}$.

We shall now see that a function $u \in H^{1,p}(I)$, $p \geq 1$, does possess a *generalized derivative*, also called the *strong L^p-derivative*. To define it for $u \in H^{1,p}(I)$, we consider a sequence $\{u_k\}$ in $C^1(I) \cap H^{1,p}(I)$ converging in $H^{1,p}(I)$ to u; $\{u_k\}$ is in particular a Cauchy sequence with respect to (2.6). Then it follows that there exists a function $v \in L^p(I)$ such that

$$\lim_{k\to\infty} \|u_k' - v\|_{L^p(I)} = 0.$$

It is easily seen that the function v does not depend on the chosen approximating sequence $\{u_k\}$; it is uniquely determined by u and coincides with the classical derivative of u provided that $u \in C^1(I) \cap H^{1,p}(I)$. By definition the function v is the *strong L^p-derivative*, u', of u.

There is another notion of a generalized derivative, which appears in the theory of distributions: We say that a function $u \in L^p(I)$, $p \geq 1$, has a function $v \in L^q$, $q \geq 1$, as a *weak derivative* if

$$\int_I u\varphi'\,dx = -\int_I v\varphi\,dx \quad \forall \varphi \in C_c^\infty(I).$$

The weak derivative is again denoted by u', and one easily sees that, if it exists, it is unique. Considering again a strong approximating sequence $\{u_k\}$ of $u \in H^{1,p}(I)$, we obtain, integrating by parts,

$$\int_I u_k\varphi'\,dx = -\int_I u_k'\varphi\,dx \quad \text{for all } \varphi \in C_c^\infty(I).$$

Hence, by passing to the limit, we find that *every $u \in H^{1,p}(I)$ has a weak L^p-derivative which coincides with its strong derivatives.*

It is a natural question to ask whether an L^p-function in I which possesses a weak derivative in $L^p(I)$ also has strong derivatives, i.e. belongs to $H^{1,p}(I)$. This is in fact true, i.e. *the notions of a weak derivative in $L^p(I)$ and of a strong derivative in L^p coincide*. Before verifying this assertion, let us prove the following theorem which states that functions in $H^{1,p}(I)$ are actually continuous functions.

Theorem 2.2 *We have*

(i) *Every function in $H^{1,1}(I)$ is uniformly continuous in I, in particular*

$$H^{1,1}(I) \subset C^0(\bar{I})$$

and

$$\sup_I |u| \leq \frac{1}{\text{meas } I}\int_I |u|\,dx + \int_I |u'|\,dx. \tag{2.7}$$

Moreover, the fundamental theorem of calculus holds, i.e. for all $x, y \in \bar{I}$

$$u(x) - u(y) = \int_y^x u'(t)\,dt.$$

(ii) *If* $u \in H^{1,p}(I)$, $p > 1$, *then* $u \in C^{0,1-1/p}(I)$ *and*

$$\sup_I |u| \le \left(\frac{1}{\text{meas } I}\int_I |u|^p\,dx\right)^{1/p} + \left(\int_I |u'|^p\,dx\right)^{1/p} (\text{meas } I)^{1-1/p}. \tag{2.8}$$

Moreover, for all $x, y, \in \overline{I}$ *we have*

$$|u(x) - u(y)| \le \left(\int_I |u'|^p\,dx\right)^{1/p} |x - y|^{1-1/p}.$$

Proof Let $\{u_k\}$ be a sequence in X which converges strongly in $H^{1,1}(I)$ to u. Clearly $\{u_k\}$ is equibounded in $H^{1,1}(I)$ and, by the absolute continuity of Lebesgue's integral, the set functions

$$E \mapsto \int_E |u_k'|\,dx, \quad E \subset I,$$

are equiabsolutely continuous, i.e. for any $\epsilon > 0$ there exists a positive δ such that if meas $E < \delta$ then

$$\int_E |u_k'|\,dx < \epsilon \tag{2.9}$$

for all k. In fact, for all ϵ, we can find δ so that

$$\int_E |u'|\,dx < \epsilon/2$$

whenever meas $E < \delta$, and we have

$$\int_I |u_k' - u'|\,dx < \epsilon/2$$

for all k larger than some $k(\epsilon)$ depending on ϵ. From this it follows at once that (2.9) holds for $k > k(\epsilon)$; thus to finish the proof it suffices to adjust δ in such a way that it works also for the finite set of functions $u_1, \dots, u_{k(\epsilon)}$. Now for all $x, y \in \overline{I}$ we have

$$u_k(x) - u_k(y) = \int_y^x u_k'(t)\,dt \tag{2.10}$$

and, in particular,

$$|u_k(x) - u_k(y)| \le \left|\int_y^x |u_k'(t)|\,dt\right| \tag{2.11}$$

and

$$|u_k(x)| \le |u_k(y)| + \int_I |u_k'(t)|\,dt. \tag{2.12}$$

Integrating inequality (2.12) with respect to y on I, we obtain

$$|u_k(x)| \le \frac{1}{\text{meas } I}\int_I |u_k(t)|\,dt + \int_I |u'(t)|\,dt \tag{2.13}$$

and we conclude, taking (2.11) also into account, that $\{u_k\}$ is a sequence of equibounded and equicontinuous functions. The Arzelà–Ascoli theorem then yields that a suitable subsequence of $\{u_k\}$ converges to a continuous function and therefore u is continuous. Passing in (2.10) and (2.13) to the limit for $k \to \infty$ the statement (i) follows at once. The second part of the theorem follows by applying Hölder's inequality to (2.11) and (2.13). For example,

$$\begin{aligned}|u(x) - u(y)| &\le \left|\int_y^x |u'(t)|\,dt\right| \\ &\le \left(\int_y^x |u'(t)|^p\,dt\right)^{1/p} |x-y|^{1-1/p} \\ &\le \left(\int_I |u'|^p\,dt\right)^{1/p} |x-y|^{1-1/p}. \qquad \square\end{aligned}$$

We note that, according to our convention, we have stated in Theorem 2.2 that u is continuous: more precisely we should have said that in the equivalence class of u there is a continuous function: that is, if u is a function in the equivalence class of u, by changing the values of u on a set of zero measure, u becomes continuous in I. But we shall stay with our convention and we shall freely use the 'values of u at points'. In particular, since every function in $H^{1,p}(I)$, $p \ge 1$, is uniformly continuous, *the trace of u on ∂I is well defined* and, for $I = (a, b)$ and $u \in H^{1,p}(I)$, $p \ge 1$, we can define $u(a)$ and $u(b)$ as $\lim_{x\to a+0} u(x)$ and $\lim_{x\to b-0} u(x)$ respectively.

Theorem 2.3 *Let $I = (a, b)$ be a bounded interval of $\mathbb{R}$, and suppose that u is of class $L^p(I)$ and has a weak derivative u' belonging to $L^p(I)$. Then*

(i) *There exists a function $U \in L^p(\mathbb{R})$ which has a weak derivative U' in $L^p(\mathbb{R})$ and satisfies $U = u$ in (a, b);*
(ii) $u \in H^{1,p}(I)$.

Proof

(i) Let a' and b' be such that $a < a' < b' < b$, and let η be a function in $C^1(\mathbb{R})$ such that

$$0 \le \eta \le 1, \quad \eta = 1 \text{ in } (-\infty, a'),\ \eta = 0 \text{ in } (b', \infty).$$

We write u as $u = \eta u + (1-\eta)u$ and observe that ηu and $(1-\eta)u$ are functions in $L^p(a, \infty)$ and $L^p(-\infty, b)$ respectively with weak derivatives in $L^p(a, \infty)$ and $L^p(-\infty, b)$. In fact, for any $\varphi \in C_c^\infty(a, \infty)$ we obtain

$$\begin{aligned}\int_a^\infty \eta u\varphi'\,dx &= \int_a^b \eta u\varphi'\,dx = \int_a^b u[(\eta\varphi)' - \eta'\varphi]\,dx \\ &= -\int_a^b u'\eta\varphi\,dx - \int_a^b u\eta'\varphi\,dx = -\int_a^\infty (u'\eta + u\eta')\varphi\,dx,\end{aligned}$$

i.e.

$$(\eta u)' = u'\eta + u\eta' \in L^p(a, \infty).$$

and similarly we can argue for $(1-\eta)u$. We now define U_1 and U_2 as

$$U_1(a+t) := \begin{cases} (\eta u)(a+t) & \text{if } t \geq 0 \\ \eta u(a-t) & \text{if } t < 0 \end{cases}$$

$$U_2(a+t) := \begin{cases} [(1-\eta)u](a+t) & \text{if } t \leq 0 \\ [(1-\eta)u](a-t) & \text{if } t > 0. \end{cases}$$

Then we immediately see that $U := U_1 + U_2$ satisfies the claim.

(ii) If S_ϵ is the mollifying convolution operator defined by

$$(S_\epsilon U)(x) = \frac{1}{\epsilon}\int U(x-y)\varphi(y/\epsilon)\,dy \quad (x \in \mathbb{R})$$

where φ is a non-negative $C_c^\infty(-1,1)$ function with $\int \varphi(y)\,dy = 1$, we readily conclude that $S_\epsilon U$ converges in $H^{1,p}(I)$ to U; hence $U \in H^{1,p}(I)$ and consequently we obtain that $u \in H^{1,p}(I)$. □

We can actually state something more.

Theorem 2.4 *Let $I = (a,b)$ be a bounded interval of $\mathbb{R}$. Then every function u in $H^{1,p}(I)$ can be approximated in $H^{1,p}(I)$ by a sequence $\{u_k\}$ of functions in $C^1(\bar{I})$ such that $u_k(a) = u(a)$ and $u_k(b) = u(b)$.*

Proof For every $\epsilon > 0$ consider an affine transformation λ_ϵ such that $\lambda_\epsilon(a,b) = (a+\epsilon, b-\epsilon)$ and extend $v_\epsilon(x) := u(\lambda_\epsilon^{-1}(x))$ to (a,b) setting it equal to $u(a)$ for $x \leq a+\epsilon$ and $u(b)$ for $x \geq b-\epsilon$. Then one readily sees that the sequence $u_k = S_{1/k}v_{2/k}$ has the desired properties and converges in $H^{1,p}(I)$ to u. □

Consider now the closed linear subspace $H_0^{1,p}(a,b)$. Clearly it coincides with the subspace of $H^{1,p}(a,b)$ consisting of functions u such that $u(a) = u(b) = 0$. Moreover, one readily sees that if $u \in H_0^{1,p}(a,b)$ and $(a,b) \subset I$, then the function $\tilde{u}$ defined as $u(x)$ in (a,b) and zero in $I-(a,b)$ is a function in $H_0^{1,p}(I)$. Moreover, if $\tilde{I} \subset I$, $u \in H^{1,p}(I)$, $v \in H^{1,p}(\tilde{I})$, and $u - v \in H_0^{1,p}(\tilde{I})$ then

$$U(x) := \begin{cases} v(x) & \text{in } \tilde{I} \\ u(x) & \text{in } I-\tilde{I} \end{cases}$$

belongs to $H^{1,p}(I)$, while $U - u \in H_0^{1,p}(I)$ and $U' = v'$ in $\tilde{I}$, $U' = u'$ in $I - \tilde{I}$.

Theorem 2.5 *We have $H_0^{1,p}(\mathbb{R}) = H^{1,p}(\mathbb{R})$, $p \geq 1$. Moreover, for every $u \in H^{1,p}(\mathbb{R})$, $p \geq 1$, it follows that*

$$\lim_{x\to\pm\infty} u(x) = 0.$$

Proof The first claim follows by considering a function $\varphi \in C_c^\infty(\mathbf{R})$ with $\varphi = 1$ in $(-1, 1)$, $\varphi = 0$ for $|x| > 2$, and observing that the functions u_r defined by

$$u_r(x) := u(x)\varphi\left(\frac{x}{r}\right)$$

converge strongly to u in $H^{1,p}$ (as $r \to \infty$) if $u \in H^{1,p}(\mathbf{R})$. Suppose now that $u \in H^{1,p}(\mathbf{R})$ and that $\{u_k\}$ is a sequence in $C_c^\infty(\mathbf{R})$ which converges strongly to u. The functions $|u_k|^{p-1}u_k$ belong to $C_c^1(\mathbf{R})$ and

$$|u_k(x)|^p \le \int_{\mathbf{R}} |(|u_k|^{p-1}u_k)'|\,dx = p\int_{\mathbf{R}} |u_k|^{p-1}|u_k'|\,dx \le p\|u_k\|^{p-1}_{L^p(\mathbf{R})}\|u_k'\|_{L^p(\mathbf{R})};$$

compare (2.12). Using Young's inequality

$$ab \le \frac{1}{p'}a^{p'} + \frac{1}{p}b^p, \quad p' = \frac{p}{p-1},$$

we therefore conclude that

$$\sup_{\mathbf{R}} |u_k(x)| \le p^{1/p}\|u_k\|_{H^{1,p}(\mathbf{R})}. \tag{2.14}$$

Consequently the same inequality is valid for $u \in H^{1,p}(\mathbf{R})$. In particular, we infer that

$$\sup_{\mathbf{R}} |u(x) - u_r(x)| \to 0 \quad \text{as } r \to \infty.$$

Since $u_r = 0$ near infinity, this shows the second part of the theorem. □

Remark 1 A simple consequence of estimate (2.14) is that Theorems 2.2, 2.3, 2.4 hold for any open interval in $\mathbf{R}$, bounded or not, provided that we replace for instance $u(a)$ with $\lim_{x\to-\infty} u(x)$ if $a = -\infty$, and the sup-estimates in (2.7), (2.8) by

$$\sup_I |u(x)| \le p^{1/p}\|u\|_{H^{1,p}(I)}. \tag{2.15}$$

Remark 2 Analogously to $H^{1,p}(I)$ and $H_0^{1,p}(I)$ we can define the Sobolev spaces $H^{1,p}(I, \mathbf{R}^N)$ and $H_0^{1,p}(I, \mathbf{R}^N)$ of mappings with values in $\mathbf{R}^N$. One readily sees that

$$H^{1,p}(I, \mathbf{R}^N) = \{u = (u^1, \dots, u^N) : u^i \in H^{1,p}(I), i = 1\dots N\} = (H^{1,p}(I))^N$$
$$H_0^{1,p}(I, \mathbf{R}^N) = \{u = (u^1, \dots, u^N) : u^i \in H_0^{1,p}(I), i = 1\dots N\} = (H_0^{1,p}(I))^N.$$

Therefore results stated in $H^{1,p}(I)$ can be transferred to mappings $u \in H^{1,p}(I, \mathbf{R}^n)$ simply by reading them on the components of u.

A special feature of Sobolev spaces in one dimension is that they are *Banach algebras*, i.e. *if* $u, v \in H^{1,p}(I)$ *then* $uv \in H^{1,p}(I)$ and the usual derivation rule for the product holds. In particular, if $\varphi \in C_0^1(I)$ and $u \in H^{1,p}(I)$, then $u\varphi \in H_0^{1,p}(I)$.

Finally we note that

(a) *Sobolev spaces are 'local' spaces*: This means that if $I_1, \dots, I_l$ and I are open intervals such that $I \subset \bigcup_{i=1}^{l} I_i$ and if $u \in H^{1,p}(I_i)$ for all $i = 1, \dots, l$, then $u \in H^{1,p}(I)$. This is easily seen by means of a partition of unity.

(b) *Sobolev spaces are invariant under diffeomorphisms*: If $g : I \to I'$ is a C^1-diffeomorphism then the mapping

$$g_*: H^{1,p}(I') \to H^{1,p}(I)$$

defined by

$$g_*(u)(x) = u(g(x))$$

is an isomorphism of Banach spaces.

Properties (a) and (b) clearly permit us to consider Sobolev spaces on one-dimensional manifolds M: if $\{(U_i, \varphi_i)\}$ is an atlas of M and $\{\psi_i\}$ a decomposition of unity associated with the covering $\{U_i\}$, then $u \in H^{1,p}(M)$ if and only if $(\psi_i u) \circ \varphi_i^{-1} \in H^{1,p}$ for all i. Actually, we can require in (b) that g be only one-to-one and Lipschitz-continuous with its inverse.

The map

$$u \mapsto (u, u')$$

clearly defines an isomorphism of $H^{1,p}(I)$ into a closed subspace of $L^p(I) \times L^p(I) =: L^p(I, \mathbb{R}^2)$; thus functional properties of L^p-spaces can be transferred at once into analogous properties of Sobolev spaces. In particular we have

Theorem 2.6 *$H^{1,p}(I)$ is a separable Banach space for all $p \geq 1$.*

Consequently we can find a sequence of finite dimensional subspaces V_k such that every $u \in H^{1,p}(I)$ can be written as

$$u = \sum_{k=1}^{\infty} a_k u_k$$

where $a_k \in \mathbb{R}$ and $u_k \in V_k$.

Recall that, for $p > 1$, $L^p(I)$ is a *reflexive Banach space*. This means that the unit ball of $L^p(I)$, $p > 1$, is weakly compact. In other words, from every sequence $\{u_k\} \subset L^p(I)$ with $\sup_k \|u_k\|_{L^p} < \infty$ we can extract a subsequence $\{u_{k_l}\}$ which converges weakly to some $u \in L^p(I)$, i.e.

$$\int_I u_{k_l} \varphi \, dx \to \int_I u\varphi \, dx \tag{2.16}$$

for all φ in the dual space $L^{p'}(I)$, $p' = p/(p-1)$, of $L^p(I)$. Actually, since $C_c^\infty(I)$ is dense in $L^p(I)$ for $p < \infty$, it suffices to know (2.16) for all functions φ of class $C_c^\infty(I)$. From this we deduce at once the following result.

Theorem 2.7 *For $p > 1$ the space $H^{1,p}(I)$ is a reflexive Banach space.*

In particular, from every bounded sequence $\{u_k\}$ in $H^{1,p}(I)$, $p > 1$, we can extract a subsequence $\{u_{k_i}\}$ converging weakly in L^p to a function $u \in H^{1,p}(I)$ with weak derivative u'_{k_i} weakly converging in L^p to the weak derivative u' of u.

Since $L^1(I)$ is not reflexive, $H^{1,1}(I)$ is also not reflexive. However, $H^{1,1}(I)$ is closed with respect to the weak convergence. That is, if $u_k \in H^{1,1}(I)$ and[6]

$$\int_I u_k\varphi\,dx \to \int_I u\varphi\,dx, \quad \int_I u'_k\varphi\,dx \to \int_I v\varphi\,dx \qquad \text{for all } \varphi \in L^\infty(I),$$

then $u \in H^{1,1}(I)$, but in general bounded sequences in $H^{1,1}(I)$ do not have weakly converging subsequences. As we shall see at the end of this subsection, bounded sequences $\{u_k\}$ in $H^{1,1}(I)$ do have subsequences converging in L^p to some function u, but with derivatives converging only in the sense of measures to a measure. In particular, while from Theorem 2.2 we readily deduce that if I is bounded, the immersion

$$H^{1,p}(I) \hookrightarrow C(\bar{I})$$

is compact for $p > 1$, we have that the continuous immersion

$$H^{1,1}(I) \hookrightarrow C(\bar{I})$$

is not compact. However, we obtain the following result:

Theorem 2.8 *Let I be a bounded interval of $\mathbb{R}$. Then the immersion*

$$H^{1,1}(I) \hookrightarrow L^q(I)$$

is compact for any $q \in (1, \infty)$.

In proving this theorem we shall use an inequality which plays an important role in many instances; thus we state it separately.

Poincaré's inequality. *Let (a, b) be a bounded interval and let $u \in H_0^{1,p}(I)$. Then we have*

$$\int_a^b |u|^p\,dx \le (b-a)^p \int_a^b |u'|^p\,dx. \tag{2.17}$$

In particular, the expression

$$|u|_{1,p} := \left(\int_a^b |u'|^p\,dx\right)^{1/p}$$

[6]Notice that this time one cannot replace $\varphi \in L^\infty(I)$ with $\varphi \in C_c^\infty(I)$. In fact, if $I = (0, 1)$ and $u_k = k\chi_k$, χ_k being the characteristic function of the interval $(1/k, 2/k)$, we have $\int_0^1 u_k\varphi\,dx \to 0$ for all $\varphi \in C_c^\infty(0, 1)$ while, for example, for $\varphi = 1$ we obtain $1 = \int_0^1 u_k\varphi\,dx \to 1$ as $k \to \infty$.

yields a norm in $H_0^{1,p}(a,b)$ *which is equivalent to* $\|\cdot\|_{1,p}$. *More generally, for all* $u \in H^{1,p}(a,b)$ *and all* $x_0 \in [a,b]$ *we have*

$$\int_a^b |u(x) - u(x_0)|^p\, dx \le (b-a)^p \int_a^b |u'|^p\, dx, \tag{2.18}$$

and in particular

$$\int_a^b |u(x) - u_I|^p\, dx \le (b-a)^p \int_a^b |u'|^p\, dx \tag{2.19}$$

where u_I *denotes the average of* u *on* I *defined by*

$$u_I := \frac{1}{\operatorname{meas} I} \int_I u\, dx =: \fint_I u\, dx.$$

Proof For all $x_0 \in [a,b]$ we have

$$u(x) - u(x_0) = \int_{x_0}^x u'(t)\, dt;$$

thus,

$$|u(x) - u(x_0)| \le \int_a^b |u'(t)|\, dt.$$

Integrating over (a,b) and using Hölder's inequality, (2.18) follows at once. If u is zero at one point x_0, in particular if $u \in H_0^{1,p}(a,b)$, then (2.17) follows. Finally we deduce (2.19) since, u being continuous, there is a point $x_0 \in [a,b]$ such that $u_I = u(x_0)$. □

Notice that (2.18) yields at once that for any fixed $x_0 \in [a,b]$

$$|u(x_0)| + \left(\int_a^b |u'|^p\, dx\right)^{1/p}$$

is a norm in $H^{1,p}(a,b)$ which is equivalent to $\|\cdot\|_{1,p}$.

Proof of Theorem 2.7 We must prove that bounded sets in $H^{1,1}(I)$ are relatively compact in $L^q(I)$. Clearly, it will be sufficient to show that if $\|u_k\|_{1,1} < c$ for all k, then the sequence $\{u_k\}$ has a subsequence $\{u_{k_i}\}$ which converges strongly in $L^q(I)$. We recall that a subset E of a complete metric space X is relatively compact if and only if, for all $\epsilon > 0$, there exists an *ϵ-net*, i.e., a finite family $\{x_1^{(\epsilon)}, \dots, x_s^{(\epsilon)}\}$ of points $x_i^{(\epsilon)}$, such that E is contained in the union of the balls $B(x_i^{(\epsilon)}, \epsilon)$. We shall now construct such an ϵ-net in $L^q(I)$. Let $l = b - a$ be the size of I. For a fixed $\epsilon > 0$, let us consider a subdivision of

I in a family of intervals $I_1, \ldots, I_s$ such that length $I_j = \sigma < (\epsilon/(4c))^q$ for $1 \le j \le s$ and $\mathring{I}_j \cap \mathring{I}_k = \phi$ for $j \neq k$. Then, for all k and j, the mean values

$$u_{k,I_j} = \fint_{I_j} u_k \, dx$$

satisfy

$$|u_{k,I_j}| \le \frac{c}{\sigma}.$$

Consider the finite family $\mathcal{G}$ of simple functions of the type

$$g(x) = n_1 \epsilon \chi_1 + \cdots + n_s \epsilon \chi_s$$

where $n_1, \ldots, n_s$ are integers running in $(-N, N)$ with $N > c/(\epsilon \cdot \sigma)$, and χ_j is the characteristic function of I_j. We shall now show that every u_k has an L^q-distance less than ϵ from some element $g \in \mathcal{G}$. For this purpose, we define the function

$$u_k^* := \sum_{j=1}^{s} u_{k,I_j} \chi_j.$$

From Poincaré's inequality and relation (2.7) we deduce that

$$\begin{aligned}
\int_I |u_k - u_k^*|^q \, dx &\le \sum_{j=1}^{s} \int_{I_j} |u_k - u_k^*|^q \, dx \\
&\le \sum_{j=1}^{s} (\sup_{I_j} |u_k - u_{k,I_j}|)^{q-1} \int_{I_j} |u_k - u_{k,I_j}| \, dx \\
&\le \sum_{j=1}^{s} \left(\frac{1}{\sigma} \int_{I_j} |u_k - u_{k,I_j}| \, dx + \int_{I_j} |u_k'| \, dx \right)^{q-1} \sigma \int_{I_j} |u_k'| \, dx \\
&\le (2c)^{q-1} \sigma \int_I |u_k'| \, dx = 2^{q-1} c^q \sigma.
\end{aligned}$$

On the other hand by the definition of $\mathcal{G}$ we can find $g \in \mathcal{G}$ such that

$$|g(x) - u_k^*(x)| \le \frac{\epsilon}{2l^{1/q}} \quad \text{for all } x \in I.$$

Therefore we infer that

$$\|u_k - g\|_{L^q(I)} \le \|u_k - u_k^*\|_{L^q(I)} + \|u_k^* - g\|_{L^q(I)} \le 2c\sigma^{1/q} + \frac{\epsilon}{2} < \epsilon.$$

□

For the reader's convenience we shall now give different characterizations of Sobolev spaces.

Theorem 2.9 *Let $p > 1$ and $u \in L^p(I)$. Then the following properties are equivalent:*

(i) $u \in H^{1,p}(I)$.

(ii) *There exists a constant c such that*

$$\left|\int_I u\varphi'\,dx\right| \le c\|\varphi\|_{L^{p'}(I)} \quad \text{for all } \varphi \in C_c^\infty(I),\ p' = \frac{p}{p-1}.$$

(iii) *There exists a constant c such that, for all intervals $\tilde{I} \subset\subset I$ and all $h \in \mathbb{R}$ with $|h| < \operatorname{dist}(\tilde{I}, \partial I)$, we have*

$$\|u(x+h) - u(x)\|_{L^p(\tilde{I})} \le c|h|.$$

Moreover, we can take $c = |u'|_{L^p(\tilde{I})}$.

Proof Trivially (i) implies (ii). Let us prove that (ii) implies (i). By the Hahn–Banach theorem, the linear form $\varphi \mapsto \int_I u\varphi'\,dx$ defined for $\varphi \in C_c^\infty(I) \subset L^{p'}(I)$ can be extended to a bounded linear form $F(\varphi)$ on $L^{p'}(I)$, and the Riesz representation theorem yields the existence of an element $g \in L^p(I)$ such that

$$F(\varphi) = \int_I g\varphi\,dx.$$

In particular we find

$$\int_I u\varphi'\,dx = \int_I g\varphi\,dx \quad \text{for all } \varphi \in C_c^\infty(I);$$

hence $u \in H^{1,p}(I)$ by Theorem 2.3.

Let us now prove that (i) implies (iii). For $x \in \tilde{I}$ we have

$$u(x+h) - u(x) = \int_x^{x+h} u'(t)\,dt = h\int_0^1 u'(x+sh)ds;$$

hence

$$|u(x+h) - u(x)| \le |h|\int_0^1 |u'(x+sh)|ds.$$

Applying Hölder's inequality we infer that

$$|u(x+h) - u(x)|^p \le |h|^p \int_0^1 |u'(x+sh)|^p ds.$$

and integrating over $\tilde{I}$ we deduce that

$$\begin{aligned}\int_{\tilde{I}} |u(x+h) - u(x)|^p\,dx &\le |h|^p \int_{\tilde{I}} dx \int_0^1 |u'(x+sh)|^p ds \\ &= |h|^p \int_0^1 ds \int_{\tilde{I}} |u'(x+sh)|^p\,dx.\end{aligned}$$

Then we obtain (iii) by observing that

$$\int_{\tilde{I}} |u'(x+sh)|^p \, dx \le \int_{\tilde{I}+sh} |u'(y)|^p \, dx \quad \text{for } 0 < s < 1.$$

Finally we prove that (iii) implies (ii). Let $\varphi \in C_c^\infty(I)$. We choose $\tilde{I}$ so that spt $\varphi \subset \tilde{I}$ and observe that, for $|h| < \operatorname{dist}(\tilde{I}, \partial I)$, we have

$$\int_I u(x)[\varphi(x-h) - \varphi(x)] \, dx = \int_I [u(x+h) - u(x)]\varphi(x) \, dx.$$

Using Hölder's inequality and (iii), we then deduce that

$$\left| \int_I u(x)[\varphi(x-h) - \varphi(x)] \, dx \right| \le c|h| \|\varphi\|_{L^{p'}(I)}$$

whence we arrive at (ii), letting h tend to zero. □

Remark 3 We observe that the previous proof also yields that (ii) and (iii) are even equivalent if $p = 1$, but they are not equivalent to (i) as one can easily check by considering the Heaviside function

$$H(x) := \begin{cases} 1 & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

which satisfies (ii), but, not being continuous, does not belong to $H^{1,1}(\mathbf{R})$. In fact, going through the proof of Theorem 2.9, one can see that, for $p = 1$, both (ii) and (iii) are equivalent to

(i*) *The distributional derivative of u is a measure of bounded variation on I.*

Using the characterization (iii) of Theorem 2.9, the result of Theorem 2.1 is often proved by means of the so-called *Kolmogorov strong compactness criterion L^p*. We leave the proof of this fact to the reader, but we shall now state and prove Kolmogorov's criterion.

Theorem 2.10 *Let Ω be a bounded open set in $\mathbf{R}^n$, and let C be a subset of $L^p(\Omega)$, $p \ge 1$. Then C is relatively compact in $L^p(\Omega)$ if and only if the following two conditions hold:*

(i) *C is bounded in $L^p(\Omega)$, i.e.*

$$\sup_{u \in C} \|u\|_{L^p(\Omega)} < \infty;$$

(ii) *the functions of C are equicontinuous in the mean, i.e. for every $\epsilon > 0$, there exists a positive δ such that*

$$\int_\Omega |u(x+z) - u(x)|^p \, dx \le \epsilon^p$$

for all z with $|z| < \delta$, if we extend every $u \in C$ by zero outside of Ω.

Proof Let us first show that (i) and (ii) are sufficient for relative compactness. This is consequence of the Arzelà–Ascoli theorem in conjunction with the mollifying procedure. As is well known, the mollified functions $S_\epsilon u$, already introduced in the proof of Theorem 2.3, converge in $L^p(\Omega)$ to u and, owing to equicontinuity in the mean, we have

$$\|S_\eta u - u\|_{L^p(\Omega)} < \epsilon \quad \text{for all } u \in \mathcal{C} \text{ and for all } \eta < \delta. \tag{2.20}$$

In fact, if

$$S_\eta u(x) := \int u(x-y)\psi_\eta(y)\,dy,$$
$$\psi_\eta(y) := \frac{1}{\eta}\psi(y/\eta),$$

we have

$$\left|S_\eta u(x) - u(x)\right| \le \left(\int \left|u(x-y) - u(x)\right|^p \psi_\eta(y)\,dy\right)^{1/p};$$

hence

$$\int_\Omega \left|S_\eta u(x) - u(x)\right|^p dx \le \int_{B(0,\eta)} \psi_\eta(y)\,dy \int_\Omega \left|u(x-y) - u(x)\right|^p dx < \epsilon^p.$$

Moreover, for a fixed η, the family $\{S_\eta u : u \in \mathcal{C}\}$ is equicontinuous and equibounded: in fact,

$$\sup\left|S_\eta u\right| \le \sup\left|\psi_\eta\right| \|u\|_{L^1(\Omega)}$$

and

$$\left|S_\eta u(x) - S_\eta u(y)\right| \le \left|\psi_\eta\right|_{\mathrm{Lip}} \left|x-y\right| \|u\|_{L^1(\Omega)}.$$

Hence, by the Arzelà–Ascoli theorem, the family $\{S_\eta u : u \in \mathcal{C}\}$, with η fixed, is relatively compact in $C^0(\overline{\Omega})$, and consequently also in $L^p(\Omega)$. Therefore we can cover it by a finite number of balls of radius ϵ in L^p. By (2.20), it is clear that, if we double the radius, the same balls cover $\mathcal{C}$. This concludes the proof of sufficiency. Let us prove the necessity of (i) and (ii). The condition (i) trivially follows from the existence of an ϵ-net. To prove (ii), we first show that *every function u in $L^p(\Omega)$ is continuous in the mean*. Then, again by using the ϵ-net, we readily find that if $\mathcal{C}$ is relatively compact in L^p, then the functions of $\mathcal{C}$ are equicontinuous in the mean. Let u be of class $L^p(\Omega)$. By the absolute continuity of Lebesgue's integral, we can find for $\epsilon > 0$ some $\tilde{\delta} > 0$ such that

$$\left(\int_{\tilde{\Omega}} |u(x)|^p\,dx\right)^{1/p} < \epsilon/3$$

provided that meas $\tilde{\Omega} < \tilde{\delta}$. Lusin's theorem yields a closed set $A \subset \Omega$ such that meas $A >$ meas $\Omega - \frac{1}{2}\tilde{\delta}$ and u is continuous on A. Consequently we find some $\delta > 0$ such that, if

$|z| < \delta$ and $x, x + z \in A$, then

$$|u(x) - u(x+z)| \leq \frac{\epsilon}{3(\text{meas}\,\Omega)^{1/p}}.$$

For a fixed z with $|z| < \delta$ we now define

$$\begin{aligned} H_z &:= \{y : y = x + z, x \in A\}, \\ A_z &:= A \cap H_z = A \backslash (A \backslash H_z), \end{aligned}$$

and we choose δ so small that $H_z \subset \Omega$. Obviously, we have

$$\text{meas}\,A_z > \text{meas}\,\Omega - \frac{\tilde{\delta}}{2} - \left[\text{meas}\,\Omega - \left(\text{meas}\,\Omega - \frac{\tilde{\delta}}{2}\right)\right] = \text{meas}\,\Omega - \tilde{\delta};$$

hence

$$\text{meas}(\Omega - A_z) < \tilde{\delta}.$$

It follows that

$$\begin{aligned} &\left(\int_\Omega |u(x+z) - u(x)|^p\,dx\right)^{1/p} \\ &\leq \left(\int_{A_z} |u(x+z) - u(x)|^p\,dx\right)^{1/p} + \left(\int_{\Omega - A_z} |u(x+z) - u(x)|^p\,dx\right)^{1/p} \\ &\leq \frac{\epsilon(\text{meas}\,A_z)^{1/p}}{3(\text{meas}\,\Omega)^{1/p}} + \left(\int_{\Omega - A_z} |u(x+z)|^p\,dx\right)^{1/p} + \left(\int_{\Omega - A_z} |u(x)|^p\,dx\right)^{1/p} \\ &\leq 2\epsilon. \end{aligned}$$

□

Remark 4 It is easily seen from the previous proof that a similar theorem holds also if Ω is not bounded; more precisely, we have:

A subset C of $L^p(\Omega)$ is relatively compact in $L^p(\Omega)$, $p \geq 1$, if and only if

(i) *C is bounded in L^p;*

(ii) *the functions in C are equicontinuous in the mean;*

(iii) *for every $\epsilon > 0$ there is an open set $\tilde{\Omega} \subset\subset \Omega$ such that $\int_{\Omega - \tilde{\Omega}} |u|^p\,dx < \epsilon^p$ for all $u \in C$.*

As we have mentioned, the space $H^{1,1}(a, b)$ is not reflexive. In the remainder of this section we shall discuss the weak convergence in $H^{1,1}(a, b)$ in more detail since this is particularly relevant to the calculus of variations.

The key point is to understand weak convergence in $L^1(a, b)$, and the first question is *whether a bounded sequence in $L^1(a, b)$ always contains a subsequence converging in some sense.*

We need some tools from measure theory; we shall discuss measures in more detail in Section 3.3, but for a complete presentation and further details on measure theory the interested reader is referred to one of several good books dealing with this subject, e.g. Federer [99], Hewitt–Stromberg [139], Rudin [229], Yosida [293].

We recall that a *Radon measure* μ on a topological space X is an *outer measure* with the following properties:

(i) μ is regular with respect to the family of open sets, i.e.

$$\mu(E) = \inf\{\mu(A) : A \text{ open}, \ E \subset A\};$$

(ii) $\mu(K) < \infty$ for all compact sets K;
(iii) for all open sets A we have

$$\mu(A) = \sup\{\mu(K) : K \text{ compact}, \ K \subset A\}.$$

Riesz's theorem identifies Radon measures with continuous linear functionals α on $C_c^0(X)$, in the sense that every continuous linear mapping α of $C_c^0(X)$ into $\mathbf{R}$ can be represented in the form

$$\alpha(f) = \int f \, d\mu_\alpha \quad \text{for all } \varphi \in C_c^0(X) \tag{2.21}$$

where μ_α is a Radon measure, and, vice versa, by (2.21) every Radon measure μ_α defines a continuous linear functional on $C_c^0(X)$. The *total variation of* μ_α is defined as the norm of α, i.e.

$$\|\mu_\alpha\| := \sup\{\alpha(f) : f \in C_c^0(X), \ |f(x)| \le 1 \text{ for all } x \in X\}. \tag{2.22}$$

One verifies that the space of Radon measures with bounded total variation is a Banach space with the norm $\|\cdot\|$ in (2.22), and that from every sequence μ_k of Radon measures with equibounded total variation we can extract a subsequence μ_{k_i} weakly converging *in the sense of measures* to a Radon measure μ, i.e.

$$\langle \mu_{k_i}, \varphi \rangle \to \langle \mu, \varphi \rangle \quad \text{for all } \varphi \in C_c^0(X),$$

where we have used the notation

$$\langle \mu, \varphi \rangle = \int \varphi \, d\mu.$$

Every function $u \in L^1(a, b)$ (or, equivalently, the Radon measure $u\,dx$) clearly defines a continuous linear functional

$$\varphi \mapsto \int_a^b u\varphi \, dx$$

on $C_c^0(a, b)$. One easily computes that the total variation of $u\,dx$ on (a, b) is just the L^1-norm of u in (a, b). Thus we can conclude that, passing to a subsequence, *every bounded* *sequence $\{u_k\}$ in $L^1(a,b)$ converges in the sense of measures to some Radon measure μ, i.e.*

$$\int_a^b u_k\varphi\,dx \to \langle\mu,\varphi\rangle \quad \textit{for all } \varphi \in C_c^0(a,b).$$

Note that, even if μ has a density u with respect to the Lebesgue measure (which is not the case in general as we see by looking at Dirac's measure), we do not have in general that

$$\int u_k\varphi\,dx \to \int u\varphi\,dx \quad \text{for all } \varphi \in L^\infty(a,b)$$

which would mean that the u_k converge weakly in L^1; compare the footnote on page 65.

The next theorem characterizes the bounded sets of $L^1(a,b)$ which are sequentially weakly compact in $L^1(a,b)$.

Theorem 2.11 *Let Ω be a bounded open set in $\mathbb{R}^n$, and let $\{u_k\}$ be a sequence in $L^1(\Omega)$ such that*

(i) $\sup_k \|u_k\|_{L^1(\Omega)} < \infty$;

(ii) *the set functions*

$$E \mapsto \int_E |u_k|\,dx, \quad E \subset \Omega,$$

are equiabsolutely continuous, i.e. for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\int_E |u_k|\,dx < \epsilon$$

for all k, provided that meas $E < \delta$.

Then there exists a subsequence of $\{u_k\}$ which converges weakly in $L^1(\Omega)$. Moreover, if $\{u_k\}$ converges weakly in $L^1(\Omega)$, then (i) and (ii) hold true.

Proof Suppose that (i) and (ii) are true. As we have seen, there exists a subsequence $\{u_{k_i}\}$ and a Radon measure α such that

$$\int u_{k_i}\varphi\,dx \to \alpha(\varphi) \quad \text{for all } \varphi \in C_c^0(\Omega).$$

We shall now show that for such a subsequence the limit

$$\lim_{i\to\infty} \int_B u_{k_i}\,dx = \gamma(B) \tag{2.23}$$

exists for all measurable sets B contained in Ω. It is not difficult to show that this implies, by assumptions (i) and (ii), that $\gamma(B) = \alpha(B)$ for every measurable subset B of Ω, so that α turns out to be absolutely continuous with respect to the Lebesgue measure; hence, by the Radon–Nikodym theorem, it is represented by a function $u \in L^1(\Omega)$. Since step functions are dense in $L^\infty(\Omega)$, equality (2.23) implies that $\{u_{k_i}\}$ tends weakly to u in $L^1(\Omega)$. Let us now prove that limit (2.23) exists. To this end we show that $\{\int_B u_{k_i}\,dx\}$ is a Cauchy sequence. Since the characteristic function χ_B of B is measurable and bounded by 1, we can find by Lusin's theorem a sequence $\{\varphi_h\}$ in $C^0_c(\Omega)$ such that $\|\varphi_h\|_{L^\infty(\Omega)} \leq 1$ and $\varphi_h(x) \to \chi_B(x)$ for $x \in \Omega$ a.e. In correspondence to the δ in (ii), by Egoroff's theorem we find an open set $B_\delta \subset \Omega$ such that meas $B_\delta < \delta$ and that the φ_h converge uniformly on $\Omega - B_\delta$ to χ_B. Now we have

$$\begin{aligned}\left|\int_B (u_{k_i} - u_{k_j})\,dx\right| &= \left|\int_\Omega (u_{k_i} - u_{k_j})\chi_B\,dx\right| \\ &\leq \left|\int_{B_\delta} (u_{k_i} - u_{k_j})(\chi_B - \varphi_h)\,dx\right| \\ &\quad + \left|\int_\Omega (u_{k_i} - u_{k_j})\varphi_h\,dx\right| + \left|\int_{\Omega\setminus B_\delta} (u_{k_i} - u_{k_j})(\chi_B - \varphi_h)\,dx\right| \\ &\leq 2\int_{B_\delta} (|u_{k_i}| + |u_{k_j}|)\,dx + \sup_{\Omega\setminus B_\delta} |\chi_B - \varphi_h| \int_\Omega (|u_{k_i}| + |u_{k_j}|)\,dx \\ &\quad + \left|\int_\Omega (u_{k_i} - u_{k_j})\varphi_h\,dx\right|.\end{aligned}$$

For any $\epsilon > 0$ we now find some h_0 such that $\sup_{\Omega - B_\delta} |\chi_B - \varphi_h| < \epsilon$ for all $h \geq h_0$. Since $\varphi_{h_0} \in C^0_c(\Omega)$, the sequence

$$\int_\Omega u_{k_i}\varphi_{h_0}\,dx$$

is a Cauchy sequence, and therefore

$$\left|\int_\Omega (u_{k_i} - u_{k_j})\varphi_{h_0}\,dx\right| < \epsilon$$

for all i, j larger than some k_0 depending on h_0 and ϵ. In conclusion, we obtain, for a suitable constant $K > 0$,

$$\left|\int_B (u_{k_i} - u_{k_j})\,dx\right| \leq 4\epsilon + 2\epsilon K + \epsilon = (5 + 2K)\epsilon$$

for all $i, j \geq k_0$. This completes the proof of the first part of the theorem.

Suppose now that $\{u_k\}$ converges weakly in L^1 to u, and suppose for simplicity that $u = 0$ (otherwise we would consider the sequence $\{u_k - u\}$). Then (i) follows from the Banach–Steinhaus theorem. In order to verify (ii), we first prove the following claim due to Lebesgue:

If (ii) *does not hold, then there exist a positive number z, a sequence of disjoint measurable sets $E_i \subset \Omega$, and an increasing sequence of integers ν_i such that*

$$\int_{E_i} |u_{\nu_i}|\,dx \geq z \quad \text{for all } i \in \mathbb{N}.$$

By assumption, there exists an $\epsilon > 0$ such that, for all $\delta > 0$, we can find some set $F \subset \Omega$ with meas $F < \delta$, and some $\nu \in \mathbb{N}$ which may be taken arbitrarily large such that

$$\int_F |u_\nu|\,dx \geq \epsilon.$$

Since $u_\nu \in L^1(\Omega)$, for every $\sigma > 0$ there is some $\eta > 0$ such that

$$\int_B |u_\nu|\,dx < \sigma$$

holds true for all measurable sets B with meas $B < \eta$.

Choose now $\sigma = \sigma_1 = \epsilon/2$, $\delta = \delta_1 = \operatorname{meas}\Omega$; then we find $\eta_1 > 0$, $F_1 \subset \Omega$, and $\nu_1 \in \mathbb{N}$ such that

$$\operatorname{meas} F_1 < \delta_1, \qquad \int_{F_1} |u_{\nu_1}|\,dx \geq \epsilon,$$

and

$$\int_B |u_{\nu_1}|\,dx < \sigma_1 \quad \text{for all } B \text{ with meas } B < \eta_1.$$

Next we choose $\sigma_2 = \frac{1}{4}\epsilon$, $\sigma_2 = \frac{1}{2}\eta_1$; then we find η_2, $F_2 \subset \Omega$, $\nu_2 \in \mathbb{N}$ such that

$$\operatorname{meas} F_2 < \delta_2, \qquad \int_{F_2} |u_{\nu_2}|\,dx \geq \epsilon,$$

and

$$\int_B |u_{\nu_2}|\,dx < \sigma_2 \quad \text{for all } B \text{ with meas } B < \eta_2.$$

Similarly, for all $i > 2$ we choose $\sigma_i = \frac{1}{2^i}\epsilon\delta_i = \min\{\frac{\eta_1}{2^{i-1}}, \dots, \frac{\eta_{i-1}}{2}\}$; then we find $\eta_i > 0$, $F_i \subset \Omega$, $\nu_i > \nu_{i-1}$ such that

$$\operatorname{meas} F_i < \delta_i, \quad \int_{F_i} |u_{\nu_i}| \geq \epsilon, \quad \int_B |u_{\nu_i}|\,dx < \sigma_i \qquad \text{for all } B \text{ with meas } B < \eta_i.$$

Now set

$$E_i = F_i - \bigcup_{q>i} F_q.$$

We have

$$\text{meas}\left(\bigcup_{q>i} F_q\right) \le \sum_{q>i} \text{meas}\, F_q \le \sum_{q>i} \frac{\eta_i}{2^{q-i}} = \eta_i$$

and also

$$\int_{E_i} |u_{\nu_i}|\, dx = \int_{F_i - \bigcup\limits_{q>i} F_q} |u_{\nu_i}|\, dx \ge \epsilon - \frac{\epsilon}{2i} \ge \epsilon/2 \quad \text{for all } i \ge 1.$$

Since the E_i are disjoint, the claim is proved with $z = \epsilon/2$.

We now observe that by replacing E_i by $E_i \cap \{x : u_{\nu_i}(x) \ge 0\}$ or by $E_i \cap \{x : u_{\nu_i}(x) \le 0\}$ and z by $z/2$ we can modify the claim as follows. There exist $E_i \subset \Omega$ and ν_i such that

$$\left|\int_{E_i} u_{\nu_i}\, dx\right| \ge z \quad \text{for all } i \in \mathbb{N}.$$

We shall finally construct a function $\varphi \in L^\infty(\Omega)$ for which the sequence $\int u_k \varphi\, dx$ does not converge to zero. This will conclude the proof. The function φ will be defined as one on the union of suitable E_i and zero outside.

We set $E^{(1)} := E_1$ and $\nu^{(1)} := \nu_1$, and we choose $\epsilon^{(1)}$ so that

$$\int_B |u_{\nu^{(1)}}|\, dx < z/3 \quad \text{for all } B \text{ with meas } B < \epsilon^{(1)}.$$

If $\int_{E^1} u_k\, dx$ does not converge to zero, the proof is complete as we can take $\varphi = \chi_{E^{(1)}}$. Otherwise we choose $E^{(2)}$ as the first E_i for which the remaining ones, i.e. the E_j with $j > i$, have total measure less than $\epsilon^{(1)}$ (this is possible since the E_i are disjoint) and the corresponding index ν_i is such that

$$\left|\int_{E^{(1)}} u_k\, dx\right| < z/3 \quad \text{for all } k \ge \nu_i$$

(this is possible since $\int_{E^1} u_k\, dx \to 0$). Denoting by $\nu^{(1)}$ the index corresponding to $E^{(2)}$, we choose $\epsilon^{(2)} > 0$ such that

$$\int_B |u_{\nu^{(2)}}|\, dx < z/3 \quad \text{for all } B \text{ with meas } B < \epsilon^{(2)}.$$

If $\int_{E^{(1)} \cup E^{(2)}} u_k\, dx$ does not converge to zero, the proof is complete. Otherwise we proceed as before and in general we find $E^{(k)}$, $\nu^{(k)}$, $\epsilon^{(k)}$ such that

$$\text{meas} \bigcup_{i \ge k} E_i < \epsilon^{(k-1)}, \qquad \left|\int_{\bigcup\limits_{i>k} E^{(i)}} u_h\, dx\right| > z/3 \qquad \text{for all } h \ge \nu^{(k)}$$

and

$$\int_B |u_{\nu(k)}| < \varepsilon/3 \quad \text{for all } B \text{ with meas } B < \epsilon^{(k)}.$$

Finally, set

$$\varphi := \chi_{\bigcup_{k\geq 1} E^{(k)}}.$$

Then we find

$$\begin{aligned}\left|\int_\Omega u_{\nu(k)}\varphi\,dx\right| &= \left|\int_{\bigcup_{i<k} E^{(i)}} u_{\nu(k)}\,dx + \int_{E^{(k)}} u_{\nu(k)}\,dx + \int_{\bigcup_{i>k} E^{(i)}} u_{\nu(k)}\,dx\right| \\ &\geq \left|\int_{E^{(k)}} u_{\nu(k)}\,dx\right| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \geq \frac{\varepsilon}{3} \quad \text{for all } k \in \mathbb{N},\end{aligned}$$

and this gives a contradiction to the weak convergence of u_k to zero. □

Often Theorem 2.11 is referred to as the *Dunford–Pettis theorem*, although it was first proved by Lebesgue.

We may now ask: what are the weakest 'integral conditions' replacing the boundedness in L^1 to ensure sequential weak compactness in L^1? The following theorem gives a satisfactory answer to this question and collects criteria for the sequential weak compactness in L^1.

Theorem 2.12 *Let C be a subset of $L^1(\Omega)$. Then the following claims are equivalent:*

(i_1) *C is sequentially weakly compact in $L^1(\Omega)$;*

(i_2) *the functions u in C are equibounded in $L^1(\Omega)$ and the set functions*

$$E \to \int_E |u|\,dx, \quad E \subset \Omega,\ u \in C,$$

are equiabsolutely continuous;

(i_3) *the functions $u \in C$ are uniformly integrable, i.e. the integrals*

$$\int_{\{x\in\Omega:|u(x)|>c\}} |u(x)|\,dx$$

tend to zero as the positive number c tends to ∞ uniformly for $u \in C$;

(i_4) *there exists a function Θ: $(0, +\infty) \to \mathbb{R}$ (that can be taken as convex and increasing) such that*

$$\lim_{t\to\infty} \frac{\Theta(t)}{t} = \infty$$

$$\sup_{u\in C} \int_\Omega \Theta(|u|)\,dx < \infty.$$

Proof We have proved in Theorem 2.11 that (i_1) and (i_2) are equivalent. We shall now prove that (i_2) is equivalent to (i_3) and that (i_3) is equivalent to (i_4). Suppose that the functions in $\mathcal{C}$ are uniformly integrable; then for any measurable set $E \subset \Omega$ we have

$$\int_E |u|\,dx \le c \operatorname{meas} E + \int_{\{x\in\Omega:|u(x)|>c\}} |u(x)|\,dx.$$

Choosing c so large that the second term on the right-hand side is less than $\epsilon/2$, we obtain equiboundedness of $\mathcal{C}$ in L^1 taking $E = \Omega$, and equiabsolute continuity of the functions in $\mathcal{C}$ by choosing $\delta = \epsilon/(2c)$. Conversely, choosing $c = 1/\delta \sup_{\mathcal{C}} \|u\|_{L^1(\Omega)}$ where δ is the number appearing in the equiabsolute continuity condition, and taking $E := \{x \in \Omega\colon |u(x)| > c\}$, we find at once that the functions in $\mathcal{C}$ are uniformly integrable on account of the inequality

$$\int_{\{x\in\Omega:|u(x)|>c\}} |u(x)|\,dx \le \frac{1}{c}\int_\Omega |u(x)|\,dx.$$

Let us now show that the existence of Θ in (i_4) implies uniform integrability. Set

$$M := \sup_{u\in\mathcal{C}} \int_\Omega \Theta(|u(x)|)\,dx.$$

For a given $\epsilon > 0$, we choose c so large that $\Theta(t)/t \ge M/\epsilon$ for all $t \ge c$. Then we have $|u| \le \epsilon\Theta(|u|)/M$ on the set $\{x : |u(x)| > c\}$, and consequently

$$\int_{\{x:|u(x)|>c\}} |u|\,dx \le \frac{\epsilon}{M}\int_{\{x:|u(x)|>c\}} \Theta(|u|)\,dx \le \epsilon \quad \text{for all } u \in \mathcal{C}.$$

In order to prove the converse, we now construct a function Θ of the form $\int_0^t g(s)ds$ where g is an increasing function with $g(0) = 0$ tending to infinity as $s \to \infty$ and which assumes constant values on each interval $(n, n+1)$, $n \in \mathbb{N}$. For all $u \in \mathcal{C}$ we write

$$a_n(u) := \int_{\{x\in\Omega:|u(x)|>n\}} |u(x)|\,dx.$$

Since $g_0 = 0$, we have

$$\int_\Omega |u|\,dx \le g_1 \int_{\{1\le|u|<2\}} |u|\,dx + (g_1+g_2)\int_{\{2\le|u|<3\}} |u|\,dx + \cdots = \sum_{n=1}^\infty g_n a_n(u).$$

Therefore it remains to show that it is possible to choose a sequence of values g_n tending to ∞ so that the $\sum_{n=1}^\infty g_n a_n(u)$ are uniformly bounded for $u \in \mathcal{C}$. According to our assumption of uniform integrability we can choose integers $c_n \uparrow \infty$ such that

$$\int_{\{x:|u(x)|>c_n\}} |u(x)|\,dx \le 2^{-n} \quad \text{for all } u \in \mathcal{C}.$$

We have

$$\int_{\{|u|>c_n\}} |u|\,dx \geq \sum_{k=c_n}^{\infty} k \int_{\{k\leq|u|<k+1\}} |u|\,dx \leq \sum_{k=c_n}^{\infty} \int_{\{|u|>k\}} |u|\,dx = \sum_{k=c_n}^{\infty} a_k(u).$$

By the choice of c_n it follows that the numbers

$$\sum_n \sum_{k=c_n}^{\infty} a_k(f)$$

are uniformly bounded for $u \in C$, but this sum is of the form $\sum_k g_k a_k(u)$ where g_k denotes the number of integers n such that $c_n < k$. Thus the theorem is proved. □

An immediate consequence of Theorem 2.12 is the following *sequential weak compactness criterion in* $H^{1,1}(a,b)$.

Theorem 2.13 *Let* $\{u_k\}$ *be a sequence in* $H^{1,1}(a,b), a, b \in \mathbf{R}$. *Suppose that*

(i) $\sup_k \|u_k\|_{1,1} = K < \infty$;

(ii) *the set functions* $E \mapsto \int_E |Du_k|\,dx$, $E \subset (a,b)$, *are equiabsolutely continuous.*

Then there exists a subsequence $\{u_{k_i}\}$ *which converges weakly in* $H^{1,1}(a,b)$ *to some function* $u \in H^{1,1}(a,b)$. *Conversely, if* $\{u_k\}$ *converges weakly in* $H^{1,1}(a,b)$ *to some* $u \in H^{1,1}(a,b)$, *then both* (i) *and* (ii) *hold true. Finally, the conditions* (i) *and* (ii) *are both satisfied if and only if* $\{u_k\}$ *is equibounded in* $L^1(a,b)$ *and there exists a function* $\Theta : [0,\infty) \to \mathbf{R}$ *such that*

$$\frac{\Theta(t)}{t} \to \infty \quad \text{as } t \to \infty$$

and

$$\sup_k \int_a^b \Theta(|u_k'(x)|)\,dx < +\infty.$$

Proof From (i) and Theorem 2.1 we infer that (a suitable subsequence of) $\{u_k\}$ converges strongly in L^1 to some function u; actually, taking (ii) into account, a subsequence of $\{u_k\}$ converges uniformly to u. By Theorem 2.11, passing to another subsequence, $\{u_k'\}$ converges weakly in L^1 to some function $w \in L^1(a,b)$. On the other hand,

$$-\int_a^b u_k\varphi'\,dx = \int_a^b u_k'\varphi\,dx \to \int_a^b w\varphi \quad \text{for all } \varphi \subset C_c^1(a,b)$$

and

$$-\int_a^b u_k\varphi'\,dx \to \int_a^b u\varphi'\,dx;$$

hence $u' = w$ in the sense of distributions, and u_k converge to u weakly in $H^{1,1}$. Since the other claims are a trivial consequence of Theorem 2.10 the proof is complete. □

Let us point out that, modulo a passage to subsequences, *weak convergence of* $\{u_k\}$ *in* $H^{1,1}(a, b)$ *is equivalent to uniform convergence plus the equiabsolute continuity of the set functions* $E \mapsto \int_E |u'_k|\, dx$.

2.2 Absolutely continuous functions

As we have seen, a basic idea in introducing Sobolev spaces is the notion of a weak or distributional derivative. For smooth functions, the weak derivative coincides with the classical one, but, for non-smooth functions, for instance for functions which are almost everywhere differentiable in a classical sense, the two derivatives might be different. It is the aim of this section to discuss the relationship between classical derivatives almost everywhere and weak derivatives. This will be done by means of *absolutely continuous* functions, a concept introduced by Vitali and extensively used by Tonelli in the calculus of variations.

Let $u \in H^{1,1}(a, b)$. As we have seen, for all $x, y \in [a, b]$ we have

$$u(x) - u(y) = \int_y^x u'(t)\, dt.$$

Using the Lebesgue differentiation theorem, we deduce from

$$\frac{u(x+h) - u(x)}{h} = \frac{1}{h}\int_x^{x+h} u'(t)\, dt$$

that, for almost every x in $[a, b]$, $u(x)$ is differentiable in the classical sense, i.e.

$$\lim_{h\to 0} \frac{u(x+h) - u(x)}{h} = [u'(x)]$$

exists almost everywhere in $[a, b]$, and we see that the classical derivative, denoted by $[u'(x)]$, coincides a.e. with the distributional derivative, i.e.

$$u'(x) = [u'(x)] \quad \text{a.e. in } [a, b].$$

We can then collect our information as

Theorem 2.14 *Let* $u \in H^{1,1}(a, b)$. *Then, by possibly changing* u *on a set of measure zero, we have that* u *is a function of class* $C^0([a, b])$ *which is almost everywhere differentiable in the classical sense, and its classical derivative* $[u']$ *coincides almost everywhere with the weak* L^1*-derivative* u'. *Moreover, for all* $x, y \in [a, b]$, *the fundamental theorem of calculus holds:*

$$u(x) - u(y) = \int_y^x u'(t)\, dt. \tag{2.24}$$

Since functions of class $L^1(a, b)$, or even of class $L^\infty(a, b)$, which are almost everywhere differentiable in a classical sense need not be continuous, they in general do not belong to $H^{1,1}(a, b)$.

$\boxed{1}$ ***The Heaviside function.*** The function

$$H(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

is almost everywhere differentiable, indeed everywhere except at 0, with a zero classical derivative everywhere except at 0. But its restriction, for example, to $(-1, 1)$ does not belong to any Sobolev space $H^{1,p}(-1, 1)$, $p \geq 1$, and one can easily check that its distributional derivative is the Dirac measure at zero, i.e. the linear continuous functional

$$\delta_0 : \varphi \mapsto \varphi(0) \quad \text{for all } \varphi \in C_c^0(-1, 1).$$

In fact,

$$\int_{-1}^{1} H(x)\varphi'(x)\, dx = \int_0^1 \varphi'(x)\, dx = -\varphi(0) = -\langle \delta_0, \varphi \rangle.$$

Suppose now that u is of class $C^0([a, b])$ and that, moreover, it is almost everywhere differentiable in the classical sense; finally, assume that the classical derivative $[u']$ of u belongs to $L^1(a, b)$, or even to $L^\infty(a, b)$. Does this imply that u belongs to $H^{1,1}(a, b)$, i.e. that $u' = [u']$? The answer to this question is still *no*, as shown by the so-called *Cantor function* (or *Cantor–Vitali function*) which is continuous and non-decreasing in $[0, 1]$, $f(0) = 0$, $f(1) = 1$, and which has a classical derivative equal to zero a.e.[7]

$\boxed{2}$ *The Cantor–Vitali function.* Choose a strictly decreasing sequence $1 = \delta_0 > \delta_1 > \cdots > \delta_n > \cdots$ converging to zero, and set $E_0 := [0, 1]$. We now define by induction subsets E_n with $E_0 \supset E_1 \supset \cdots$, see Fig. 2.1. For $n \geq 0$, suppose E_n is constructed in such a way that E_n is the union of 2^n disjoint closed intervals, each of which is of length $2^{-n}\delta_n$. Delete a segment in the centre of each of these 2^n intervals, so that each of the remaining $2 \cdot 2^n$ intervals has the length $2^{-n-1}\delta_{n+1}$ (this is possible since $\delta_{n+1} < \delta_n$), and let E_{n+1} be the union of these 2^{n+1} intervals. Then, clearly,

$$E_0 \supset E_1 \supset E_2 \supset \ldots$$

and

$$\text{meas } E_n = \delta_n.$$

Now set

$$E := \bigcap_{n=0}^{\infty} E_n.$$

The set E is compact, and meas $E = 0$. Next we define

$$G_n(t) := \begin{cases} 1/\delta_n & \text{if } x \in E_n \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_n(x) := \int_0^x g_n(t)\, dt, \quad n = 0, 1, \ldots.$$

[7]For an example of a function with the same properties which, in addition, is *strictly increasing*, compare Hewitt and Stromberg [139] p. 278.

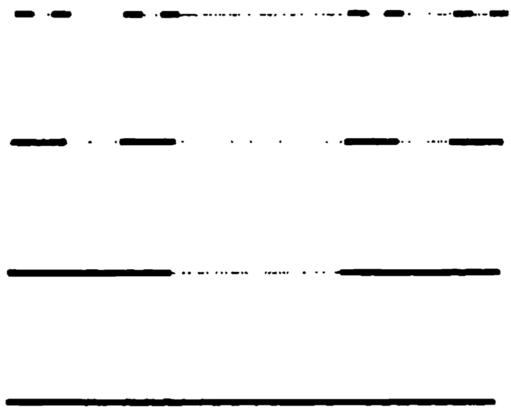

FIG. 2.1.

Clearly $f_n(0) = 0$, $f_n(1) = 1$, and each f_n is a monotone function which is constant on each segment in the complement of E_n, see Fig. 2.2. If I is one of the 2^n intervals whose union is E_n, then

$$\int_I g_n(t)\,dt = \int_I g_{n+1}(t)\,dt = 2^{-n},$$

whence

$$f_{n+1}(x) = f_n(x), \quad \text{for } x \notin E_n$$

and

$$|f_n(x) - f_{n+1}(x)| \le \int_0^1 |g_n - g_{n+1}|\,dt < 2^{-n+1}.$$

Therefore it follows that $\{f_n\}$ converges uniformly to a continuous and non-decreasing function f with $f(0) = 0$, $f(1) = 1$, and $[f'(x)] = 0$ for all $x \notin E$. Consequently, since meas $E = 0$, we have $[f'(x)] = 0$ a.e. In particular, (2.24) cannot hold. If $\delta_n = \left(\frac{2}{3}\right)^n$, the set E is called *Cantor's middle thirds*, or simply *Cantor's set*.

We shall now give a characterization of $H^{1,1}$-functions by means of classical notions, without using distributional derivatives.

Definition 2.15 (Absolutely continuous functions) *A function $f : (a, b) \to \mathbf{R}$ is said to be* absolutely continuous[8] *if, for every $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\sum_{i=1}^{N} (\beta_i - \alpha_i) < \delta \quad \textit{implies} \quad \sum_{i=1}^{N} |f(\beta_i) - f(\alpha_i)| < \epsilon \tag{2.25}$$

whenever $(\alpha_1, \beta_1), \dots, (\alpha_N, \beta_N)$ are disjoint segments in (a, b). The class of absolutely continuous functions is denoted by $AC(a, b)$.

[8] In the sense of Vitali.

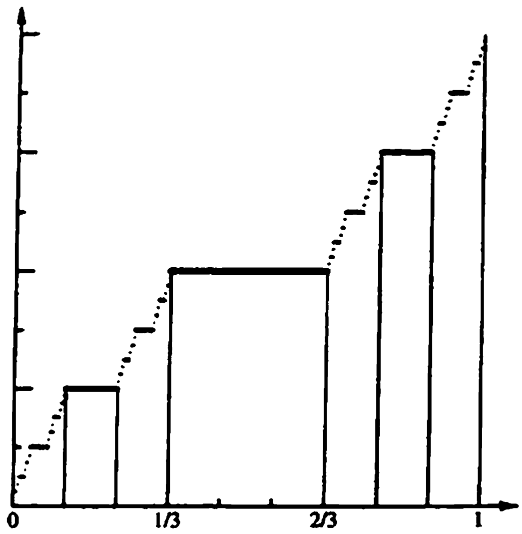

FIG. 2.2.

Clearly, every $u \in AC(a, b)$ is uniformly continuous in (a, b); therefore we can always extend it as a continuous function to the closure of (a, b).

It is also clear that every function u which is Lipschitz-continuous on (a, b) belongs to $AC(a, b)$. Indeed, if

$$|u(x) - u(y)| \leq k|x - y| \quad \text{for every } x, y \in (a, b),$$

then (2.25) is fulfilled for $\delta = \epsilon / k$.

The *total variation* of a function $u : (a, b) \to \mathbb{R}$ is defined by

$$V_a^b(u) = \sup \sum_{i=1}^{N} |u(x_i) - u(x_{i-1})| \tag{2.26}$$

where the supremum is taken over all integers N and over all choices of $\{x_i\}$ such that

$$a < x_0 < x_1 < \cdots < x_N < b.$$

Proposition 2.16 *For every $u \in AC(a, b)$ we have $V_a^b(u) < +\infty$.*

Proof Take $\epsilon = 1$ and let $\delta > 0$ be such that (2.25) holds. Now let $y_0 < y_1 < \cdots < y_m$ be points in (a, b) such that $y_0 = a$, $y_m = b$, and $y_i - y_{i-1} < \delta$. If x_j, $j = 1, \ldots, N$, are points of (a, b) with

$$a < x_0 < x_i < \cdots < x_N < b,$$

denote by t_k, $k = 1, \ldots, n$, the finite family of points of (a, b) obtained by adding the points x_j, $j = 1, \ldots, N$, to the points y_i, $i = 1, \ldots, m - 1$. We have $n \leq N + m - 1$, and the family $\{t_k\}$ generates m groups of consecutive intervals, each group covering an interval of length less than or equal to δ. Therefore by (2.25),

$$\sum_{i=1}^{N} \left|u(x_i) - u(x_{i-1})\right| \le \sum_{k=1}^{n} \left|u(t_k) - u(t_{k-1})\right| \le m.$$

Since we can choose $m \le 1 + (b-a)/\delta$, we obtain by taking the supremum over all families $\{x_j\}$

$$V_a^b(u) \le 1 + \frac{b-a}{\delta} < \infty.$$

□

The following theorem gives a 'classical' characterization of $H^{1,1}$-functions and a characterization of absolutely continuous functions.

Theorem 2.17 *We have*

$$AC(a,b) = H^{1,1}(a,b).$$

More precisely, every $u \in AC(a,b)$ has an almost everywhere classical derivative $[u']$ which belongs to $L^1(a,b)$, and viewed as an element of L^1, $[u']$ is the weak derivative of u. Conversely, every $u \in H^{1,1}(a,b)$, modulo a modification on a set of measure zero, is an absolutely continuous function. Finally, $u \in AC(a,b)$ if and only if u is almost everywhere differentiable in a classical sense, $[u']$ belongs to $L^1(a,b)$, and the fundamental theorem of calculus holds true, i.e. for all x, y in (a,b) we have

$$u(x) - u(y) = \int_y^x [u'(t)]\,dt. \tag{2.27}$$

We can therefore say that $H^{1,p}$-functions are the primitives of L^p-functions.

Before proving Theorem 2.17, it is convenient to state separately some simple remarks that are interesting by themselves.

Lemma 2.18 *Let $u \in C^1(a,b)$. Then the total variation of u is finite, i.e.*

$$V_a^b(u) < \infty$$

if and only if

$$\int_a^b |u'|\,dx < \infty.$$

In this case we have

$$V_a^b(u) = \int_a^b |u'|\,dx. \tag{2.28}$$

Proof Take a family $\{x_j\}$ in (a, b) with $a < x_0 < x_1 < \cdots < x_N < b$; then we have for every $j = 1, \ldots, N$

$$|u(x_j) - u(x_{j-1})| = \left| \int_{x_{j-1}}^{x_j} u'(x)\,dx \right| \le \int_{x_{j-1}}^{x_j} |u'|\,dx$$

so that

$$\sum_{j=1}^{N} |u(x_j) - u(x_{j-1})| \le \int_{x_0}^{x_N} |u'|\,dx \le \int_a^b |u'|\,dx.$$

Passing to the supremum over all families $\{x_j\}$ we obtain, by definition (3) of the total variation,

$$V_a^b(u) \le \int_a^b |u'|\,dx.$$

Let us now prove the opposite inequality. Fix $a' > a$ and $b' < b$; since $u \in C^1(a, b)$ we have that u' is uniformly continuous on $[a', b']$; therefore for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|u'(x) - u'(y)| < \epsilon \quad \text{whenever } x, y \in [a', b'] \text{ with } |x - y| < \delta.$$

Let $x_0 < x_1 < \cdots < x_N$ be points in $[a', b']$ such that $x_j - x_{j-1} < \delta$ for every $j = 1, \ldots, N$; then we have for every $x \in [x_{j-1}, x_j]$

$$u(x_j) - u(x_{j-1}) = \int_{x_{j-1}}^{x_j} u'(y)\,dy = \int_{x_{j-1}}^{x_j} (u'(y) - u'(x))\,dy + (x_j - x_{j-1})u'(x)$$

so that

$$\begin{aligned} |u'(x)| &\le \frac{|u(x_j) - u(x_{j-1})|}{x_j - x_{j-1}} + \frac{1}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} |u'(x) - u'(y)|\,dy \\ &\le \frac{|u(x_j) - u(x_{j-1})|}{x_j - x_{j-1}} + \epsilon. \end{aligned}$$

Integrating over $[x_{j-1}, x_j]$ and taking the sum for $j = 1, \ldots, N$ we obtain

$$\int_{x_0}^{x_N} |u'|\,dx \le \sum_{j=1}^{N} |u(x_j) - u(x_{j-1})| + \epsilon(x_N - x_0) \le V_a^b(u) + \epsilon(b - a).$$

Taking the supremum over all families $\{x_j\}$ in $[a', b']$ and then over all $a' > a$ and $b' < b$ yields

$$\int_a^b |u'|\,dx \le V_a^b(u) + \epsilon(b - a).$$

Finally, the desired inequality follows by taking $\epsilon \to 0$. □

From Lemma 2.18 and Proposition 2.16 we obtain

Lemma 2.19 *For a function $u \in C^1(a, b)$ the following conditions are equivalent:*

(i) $u \in AC(a, b)$;

(ii) $V_a^b(u) < +\infty$;

(iii) $\int_a^b |u'|\, dx < +\infty$;

(iv) *the set function*

$$E \mapsto \int_E |u'|\, dx, \quad E \text{ measurable}, \ E \subseteq (a, b),$$

is absolutely continuous, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |u'|\, dx < \epsilon \quad \textit{whenever} \operatorname{meas} E < \delta.$$

Proof The implication (i) $\Longrightarrow$ (ii) follows from Proposition 2.16. The implication (ii) $\Longrightarrow$ (iii) follows from Lemma 2.18. The implication (iii) $\Longrightarrow$ (iv) is well known from measure theory and follows from the fact that $u' \in L^1(a, b)$. The implication (iv) $\Longrightarrow$ (i) follows from the fact that, if $\sum_{i=1}^N (\beta_i - \alpha_i) < \delta$ then, setting $E = \bigcup_{i=1}^N (\alpha_i, \beta_i)$, we have

$$\sum_{i=1}^N |u(\beta_i) - u(\alpha_i)| \le \sum_{i=1}^N \int_{\alpha_i}^{\beta_i} |u'|\, dx = \int_E |u'|\, dx < \epsilon. \qquad \square$$

Lemma 2.19 now justifies the following definition.

Definition 2.20 *A family of absolutely continuous functions in (a, b) is called* equiabsolutely continuous *if, for all $\epsilon > 0$, there is a $\delta > 0$ such that* (2.25) *holds for all elements of the family.*

Remark 1 A family of equiabsolutely continuous functions is clearly equibounded and equicontinuous; thus, by the Arzelà–Ascoli theorem, it is relatively compact with respect to uniform convergence.

Consider now a function $u \in AC(a, b)$ and extend it on a larger interval $(a - \tau, b + \tau)$ setting $u(x) = u(a)$ for $x \in (a - \tau, a)$ and $u(x) = u(b)$ for $x \in (b, b + \tau)$. Clearly this extension does not change the total variation of u nor its maximum modulus. The mollified functions $\mathcal{S}_\epsilon u$, already introduced in the proof of Theorem 2.3 of Section 2.1, converge uniformly to u in (a, b), and satisfy

$$\mathcal{S}_\epsilon u(y) - \mathcal{S}_\epsilon u(x) = \frac{1}{\epsilon} \int [u(y - t) - u(x - t)] \varphi(t/\epsilon)\, dt$$

for every $x, y \in (a, b)$. Therefore

$$V_\alpha^\beta(\mathcal{S}_\epsilon u) \le V_{\alpha-\epsilon}^{\beta+\epsilon}(u) \quad \text{for every } (\alpha, \beta) \subseteq (a, b).$$

Taking Lemma 2.19 into account, this yields at once

Lemma 2.21 *Let $u \in AC(a,b)$. Then there exists a sequence of functions $\{u_k\}$ in $C^1(a,b) \cap AC(a,b)$ which are equiabsolutely continuous and converge uniformly to u. Moreover, since the set functions*

$$E \mapsto \int_E |u_k'|\,dx$$

are also equiabsolutely continuous, we can assume (compare Theorem 2.12 of Section 2.1) that u_k' converge weakly in $L^1(a,b)$ to u'.

Proof of Theorem 2.17 By Theorem 2.2 of Section 2.1 every function $u \in H^{1,1}(a,b)$ belongs to $AC(a,b)$, i.e. $H^{1,1}(a,b) \subseteq AC(a,b)$. The opposite inclusion follows immediately from Lemma 2.21. Therefore we conclude that every function $u \in AC(a,b)$ has almost everywhere a classical derivative which coincides almost everywhere with the distributional derivative, and (2.27) holds true. Finally it is clear that, if (2.27) holds, then $u \in AC(a,b)$ and this concludes the proof of Theorem 2.17. □

If u is continuous and almost everywhere differentiable with $[u'] \in L^1$, we have seen that in general u does not belong to AC; the following proposition shows instead that differentiability *everywhere* yields that $u \in AC$.

Proposition 2.22 *Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable at every point of $[a,b]$, and assume that $[f']$ belongs to $L^1(a,b)$. Then*

$$f(x) - f(a) = \int_a^x [f'(t)]\,dt \quad \textit{for all } x \in [a,b]. \tag{2.29}$$

Proof Clearly it suffices to show (2.29) for $x = b$. Since $[f'] \in L^1(a,b)$, for every $\epsilon > 0$ we can find a simple function $g = \sum_i c_i \chi_{E_i}$, with $c_i \in \mathbb{R}$ and E_i measurable, such that $g \geq f$ and

$$\int_a^b g(x)\,dx < \int_a^b [f'(x)]\,dx + \epsilon.$$

Modifying every E_i corresponding to a $c_i \geq 0$ into an open set $A_i \supseteq E_i$ and every E_i corresponding to a $c_i < 0$ into a compact set $K_i \subseteq E_i$, we may assume that g is lower semicontinuous; moreover, by possibly adding a small constant to g we have

$$[f'(x)] < g(x) \quad \text{for every } x \in [a,b]$$
$$\int_a^b g(x)\,dx < \int_a^b [f'(x)]\,dx + \epsilon.$$

Now for a fixed $\eta > 0$ consider the continuous function

$$F_\eta(x) := \int_a^x g(t)\,dt - f(x) + f(a) + \eta(x-a);$$

we have $F_\eta(a) = 0$, and we shall show that $F_\eta(x) \geq 0$ for all $x \in [a, b]$. Consider

$$x_0 = \sup\{x \in (a, b) \colon F_\eta(t) \geq 0 \ \text{ for all } t < x\};$$

it suffices to show that $x_0 = b$. Suppose on the contrary that this is not the case; then $F_\eta(x_0) = 0$ and there exists a sequence $x_n \downarrow x_0$ such that $F_\eta(x_n) < 0$, i.e.

$$0 > F_\eta(x_n) - F_\eta(x_0) = \int_{x_0}^{x_n} g(t)\,dt - f(x_n) + f(x_0) + \eta(x_n - x_0). \tag{2.30}$$

Since g is lower semicontinuous and $g(x_0) > [f'(x_0)]$, for n large enough we have

$$g(t) \geq [f'(x_0)] \quad \text{for every } t \in [x_0, x_n],$$

so that by (2.30)

$$0 > [f'(x_0)] - \frac{f(x_n) - f(x_0)}{x_n - x_0} + \eta$$

which gives a contradiction as $n \to \infty$. Since $F_\eta \geq 0$ in $[a, b]$ for all $\eta > 0$, we obtain for $\eta \to 0$ and $x = b$ that

$$f(b) - f(a) \leq \int_a^b g(t)\,dt \leq \int_a^b [f'(t)]\,dt + \epsilon$$

and, since ϵ was arbitrary, it follows that

$$f(b) - f(a) \leq \int_a^b [f'(t)]\,dt.$$

The opposite inequality follows by repeating the argument above for the function $-f$ which also satisfies the assumptions in the statement. □

We shall now derive some useful consequences of Theorem 2.17.

Since every Lipschitz function is clearly an AC function, we infer from Theorem 2.17 that

Corollary 2.23 *Every Lipschitz function u in (a, b) has a classical derivative $[u']$ almost everywhere in (a, b) as well as a distributional derivative u', and both derivatives are equally viewed as functions in $L^\infty(a, b)$. In particular, u belongs to all Sobolev spaces $H^{1,p}(a, b)$.*

Our next theorem shows that the standard chain rule continues to hold in Sobolev spaces.

Theorem 2.24 (Chain rule) *Let θ be a Lipschitz function on $\mathbb{R}$, and let u be of class $H^{1,p}(a,b)$ for some $p \geq 1$. Then the function $\theta(u) = \theta \circ u$ belongs to $H^{1,p}(a,b)$ and*

$$D\theta(u(x)) = \theta'(u(x))Du(x). \tag{2.31}$$

The proof of this theorem relies on the following lemma.

Lemma 2.25 (De La Vallée Poussin) *Let $g \in AC(I)$, and suppose that, for a set $E \subset I$, the Lebesgue measure of $g(E)$ is zero. Then we have $g' = 0$ almost everywhere in E.*

Proof Let B be the subset of E where $|g'(t)| > 0$; define

$$B_n := \left\{ t \in B \colon |g(s) - g(t)| > \frac{|s-t|}{n} \text{ for all } |s-t| < \frac{1}{n} \right\}.$$

We have $B = \bigcup_n B_n$. Fix n and for any interval J of length less than $1/n$ consider the set

$$A := J \cap B_n.$$

We shall now show that meas $A = 0$, which will imply that meas $B_n = 0$ and, in turn, that meas $B = 0$. Since meas $g(A) = 0$, for each $\epsilon > 0$ we can choose a sequence of intervals I_k such that

$$g(A) \subset \bigcup_k I_k \quad \text{and} \quad \sum_k \text{meas } I_k < \epsilon.$$

Now set

$$A_k := g^{-1}(I_k) \cap A.$$

Since $\bigcup_k A_k$ certainly covers A, we have

$$\text{meas}^*(A) \leq \sum \text{meas}^*(A_k) \leq \sum_k \sup_{s,t \in A_k} |s-t|$$

where meas* denotes the Lebesgue outer measure.

Because $A_k \subset J \cap B_n$, we know that

$$\sup_{t,s \in A_k} |s-t| \leq n \sup_{s,t \in A_k} |g(s) - g(t)|,$$

and, since $g(A_k) \subset I_k$ and I_k is an interval, it follows that

$$\sup_{t,s \in A_k} |g(s) - g(t)| \leq \text{meas } I_k.$$

Thus we conclude:

$$\text{meas}^*(A) \leq n \sum_k \text{meas } I_k \leq n\epsilon.$$

However, n is fixed while ϵ can be chosen arbitrarily small; hence meas $A = 0$. □

An immediate corollary of Lemma 2.25 is the following result that could also be proved directly.

Corollary 2.26 *If g is absolutely continuous, then $g' = 0$ almost everywhere in any subset where g is constant.*

Proof of Theorem 2.24 Set

$$Z := \{x : \theta' \text{ does not exist at } x\}$$
$$S := u^{-1}(z)$$

and notice that $\operatorname{meas} u(S) = 0$ and $\operatorname{meas}(\theta(u(s))) = 0$, since the Lipschitz image of a set of measure zero has measure zero. Clearly $\theta(u(x))$ is absolutely continuous, so we have only to show that its derivative is in L^p and that (2.31) holds. We have

$$D\theta(u)(x) = \theta'(u(x))Du(x)$$

for every $x \in (a, b)\backslash S$ where u is differentiable. On the other hand, as a consequence of Lemma 2.25, it follows that

$$\begin{aligned} D\theta(u)(x) &= 0 \quad \text{a.e. in } S, \\ Du(x) &= 0 \quad \text{a.e. in } S, \end{aligned}$$

whence we have

$$D\theta(u)(x) = \theta'(u(x))Du(x) \quad \text{a.e. in } (a, b),$$

and this concludes the proof. □

2.3 Functions of bounded variation

A more refined analysis of absolutely continuous functions can be carried out in terms of the so-called *functions with bounded variation* introduced by Jordan.

To each function $f : (a, b) \to \mathbf{R}$, not necessarily continuous, we associate its *total variation function* defined, for $x \in (a, b)$, by

$$T_f(x) := \sup \sum_{j=1}^{N} |f(x_j) - f(x_{j-1})|$$

where the supremum is taken over all N and over all choices of $\{x_j\}$ such that

$$a < x_0 < \cdots < x_N < x.$$

Clearly, for $x < y$, we have

$$0 \le T_f(x) \le T_f(y) \le \infty, \tag{2.32}$$

whence

$$V_a^b(f) = \lim_{x \to b-0} T_f(x)$$

exists; if it is finite we say that f is *a function of bounded variation*, and we shall denote by $BV(a, b)$ the class of all such f.

Since f is not a priori continuous, the previous definition has the disadvantage that, if we modify f just at one point, then both $T_f(x)$ and $V_a^b(f)$ change. Hence it would be convenient to find 'normalized representatives' of functions of bounded variation. To this end we first observe that

(a) *If* $f \in BV(a, b)$ *and* $x < y$, *then*

$$|f(y) - f(x)| \leq T_f(y) - T_f(x).$$

In fact, for all $\epsilon > 0$, there are points $a < x_0 < \cdots < x_N = x$ such that

$$\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| > T_f(x) - \epsilon,$$

whence

$$\begin{aligned} T_f(y) &\geq |f(y) - f(x)| + \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \\ &> |f(x) - f(y)| + T_f(x) - \epsilon, \end{aligned}$$

from which the assertion follows at once.

Statement (a) in particular says that $\{f(x_i)\}$ is a Cauchy sequence if $\{T_f(x_i)\}$ is a Cauchy sequence. On the other hand, since T_f is a monotone function, and monotone functions have right and left limits at all points, and have at most countably many discontinuities, we conclude that the same is true for f. Therefore we can define

$$c := \lim_{t \to a+0} f(t) \quad \text{and} \quad g(x) := f(x - 0) - c.$$

Clearly, $g(x)$ is a left-continuous function and

$$V_a^b(g) \leq V_a^b(f).$$

In conclusion we have shown

(b) *Let* $f \in BV(a, b)$. *Then* $f(x - 0)$ *exists at every point of* $(a, b]$ *and* $f(x + 0)$ *exists at every point of* $[a, b)$; *the set of points at which* f *is discontinuous is at most countable, and there is a unique constant* c *and a unique*[9] *function* g *with bounded variation, which is left-continuous and satisfies*

$$\lim_{x \to a+0} g = 0,$$

so that

$$f(x) = c + g(x)$$

at all points of continuity of f.

[9]The uniqueness follows from the fact that, if two left-continuous functions coincide on a dense set, then they are equal.

Using (b) we *normalize* f *to* $c + g(x)$. The class of normalized bounded variation functions is denoted by $NBV(a, b)$.

It is not difficult to show that

(c) *If* $f \in NBV(a, b)$, *then* $T_f(x) \in NBV(a, b)$.

We shall see that it is possible to associate to every function $u \in NBV(a, b)$ a Borel measure; therefore we recall here some properties of measures that will be used later, referring to one of the books on measure theory for any further detail.

A Borel $\mathbf{R}^n$-valued measure μ on (a, b) will be simply a countably additive set function $\mu : \mathcal{B} \to \mathbf{R}^n$ where $\mathcal{B}$ denotes the family of all Borel subsets of (a, b). For every Borel $\mathbf{R}^n$-valued measure μ on (a, b) and every $B \in \mathcal{B}$ the *total variation* of μ on B is defined by

$$|\mu|(B) = \sup\left\{ \sum_{j=1}^{\infty} |\mu(B_j)| : B_j \in \mathcal{B} \text{ pairwise disjoint, } \bigcup_{j=1}^{\infty} B_j \subseteq B \right\}; \tag{2.33}$$

in this way the set function $B \mapsto |\mu|(B)$ turns out to be a non-negative Borel measure on (a, b), which will be denoted by $|\mu|$.

The space $\mathcal{M}(a, b; \mathbf{R}^n)$ of all Borel $\mathbf{R}^n$-valued measures on (a, b) with finite total variation on (a, b) can be endowed with the norm

$$\|\mu\|_{\mathcal{M}} := |\mu|(a, b) \tag{2.34}$$

which makes it a Banach space.

Another equivalent way to construct the space $\mathcal{M}(a, b; \mathbf{R}^n)$ is the following. Consider the space $C_0(a, b; \mathbf{R}^n)$ of all uniformly continuous functions on (a, b) vanishing at a and at b, endowed with the sup norm; it is a separable Banach space. The space $\mathcal{M}(a, b; \mathbf{R}^n)$ can be equivalently defined as the dual space of $C_0(a, b; \mathbf{R}^n)$, with the duality

$$\langle \mu, u \rangle := \int_a^b u \, d\mu. \tag{2.35}$$

In this way the norm $\|\mu\|_{\mathcal{M}}$ will be the usual dual norm

$$\|\mu\|_{\mathcal{M}} = \sup\{\langle \mu, u \rangle : \bar{u} \in C_0(a, b; \mathbf{R}^n), \|u\| \le 1\},$$

and $\mathcal{M}(a, b; \mathbf{R}^n)$ can be endowed with the weak* topology induced by the duality (2.35). In particular, a sequence (μ_h) in $\mathcal{M}(a, b; \mathbf{R}^n)$ is said to converge weakly* to a measure $\mu \in \mathcal{M}(a, b; \mathbf{R}^n)$ if and only if

$$\lim_{h \to \infty} \langle \mu_h, u \rangle = \langle \mu, u \rangle$$

for every $u \in C_0(a, b; \mathbf{R}^n)$. By the Alaoglu compactness theorem we have that from every bounded sequence $\{\mu_h\}$ in $\mathcal{M}(a, b; \mathbf{R}^n)$ there exists a subsequence $\{\mu_{h_k}\}$ which converges weakly* to some measure $\mu \in \mathcal{M}(a, b; \mathbf{R}^n)$.

Proposition 2.27 *Let $\mu_h \to \mu$ weakly* in $\mathcal{M}(a,b;\mathbf{R}^n)$. Then*

$$\lim_{h\to\infty} \mu_h(A) = \mu(A)$$

for every Borel set A that is relatively compact in (a,b) and satisfies $|\mu|(\partial A) = 0$.

Proof By arguing componentwise we may assume that $n = 1$, and by decomposing every $\mathbf{R}$-valued measure ν as $\nu^+ - \nu^-$, where the non-negative measures ν^+ and ν^- are defined by

$$\nu^+(E) := \sup\{\nu(B) \colon B \in \mathcal{B}, B \subseteq E\}$$
$$\nu^-(E) := -\inf\{\nu(B) \colon B \in \mathcal{B}, B \subseteq E\}$$

we may assume that all μ_h and μ are non-negative. For every continuous function φ with compact support in A and with values in $[0, 1]$ we have

$$\liminf_{h\to\infty} \mu_h(A) \geq \liminf_{h\to\infty} \int \varphi\, d\mu_h = \int \varphi\, d\mu$$

so that, taking the supremum with respect to φ, we get

$$\liminf_{h\to\infty} \mu_h(A) \geq \mu(\mathring{A}) \tag{a}$$

where $\mathring{A}$ denotes the interior of A. Analogously, for every continuous function ψ with compact support in (a,b), with values in $[0, 1]$, and such that $\psi = 1$ on A we have

$$\limsup_{h\to\infty} \mu_h(A) \leq \limsup_{h\to\infty} \int \psi\, d\mu_h = \int \psi\, d\mu$$

so that, taking the infimum with respect to ψ, we obtain

$$\limsup_{h\to\infty} \mu_h(A) \leq \mu(\overline{A}) \tag{b}$$

where $\overline{A}$ denotes the closure of A. The conclusion now follows from (a) and (b) above. □

Definition 2.28 *Given a non-negative finite Borel measure ν on (a,b) we say that $\mu \in \mathcal{M}(a,b;\mathbf{R}^n)$ is*

(i) absolutely continuous with respect to ν *(and we write $\mu \ll \nu$) if*

$$|\mu|(B) = 0 \quad \textit{whenever } \nu(B) = 0;$$

(ii) singular with respect to ν (*and we write* $\mu \perp \nu$) *if there exists* $B \in \mathcal{B}$ *with* $\nu(B) = 0$ *such that* $|\mu|((a,b)\setminus B) = 0$.

For instance, if ν is the Lebesgue measure, then for every function $u \in L^1(a,b;\mathbf{R}^n)$ the measure

$$\mu(B) := \int_B u\,dx$$

is absolutely continuous with respect to the Lebesgue measure, while for every $x_0 \in (a,b)$ the *Dirac measure* δ_{x_0} defined by

$$\delta_{x_0}(B) := \begin{cases} 1 & \text{if } x_0 \in B \\ 0 & \text{if } x_0 \notin B \end{cases}$$

is singular with respect to the Lebesgue measure. In the following, given $u \in L^1_\nu(a,b;\mathbf{R}^n)$, we shall denote by $u\nu$ (or simply by u when no confusion is possible) the measure of $\mathcal{M}(a,b;\mathbf{R}^n)$ defined by

$$(u\nu)(B) := \int_B u\,d\nu \quad (B \in \mathcal{B});$$

moreover, if $u : (a,b) \to \mathbf{R}$ is a bounded Borel function and $\mu \in \mathcal{M}(a,b;\mathbf{R}^n)$, we shall denote by $u\mu$ the measure of $\mathcal{M}(a,b;\mathbf{R}^n)$ defined by

$$(u\mu)(B) := \int_B u\,d\mu \quad (B \in \mathcal{B}).$$

It is well known that every measure $\mu \in \mathcal{M}(a,b;\mathbf{R}^n)$ which is absolutely continuous with respect to ν is representable in the form $\mu = u\nu$ for a suitable function $u \in L^1_\nu(a,b;\mathbf{R}^n)$; moreover, according to the Radon–Nikodym theorem, the function u can be obtained by the formula

$$u(x) = \lim_{\epsilon\to+0} \frac{\mu(x-\epsilon,x+\epsilon)}{\nu(x-\epsilon,x+\epsilon)} \quad \text{for } \nu - \text{almost all } x \in (a,b).$$

The following Lebesgue–Nikodym decomposition result for measures of $\mathcal{M}(a,b;\mathbf{R}^n)$ holds true.

Theorem 2.29 *For every* $\mu \in \mathcal{M}(a,b;\mathbf{R}^n)$ *there exists a unique function* $u \in L^1_\nu(a,b;\mathbf{R}^n)$ *and a unique measure* $\mu^s \in \mathcal{M}(a,b;\mathbf{R}^n)$ *such that*

(i) $\mu = u\nu + \mu^s$;

(ii) μ^s *is singular with respect to* ν.

The measures $u\nu$ *and* μ^s *are called* absolutely continuous part *and* singular part of μ *with respect to* ν, *and the function* u *is often denoted by* $d\mu/d\nu$.

The following theorem identifies functions in $NBV(a,b)$ with measures of $\mathcal{M}(a,b)$.

Theorem 2.30 *We have*

(i) *If μ is a Borel measure of $\mathcal{M}(a, b)$ and if we set*

$$f(x) := \mu((a, x)), \quad x \in (a, b), \tag{2.36}$$

then $f \in NBV(a, b)$.

(ii) *Conversely, to every $f \in NBV(a, b)$ there corresponds a unique Borel measure $\mu \in \mathcal{M}(a, b)$ such that* (2.36) *holds; moreover for this μ we have*

$$T_f(x) = |\mu|((a, x)).$$

(iii) *Finally, if* (2.36) *holds, then f is continuous precisely at those points x where $\mu(\{x\}) = 0$.*

Proof (i) From the properties of Borel measures we infer: if $\{x_n\} \uparrow x$, then

$$\mu((a, x)) = \lim_{n\to\infty} \mu((a, x_n));$$

thus $f(x_n) \to f(x)$. If $\{x_n\} \downarrow a$, then

$$\mu((a, x_n)) \to 0,$$

and therefore $f(x_n) \to 0$. Finally, for $a < x_0 < \cdots < x_N = x$,

$$\begin{aligned}\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| &= \sum_{i=1}^{N} |\mu([x_{i-1}, x_i))| \\ &= \sum_{i=1}^{N} |\mu([x_{i-1}, x_i))| \leq |\mu|(a, x).\end{aligned}$$

whence

$$T_f(x) \leq |\mu|(a, x) \quad \text{for every } x \in (a, b). \tag{2.37}$$

(ii) First we notice that f can always be written as the difference of two non-decreasing functions in $NBV(a, b)$. In fact

$$f = \tfrac{1}{2}(T_f + f) - \tfrac{1}{2}(T_f - f)$$

and, from the properties of the function T_f, it follows that the functions

$$u := \tfrac{1}{2}(T_f + f), \qquad v := \tfrac{1}{2}(T_f - f)$$

are in $NBV(a, b)$ and non-decreasing. Now with a non-decreasing function u (and v) we associate a Borel measure μ_u (and μ_v) in the following way. To each $x \in (a, b)$ we consider the following set E_x:

$$E_x := \begin{cases} \{u(x)\} & \text{if } u \text{ is continuous at } x \\ [u(x), u(x^+)] & \text{if } u(x^+) > u(x), \end{cases}$$

and with a set $E \subset (a, b)$ we associate the set $\bigcup_{x \in E} E_x$; then we define

$$\mu_u(E) := \mathcal{L}^1\left(\bigcup_{x \in E} E_x\right)$$

where $\mathcal{L}^1$ is the Lebesgue one-dimensional measure. Now set

$$\mu := \mu_u - \mu_v,$$

and denote by λ the measure μ_{T_f} associated in the same way with T_f. It is not difficult to show that μ is a Borel measure and that for all intervals $[\alpha, \beta)$

$$\mu([\alpha, \beta)) = f(\beta) - f(\alpha);$$

moreover, we have

$$\lambda([\alpha, \beta)) = T_f(\beta) - T_f(\alpha).$$

Hence we get

$$|\mu(E)| \le \lambda(E),$$

i.e.

$$|\mu|(a, x) \le \lambda((a, x)) \le T_f(x)$$

which together with (2.37) concludes the proof of (ii). Finally (iii) follows at once. □

Let us now go back for a while to absolutely continuous functions. From their definition, it is clear that, if $u \in AC(a, b)$, then $u \in BV(a, b)$, and subtracting the value of u at a, which is well defined, we obtain $u - u(a) \in NBV(a, b)$. Consider now the measure μ which is associated to u by the definition in the proof of Theorem 2.30. It is readily seen that the definition of absolute continuity of u amounts exactly to the condition that the measure μ be absolutely continuous with respect to Lebesgue's measure $\mathcal{L}^1$. Therefore we easily conclude

Theorem 2.31 *Let $f \in NBV(a, b)$ and let μ be the measure associated to f in Theorem 2.30. Then μ is absolutely continuous with respect to Lebesgue's measure if and only if f is absolutely continuous.*

Now we can translate the results concerning the differentiation of measures into results about the differentiation of functions.

Let $g \in L^1(a,b)$ and set

$$f(x) := \int_a^x g(t)\,dt.$$

If we consider the measure $\mu = g\mathcal{L}^1$, we have

$$f(x) = \mu((a,x)) \quad \text{and} \quad \mu \ll \mathcal{L}^1.$$

From Theorem 2.30, we then deduce that $f \in NBV(a,b)$, and from Theorem 2.31 it follows that f is absolutely continuous. Moreover, the Radon–Nikodym theorem yields that $[f'(x)]$ exists almost everywhere and

$$[f'(x)] = \frac{d\mu}{d\mathcal{L}^1}(x) = g(x) \quad \text{a.e. on } (a,b).$$

More generally, for a measure μ associated with a function f in $NBV(a,b)$, the Lebesgue–Nikodym decomposition of Theorem 2.29 gives

$$\mu(E) = \int_E \frac{d\mu}{d\mathcal{L}^1}(t)\,dt + \mu^s(E);$$

hence, if we set

$$f_s(x) := \mu^s((a,x)) \quad \text{for } x \in (a,b),$$

we conclude that

$$[f_s'(x)] = 0 \qquad \text{a.e. on } (a,b)$$

$$[f'(x)] = \frac{d\mu}{d\mathcal{L}^1}(x) \quad \text{a.e. on } (a,b)$$

and

$$f(x) = f_s(x) + \int_a^x [f'(t)]\,dt.$$

Finally, $f_s = 0$ if and only if f is absolutely continuous. The function f_s is called the *singular part of* f. For Cantor's function f we just have $f = f_s$.

We can collect our previous statements as follows.

Theorem 2.32 *We have:*

(i) *If $g \in L^1(a,b)$ and if, for $x \in (a,b)$,*

$$f(x) := \int_a^x g(t)\,dt,$$

then f belongs to $NBV(a,b)$, is absolutely continuous, and

$$[f'(x)] = g(x) \quad a.e. \text{ in } (a,b).$$

(ii) *If $f \in NBV(a,b)$, then f is a.e. differentiable, $[f'] \in L^1(a,b)$, and there exists a function $f_s \in NBV(a,b)$ with $[f_s'(x)] = 0$ a.e. such that*

$$f(x) = f_s(x) + \int_a^x [f'(t)]\,dt, \quad x \in (a,b).$$

The function f_s is zero if and only if f is absolutely continuous.

(iii) *If $f \in BV(a,b)$, then f is differentiable a.e. and $[f'] \in L^1(a,b)$.*

(iv) *The relation*

$$f(x) - f(a) = \int_a^x [f'(t)]\,dt \quad \text{with } x \in (a,b),\ [f'] \in L^1(a,b),$$

holds if and only if f is absolutely continuous.

From the functional analytic point of view it is convenient to consider bounded variation functions as defined only almost everywhere, and to identify a bounded variation function u with its equivalence class of all functions which coincide with u almost everywhere. In this way, for every function $u \in BV(a,b)$ there exists a function $\tilde{u} \in NBV(a,b)$ such that

$$u(x) = \tilde{u}(x) \quad \text{for } x \in (a,b) \text{ a.e.}$$

Moreover, from Theorem 2.32 it follows that a function u belongs to $BV(a,b)$ if and only if its distributional derivative u' defined by

$$\langle u', \varphi \rangle = -\int_a^b u\varphi'\,dx \quad \text{for every } \varphi \in C_c^\infty(a,b)$$

is a measure of $\mathcal{M}(a,b)$. In this way $BV(a,b)$ can be endowed with the norm

$$\|u\|_{BV} := \|u\|_{L^1} + \|u'\|_{\mathcal{M}}$$

which makes it a Banach space. We now summarize the most important properties of the space $BV(a,b)$, referring for instance to the books by Federer [99], Giaquinta *et al.* [120], Giusti [122], Massari–Miranda [180], and Ziemer [295] for a more systematic approach.

Proposition 2.33 *The space $BV(a,b)$ is not separable.*

Proof Consider for every $x \in (a,b)$ the function

$$H_x(t) = \begin{cases} 0 & \text{if } t \le x \\ 1 & \text{if } t > x. \end{cases}$$

We have $H_x \in BV(a,b)$, and the distributional derivative H_x' coincides with the Dirac measure δ_x at x. If $BV(a,b)$ were separable, its subspace

$$\mathcal{H} = \{H_x : x \in (a,b)\}$$

would be separable too for the induced norm, but this is excluded by the fact that

$$\begin{aligned}\|H_x - H_y\|_{BV} &\geq \|H_x' - H_y'\|_{\mathcal{M}} \\ &= \|\delta_x - \delta_y\|_{\mathcal{M}} = \|\delta_x\|_{\mathcal{M}} + \|\delta_y\|_{\mathcal{M}} = 2\end{aligned}$$

whenever $x \neq y$. □

The functions H_x above are not absolutely continuous; therefore $H^{1,1}(a,b) \subset BV(a,b)$ is a strict inclusion. Moreover, by a proof similar to that of the proposition above it can be shown that $H^{1,1}(a,b)$ is not dense in $BV(a,b)$. Indeed, if u is an absolutely continuous function and $x \in (a,b)$ we have

$$\begin{aligned}\|u - H_x\|_{BV} &\geq \|u' - H_x'\|_{\mathcal{M}} \\ &= \|u' - \delta_x\|_{\mathcal{M}} = \|u'\|_{L^1} + \|\delta_x\|_{\mathcal{M}} \geq 1\end{aligned}$$

which proves that H_x cannot be approximated in the BV-norm by absolutely continuous functions.

However, the following approximation result by smooth functions holds true.

Proposition 2.34 *For every $u \in BV(a,b)$ there exists a sequence $\{u_n\}$ of functions in $C^\infty(\mathbf{R})$ converging to u strongly in $L^1(a,b)$ and such that*

$$\lim_{n\to\infty} \int_a^b |u_n'|\,dx = |u'|(a,b).$$

Proof Fix $u \in BV(a,b)$ and extend it to all of $\mathbf{R}$ by setting

$$u(x) := \begin{cases} u(a+0) & \text{if } x \leq a \\ u(b-0) & \text{if } x \geq b. \end{cases}$$

If $\{\rho_n\}$ is a sequence of mollifiers, consider

$$u_n = u * \rho_n;$$

then the functions u_n are in $C^\infty(\mathbf{R})$ and converge to u in $L^1(a,b)$. Moreover, by the properties of the convolution operator, we have

$$\int_a^b |u_n'|\,dx \leq \|u_n'\|_{\mathcal{M}(\mathbf{R})} \leq \|u'\|_{\mathcal{M}(\mathbf{R})} = |u'|(a,b).$$

The opposite inequality

$$|u'|(a,b) \leq \liminf_{n\to\infty} \int_a^b |u_n'|\,dx$$

follows from the fact that u_n' tends to u' weakly* in $\mathcal{M}(a,b)$, and from the lower semicontinuity of the norm in $\mathcal{M}(a,b)$ with respect to the weak* convergence. □

Corollary 2.35 *By the proposition above it follows that the function*

$$d(u, v) := \int_a^b |u - v| \, dx + \left| \|u'\|_{\mathcal{M}(a,b)} - \|v'\|_{\mathcal{M}(a,b)} \right|$$

is a distance on $BV(a, b)$ with respect to which $C^\infty(\mathbf{R})$ is dense.

Furthermore, BV is embedded in L^∞; more precisely we have

Proposition 2.36 *Every function $u \in BV(a, b)$ is in $L^\infty(a, b)$ and*

$$\|u\|_{L^\infty(a,b)} \le \frac{1}{b-a} \|u\|_{L^1(a,b)} + \|u'\|_{\mathcal{M}(a,b)}.$$

Proof Let $u \in BV(a, b)$; by modifying u on a Lebesgue null set we may assume that $u \in NBV(a, b)$ so that, if u' is the distributional derivative of u, we have

$$u(y) - u(x) = \int_{[x,y)} du'$$

for every $x, y \in (a, b)$. Therefore

$$|u(y)| \le |u(x)| + |u'|([x, y)) \le |u(x)| + \|u'\|_{\mathcal{M}(a,b)},$$

and an integration with respect to x gives

$$|u(y)| \le \frac{1}{b-a} \int_a^b |u| \, dx + \|u'\|_{\mathcal{M}(a,b)}.$$

Taking the essential supremum with respect to y yields the desired inequality. □

Proposition 2.37 *For every bounded sequence in $BV(a, b)$ there exists a subsequence converging almost everywhere on (a, b), i.e. everywhere except for a Lebesgue null set.*

Proof Let $\{u_n\}$ be a sequence bounded in the $BV(a, b)$ norm; then by Proposition 2.36 $\{u_n\}$ is bounded in $L^\infty(a, b)$ and also the norms $\|u_n'\|_{\mathcal{M}(a,b)}$ are bounded. Therefore we may extract a subsequence (which for simplicity we still denote by $\{u_n\}$) such that

(i) $u_n' \to \mu$ weakly* in $\mathcal{M}(a, b)$, for a suitable measure $\mu \in \mathcal{M}(a, b)$;

(ii) $u_n(x_0) \to \xi$ for a suitable $\xi \in \mathbf{R}$ and an $x_0 \in (a, b)$ such that $u_n'(\{x_0\}) = \mu(\{x_0\}) = 0$.

Now set for every $x \in (a, b)$

$$u(x) := \begin{cases} \xi + \mu((x_0, x)) & \text{if } x \ge x_0 \\ \xi - \mu((x, x_0)) & \text{if } x < x_0. \end{cases}$$

Since every u_n can be written for almost every $x \in (a, b)$ as

$$u_n(x) = \begin{cases} u_n(x_0) + u_n'((x_0, x)) & \text{if } x \geq x_0 \\ u_n(x_0) + u_n'((x, x_0)) & \text{if } x < x_0, \end{cases}$$

by (i) and (ii) above and by Proposition 2.27 we obtain that

$$\lim_{n \to \infty} u_n(x) = u(x)$$

for $x \in (a, b)$ a.e. with $\mu(\{x\}) = 0$, i.e. Lebesgue almost everywhere. □

Corollary 2.38 *By Propositions 2.36 and 2.37 it follows that for every $p < \infty$ the embedding $BV(a, b) \to L^p(a, b)$ is compact. Moreover, the convergence on $BV(a, b)$ (often called BV-weak* convergence) defined by*

$$\begin{cases} u_n \to u & \textit{strongly in } L^1(a, b) \\ u_n' \to u' & \textit{weakly* in } \mathcal{M}(a, b) \end{cases}$$

is such that norm bounded sequences in BV are BV-weakly compact.*

We conclude this section with two inequalities of Poincaré type.

Proposition 2.39 *For every $u \in NBV(a, b)$ we have*

$$\int_a^b |u|\, dx \leq (b - a)\left[|u'|(a, b) + |u(a^+)|\right].$$

Proof From the equality

$$u(x) - u(a^+) = \int_{(a,x)} du' \quad \text{for every } x \in (a, b)$$

we obtain

$$|u(x)| \leq |u(a + 0)| + |u'|(a, b) \quad \text{for every } x \in (a, b).$$

Integrating on (a, b) gives the desired inequality. □

Proposition 2.40 *For every $u \in BV(a, b)$ we have*

$$\int_a^b |u - \overline{u}|\, dx \leq \frac{b - a}{2} |u'|(a, b)$$

where $\overline{u}$ denotes the average of u on (a, b),

$$\overline{u} = \frac{1}{b - a} \int_a^b u\, dx.$$

Proof Let us first consider the case of a smooth function u; then for every $x, y \in (a, b)$ we have

$$|u(x) - u(y)| = \left| \int_x^y u' \, dt \right| \le \begin{cases} \int_x^y |u'| \, dt & \text{if } x \le y \\ \int_y^x |u'| \, dt & \text{if } y \le x \end{cases}$$

so that

$$\begin{aligned} \int_a^b |u - \bar{u}| \, dx &\le \frac{1}{b-a} \iint_{(a,b)\times(a,b)} |u(x) - u(y)| \, dx \, dy \\ &\le \frac{2}{b-a} \iint_{\{x \le y\}} \left(\int_x^y |u'| \, dt \right) dx \, dy. \end{aligned}$$

Setting

$$F(x) = \int_a^x |u'| \, dt$$

the last term becomes

$$\frac{2}{b-a} \int_a^b \left(\int_a^y (F(y) - F(x)) \, dx \right) dy = \frac{2}{b-a} \int_a^b \left(\int_a^b (x - a)|u'|(x) \, dx \right) dy$$

where the last equality has been obtained by integrating by parts. A further integration by parts yields for this term

$$\frac{2}{b-a} \int_a^b (y - a)(b - y)|u'|(y) \, dy$$

and, since $(y - a)(b - y) \le (b - a)^2/4$, we finally obtain

$$\int_a^b |u - \bar{u}| \, dx \le \frac{b-a}{2} \int_a^b |u'| \, dx.$$

Once the inequality is obtained for smooth functions we may pass to all $u \in BV(a, b)$ by using the density result of Proposition 2.34. □

Remark 1 The inequalities of Propositions 2.39 and 2.40 cannot be improved; indeed, taking (with $(a, b) = (0, 1)$) for every $\epsilon > 0$

$$u_\epsilon(x) := \begin{cases} x/\epsilon & \text{if } x \le \epsilon \\ 1 & \text{if } x > \epsilon \end{cases}$$

we obtain

$$\int_0^1 u_\epsilon \, dx = 1 - \frac{\epsilon}{2}, \qquad \int_0^1 u'_\epsilon \, dx = 1$$

which proves, as $\epsilon \to 0$, the sharpness of the inequality of Proposition 2.39. In order to prove the sharpness of the inequality of Proposition 2.40 it is enough to consider the function

$$u(x) = \begin{cases} -1 & \text{if } x \le 1/2 \\ 1 & \text{if } x > 1/2 \end{cases}$$

(again with $(a, b) = (0, 1)$) which gives

$$\overline{u} = 0, \qquad \int_0^1 |u|\, dx = 1, \qquad |u'|(0, 1) = 2.$$

3

SEMICONTINUITY AND EXISTENCE RESULTS

According to our previous considerations, in order to apply the direct methods to integral functionals $\mathcal{F}(u)$ on Sobolev spaces or in the class of absolutely continuous functions, the key point is a semicontinuity theorem with respect to weak convergence in Sobolev spaces. In the following sections we prove some fairly general semicontinuity theorems. Then, under suitable coerciveness assumptions, the direct methods lead to various existence results.

3.1 A lower semicontinuity theorem

Let I be a bounded interval in $\mathbb{R}$, $I = (a, b)$, and let $F(x, u, p)$ be a *continuous Lagrangian* from $I \times \mathbb{R}^N \times \mathbb{R}^N$ into $\mathbb{R}$. Consider the variational integral

$$\mathcal{F}(u) = \int_I F(x, u(x), u'(x))\, dx. \tag{3.1}$$

Then $\mathcal{F}$ is defined for every absolutely continuous function on I. In fact we have

Proposition 3.1 *Suppose that F is non-negative or bounded from below by an L^1-function. Then the variational integral $\mathcal{F}(u)$ is well defined, with the possible value ∞, for all functions u in the Sobolev space $H^{1,m}(I, \mathbb{R}^N)$, $m \geq 1$.*

This is an immediate consequence of the following lemma, stating the same result if we replace continuity of $F(x, u, p)$ by measurability in x and continuity in (u, p) for almost every x.

Lemma 3.2 *Let $h(x, y)$ be a function of the real variables $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$ which is measurable in x for all y and continuous in y for almost all x. Then, for any measurable function $w : \mathbb{R}^n \mapsto \mathbb{R}^k$ the function $x \to h(x, w(x))$ is measurable.*

Proof Let $\{s_j\}$ be a sequence of simple functions such that

$$w(x) = \lim_{i \to \infty} s_i(x) \quad \text{a.e. in } \mathbb{R}.$$

Each s_j has the form

$$s_j(x) = \sum_{i=1}^{l_j} \lambda_i \chi_{A_i}$$

where $A_1 \dots A_{l_j}$ are measurable sets which can assumed to be disjoint. Observing that for any real number a

$$\{x \in \mathbf{R}^n : h(x, s_i(x)) > a\} = \bigcup_{j=1}^{l_i} [\{x \in \mathbf{R}^n : h(x, \lambda_i) > a\} \cap A_j]$$

we infer that $h(x, s_i(x))$ is measurable. By the continuity of $h(x, y)$ in y we obtain

$$h(x, s_i(x)) \to h(x, w(x)) \quad \text{for } x \text{ a.e.},$$

whence we conclude that $h(x, w(x))$ is measurable. □

Our main question is now: under which conditions on the integrand $F(x, u, p)$ is the integral $\mathcal{F}(u)$ sequentially lower semicontinuous with respect to the weak convergence in $H^{1,m}(I), m \geq 1$? Observing that if $\{u_k\}$ is a sequence of Lipschitz functions with equibounded Lipschitz constants which converges uniformly to a function u, then $\{u_k\}$ weakly converges to u in any $H^{1,m}(I), m \geq 1$, a necessary condition for lower semicontinuity is given by the following theorem.

Theorem 3.3 *If $\mathcal{F}(u)$ is sequentially lower semicontinuous with respect to the uniform convergence of equi-Lipschitzian functions, i.e.*

$$\mathcal{F}(u) \leq \liminf_{k\to\infty} \mathcal{F}(u_k) \tag{3.2}$$

for all u_k with $\sup_k |u_k|_{Lip} < +\infty$ and $u_k \to u$ uniformly, then for all $x_0 \in I$, $u_0 \in \mathbf{R}^N$, $p_0 \in \mathbf{R}$, and for all $\varphi \in C_c^\infty(I, \mathbf{R}^N)$ we have

$$\int_I F(x_0, u_0, p_0 + \varphi'(x))\, dx \geq F(x_0, u_0, p_0) \operatorname{meas} I. \tag{3.3}$$

In particular, the integrand $F(x, u, p)$ is convex in p for all fixed $x \in I$ and $u \in \mathbf{R}^N$.

Proof To make the proof transparent, we first consider the case in which the integrand F does not depend explicitly on x and u, i.e. $F = F(p)$. Without loss of generality we may assume that I is the unit interval centred at zero.

Given $\varphi \in C_c^\infty(I, \mathbf{R}^N)$, we extend it periodically to $\mathbf{R}$ and define φ_ν for all positive integers ν by

$$\varphi_\nu(x) := \nu^{-1}\varphi(\nu x).$$

Now set

$$u_0(x) := u_0 + p_0 x$$
$$u_\nu(x) := u_0(x) + \varphi_\nu(x).$$

The sequence $\{u_\nu\}$ has equibounded Lipschitz constants and converges uniformly to u_0. Thus, by assumption,

$$F(p_0)\,\text{meas}\, I \le \liminf_{\nu\to\infty} \int_I F(p_0 + D\varphi_\nu(x))\,dx.$$

Now $D\varphi_\nu(x) = \varphi'(\nu x)$; hence, if we change variables in the integral $\nu x = y$ we get

$$\begin{aligned} F(p_0)\,\text{meas}\, I &\le \liminf_{\nu\to\infty} \frac{1}{\nu}\int_{\nu I} F(p_0+\varphi'(y))dy \\ &= \int_I F(p_0+\varphi'(y))dy, \end{aligned}$$

if we take the periodicity of φ into account. Inequality (3.3) is then proved in the case $F = F(p)$.

Let us now consider the general case. We again assume for the sake of simplicity that I is the unit interval centred at zero and that φ is extended periodically to $\mathbf{R}$. For $x_0 \in I$, we consider the interval $R := (x_0, x_0 + h)$, choosing h so small that $R \subset I$, and, similarly to the above, we set

$$\begin{aligned} \varphi_\nu(x) &:= \nu^{-1}h\varphi(\nu h^{-1}(x - x_0)) \\ u_0(x) &:= u_0 + p_0(x - x_0) \\ u_\nu(x) &:= u_0(x) + \varphi_\nu(x). \end{aligned}$$

For each ν, the integral $\mathcal{F}(u_\nu, R)$ can be written as a sum of integrals over ν subintervals I_i of R of size $\nu^{-1}h$, given by

$$I_i := (x_i, x_{i+1}), \quad x_i = x_0 + i\nu^{-1}h, \qquad i = 0, \dots, \nu - 1.$$

On each of these intervals the integral is given by

$$\int_{I_i} F(x, u_\nu(x), u_\nu'(x))\,dx = \nu^{-1}h \int_I F(x_i + \nu^{-1}hy, u_\nu(x_i + \nu^{-1}hy), p_0 + \varphi'(y))dy;$$

thus, passing to the limit for $\nu \to \infty$ and using the continuity of F, we obtain

$$\lim_{\nu\to\infty} \mathcal{F}(u_\nu, R) = \int_R dx \int_I F(x, u_0(x), p_0 + \varphi'(y))dy.$$

and, by assumption,

$$\int_R dx \int_I F(x, u_0(x), p_0 + \varphi'(y))dy \ge \int_R F(x, u_0(x), p_0)\,dx.$$

Dividing by h and letting h tend to zero, we then deduce (3.3).

Finally, let us prove that (3.3) implies that F is convex in p. Clearly (3.3) holds also for all Lipschitz functions φ with zero boundary values on ∂I; also, as before, we can assume without loss of generality that $I = (0, 1)$. Set

$$\xi := \lambda\xi_1 + (1-\lambda)\xi_2, \quad \lambda \in (0, 1), \ \ \xi_1, \xi_2 \in \mathbf{R}^N,$$

and consider a Lipschitz function $\tilde{\varphi}(x)$ from $(0, 1) \subset \mathbf{R}$ into $\mathbf{R}^N$ satisfying

$$\tilde{\varphi}'(x) = \begin{cases} \xi_1 & \text{if } x \in (0, \lambda) \\ \xi_2 & \text{if } x \in (\lambda, 1); \end{cases}$$

obviously,

$$\tilde{\varphi}(1) = \tilde{\varphi}(0) + \int_0^1 \tilde{\varphi}'(t)dt = \tilde{\varphi}(0) + \int_0^\lambda \xi_1 dt + \int_\lambda^1 \xi_2\, dx = \tilde{\varphi}(0) + \xi.$$

Therefore, if we define $\varphi(x) := \tilde{\varphi}(x) - \tilde{\varphi}(0) - \xi x$, we have $\varphi(0) = \varphi(1) = 0$. Hence (3.3) yields

$$\begin{aligned} F(x_0, u_0, \xi) &\leq \int_0^1 F(x_0, u_0, \varphi')\, dx \\ &= \int_0^\lambda F(x_0, u_0, \xi_1)\, dx + \int_\lambda^1 F(x_0, u_0, \xi_2)\, dx \\ &\leq \lambda F(x_0, u_0, \xi_1) + (1-\lambda)F(x_0, u_0, \xi_2). \end{aligned}$$

□

We say that the integrand $F(x, u, p)$ is *quasiconvex* (in the sense of Morrey) if (3.3) holds. We remark that quasiconvexity is equivalent to saying that linear functions $l(x)$ are minimizers in the class of functions u with $u(x) = l(x)$ on ∂I of the 'frozen functional'

$$\mathcal{F}_0(u) := \int_I F(x_0, u_0, Du)\, dx$$

for all $x_0 \in I$ and $u_0 \in \mathbf{R}^N$. In particular, if F is of class C^2 in p, this at once implies convexity of F in p, i.e. $F_{pp}(x_0, u_0, p) \geq 0$ for every p, as we have seen in Theorem 3.3.

Remark 1 In fact, for every continuous $F(x, u, p)$, we have seen in the proof of Theorem 3.3 that quasiconvexity of F implies convexity in p. We note that convexity trivially implies quasiconvexity; hence, in dimension 1 quasiconvexity and convexity are equivalent. It can be seen that for multiple integrals defined on scalar functions (i.e. when $N = 1$), again convexity and quasiconvexity are equivalent, while for multiple integrals defined on vector-valued functions quasiconvexity is a *strictly weaker* condition than convexity. We refer the interested reader to the books by Morrey [193] and Dacorogna [75] for a detailed discussion of convexity and quasiconvexity in variational problems for multiple integrals; see also Giaquinta *et al.* [120].

We also deduce from Theorem 3.3 simply by changing signs that sequential upper semicontinuity of $\mathcal{F}(u)$ with respect to the weak convergence in $H^{1,m}(I)$, $m \geq 1$, implies that the integrand $F(x, u, p)$ is concave with respect to p. Thus, if $\mathcal{F}(u)$ is continuous with respect to the weak convergence of $H^{1,m}(I)$, $m \geq 1$, the integrand F must be both convex and concave in p; that is, $F(x, u, p)$ must be linear in p,

$$F(x, u, p) = A(x, u) + B(x, u)p, \tag{3.4}$$

for suitable functions A and B. On the other hand one can readily verify that integral functionals with linear Lagrangians in p are sequentially weakly continuous in $H^{1,m}(I)$, $m \geq 1$. Thus we obtain the following result:

Proposition 3.4 *The integral $\mathcal{F}(u)$ in (3.1) is continuous with respect to sequential weak convergence of $H^{1,m}(I)$, $m \geq 1$ (or equivalently with respect to the uniform convergence of sequences $\{u_k\}$ with equibounded L^1-norms of the gradients), if and only if its Lagrangian $F(x, u, p)$ is linear in p, i.e. if and only if $F(x, u, p)$ has the form* (3.4).

The next theorem shows that convexity in p is also a sufficient condition for the semicontinuity of $\mathcal{F}(u)$.

Theorem 3.5 (Tonelli's semicontinuity theorem) *Let I be a bounded open interval in $\mathbf{R}$ and let $F(x, u, p)$ be a Lagrangian satisfying the following conditions:*

(i) *F and F_p are continuous in (x, u, p);*
(ii) *F is non-negative or bounded from below by an L^1-function;*
(iii) *F is convex in p.*

Then the functional $\mathcal{F}(u)$ in (3.1) is sequentially weakly lower semicontinuous in $H^{1,m}(I, \mathbf{R}^N)$ for all $m \geq 1$, i.e. if $\{u_k\}$ converges weakly in $H^{1,m}(I, \mathbf{R}^N)$ to u, then

$$\mathcal{F}(u) \leq \liminf_{k\to\infty} \mathcal{F}(u_k). \tag{3.5}$$

Equivalently we can say that (3.5) holds if $\{u_k\}$ converges uniformly to u and the L^1-norms of u'_k are equibounded.

Proof It suffices to consider only the case $m = 1$, since if $\{u_k\}$ converges weakly to u in $H^{1,m}(I, \mathbf{R}^N)$ for some $m > 1$, it also converges to u in $H^{1,1}(I, \mathbf{R}^N)$.

Let $\{u_k\}$ be a sequence which converges weakly to u in $H^{1,1}(I, \mathbf{R}^N)$. Passing to a subsequence we can assume that $\{u_k\}$ converges to u in $L^q(I, \mathbf{R}^N)$ for every $q \geq 1$, hence almost everywhere (and even uniformly, but this is not needed). Assume now that $\mathcal{F}(u)$ is finite. For any positive ϵ we can find a compact subset $K \subset I$ such that, by Egorov's theorem,

$$u_k \to u \quad \text{uniformly in } K,$$

and by Lusin's theorem,

$$u \text{ and } u' \quad \text{are continuous in } K,$$

and by Lebesgue's absolute continuity theorem

$$\int_K F(x,u,u')\,dx \geq \int_I F(x,u,u')\,dx - \epsilon$$

(if $\mathcal{F}(u) = +\infty$, we can assume that $\mathcal{F}(u,K) > 1/\epsilon$).

Since F is convex in p, we obtain

$$\begin{aligned}
\mathcal{F}(u_k) &\geq \int_K F(x,u_k,u_k')\,dx \\
&\geq \int_K F_p(x,u_k,u')(u_k'-u')\,dx + \int_K F(x,u_k,u')\,dx \\
&= \int_K F(x,u_k,u')\,dx + \int_K F_p(x,u,u')(u_k'-u')\,dx \\
&\quad + \int_K [F_p(x,u_k,u') - F_p(x,u,u')](u_k'-u')\,dx.
\end{aligned}$$

Since u and u' are continuous on K, the function $F_p(x,u,u')$ is bounded; hence we infer that

$$\int_K F_p(x,u,u')(u_k'-u')\,dx \to 0 \quad \text{as } k \to \infty.$$

Since $(u_k' - u')$ are equibounded in $L^1(I)$ and $F_p(x,u_k,u') - F_p(x,u,u')$ converge uniformly to zero on K as $k \to \infty$, we also obtain

$$\int_K [F_p(x,u_k,u') - F_p(x,u,u')](u_k'-u')\,dx \to 0 \quad \text{as } k \to \infty.$$

Therefore we conclude that

$$\liminf_{k\to\infty} \int_K F(x,u_k,u_k')\,dx \geq \int_K F(x,u,u')\,dx \geq \int_I F(x,u,u')\,dx - \epsilon.$$

Since this holds for all ϵ, the result follows at once, taking (ii) into account. □

Remark 2 The assumptions of Theorem 3.5 can be weakened considerably; for the sake of completeness we present here another proof in which the integrand $F(x,u,p)$ is only assumed to be lower semicontinuous in (u,p) and convex in p. We want to mention that the first proof, in which condition (i) of Theorem 3.5 is replaced by

(i′) *$F(x,u,p)$ is a Carathéodory function, i.e. F is measurable in x for all u and p, and continuous in (u,p) for almost every x,*

was obtained by De Giorgi in 1968 in an unpublished paper [78]. Later, Olech in 1976 [208] and Ioffe in 1977 [146] independently generalized Theorem 3.5 to the case of integrands $F(x,u,p)$ which are only lower semicontinuous in (u,p). The proof which we present here follows the scheme of Ioffe's proof.

For further generalizations weakening also the lower semicontinuity assumption with respect to u we refer the interested reader to the papers by De Giorgi *et al.* [80], Ambrosio [8], and to the book by Buttazzo [47] for a general discussion on the subject.

We consider functionals of the form

$$\mathcal{F}(u, v) := \int_\Omega F(x, u(x), v(x))d\mu(x) \tag{3.6}$$

where $(\Omega, \mathcal{A}, \mu)$ is a measure space with a non-negative and finite measure μ, $F : \Omega \times \mathbf{R}^m \times \mathbf{R}^n \mapsto [0, +\infty]$ is an $\mathcal{A} \otimes \mathcal{B}_m \otimes \mathcal{B}_n$-measurable function ($\mathcal{B}_m$ and $\mathcal{B}_n$ respectively denote the σ-algebras of Borel subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$), and (u, v) varies in $L^1_\mu(\Omega; \mathbf{R}^m) \times L^1_\mu(\Omega; \mathbf{R}^n)$.

Let $\Omega = (a, b)$, μ be the Lebesgue measure, and $n = m = 1$. As soon as we have a sequential lower semicontinuity result for $\mathcal{F}$ above with respect to the strong $L^1_\mu(\Omega; \mathbf{R}^m)$ convergence on u and the weak $L^1_\mu(\Omega; \mathbf{R}^n)$ convergence on v, we immediately have a sequential lower semicontinuity result for the functional

$$\int_a^b F(x, u(x), u'(x))\, dx$$

with respect to the weak $H^{1,1}(a, b; \mathbf{R}^N)$ convergence.

Theorem 3.6 *Assume that the function F satisfies the following conditions:*

(i) *For μ-a.e. $x \in \Omega$ the function $F(x, \cdot, \cdot)$ is lower semicontinuous on $\mathbf{R}^m \times \mathbf{R}^n$.*

(ii) *For μ-a.e. $x \in \Omega$ and for every $u \in \mathbf{R}^m$ the function $F(x, u, \cdot)$ is convex on $\mathbf{R}^m \times \mathbf{R}^n$.*

Then the functional $\mathcal{F}$ defined in (3.6) is sequentially lower semicontinuous on the space $L^1_\mu(\Omega; \mathbf{R}^m) \times L^1_\mu(\Omega; \mathbf{R}^n)$ endowed with the strong topology on $L^1_\mu(\Omega; \mathbf{R}^m)$ and the weak topology on $L^1_\mu(\Omega; \mathbf{R}^n)$.

Proof Let $u_h \to u$ strongly in $L^1_\mu(\Omega; \mathbf{R}^m)$ and $v_h \to v$ weakly in $L^1_\mu(\Omega; \mathbf{R}^n)$; we have to prove that

$$\mathcal{F}(u, v) \le \liminf_{h\to\infty} \mathcal{F}(u_h, v_h). \tag{3.7}$$

Without loss of generality, passing to subsequences we may assume that the lim inf on the right-hand side of (3.7) is a finite limit, i.e.

$$\lim_{h\to\infty} \mathcal{F}(u_h, v_h) = c \in \mathbf{R}. \tag{3.8}$$

Since $\{v_h\}$ is weakly compact in $L^1_\mu(\Omega; \mathbf{R}^n)$, by the Dunford–Pettis theorem (compare Theorems 2.11 and 2.12 of Section 2.1) there exists a function $\vartheta : [0, \infty) \to [0, \infty)$ which can be taken as convex and strictly increasing, such that

$$\lim_{t\to\infty} \frac{\vartheta(t)}{t} = \infty \quad \text{and} \quad \sup_{h\in\mathbf{N}} \int_\Omega \vartheta(|v_h|)d\mu \le 1. \tag{3.9}$$

Setting

$$H(t) = \sqrt{t\vartheta(t)}$$

$$\Phi(t) = \vartheta(H^{-1}(t))$$

$$\xi_h(x) = H\left(|v_h(x)|\right)$$

it is easy to see that

(i) H is strictly increasing and $H(t)/t \to \infty$ as $t \to \infty$;
(ii) Φ is strictly increasing and $\Phi(t)/t \to \infty$ as $t \to \infty$;
(iii) $\vartheta(t)/H(t) \to \infty$ as $t \to \infty$;
(iv) $\Phi(\xi_h(x)) = \vartheta\left(|v_h(x)|\right)$.

Therefore, by (3.9) we have

$$\sup_{h\in\mathbb{N}} \int_\Omega \Phi(\xi_h)d\mu \leq 1$$

so that, again by the Dunford–Pettis theorem, the sequence $\{\xi_h\}$ is weakly compact in $L^1_\mu(\Omega)$. By Mazur's theorem a suitable sequence of convex combinations of (ξ_h, v_h) is strongly convergent in $L^1_\mu(\Omega) \times L^1_\mu(\Omega; \mathbf{R}^n)$. More precisely, there exist $N_h \to \infty$ and $\alpha_{i,h} \geq 0$ with

$$\sum_{i=N_{h+1}}^{N_{h+1}} \alpha_{i,h} = 1$$

such that

$$\nu_h = \sum_{i=N_{h+1}}^{N_{h+1}} \alpha_{i,h}v_i \to \nu \quad \text{strongly in } L^1_\mu(\Omega; \mathbf{R}^n)$$

$$\eta_h = \sum_{i=N_{h+1}}^{N_{h+1}} \alpha_{i,h}\xi_i \to \eta \quad \text{strongly in } L^1_\mu(\Omega).$$

By passing to suitable subsequences we may assume that $\nu_h \to \nu$ μ-a.e. on Ω, $\eta_h \to \eta$ μ-a.e. on Ω, and also $u_h \to u$ μ-a.e. on Ω. Let $x \in \Omega$ be a point where all the convergences above occur, and set

$$\epsilon_h := \max\left\{|u(x) - u_i(x)| : N_h < i \leq N_{h+1}\right\}$$

$$\lambda_h := \sum_{i=N_h+1}^{N_{h+1}} \alpha_{i,h}F(x, u_i(x), v_i(x))$$

$$\mathcal{A}_h := \{(\nu, \eta, \lambda) \in \mathbf{R}^{n+2} : \eta = H(\|\nu\|), \lambda \geq F(x, s, \nu)$$
$$\text{for some } s \in \mathbf{R}^m \text{ with } |s - u(x)| \leq \epsilon_h\}.$$

We have $\epsilon_h \to 0$ and, by definition of ν_h, η_h, λ_h, the point $(\nu_h(x), \eta_h(x), \lambda_h(x))$ is contained in the convex hull of $\mathcal{A}_h$.

Since $\mathcal{A}_h \subset \mathbf{R}^{n+2}$, Carathéodory's theorem on convex hulls in Euclidean spaces says that the vector $(\nu_h(x), \eta_h(x), \lambda_h(x))$ can be written as a convex combination of $n+3$ elements of $\mathcal{A}_h$; that is, there exist

$$\beta_{i,h} \geq 0, \quad \nu_{i,h} \in \mathbf{R}^n, \quad \eta_{i,h} \geq 0, \quad \lambda_{i,h} \geq 0 \qquad (i = 1, \dots, n+3)$$

such that $(\nu_{i,h}, \eta_{i,h}, \lambda_{i,h}) \in \mathcal{A}_h$ for every i, and

$$\sum_{i=1}^{n+3} \beta_{i,h} = 1, \qquad \sum_{i=1}^{n+3} \beta_{i,h}\nu_{i,h} = \nu_h(x),$$
$$\sum_{i=1}^{n+3} \beta_{i,h}\eta_{i,h} = \eta_h(x), \qquad \sum_{i=1}^{n+3} \beta_{i,h}\lambda_{i,h} = \lambda_h(x).$$

Therefore, for suitable $s_{i,h} \in \mathbf{R}^m$ with $|s_{i,h} - u(x)| \leq \epsilon_h$ we have

$$\lambda_{i,h} \geq F(x, s_{i,h}, \nu_{i,h}).$$

Let us denote by I the set of indices i such that the sequence $|\nu_{i,h}|$ does not tend to ∞ as $h \to \infty$; since

$$\sum_{i=1}^{n+3} \beta_{i,h} H\left(|\nu_{i,h}|\right) = \eta_h(x) \to \eta(x)$$

the set I cannot be empty. By passing to subsequences we may assume that

$$\begin{aligned} \nu_{i,h} &\to \nu_i && \text{for all } i \in I \\ |\nu_{i,h}| &\to +\infty && \text{for all } i \notin I \\ \beta_{i,h} &\to \beta_i && \text{for all } i = 1, \dots, n+3. \end{aligned}$$

From the relation

$$\sum_{i=1}^{n+3} \beta_{i,h}\nu_{i,h} = \nu_h(x) \to \nu(x)$$

we obtain that $\beta_i = 0$ for every $i \notin I$. Moreover, from

$$\eta_h(x) = \sum_{i=1}^{n+3} \beta_{i,h}\eta_{i,h} \geq \sum_{i \notin I} \beta_{i,h}\eta_{i,h} = \sum_{i \notin I} \beta_{i,h}|\nu_{i,h}| \frac{H\left(|\nu_{i,h}|\right)}{|\nu_{i,h}|}$$

we infer that

$$\beta_{i,h}|\nu_{i,h}| \to 0 \quad \text{for all } \quad i \notin I,$$

so that

$$\sum_{i\in I}\beta_i = 1, \qquad \sum_{i\in I}\beta_i v_i = v(x).$$

Finally, by using the assumptions of the theorem,

$$\begin{aligned} F(x, u(x), v(x)) &\le \sum_{i\in I}\beta_i F(x, u(x), v_i) \\ &\le \liminf_{h\to\infty}\sum_{i\in I}\beta_{i,h}F(x, s_{i,h}, v_{i,h}) \\ &\le \liminf_{h\to\infty}\sum_{i=1}^{n+3}\beta_{i,h}F(x, s_{i,h}, v_{i,h}) \\ &\le \liminf_{h\to+\infty}\lambda_h(x) \end{aligned}$$

and by Fatou's lemma

$$\begin{aligned} \int_\Omega F(x, u, v)d\mu &\le \liminf_{h\to\infty}\int_\Omega \lambda_h(x)d\mu \\ &= \liminf_{h\to\infty}\sum_{i=N_h+1}^{N_{h+1}}\alpha_{i,h}\int_\Omega F(x, u_i, v_i)d\mu. \end{aligned} \tag{3.10}$$

If we fix $\epsilon > 0$ by (3.8) we obtain for sufficiently large h that

$$\int_\Omega F(x, u_i, v_i)d\mu \le c + \epsilon \quad \text{for all } i \in [N_h + 1, N_{h+1}].$$

By virtue of (3.10) we arrive at

$$\mathcal{F}(u, v) \le c + \epsilon.$$

The conclusion now follows by letting $\epsilon \to +0$. □

Remark 3 It is easy to see that Theorem 3.6 remains true if the measure μ is only assumed to be σ-finite.

Remark 4 For $\mathcal{A}\otimes\mathcal{B}_m\otimes\mathcal{B}_n$-measurable functions $F : \Omega\times\mathbf{R}^m\times\mathbf{R}^n \to [0, +\infty]$, and when μ is a non-atomic measure (such as the Lebesgue measure), it can be proved that conditions (i) and (ii) of Theorem 3.6 are actually necessary for the sequential lower semicontinuity of the functional $\mathcal{F}$ defined in (3.6) with respect to the strong topology of $L^1_\mu(\Omega, \mathbf{R}^m)$ and the weak topology of $L^1_\mu(\Omega; \mathbf{R}^n)$, provided $\mathcal{F}$ is not identically equal to $+\infty$. (See the original paper by Ioffe [146] or the book by Buttazzo [47].)

3.2 Existence results in Sobolev spaces

Let $I = (a, b)$ be a bounded interval in $\mathbb{R}$ and $F(x, u, p)$ a Lagrangian defined on $I \times \mathbb{R}^N \times \mathbb{R}^N$, $N \geq 1$. We say that $F(x, u, p)$ has a *superlinear growth* if there exists a function $\theta(p)$ such that

$$\begin{cases} F(x, u, p) \geq \theta(p) & \text{for all } x, u, p \\ \theta(p)/|p| \to \infty & \text{as } |p| \to \infty. \end{cases} \tag{3.11}$$

We say that the Lagrangian $F(x, u, p)$ has a *polynomial growth m*, if there are positive constants c_0, c_1, c_2 and a constant $m \geq 1$ such that

$$c_0 |p|^m \leq F(x, u, p) \leq c_1 |p|^m + c_2 \quad \text{for all } x, u, p. \tag{3.12}$$

Obviously an integrand with a polynomial growth $m > 1$ is of superlinear growth.

An immediate consequence of Theorem 3.5 of Section 3.1 and of the weak compactness criterion in $H^{1,1}(a, b)$ (cf. Theorem 2.12 of Section 2.1) is the following result.

Theorem 3.7 (Tonelli's existence theorem) *Suppose that the Lagrangian $F(x, u, p)$ satisfies the following conditions:*

(i) *$F(x, u, p)$ and $F_p(x, u, p)$ are continuous in (x, u, p);*
(ii) *$F(x, u, p)$ is convex in p;*
(iii) *$F(x, u, p)$ has a superlinear growth.*

Then there exists a minimizer of

$$\mathcal{F}(u) := \int_I F(x, u, u')\, dx$$

in the class

$$C(\alpha, \beta) := \{u \in H^{1,1}((a, b), \mathbb{R}^N) : u(a) = \alpha,\ u(b) = \beta\}$$

where α, β are fixed vectors in $\mathbb{R}^N$.

Proof By (3.2) the functional $\mathcal{F}$ is bounded from below. Let $\{u_k\}$ be a minimizing sequence in (α, β). We may assume that $\inf \mathcal{F}(u_k) < +\infty$, otherwise $\mathcal{F}(u) \equiv +\infty$ identically on $C(\alpha, \beta)$. Since F has a superlinear growth, the sequence $\{u_k\}$ is equibounded in $H^{1,1}(I, \mathbb{R}^N)$; moreover, the integrals

$$\int_I \theta(u_k')\, dx$$

are equibounded. Then applying Theorem 2.12 of Section 2.1 (a subsequence of) $\{u_k\}$ converges weakly in $H^{1,1}(I)$ and uniformly on $\bar{I}$ to some function $u \in H^{1,1}(I, \mathbb{R}^N)$ with values α and β respectively at a and b. The semicontinuity Theorem 3.5 of Section 3.1 then yields

$$\mathcal{F}(u) \leq \liminf_{k \to \infty} \mathcal{F}(u_k),$$

i.e. u is a minimizer of $\mathcal{F}$ on (α, β). □

Remark 1 Taking Remark 2 of Section 3.1 into account one readily sees that Theorem 3.7 remains true if, instead of (i), we require that $F(x, u, p)$ is a Carathéodory function, or more generally, by using Theorem 3.6 of Section 3.1, that F is $\mathcal{L} \otimes \mathcal{B}_N \otimes \mathcal{B}_N$-measurable ($\mathcal{L}$ denotes the σ-algebra of all Lebesgue measurable subsets of I) and that

for a.e. $x \in I$ *the function* $F(x, \cdot, \cdot)$ *is lower semicontinuous on* $\mathbf{R}^N \times \mathbf{R}^N$;
for a.e. $x \in I$ *and for every* $u \in \mathbf{R}^N$ *the function* $F(x, u, \cdot)$ *is convex on* $\mathbf{R}^N$.

Remark 2 We note that, if F has a polynomial growth $m > 1$, then the minimizer in Theorem 3.7 is of class $H^{1,m}(I, \mathbf{R}^N)$; moreover, any minimizer of $\mathcal{F}$ in $C(\alpha, \beta)$ is a minimizer of $\mathcal{F}$ in

$$C_m(\alpha, \beta) := \{u \in H^{1,m}(I, \mathbf{R}^N) : u(a) = \alpha, \ u(b) = \beta\}$$

and vice versa. Actually, in this case proving the existence of a minimizer in $C_m(\alpha, \beta)$, and consequently in $C(\alpha, \beta)$, is slightly easier, since $H^{1,m}(I, \mathbf{R}^N)$ is reflexive, and we do not need the weak compactness criterion in $H^{1,1}$.

Remark 3 By a simple use of Poincaré's inequalities one immediately sees that the boundary conditions $u(a) = \alpha$, $u(b) = \beta$ in the definition of $C(\alpha, \beta)$ can be replaced by $u(a) = \alpha$ and no condition on b, or by no condition in a nor in b but requiring $\int_a^b u\, dx = 0$, and there will still be a minimizer in the corresponding classes.

Remark 4 Tonelli proved several extensions of Theorem 3.7. We mention here only one of them: in Theorem 3.7 it suffices to require F to have a superlinear growth in p outside a small set of (x, u)-values, for instance outside the graph of a curve of finite length.

$\boxed{1}$ Consider for every $\alpha \in \mathbf{R}$ and $p > 1$ the functional

$$\mathcal{F}_{\alpha,p}(u) := \int_0^1 x^\alpha |u'|^p \, dx$$

defined for every $u \in H^{1,1}(0, 1)$, and the associated minimum problem

$$\min\{\mathcal{F}_{\alpha,p}(u) : u \in H^{1,1}(0, 1), \ u(0) = a, \ u(1) = b\} \tag{3.13}$$

where a, b are two real numbers with $a \neq b$. Weierstrass observed that when $\alpha = p = 2$ the minimum problem above does not admit any solution whenever $a \neq b$, because taking the sequence of functions

$$u_h(x) = a + (b - a)\frac{\arctan(hx)}{\arctan h}$$

we obtain

$$\mathcal{F}_{2,2}(u_h) = \frac{(b-a)^2}{2h \arctan^2 h}\left(\arctan h - \frac{h}{1+h^2}\right)$$

which tends to zero as $h \to \infty$. Therefore

$$\inf\{\mathcal{F}_{2,2}(u) : u \in H^{1,1}(0,1), u(0) = a, u(1) = b\} = 0.$$

On the other hand, no function $u \in H^{1,1}(0,1)$ with $u(0) \neq u(1)$ may have $\mathcal{F}_{2,2}(u) = 0$ because this would imply $u' = 0$ almost everywhere on $(0,1)$ and so $u(0) = u(1)$.

We now consider the general case $\mathcal{F}_{\alpha,p}$. When $\alpha \leq 0$ the assumptions of the existence theorem 3.7 (see also Remark 1) are fulfilled and so problem (3.13) admits a solution for every a and b. The solution is also unique because of the strict convexity of the functional $\mathcal{F}_{\alpha,p}$ and, from the Euler–Lagrange equation, we obtain

$$x^\alpha |u'|^{p-2} u' = c \quad (c = \text{constant})$$

whence

$$u(x) = a + (b-a)x^{(p-1-\alpha)/(p-1)}.$$

When $\alpha > 0$, the functional $\mathcal{F}_{\alpha,p}$ still remains sequentially weakly lower semicontinuous on $H^{1,1}(0,1)$ but the integrand $F(x,z) = x^\alpha |z|^p$ no longer verifies the superlinear growth condition of Theorem 3.7. However, we shall see that for certain values of α and p, we still have the existence of a solution for problem (3.13). More precisely this happens if and only if $p > \alpha + 1$. Indeed, let q be a real number such that $1 < q < p/(\alpha+1)$; we have by Hölder's inequality

$$\begin{aligned}
\int_0^1 |u'|^q \, dx &= \int_0^1 \left(|u'|^q x^{\alpha q/p}\right) x^{-\alpha q/p} \, dx \\
&\leq \left(\int_0^1 x^\alpha |u'|^p \, dx\right)^{q/p} \left(\int_0^1 x^{-\alpha q/(p-q)} \, dx\right)^{(p-q)/p} \\
&= (F_{\alpha,p}(u))^{q/p} \left(\frac{p-q}{p-q(\alpha+1)}\right)^{(p-q)/p}.
\end{aligned}$$

Therefore, a sequence $\{u_h\}$ which has $\mathcal{F}_{\alpha,p}(u_n)$ bounded is bounded in $H^{1,q}(0,1)$ and hence weakly relatively compact in $H^{1,1}(0,1)$, and this, together with the lower semicontinuity of $\mathcal{F}_{\alpha,p}$, furnishes the existence result for problem (3.13). As before, the solution is unique, and it is given by the function

$$u(x) = a + (b-a)x^{(p-1-\alpha)/(p-1)}.$$

On the contrary, when $p \leq \alpha + 1$, we have as in the Weierstrass case that

$$\inf\{\mathcal{F}_{\alpha,p}(u) : u \in H^{1,1}(0,1), u(0) = a, u(1) = b\} = 0, \tag{3.14}$$

and so, by the same argument used before, no solution of problem (3.13) is possible when $a \neq b$. Since $\mathcal{F}_{\alpha,p} \geq \mathcal{F}_{\beta,p}$ whenever $\alpha \leq \beta$, it is enough to show (3.14) in the case $p = \alpha + 1$. To do this, for every $\epsilon > 0$ consider the function

$$u_\epsilon(x) = a + (b - a)\frac{\log(1 + x/\epsilon)}{\log(1 - 1/\epsilon)}$$

which is in $H^{1,1}(0, 1)$ and satisfies the boundary conditions $u(0) = a$ and $u(1) = b$. We have

$$u_\epsilon'(x) = \frac{1}{\epsilon + x}\;\frac{b - a}{\log(1 + 1/\epsilon)}$$

so that

$$\begin{aligned}\mathcal{F}_{\alpha,p}(u_\epsilon) &= \frac{|b - a|^p}{|\log(1 + 1/\epsilon)|^p}\int_0^1 \left(\frac{x}{\epsilon + x}\right)^p \frac{1}{x + \epsilon}\,dx\\ &\le \frac{|b - a|^p}{|\log(1 + 1/\epsilon)|^p}\int_0^1 \frac{1}{x + \epsilon}\,dx = \frac{|b - a|^p}{|\log(1 + 1/\epsilon)|^{p-1}}\end{aligned}$$

which tends to zero as $\epsilon \to +0$.

The reader can show a similar result in the more general case of minimum problems like

$$\min\left\{\int_0^1 \alpha(x)|u'|^p\,dx : u \in H^{1,1}(0, 1),\ u(0) = a,\ u(1) = b\right\}. \tag{3.15}$$

More precisely, problem (3.15) admits a solution for arbitrary boundary data a and b if and only if the function $(\alpha(x))^{1/(1-p)}$ is in $L^1(0, 1)$. In this case the solution is unique, and it is given by

$$u(x) = a + (b - a)\left(\int_0^1 \alpha(t)^{1/(1-p)}dt\right)^{-1}\int_0^x \alpha(t)^{1/(1-p)}dt.$$

$\boxed{2}$ The convexity assumption (ii) in Tonelli's existence theorem (cf. Theorem 3.7) cannot be eliminated. Indeed, consider the functional

$$\mathcal{F}(u) = \int_0^1 [(1 - |u'|^2)^2 + u^2]\,dx$$

whose integrand

$$F(u, p) = (1 - |p|^2)^2 + u^2$$

satisfies all conditions of Theorem 3.7 except convexity with respect to p; the minimum problem

$$\min\{\mathcal{F}(u) : u \in H^{1,1}(0, 1),\ u(0) = u(1) = 0\}$$

has no solution. In fact, if $\varphi(x)$ denotes the function defined by $1/2 - |x - 1/2|$ on $[0, 1]$ and periodically extended to all of $\mathbf{R}$, the functions

$$u_h(x) := \frac{1}{h}\varphi(hx), \quad (h \in \mathbb{N}),$$

are in $H^{1,1}(0, 1)$, fulfil the boundary conditions $u_h(0) = u_h(1) = 0$, and give

$$\mathcal{F}(u_h) = \frac{1}{h^2}\int_0^1 \varphi^2(x)\,dx = \frac{1}{12h^2}$$

because $|u_h'(x)| = 1$ almost everywhere. Therefore the infimum of $\mathcal{F}$ on the admissible class is zero, but no function u satisfies $\mathcal{F}(u) = 0$, because this would imply $\int_0^1 u^2\,dx = 0$; hence $u = 0$, and then $\mathcal{F}(u) = 1$.

$\boxed{3}$ Consider now a slightly different example:

$$\mathcal{F}(u) := \int_0^1 \left[\left(1 - |u'|^2\right)^2 + u\right] dx.$$

In this case we will show that the minimum problem

$$\min\{\mathcal{F}(u) : u \in H^{1,1}(0, 1), u(0) = a, u(1) = b\} \tag{3.16}$$

admits a solution for every $a, b \in \mathbf{R}$, though the integrand is not convex with respect to u'. Indeed, consider the functional

$$\mathcal{G}(u) = \int_0^1 [\varphi(u') + u]\,dx$$

where the function φ is defined as the convex envelope of the function $(1 - |p|^2)^2$, i.e.

$$\varphi(p) = \begin{cases} \left(1 - |p|^2\right)^2 & \text{if } |p| \geq 1 \\ 0 & \text{if } |p| < 1. \end{cases}$$

The functional $\mathcal{G}$ satisfies all assumptions of the existence theorem, and so there exists a solution $\overline{u}$ of the minimum problem

$$\min\{\mathcal{G}(u) : u \in H^{1,1}(0, 1), u(0) = a, u(1) = b\}.$$

The Euler–Lagrange equation gives

$$\varphi'(\overline{u}') = x + c \quad \text{a.e. on } (0, 1) \quad (c = \text{const}),$$

and since $\varphi'(p) = 0$ for all $p \in (-1, 1)$, we obtain $|\overline{u}'(x)| \geq 1$ almost everywhere, so that

$$\varphi(\overline{u}') = (1 - |\overline{u}'|^2)^2 \quad \text{a.e. on } (0, 1).$$

Hence we have

$$\inf \mathcal{F} \leq \mathcal{F}(\overline{u}) = \mathcal{G}(\overline{u}) = \min \mathcal{G} \leq \inf \mathcal{F},$$

and this proves that $\overline{u}$ is a solution of problem (3.16). It is not difficult to show that $\overline{u}$ is actually the unique solution of problem (3.16). The proof is left to the reader.

$\boxed{4}$ As an example[10] we apply direct methods in the calculus of variations to discuss the existence of a solution of

$$w'' = f(x) - h(x)e^{w} \quad \text{on } 0 < x < 1 \quad \text{with } w'(0) = w'(1) = 0. \tag{3.17}$$

The first step is to reduce to the special case where $f \equiv$ const by letting z be a solution of $z'' = f - c$, $z'(0) = z'(1) = 0$, where $c := \overline{f}$ is the average of f, and letting $u := w - z$. Then we want to solve

$$u'' = c - h(x)e^{u} \quad \text{on } 0 < x < 1 \quad \text{with } u'(0) = u'(1) = 0. \tag{3.18}$$

Integrating (3.18) one immediately gets for $I = (0, 1)$ that

$$\int_I he^u \, dx = c, \tag{3.19}$$

which imposes a necessary sign condition on h. For instance, if $c > 0$, the function h must be positive somewhere. The three cases $c < 0$, $c = 0$, and $c > 0$ are strikingly different. As model examples we shall discuss the cases $c = -1$, $c = 0$, $c = 1$.

Case 1: $c = -1$, so $u'' = -1 - h(x)e^u$.
The obvious functional to try is

$$J(u) := \int_I \left(\frac{1}{2}u'^2 - u - he^u \right) dx.$$

If $h <$ const < 0 then $-u - he^u >$ const, so the Tonelli theory applies.

If h is zero or positive somewhere one must work harder. A key difficulty is to ensure that the functional J is bounded from below. If one lets $u = k =$ const, it becomes clear that the functional is unbounded from below unless $\int h \, dx < 0$. In fact, this is also a necessary condition for a solution of the differential equation to exist. To see this, multiply the equation by e^{-u} and integrate by parts to find that

$$-\int_I h \, dx = \int_I (e^{-u}u'^2 + e^{-u}) \, dx > 0. \tag{3.20}$$

With further work one can show that this is not sufficient. In addition, it turns out that for any h there is a constant η_0 so that one can solve the equation $u'' = -1 - [h(x) + \eta]e^u$ if $\eta < \eta_0$ but not if $\eta > \eta_0$. The calculus of variations turns out not to be the best technique for this case (the method of sub- and supersolutions is better).

Case 2: $c = 0$, so $u'' = -h(x)e^u$.
Let us assume that $h \not\equiv 0$. From (3.19), one necessary condition is that h change sign. There is another necessary condition: just as for (3.20), multiplying the equation by e^{-u} and integrating by parts we find that $-\int h \, dx = \int e^{-u}u'^2 \, dx > 0$.

[10] This example has kindly been communicated to the authors by Jerry Kazdan.

To set up the variational problem, write $u = \bar{u} + v$, where $\bar{u} = \int u\,dx$ is the average of u and $v \perp 1$. With the constraint (3.19) in mind, seek a minimum of

$$J(v) = \frac{1}{2}\int v'^2\,dx$$

where $v \in H^1$ satisfies the two constraints

$$\int he^v\,dx = 0 \quad \text{and} \quad \int v\,dx = 0.$$

Before going further we check that this solves the problem. The Euler equation is

$$v'' = \alpha he^v + \beta,$$

where α and β are Lagrange multipliers. Integrating this and using the first constraint we find that $\beta = 0$. Similarly, multiply this by e^{-u} and integrate by parts, using $\int h\,dx < 0$ to conclude that $\alpha > 0$. Thus $\alpha = e^\gamma$, and so $u := v + \gamma$ is the desired solution of $u'' = -he^u$.

It is routine to prove that this variational problem has a minimum since weak convergence in H^1 implies uniform convergence, so the weak limit also satisfies the constraints. Thus, a necessary and sufficient condition to solve the equation is that h change sign and $\int h\,dx < 0$. We do not know any way to prove this except the above proof by the calculus of variations. This identical proof still works for the partial differential equation

$$-\Delta u = h(x)e^u \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Omega$$

in the case of a bounded two-dimensional region Ω, although one does not have the uniform convergence. In dimension 3 or higher, except for the necessary conditions, nothing is known.

Case 3: $c = 1$, so $u'' = 1 - h(x)e^u$.
As above, the obvious functional to try is

$$\mathcal{J}(u) := \int \left(\frac{1}{2}u'^2 + u - he^u\right) dx,$$

and we write $u = \bar{u} + v$, where $v \perp 1$. Then we can solve (3.19) for $\bar{u} = -\log(\int he^v\,dx)$ and get

$$\mathcal{F}(v) := \mathcal{J}(u) = \int \frac{1}{2}v'^2\,dx - \log\left(\int he^v\,dx\right) - 1.$$

We seek a minimum of this among all functions $v \in H^1$ with $\int v\,dx = 0$. Note that for this minimum we need only consider functions with $\int he^v\,dx > 0$. Because h is positive somewhere there are some admissible functions v.

First, observe that the Euler–Lagrange equation solves the correct problem. It is

$$-v'' - \frac{he^v}{\int he^v\,dx} + \lambda = 0,$$

where λ is a Lagrange multiplier from the constraint $\int v\,dx = 0$. Integrating this we see that $\lambda = 1$. Then write $\int he^v\,dx = e^\gamma$. The desired solution is $u = v - \gamma$.

To prove that the minimizing problem has a solution we show that $\mathcal{F}$ is bounded from below. By the Sobolev inequality we obtain $|v(x)| \le \|v'\|$. Thus if $|h(x)| \le M$ then

$$\int he^v\,dx \le Me^{\|v'\|}.$$

Therefore

$$\mathcal{F}(v) \ge \frac{1}{2}\int v'^2\,dx - \|v'\| - \log M - 1.$$

Since $\|v'\| \le \lambda + (1/\lambda)\|v'\|^2$ for any $\lambda > 0$, if we let $\lambda = 4$ we find that

$$\mathcal{F}(v) \ge \tfrac{1}{4}\|v'\|^2 + \text{const}.$$

Thus $\mathcal{F}$ is bounded from below, and minimizing sequences $\{v_j\}$ are bounded in H^1. A weakly convergent subsequence converges uniformly, so the remainder of the proof is routine.

This proves that the equation $u'' = 1 - he^u$ with Neumann boundary conditions on $0 < x < 1$ has a solution if and only if h is positive somewhere. On compact two-dimensional surfaces without boundary the analogous partial differential equation has been studied intensively, but except for the sphere S^2 essentially nothing is known. In higher dimensions essentially everything is open.

Let us return to the discussion before $\boxed{1}$–$\boxed{3}$. We note that Theorem 3.7 is very simple, and it uses the weakest possible assumptions in order to apply the direct methods in the class of absolutely continuous functions. Therefore we might have the impression that we have found the correct space to work with and the correct *generalization* of minimum problems for *any* variational integral whose Lagrangian has a superlinear growth.

Unfortunately, the situation is slightly more complicated. If we want to consider the minimizer of Theorem 3.7 as a *generalized solution* of the classical problem 'Minimize $\mathcal{F}$ in the class of smooth functions with $u(a) = \alpha$ and $u(b) = \beta$', we should at least expect that $\mathcal{F}(u)$ agrees with the infimum on the class of smooth functions, i.e.

$$\begin{aligned}&\inf\{\mathcal{F}(v) : v \in H^{1,1}(I, \mathbb{R}^N) : v(a) = \alpha,\ v(b) = \beta\}\\ &\quad = \inf\{\mathcal{F}(v) : v \text{ smooth}, v(a) = \alpha, v(b) = \beta\}.\end{aligned} \tag{3.21}$$

We shall see that this is true when F has a polynomial growth of order $m > 1$, but it is not true for general integrands with superlinear growth, or even satisfying the condition

$$c_0|p|^m \leq F(x, u, p), \quad m > 1, \ c_0 > 0.$$

When equality (3.21) fails we say that a *Lavrentiev phenomenon* occurs; this will be discussed later in Section 4.3. For Lagrangians with a polynomial growth we have

Proposition 3.8 *Let $F(x, u, p)$ be a Carathéodory Lagrangian that is convex in p and has a polynomial growth of order $m > 1$, i.e. (3.12) holds, and let $\{u_k\}$ be a sequence in $H^{1,m}(I, \mathbf{R}^N)$. If $\{u_k'\}$ converges strongly in $H^{1,m}$ to u, then $\mathcal{F}(u_k)$ converges to $\mathcal{F}(u)$.*

Proof Passing to a subsequence, we can assume that $u_k \to u(x)$ and $u_k'(x) \to u'(x)$ for almost all x in I, and also that

$$g(x) := \sum_{k=1}^{\infty} \left||u_k'|^p - |u_{k-1}'|^p\right|$$

is an L^1-function. In fact, since $|u_k'|^p$ converges strongly in L^1 to $|u'|^p$, we can find a subsequence $|u_{k_i}'|^p$ such that

$$\int_I \left||u_{k_i}'|^p - |u_{k_{i-1}}'|^p\right| dx \leq 2^{-i}.$$

Consequently we have

$$F(x, u_k(x), u_k'(x)) \to F(x, u(x), u'(x)) \quad \text{a.e. in } I$$
$$|F(x, u_k, u_k')| \leq c_1|u_k'|^p + c_2 \leq c_1 g(x) + c_2.$$

The claim then follows from Lebesgue's dominated convergence theorem. □

Remark 5 We note that the previous proposition actually holds under the weaker assumption

$$|F(x, u, p)| \leq c_1|p|^m + c_2.$$

As a trivial consequence of Proposition 3.8 we now obtain the following theorem when we also recollect the observations of Remark 2.

Theorem 3.9 *Let $F(x, u, p)$ be a Lagrangian which is convex in p and has a polynomial growth of order $m > 1$. Then we have:*

(i) *$\mathcal{F}(u)$ is finite if and only if $u \in H^{1,m}(I, \mathbf{R}^N)$.*

(ii) *The minimizers of $\mathcal{F}$ in $C(\alpha, \beta)$ coincide with the minimizers of $\mathcal{F}$ in $C(\alpha, \beta) \cap H^{1,m}(I, \mathbf{R}^N)$. More precisely, u is a minimizer of $\mathcal{F}$ in $C(\alpha, \beta)$ if and only if $\mathcal{F}(u) \leq \mathcal{F}(v)$ for all v in $C(\alpha, \beta) \cap H^{1,m}(I, \mathbf{R}^N)$.*

(iii) *For every u with $\mathcal{F}(u) < \infty$, i.e. for every $u \in H^{1,m}(I, \mathbf{R}^N)$, there exists a sequence of functions u_k in $C^1(\overline{I})$ with $u_k(a) = u(a)$, $u_k(b) = u(b)$ such that*

$\mathcal{F}(u_k)$ converges to $\mathcal{F}(u)$. In particular $\mathcal{F}(u)$ is the greatest lower semicontinuous extension of $\mathcal{F}$ on $C^1(\bar{I})$ or on $C(\alpha,\beta)\cap C^1(\bar{I})$; that is, we have for $u \in H^{1,1}(I,\mathbf{R}^N)$

$$\mathcal{F}(u) = \inf\{\liminf_{k\to\infty} \mathcal{F}(u_k) : u_k \in C^1(\bar{I}),\ u_k \text{ weakly converging to } u \text{ in } H^{1,1}(I,\mathbf{R}^N)\},$$

and for $u \in C(\alpha,\beta)$

$$\mathcal{F}(u) = \inf\{\liminf_{k\to\infty} \mathcal{F}(u_k) : u_k \in C(\alpha,\beta)\cap C^1(\bar{I}),\ u_k \text{ weakly converging to } u \text{ in } H^{1,1}(I,\mathbf{R}^N)\}.$$

Actually we can state more, namely

Proposition 3.10 *Suppose that $F(x,u,p)$ is a Lagrangian which is strictly convex in p and has polynomial growth of order $m > 1$. If $\{u_k\}$ converges weakly in $H^{1,m}$ to u and in energy, i.e. $\mathcal{F}(u_k) \to \mathcal{F}(u)$, then $\{u_k\}$ converges strongly in $H^{1,m}$ to u. In particular, every minimizing sequence $\{u_k\}$ in $C(\alpha,\beta)$ converges to a minimizer u in $C(\alpha,\beta)$ not only weakly in $H^{1,1}$ or $H^{1,m}$, but strongly in $H^{1,m}$.*

Proof For the proof of this proposition in its full generality we refer the interested reader to Reshetnyak [220]; here we prove Proposition 3.10 under the additional condition that F be *uniformly* strictly convex, in the sense that

for some positive μ_0, satisfying $\mu_0 < c_0$ (c_0 the constant in (3.12)), the Lagrangian

$$F(x,u,p) - \mu_0|p|^m \tag{3.22}$$

is convex.

In this case the functional

$$\int_I [F(x,u,u') - \mu_0|u'|^m]\,dx$$

is sequentially lower semicontinuous with respect to weak convergence in $H^{1,m}(I,\mathbf{R}^N)$; hence, using the assumption $\mathcal{F}(u_k) \to \mathcal{F}(u)$, we deduce

$$\begin{aligned}\int_I [F(x,u,u') - \mu_0|u'|^m]\,dx &\le \liminf_{k\to\infty} \int_I [F(x,u_k,u_k') - \mu_0|u_k'|^m]\,dx\\ &= \lim_{k\to\infty} \mathcal{F}(u_k) - \mu_0 \limsup_{k\to\infty} \int_I |u_k'|^m\,dx\\ &= \int_I F(x,u,u')\,dx - \mu_0 \limsup_{k\to\infty} \int_I |u_k'|^m\,dx,\end{aligned}$$

i.e.

$$\limsup_{k\to\infty} \int_I |u_k'|^m\,dx \le \int_I |u'|^m\,dx.$$

Therefore we conclude that

$$\int_I |u_k'|^m \, dx \to \int_I |u'|^m \, dx \tag{3.23}$$

since the inequality

$$\int_I |u'|^m \, dx \leq \liminf_{k\to\infty} \int_I |u_k'|^m$$

is always true. But it is well known that (3.23) and the weak convergence of u_k' to u' imply the strong convergence of u_k' to u' in L^p. Hence Proposition 3.10 under the extra assumption (3.22) follows at once. □

3.3 Lower semicontinuity in the space of measures

When an integrand $F(x, u, p)$ does not have a superlinear growth with respect to p, the arguments of Section 3.2 do not apply; the reason is that minimizing sequences $\{u_n\}$ for the functional $\mathcal{F}$ may not fulfil the Dunford–Pettis criterion of weak compactness in L^1 for the derivatives $\{u_n'\}$. For instance, if

$$F(x, u, p) = |p| + |u - f(x)|$$

with $f \in L^1(a, b)$, the only bound we may obtain on $\{u_n'\}$ is

$$\int_a^b |u_n'| \, dx \leq C$$

for a suitable positive constant C, and we know that bounded sequences in L^1 may converge, in the weak* sense of measures, to measures which are not absolutely continuous with respect to the Lebesgue measure, as for instance the Dirac mass δ_{x_0}. Therefore, when we deal with a problem with *linear growth*, i.e.

$$F(x, u, p) \geq \alpha |p| \quad \text{for all } x, u, p.$$

with $\alpha > 0$, we have to expect solutions u with $u' \in \mathcal{M}(a, b; \mathbf{R}^N)$: that is, $u \in BV(a, b; \mathbf{R}^N)$. Notice that, for example, the integrands $F(p) = |p|$ and $\sqrt{1 + |p|^2}$ respectively have linear growth, and so the superlinear growth assumption fails.

In order to apply the direct methods of the calculus of variations to variational problems with a linear growth, we provide a lower semicontinuity theorem on $BV(a, b; \mathbf{R}^N)$ which will be obtained by means of a lower semicontinuity theorem for functionals on the space of measures.

In the sequel Ω will be an interval of the real line $\mathbb{R}$, and μ is the Lebesgue measure on Ω. More generally Ω will be a separable locally compact metric space with a positive finite Borel measure μ. We denote by $\mathcal{B}$ the σ-algebra of all Borel subsets of Ω.

As already recalled in (2.33) of Section 2.3, for every vector-valued measure $\lambda : \mathcal{B} \to \mathbb{R}^N$ and every $B \in \mathcal{B}$ the total variation of λ on B is defined by

$$|\lambda|(B) = \sup\left\{\sum_{j=1}^{\infty} |\lambda(B_j)| : B_j \in \mathcal{B} \text{ pairwise disjoint}, \bigcup_{j=1}^{\infty} B_j \subset B\right\}.$$

In this way, the set function $B \mapsto |\lambda|(B)$ turns out to be a positive measure which will be denoted by $|\lambda|$.

We need the following localization lemma.

Lemma 3.11 *Let ν be a non-negative Borel measure on Ω and let $\{f_n\}$ be a sequence of non-negative ν-measurable functions on Ω. Setting $f = \sup\{f_n : n \in \mathbb{N}\}$ we have*

$$\int_\Omega f\,d\nu = \sup\left\{\sum_{i\in I}\int_{B_i} f_i\,d\nu\right\} \tag{3.24}$$

where the supremum on the right-hand side is taken over all finite Borel partitions $(B_i)_{i\in I}$ of Ω.

Proof Inequality $\geq$ is trivial. In order to prove the opposite inequality, we introduce the functions

$$g_n := \sup\{f_i : 1 \leq i \leq n\}, \quad n \in \mathbb{N}.$$

Clearly $\{g_n\}$ is a non-decreasing sequence of ν-measurable functions whose supremum is f. Then, by the monotone convergence theorem,

$$\int_\Omega f\,d\nu = \sup\left\{\int_\Omega g_n\,d\nu : n \in \mathbb{N}\right\}. \tag{3.25}$$

Let us now consider disjoint Borel sets $B_1, \ldots, B_n$ such that

$$\Omega = B_1 \cup \cdots \cup B_n \quad \text{and} \quad g_n = f_i \text{ on } B_i.$$

From (3.25) we get

$$\begin{aligned}\int_\Omega f\,d\nu &= \sup\left\{\sum_{i=1}^{n}\int_{B_i} f_i\,d\nu : n \in \mathbb{N}\right\}\\ &\leq \sup\left\{\sum_{i\in I}\int_{B_i} f_i\,d\nu : (B_i)_{i\in I} \text{ is a finite Borel partition of } \Omega\right\}\end{aligned}$$

which concludes the proof. □

Given a convex and lower semicontinuous function $F : \mathbb{R}^N \to [0, \infty]$ we denote by F^∞ its *recession function* (see for instance Rockafellar [228]), defined by

$$F^\infty(p) := \lim_{t\to\infty} \frac{F(p_0 + tp)}{t}$$

where p_0 is any point such that $F(p_0) < \infty$. It can be shown that the definition above does not depend on p_0, and that the function F^∞ turns out to be convex, lower semicontinuous, and positively homogeneous of degree 1 on $\mathbb{R}^N$.

We are now in a position to prove the following lower semicontinuity theorem on $\mathcal{M}(a, b; \mathbf{R}^N)$.

Theorem 3.12 *Let $F : \Omega \times \mathbf{R}^N \to [0, \infty]$ be a function such that the approximation formula*

$$F(x, p) = \sup\{a_n(x) + b_n(x) \cdot p : n \in \mathbb{N}\} \quad \text{for all } (x, p) \in \Omega \times \mathbf{R}^N \tag{3.26}$$

holds for suitable sequences $a_n \in L^1_{loc}(\Omega; \mu)$ and $b_n \in C(\Omega; \mathbf{R}^N)$. Then the functional

$$\mathcal{F}(\lambda) = \int_\Omega F\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_\Omega F^\infty\left(x, \frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s| \tag{3.27}$$

is sequentially weakly lower semicontinuous on $\mathcal{M}(a, b; \mathbf{R}^N)$, where*

$$\lambda = \frac{d\lambda}{d\mu}\mu + \lambda^s$$

is the Lebesgue–Nikodym decomposition of λ with respect to μ, and $F^\infty(x, \cdot)$ denotes the recession function of $F(x, \cdot)$ for any $x \in \Omega$.

Proof For the sake of simplicity we denote the last term on the right-hand side of (3.27) by

$$\int_\Omega F^\infty(x, \lambda^s) \quad \text{instead of} \quad \int_\Omega F^\infty\left(x, \frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s|.$$

For every $n \in \mathbb{N}$ we set

$$F_n(x, p) := [a_n(x) + b_n(x) \cdot p]^+, \quad (x, p) \in \Omega \times \mathbf{R}^N,$$

$$\mathcal{F}_n(\lambda, B) := \int_B F_n\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_B F_n^\infty(x, \lambda^s), \quad \lambda \in \mathcal{M}(a, b; \mathbf{R}^N), \; B \in \mathcal{B}(\Omega),$$

and fix $\lambda \in \mathcal{M}(a, b; \mathbf{R}^N)$. If we denote the measure $\mu + |\lambda^s|$ by ν, the μ-negligible set where λ^s is concentrated by Z, and set

$$f(x) := \begin{cases} F\left(x, \dfrac{d\lambda}{d\mu}(x)\right) & \text{if } x \in \Omega \setminus Z \\ F^\infty\left(x, \dfrac{d\lambda^s}{d|\lambda^s|}(x)\right) & \text{if } x \in Z \end{cases}$$

$$f_n(x) := \begin{cases} F_n\left(x, \dfrac{d\lambda}{d\mu}(x)\right) & \text{if } x \in \Omega \setminus Z \\ F_n^\infty\left(x, \dfrac{d\lambda^s}{d|\lambda^s|}(x)\right) & \text{if } x \in Z \end{cases}$$

we have

$$\mathcal{F}(\lambda) = \int_\Omega f(x)d\nu, \quad \mathcal{F}_n(\lambda) = \int_\Omega f_n(x)d\nu.$$

Therefore, by the lemma above,

$$\mathcal{F}(\lambda) = \sup\left\{\sum_{i\in I}\mathcal{F}_i(\lambda, B_i) : (B_i)_{i\in I} \text{ is a finite Borel partition of } \Omega\right\}. \tag{3.28}$$

Since μ and $|\lambda^s|$ are regular measures, we have

$$\mathcal{F}_i(\lambda, B_i) = \sup\{\mathcal{F}_i(\lambda, K) : K \text{ compact}, K \subset B_i\}$$

so that by (3.28)

$$\mathcal{F}(\lambda) = \sup\left\{\sum_{i\in I}\mathcal{F}_i(\lambda, K_i) : (K_i)_{i\in I} \text{ disjoint compact subsets of } \Omega\right\}. \tag{3.29}$$

Since two disjoint compact sets have a positive distance, we deduce from (3.29) that

$$\mathcal{F}(\lambda) = \sup\left\{\sum_{i\in I}\mathcal{F}_i(\lambda, A_i) : (A_i)_{i\in I} \text{ disjoint open subsets of } \Omega\right\}. \tag{3.30}$$

Thus, to conclude the proof it will be enough to show that for every $n \in \mathbb{N}$ and any open subset A of Ω the functional $\mathcal{F}_n(\cdot, A)$ is sequentially weakly* lower semicontinuous on $\mathcal{M}(a, b; \mathbf{R}^N)$.

By repeating an analogous localization argument we get

$$\begin{aligned}\mathcal{F}_n(\lambda, A) &= \sup_{\varphi\in C_c(A)}\left\{\int_A\left(a_n + b_n\cdot\frac{d\lambda}{d\mu}\right)\varphi d\mu + \int_A b_n\cdot\frac{d\lambda^s}{d|\lambda^s|}\varphi d|\lambda^s|\right\}\\ &= \sup_{\varphi\in C_c(A)}\left\{\int_A \varphi a_n d\mu + \int_A \varphi b_n\cdot d\lambda\right\}.\end{aligned}$$

Therefore, it suffices to show the lower semicontinuity of the functionals

$$\lambda \mapsto \int_A \varphi a_n d\mu + \int_A \varphi b_n \cdot d\lambda$$

for every $\varphi \in C_c(A)$, which is trivial because, owing to the assumptions made on a_n and b_n, these functionals turn out to be weakly* continuous. □

Remark 1 The approximation assumption (3.26) of the previous theorem is clearly fulfilled if $F = F(p)$ is independent of (x, u) and is convex and lower semicontinuous on $\mathbf{R}^N$, or more generally (see for instance the book by Buttazzo [47], Corollary 3.4.2), if $F(x, p)$ satisfies the following conditions:

(i) F is lower semicontinuous on $\Omega \times \mathbb{R}^N$;

(ii) for every $x \in \Omega$ the function $F(x, \cdot)$ is convex on $\mathbb{R}^N$;

(iii) there exists $u_0 \in L^\infty_\mu(\Omega; \mathbb{R}^N)$ such that $F(x, u_0(x)) \in L^\infty_\mu(\Omega)$.

As a corollary we obtain the following lower semicontinuity Reshetnyak-type theorem.

Theorem 3.13 *Let $F : \Omega \times \mathbb{R}^N \to [0, \infty]$ be a function such that*

(i) *$F(x, p)$ is lower semicontinuous in (x, p);*

(ii) *for every $x \in \Omega$ the function $F(x, \cdot)$ is convex and positively homogeneous of degree 1.*

Then the functional

$$\mathcal{F}(\lambda) = \int_\Omega F(x, \lambda)$$

is sequentially weakly lower semicontinuous on $\mathcal{M}(a, b; \mathbb{R}^N)$, and we recall that the notation $\int_\Omega F(x, \lambda)$ stands for $\int_\Omega F\big(x, d\lambda/d|\lambda|\big)\, d|\lambda|$.*

Remark 2 In Theorem 3.12, when the measure μ does not charge the points of Ω, i.e. $\mu(\{x\}) = 0$ for all $x \in \Omega$, the condition that $F^\infty(x, p)$ is lower semicontinuous in (x, p) is necessary for the lower semicontinuity of the functional $\mathcal{F}$. Indeed, for fixed $x_0 \in \Omega$, $p_0 \in \mathbb{R}^N$, and given sequences $x_n \to x_0$ in Ω and $p_n \to p_0$ in $\mathbb{R}^N$, it suffices to consider the measures $\lambda = p_0\delta_{x_0}$ and $\lambda_n = p_n\delta_{x_n}$. We have $\lambda_n \to \lambda$ weakly* in $\mathcal{M}(a, b; \mathbb{R}^N)$. Thus the weak* lower semicontinuity of $\mathcal{F}$ on $\mathcal{M}(a, b; \mathbb{R}^N)$ implies

$$\mathcal{F}(x_0, p_0) = \mathcal{F}(\lambda) \le \liminf_{n\to\infty} \mathcal{F}(\lambda_n) = \liminf_{n\to\infty} \mathcal{F}(x_n, p_n).$$

Therefore, in the Reshetnyak case of $F(x, \cdot)$ convex and positively homogeneous of degree 1, the functional $\mathcal{F}$ is sequentially weakly* lower semicontinuous on $\mathcal{M}(a, b; \mathbb{R}^N)$ if and only if the function $F(a, p)$ is lower semicontinuous in (x, p).

3.4 Existence results in the space BV

In this section we use the lower semicontinuity results of Section 3.3 in conjunction with direct methods of the calculus of variations to prove the existence of solutions of minimum problems in BV.

We recall that, given an interval $I = (a, b)$ of the real line $\mathbb{R}$, a function $u : I \to \mathbb{R}^N$ belongs to $BV(I; \mathbb{R}^N)$ if and only if it is in $L^1(I, \mathbb{R}^N)$ and its first-order distributional derivative u' belongs to $\mathcal{M}(I; \mathbb{R}^N)$. In Section 2.3 we have discussed the space $BV(I; \mathbb{R}^N)$ and its main properties. In the sequel the measure u' will often be decomposed into its absolutely continuous part u'_a with respect to the Lebesgue measure, and its singular part u'_s; moreover, we will denote by $\dot{u}$ the density of u'_a with respect to the Lebesgue measure. Therefore, the Lebesgue–Nikodym decomposition of u' reads

$$u' = \dot{u}\, dx + u'_s.$$

A straightforward application of Section 3.3 gives the following result.

Theorem 3.14 *Let $F(x,p)$ be a function satisfying the approximation formula of theorem 3.12 of Section 3.3 (compare also Remark 1 of Section 3.3). Then the functional $\mathcal{F}$ defined for every $u \in BV(I;\mathbb{R}^N)$ by*

$$\mathcal{F}(u) = \int_I F(x,\dot u)\,dx + \int_I F^\infty(x,u'_s)$$

is sequentially lower semicontinuous with respect to the weak $BV(I;\mathbb{R}^N)$ convergence.*

Remark 1 Since weak* BV convergence implies strong L^1 convergence (cf. Proposition 2.36 and Proposition 2.37 of Section 2.3), by Fatou's lemma we get that, if the integrand F satisfies the assumptions of the theorem above, then the functional

$$\mathcal{F}(u) = \int_I F(x,\dot u)\,dx + \int_I F^\infty(x,u'_s) + \int_I G(x,u)\,dx \tag{3.31}$$

is also weakly* BV sequentially lower semicontinuous, provided that $G(x,u)$ is a non-negative Borel function lower semicontinuous with respect to u.

Assume now that the integrands F and G in (3.31) satisfy the following estimates from below:

$$F(x,p) \ge \alpha|p| - a(x) \tag{3.32}$$

$$G(x,u) \ge \beta|u| - b(x) \tag{3.33}$$

for suitable positive constants α and β and integrable functions $a(x)$ and $b(x)$. Then the functional $\mathcal{F}$ in (3.31) turns out to satisfy

$$(\alpha \wedge \beta)\|u\|_{BV} \le \mathcal{F}(u) + C \quad \text{for every } u \in BV(I;\mathbb{R}^N)$$

where we have set $C = \int_I [a(x) + b(x)]\,dx$. Hence $\mathcal{F}$ is coercive for the weak* convergence on BV, and by the direct methods of the calculus of variations we get the existence of a solution of the minimum problem

$$\min\{\mathcal{F}(u) : u \in BV(I;\mathbb{R}^N)\}.$$

In order to investigate minimum problems with fixed boundary values $u(a) = \alpha$, $u(b) = \beta$, it is convenient to introduce an interval $I' = (a',b')$ with $a' < a$ and $b' > b$ and the class

$$\mathcal{A}_{\alpha,\beta} = \{u \in BV(I';\mathbb{R}^N) : u(x) = \alpha \text{ for } x < a,\ u(x) = \beta \text{ for } x > b\}.$$

The minimum problem for $\mathcal{F}$ with boundary values $u(a) = \alpha$ and $u(b) = \beta$ is then the problem

$$\min\left\{\int_a^b F(x,\dot u)\,dx + \int_{[a,b]} F^\infty(x,u'_s) + \int_a^b G(x,u)\,dx : u \in \mathcal{A}_{\alpha,\beta}\right\}. \tag{3.34}$$

Note that a function u in $\mathcal{A}_{\alpha,\beta}$ may jump at the boundary points of I; in this case we have

$$u_s'(a) = [u(a^+) - \alpha] \cdot \delta_a, \qquad u_s'(b) = [\beta - u(b^-)] \cdot \delta_b,$$

and therefore

$$\int_{[a,b]} F^\infty(x, u_s') = \int_{(a,b)} F^\infty(x, u_s') + F^\infty(a, [u(a^+) - \alpha]) + F^\infty(b, [\beta - u(b^-)]).$$

By using Proposition 2.39 of Section 2.3 we obtain that under assumption (3.32) on F there exist two positive constants c and c_0 such that

$$\int_a^b F(x, \dot{u})\,dx + \int_{[a,b]} F^\infty(x, u_s') \geq c\|u\|_{BV} - c_0$$

for every $u \in \mathcal{A}_{\alpha,\beta}$. Therefore, in this case we do not need assumption (3.33) on G in order to obtain the coerciveness estimate.

Summarizing, we have found the following existence results.

Theorem 3.15 *Let $F(x, p)$ be a function satisfying the approximation formula of Theorem 3.12 of Section 3.3 (compare also Remark 1 of Section 3.3) and inequality (3.32). Then the minimum problem with free boundary conditions,*

$$\min\left\{\int_a^b F(x, \dot{u})\,dx + \int_{(a,b)} F^\infty(x, u_s') + \int_a^b G(x, u)\,dx : u \in BV(I, \mathbb{R}^N)\right\},$$

admits a solution provided that $G(x, u)$ is a non-negative Borel function which is lower semicontinuous with respect to u and verifies inequality (3.33). On the other hand the minimum problem with fixed boundary conditions,

$$\min\left\{\int_a^b F(x, \dot{u})\,dx + \int_{[a,b]} F^\infty(x, u_s') + \int_a^b G(x, u)\,dx : u \in \mathcal{A}_{\alpha,\beta}\right\},$$

admits a solution even if the function G does not satisfy inequality (3.33).

Remark 2 If $F(a, \cdot)$ and $F(b, \cdot)$ have a superlinear growth, then

$$F^\infty(b, p) = F^\infty(a, p) = \begin{cases} \infty & \text{if } p \neq 0 \\ 0 & \text{if } p = 0 \end{cases}$$

so that problem (3.34) can be written in the usual way:

$$\min\left\{\int_a^b F(x, \dot{u})\,dx + \int_{(a,b)} F^\infty(x, u_s') + \int_a^b G(x, u)\,dx : \right.$$
$$\left. u \in BV(I; \mathbb{R}^N),\ u(a) = \alpha,\ u(b) = \beta\right\}.$$

In the following examples we will show that in general variational problems with a linear growth may admit solutions which are not in $H^{1,1}$.

$\boxed{1}$ Let I be the interval $[-1, 1]$ and let $f : I \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

For a given positive real number k we consider the functional

$$\mathcal{F}_k(u) = \int_{-1}^{1} \left[|u'| + k|u - f(x)|\right] dx.$$

The functional $\mathcal{F}_k$ can be defined for every $u \in BV(I)$ by setting

$$\mathcal{F}_k(u) = \|u'\|_{\mathcal{M}(I)} + k \int_{-1}^{1} |u - f(x)|\, dx.$$

Therefore the existence theorem 3.15 applies and furnishes a solution of the minimum problem

$$\min\{\mathcal{F}_k(u) \colon u \in BV(I)\}. \tag{3.35}$$

We will show that for $k > 1$ the only solution of problem (3.35) is given by $u(x) = f(x)$, and thus $u \notin H^{1,1}(I)$. We present the argument in three steps.

Step 1. It suffices to consider functions $u \in NBV(I)$ with $|u(x)| \leq 1$ for every $x \in I$. Indeed it is easy to see that for every $u \in NBV(I)$ the function u defined by

$$v(x) := \max\{-1, \min\{u(x), 1\}\}$$

furnishes a smaller value for the functional $\mathcal{F}_k$.

Step 2. We may restrict ourselves to functions $u \in NBV(I)$ which are non-decreasing on I. In fact if $|u| \leq 1$ on I, it is easy to see that the function v defined by

$$v(x) := \begin{cases} \inf\{u(y) \colon y \geq x\} & \text{if } x < 0 \\ \sup\{u(y) \colon y \leq x\} & \text{if } x \geq 0 \end{cases}$$

is non-decreasing and provides a smaller value for $\mathcal{F}_k$.

Step 3. If $k > 1$, the only solution of the minimum problem (3.35) is $u = f$. Indeed, setting

$$\alpha = \lim_{x \to -1} u(x), \qquad \beta = \lim_{x \to 1} u(x),$$

we have that $\alpha \geq -1$, $\beta \leq 1$, and

$$\begin{aligned} \mathcal{F}_k(u) &\geq \int_{-1}^{1} u' + k \int_{-1}^{0} |\alpha + 1|\, dx + k \int_{0}^{1} |\beta - 1|\, dx \\ &= \beta - \alpha + k(2 + \alpha - \beta) = 2k + (\beta - \alpha)(1 - k). \end{aligned}$$

Moreover, the inequality above is strict if $u \neq f$. Since $\beta \geq \alpha$ (by step 2) and $k > 1$ we obtain

$$\mathcal{F}_k(u) \geq 2k + 2(1-k) = 2$$

with a strict inequality if $u \neq f$. Finally, by the fact that $\mathcal{F}_k(f) = 2$, we obtain that for $k > 1$ the only solution of the minimum problem (3.35) is $u = f$.

With similar arguments the reader can show that for $0 < k < 1$ the only solutions of (3.35) are the constant functions $u \equiv c$ with $|c| \leq 1$, and for $k = 1$ the solutions are all functions of the form

$$u(x) = \begin{cases} c_1 & \text{if } x \leq 0 \\ c_2 & \text{if } x > 0 \end{cases}$$

with $-1 \leq c_1 \leq c_2 \leq 1$.

$\boxed{2}$ Let $a : [0, 1] \to \mathbb{R}$ be a non-negative measurable function. We claim that for suitable choices of $M > 0$ and of the function $a(x)$, the minimum problem

$$\min \left\{ \int_0^1 \sqrt{1 + a(x)|u'|^2}\, dx : u \in H^{1,1}(0, 1), u(0) = 0, u(1) = M \right\}$$

does not have any solution; see Giaquinta, Modica, Souček [114]. In fact if $u \in H^{1,1}(0, 1)$ were a solution, Euler's equation would yield that

$$\frac{a(x)u'}{\sqrt{1 + a(x)|u'|^2}} = c \tag{3.36}$$

for a suitable constant c. Thus the solution u has to be non-decreasing if $M > 0$. Moreover, we infer from (3.36) that

$$0 \leq c \leq \inf \sqrt{a} \tag{3.37}$$

where inf denotes the essential infimum on $(0, 1)$. Solving (3.36) with respect to u' we are led to

$$u'(x) = \left(\frac{c^2}{a^2(x) - c^2 a(x)} \right)^{1/2}$$

whence

$$M = \int_0^1 u'(x)\, dx = \int_0^1 \left(\frac{c^2}{a^2(x) - c^2 a(x)} \right)^{1/2} dx. \tag{3.38}$$

The function $c^2/[a^2(x) - c^2 a(x)]$ is increasing with respect to c and reaches its maximum for $c = \inf\sqrt{a}$, on account of (3.37). Because of (3.38) we obtain

$$M \leq \int_0^1 \left(\frac{\inf a}{a^2(x) - a(x) \inf a} \right)^{1/2} dx. \tag{3.39}$$

Thus, to achieve a contradiction, thereby proving the non-existence of an $H^{1,1}$-solution, we choose $a(x)$ in such a way that the right-hand side of (3.39) is finite; then (3.39) is violated if M is chosen sufficiently large.

Note that the integrand

$$F(x, p) = \sqrt{1 + a(x)|p|^2}$$

satisfies all assumptions of the lower semicontinuity theorem 3.14 if $a(x)$ is continuous; hence if we require that in addition

$$a(x) \geq \alpha > 0 \quad \text{for all } x \in (0, 1)$$

for a suitable positive constant α, the existence theorem 3.15 yields that the minimum problem

$$\min \left\{ \int_0^1 \sqrt{1 + a(x)\dot{u}^2}\, dx + \int_0^1 \sqrt{a(x)}\,|u'_s| \right. \\ \left. + \sqrt{a(0)}\,|u(0)| + \sqrt{a(1)}\,|u(1) - M| : u \in BV(0, 1) \right\} \tag{3.40}$$

has a solution. Note that problem (3.40) can also be written as

$$\min \left\{ \int_0^1 \sqrt{1 + a(x)\dot{u}^2}\, dx + \int_{[0,1]} \sqrt{a(x)}\,|u'_s| : u \in BV(\mathbb{R}), \right. \\ \left. u(x) = 0 \text{ for } x < 0, \quad u(x) = M \text{ for } x > 1 \right\}.$$

4

REGULARITY OF MINIMIZERS

In this chapter we discuss the *regularity theory* of minimizers of one-dimensional variational problems in the class of absolutely continuous functions or, more generally, in the Sobolev spaces $H^{1,m}$.

In Section 4.1 we treat variational integrals $\mathcal{F}(u)$ the Lagrangians of which have polynomial growth of order $m > 1$. We shall prove that the minimizers in $H^{1,m}$ are as regular as the integrand allows them to be; in particular they are of class C^∞ or real analytic, if the Lagrangian F is of class C^∞ or real analytic.

In Section 4.2 we prove *Tonelli's partial regularity theorem* which describes the regularity properties of minimizers of class AC.

In Section 4.3 we discuss the so-called *Lavrentiev phenomenon* that was considered in Section 3.2. Moreover, we present several examples due to Ball–Mizel [21] and Davie [76] which show that Tonelli's result is optimal. Furthermore, we see from the examples that singularities have to occur if we consider minimizers for general integrals in AC. This will conclude the discussion that we begin in Section 4.2 with Mania's example.

4.1 The regular case

The main result of this section is the following regularity theorem.

Theorem 4.1 *Let $I = (a, b)$ be a bounded integral in $\mathbb{R}$, and let $F(x, u, p)$ be a Lagrangian of class C^2 defined on $\overline{I} \times \mathbb{R}^N \times \mathbb{R}^N$, $N \geq 1$, satisfying the following conditions:*

(i) *there are constants $c_0, c_1 > 0$ such that for all $(x, z, p) \in \overline{I} \times \mathbb{R}^N \times \mathbb{R}^N$*

$$c_0 |p|^m \leq F(x, z, p) \leq c_1 (1 + |p|^m); \tag{4.1}$$

(ii) *there is a function $M(R) > 0$ such that*

$$|F_z(x, z, p)| + |F_p(x, z, p)| \leq M(R)(1 + |p|^2) \tag{4.2}$$

for all $(x, z, p) \in \overline{I} \times \mathbf{R}^N \times \mathbb{R}^N$ with $x^2 + |z|^2 \leq R^2$;

(iii) *for all $(x, z, p) \in \overline{I} \times \mathbf{R}^N \times \mathbb{R}^N$ and all $\xi \in \mathbb{R}^N - \{0\}$ we have*

$$F_{p^i p^k}(x, z, p)\xi^i \xi^k > 0. \tag{4.3}$$

Let C be the class of functions $v \in H^{1,m}(I, \mathbf{R}^N)$, $I = (a, b)$, satisfying the boundary conditions $v(a) = \alpha$ and $v(b) = \beta$, and suppose that $u \in C$ is a (local) minimizer of the variational integral

$$\mathcal{F}(u) := \int_I F(x, u(x), u'(x))\, dx \tag{4.4}$$

in C. Then u belongs to $C^2(\overline{I}, \mathbb{R}^N)$ and satisfies the Euler equation $L_F(u) = 0$. Moreover, if F is class C^k, $2 \le k \le \infty$, then $u \in C^k(\overline{I}, \mathbb{R}^N)$, and u is real analytic if $F(x, z, p)$ is real analytic.

We divide the proof of this theorem into several steps. We begin with

Proposition 4.2 *Let $u \in C^1(\overline{I}, \mathbb{R}^N)$ be a weak extremal of $\mathcal{F}$, and suppose that $F_p \in C^1(U)$ for a suitable neighbourhood U of the curve $(x, u(x), u'(x))$ and that $F_{pp}(x, u(x), u'(x))$ is invertible for all $x \in \overline{I}$. Then $u \in C^2(\overline{I}, \mathbb{R}^N)$.*

Proof We can choose the neighbourhood U of 1-graph u in such a way that $F_{pp}(x, z, p)$ is invertible for every $(x, z, p) \in U$. By Proposition 1.9 of Section 1.1 there is a constant vector $c \in \mathbb{R}^N$ such that we can write

$$F_p(x, u(x), u'(x)) = \pi(x) \quad \text{for all } x \in I$$

where

$$\pi(x) := \int_a^x F_z(t, u(t), u'(t))\, dt + c. \tag{4.5}$$

On account of the assumption on u and F we have $u \in C^1(\overline{I})$. Thus the mapping $G : \overline{I} \times \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$G(x, p) := F_p(x, u(x), p) - \pi(x)$$

is of class $C^1(\overline{I} \times \mathbb{R}^N, \mathbb{R}^N)$ and satisfies

$$\det G_p(x, u'(x)) \neq 0 \quad \text{for all } x \in I$$

since $G_p(x, u'(x)) = F_{pp}(x, u(x), u'(x))$ for all $x \in \overline{I}$. Since $p = u'(x)$, $x \in \overline{I}$, is a solution of the equation $G(x, p) = 0$ the implicit function theorem yields $u' \in C^1(\overline{I}, \mathbb{R}^N)$. □

If F is of a higher regularity class, then the implicit function theorem furnishes better regularity for u. The following result is fairly obvious.

Proposition 4.3 *Suppose that the Lagrangian is of class C^k, $2 \le k \le \infty$, on a neighbourhood U of the 1-graph of a weak extremal $u \in C^1(\overline{I}, \mathbb{R}^N)$ of $\mathcal{F}$, and that* det $F_{pp}(x, z, p) \neq 0$ *for all $(x, z, p) \in U$. Then $u \in C^k(\overline{I}, \mathbb{R}^N)$. Moreover, u is real analytic if F is real analytic on U.*

Now we want to extend the result of Proposition 4.2 to weak Lipschitz extremals of $\mathcal{F}$. Here we need to assume that F_{pp} is positive definite because our proof requires a global inverse of the map $(x, z, p) \to (x, z, y)$ with $y = F_p(x, z, p)$. This follows from the fact that $u'(x)$ is no longer continuous, and therefore we cannot merely operate in a small neighbourhood of $p_0 = u'(x_0)$ for any $x_0 \in \overline{I}$.

Proposition 4.4 *Let $u \in Lip(I, \mathbb{R}^N)$ be a weak Lipschitz extremal of $\mathcal{F}$ and suppose that F_p is of class C^1 on $\overline{I} \times \mathbb{R}^N \times \mathbb{R}^N$ and that $F_{pp}(x, z, p)$ is positive definite on $\overline{I} \times \mathbb{R}^N \times \mathbb{R}^N$. Then $u \in C^2(\overline{I}, \mathbb{R}^N)$.*

Proof Consider the mapping $\Psi : (x, z, p) \to (x, z, q)$ of $\overline{I} \times \mathbb{R}^N \times \mathbb{R}^N$ into itself defined by

$$\Psi(x, z, p) := (x, z, F_p(x, z, p)).$$

Since $F_{pp} > 0$, a standard reasoning shows that Ψ is a C^1-diffeomorphism of $\overline{I} \times \mathbb{R}^N \times \mathbb{R}^N$ onto its image $\mathcal{V} := \Psi(\overline{I} \times \mathbb{R}^N \times \mathbb{R}^N)$. By virtue of Section 1.1, Proposition 1.11, there is a constant vector $c \in \mathbb{R}^N$ such that

$$F_p(x, u(x), u'(x)) = \pi(x) \quad \text{a.e. on } I \tag{4.6}$$

where $\pi(x)$ is again defined by (4.5). Clearly we have $\pi \in AC(I, \mathbb{R}^N)$, $\pi'(x) = F_z(x, u(x), u'(x))$ a.e. on I, and $\pi' = F_z(\cdot, u, u') \in L^\infty(I, \mathbb{R}^N)$. Set

$$\sigma(x) := (x, u(x), u'(x)), \qquad e(x) := (x, u(x), \pi(x)).$$

The function $\sigma(x)$ is defined a.e. on I, whereas $e(x)$ is defined for all $x \in \overline{I}$, and (4.6) implies that

$$\Psi(\sigma(x)) = e(x) \quad \text{a.e. on } I. \tag{4.7}$$

In order to show that $\Psi^{-1}(x, u(x), \pi(x))$ is well defined for all $x \in \overline{I}$, we have to prove that $e(x) \in \mathcal{V}$ for all $x \in \overline{I}$. This is not a priori clear since (4.7) merely yields $e(x) \in \mathcal{V}$ a.e. on I, whence $e(x) \in \overline{\mathcal{V}}$ for all $x \in \overline{I}$ since $e(x)$ is continuous on $\overline{I}$.

Since $u \in Lip(I, \mathbb{R}^N)$ there is a constant $k > 0$ such that

$$|u(x) - u(\xi)| \le k|x - \xi| \quad \text{for all } x, \xi \in \overline{I}.$$

Then we obtain

$$|u'(x)| \le k \quad \text{a.e. on } I. \tag{4.8}$$

Consider the set

$$\mathcal{K} := \{\Psi(x, u(x), p) : \ x \in \overline{I} \times \mathbb{R}^N, |p| \le k\}.$$

Clearly $\mathcal{K}$ is a compact subset of $\mathcal{V}$, and by (4.7) and (4.8) we have $e(x) \in \mathcal{K}$ a.e. on I. Since $e(x)$ is continuous on $\overline{I}$ it follows that $e(x) \in \mathcal{K} \subset \mathcal{V}$ for all $x \in \overline{I}$; thus the function

$$(x, u(x), v(x)) := \Psi^{-1}(e(x)), \quad x \in \overline{I},$$

is well defined and continuous. On the other hand relation (4.7) implies that

$$(x, u(x), u'(x)) = \sigma(x) = \Psi^{-1}(e(x)) \quad \text{a.e. on } I$$

and therefore

$$u'(x) = v(x) \quad \text{a.e. on } I.$$

Then

$$u(x) = u(a) + \int_a^x u'(t)\,dt = u(a) + \int_a^x v(t)\,dt$$

and we obtain that $u \in C^1(\overline{I}, \mathbb{R}^N)$. Now we can apply Proposition 4.2 and arrive at $u \in C^2(\overline{I}, \mathbb{R}^N)$. □

Proof of Theorem 4.1 Because of (4.1) the integral $\mathcal{F}(v)$ is well defined for any $v \in \mathcal{C}$. Let $\varphi \in Lip(I, \mathbb{R}^N)$ and $\epsilon \in \mathbb{R}$, and suppose that $|\varphi(x)| \le Q$ on I and $|\varphi'(x)| \le Q$ a.e. on I. Fix some ϵ_0 with $0 < \epsilon_0 \le 1$, and let $|\epsilon| < \epsilon_0$. Then there is a number R such that

$$x^2 + |u(x) + \epsilon\varphi(x)|^2 \le R^2 \quad \text{for all } x \in \bar{I}.$$

Furthermore,

$$|u' + \epsilon\varphi'|^m \le 2^{m-1}(|u'|^m + Q^m).$$

On account of (4.2) we see that the function

$$F_z(x, u + \epsilon\varphi, u' + \epsilon\varphi') + F_p(x, u + \epsilon\varphi, u' + \epsilon\varphi')$$

is a.e. dominated by the L^1-function

$$QM(R)[1 + 2^{m-1}(Q^m + |u'|^m)].$$

Then Lebesgue's dominated convergence theorem yields that $\Phi(\epsilon) := \mathcal{F}(u + \epsilon\varphi)$ is of class C^1 on $(-\epsilon_0, \epsilon_0)$, and since u is a local minimizer of $\mathcal{F}(v)$ in $\mathcal{C}$, it follows that $\Phi(0) \le \Phi(\epsilon)$ for $|\epsilon| < \epsilon_0 \ll 1$ if $\varphi \in C_c^\infty(I, \mathbb{R}^N)$, whence $\Phi'(0) = 0$ and therefore

$$\int_I \{F_z(x, u, u') \cdot \varphi + F_p(x, u, u') \cdot \varphi'\} \, dx = 0 \tag{4.9}$$

for all $\varphi \in C_c^\infty(I, \mathbb{R}^N)$, i.e. $u \in AC(I, \mathbb{R}^N)$ with $u' \in L^m(I, \mathbb{R}^N)$ is a weak $H^{1,m}$-extremal of $\mathcal{F}$. Furthermore, by virtue of (4.2), we have $F_z(\cdot, u, u') \in L^1(I, \mathbb{R}^N)$.

Then we can proceed as in Section 1.1, Propositions 1.9 and 1.11, to show that there is a vector $c \in \mathbb{R}^N$ such that

$$F_p(x, u(x), u'(x)) = c + \int_a^x F_z(t, u(t), u'(t)) \, dt \quad \text{a.e. on } I$$

and we arrive once again at eqn (4.6).

Now we can proceed as in the proof of Proposition 4.4 provided that the image set $e(\bar{I})$ of the mapping $e : \bar{I} \to \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ lies in the range of Ψ. This is achieved by assumption (iii), which implies $F_p(x, z, \mathbb{R}^N) = \mathbb{R}^N$, whence $\mathcal{V} = \bar{I} \times \mathbb{R}^N \times \mathbb{R}^N$. Then the reasoning above yields $u \in C^2(\bar{I}, \mathbb{R}^N)$, and Proposition 4.3 yields the other statements of Theorem 4.1. □

The reasoning above actually yields the following result:

Theorem 4.5 *If $u \in H^{1,m}(I, \mathbb{R}^N)$ is a 'critical point' of $\mathcal{F}$, or more precisely, a weak $H^{1,m}$-extremal, i.e.*

$$\delta\mathcal{F}(u, \varphi) = 0 \quad \textit{for all } \varphi \in C_c^\infty(I, \mathbb{R}^N),$$

then $u \in C^2(\bar{I}, \mathbb{R}^N)$, and u satisfies the Euler equation $L_F(u) = 0$, provided that the assumptions of Theorem 4.1 hold true. Moreover, $u \in C^k$ if F is of class C^k, $2 \le k \le \infty$, and u is real analytic if F is real analytic.

Remark 1 A result analogous to Theorem 4.5 can be proved for solutions $u \in H^{1,m}(I, \mathbb{R}^N)$ of

$$\int_I \{A(x,u,u')\cdot\varphi' + B(x,u,u')\cdot\varphi\}\,dx = 0 \quad \text{for all } \varphi \in C_c^\infty(I,\mathbb{R}^N), \tag{4.10}$$

i.e. for weak solutions u of

$$\frac{d}{dx}A(x,u,u') = B(x,u,u') \quad \text{in } I, \tag{4.11}$$

provided that the assumptions of Theorem 4.1 hold, where F_p and F_u have to be replaced by A and B respectively.

We conclude this section with three examples. The first two indicate that a non-degeneracy condition of F_{pp} is necessary, while the third example shows that Theorem 4.1 is not optimal.

$\boxed{1}$ Every Lipschitz function u_0 in $(0, 1)$ with the property that u_0' takes only the values 1 and -1 is obviously a minimizer of

$$\int_0^1 (u'^2 - 1)^2\,dx$$

in the class $\{u \in H^{1,4}(0,1) : u(0) = u_0(0), u(1) = u_0(1)\}$. In this case we have $F_{pp}(p) = 12p^2 - 4$.

$\boxed{2}$ A solution of

$$\min\left\{\int_{-1}^1 u^2(2x - u')^2\,dx :\ u \in H^{1,2}(0,1),\ u(-1) = 0,\ u(1) = 1\right\}$$

is the function

$$u(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ x^2 & \text{if } 0 \le x \le 1 \end{cases}$$

which is of class $C^{1,1}([-1,1])$ but not of class C^2. Here

$$F_{pp}(x,u,p) = 2u^2 \ge 0.$$

This example was considered in Section 1.1.

$\boxed{3}$ The integrand of $\int_a^b (\dot u^2 + u^2 + e^u)\,dx$ does not satisfy the assumptions of Theorem 4.1. Nevertheless, by observing that minimizers for Dirichlet boundary conditions exist in $H^{1,2}(a,b)$ and are bounded, one can apply a modification of Theorem 4.1 inferring that minimizers are actually smooth.

4.2 Tonelli's partial regularity theorem

We shall now discuss regularity of minimizers of general variational integrals. We shall not require that the integrand F has a polynomial growth $m > 1$ as we did in Section 4.1, but, for the sake of simplicity, we only treat the scalar case.[11] The main result is the following.

Theorem 4.6 (Tonelli's partial regularity theorem) *Let $F(x, u, p)$ be a smooth Lagrangian, say, of class C^∞, satisfying $F_{pp}(x, u, p) > 0$ for every (x, u, p), and suppose that $u \in AC(a, b)$ is a strong local minimizer of the functional*

$$\mathcal{F}(u) = \int_a^b F(x, u, u')\,dx$$

in the class $\mathcal{C}$ of all absolutely continuous functions in $[a, b]$ having the same boundary values as u. Then u has a (possibly infinite) classical derivative $[u'(x)]$ at each point of $[a, b]$, and $[u'] : [a, b] \to \mathbb{R} \cup \{+\infty, -\infty\}$ is continuous. Moreover, the singular set $E := \{x \in [a, b] : [u'(x)] = \pm\infty\}$ is closed and has measure zero; finally, u is of class C^∞ outside E.

After proving Theorem 4.6, we shall state a few corollaries which yield regularity everywhere under extra assumptions. Then in the next section we shall present a result which shows that:

(a) Under the general assumptions of Theorem 4.6, and even assuming that F has a superlinear growth, the set E is in general non-empty. Indeed, any closed set with zero measure can occur as the singular set of a minimizer of a suitable variational integral.

Other examples will be discussed in the scholia showing that:

(b) Euler's equation may fail to hold for minimizers in Theorem 4.6.

The proof of Theorem 4.6 uses some classical results concerning the local solvability of Euler's equations associated with $\mathcal{F}$ and some results of field theory (cf. Section 1.2). The main ideas of the proof are the following. If at some point the difference quotients of u are equibounded we can solve classically Dirichlet's boundary value problem with data u for the Euler equation of $\mathcal{F}$ in a small neighbourhood of such a point. Moreover, since the solution $\tilde{u}$ of the Dirichlet problem can be embedded into a Mayer field and the Weierstrass condition is satisfied, $\tilde{u}$ is a minimizer of $\mathcal{F}$ in such a small interval and coincides with u; in particular u is regular in a neighbourhood of our point. Since u is almost everywhere differentiable in the classical sense, we then infer that the difference quotients are almost everywhere bounded; hence u is regular in an open set Ω_0 and $E := [a, b] - \Omega_0$ has measure zero.

Finally, before proving the theorem, we note that Theorem 4.6 obviously applies to the minimizers obtained by Tonelli's existence theorem 3.7 of Section 3.2, provided that $N = 1$.

For the reader's convenience we first collect in Lemma 4.7 some standard results from the theory of ordinary differential equations in the form to be used later.

[11] For further results and for the vector-valued case we refer to the scholia in Chap. 6

Lemma 4.7 *Let $F(x, z, p)$ be a smooth Lagrangian satisfying $F_{pp} > 0$. Moreover, let $A \subset \mathbb{R}^2$ be a bounded open set and let M and δ be positive constants. Then there exists an $\epsilon > 0$ such that, if $(x_0, u_0) \in A$, $|\alpha| \le M$, $|\beta| \le M$, the classical solution $u(x; \alpha, \beta)$ of the Euler equation*

$$-\frac{d}{dx}F_p(x, u, u') + F_z(x, u, u') = 0 \tag{4.12}$$

satisfying the initial conditions

$$u(x_0; \alpha, \beta) = u_0 + \alpha, \qquad u'(x_0; \alpha, \beta) = \beta \tag{4.13}$$

exists for $|x - x_0| < \epsilon$ and is unique. Moreover, we have

(a) *u and u' are C^1-functions of x, α, β in the set*

$$S := \{(x, \alpha, \beta) : |x - x_0| < \epsilon, |\alpha| \le M, |\beta| \le M\};$$

(b) *on the set S we have*

$$|u'(x; \alpha, \beta) - \beta| < \delta \tag{4.14}$$

$$\frac{\partial u}{\partial \alpha}(x; \alpha, \beta) > 0, \ \text{sign}\ \frac{\partial u}{\partial \beta}(x; \alpha, \beta) = \ \text{sign}(x - x_0). \tag{4.15}$$

Proof Since $F_{pp} > 0$, solving (4.12) is equivalent to solving

$$u'' = f(x, u, u')$$

where $f(x, u, p) := (-pF_{pu} - F_{px} + F_z)/F_{pp}$. The existence, uniqueness, and smoothness assertions, including smooth dependence on the data, then follow from standard results on ordinary differential equations. Since

$$\frac{\partial u}{\partial \alpha}(x_0; \alpha, \beta) = 1, \qquad \frac{\partial u}{\partial \beta}(x_0; \alpha, \beta) = 0, \qquad \frac{\partial u'}{\partial \beta}(x_0; \alpha, \beta) = 1$$

it easily follows that we can choose ϵ sufficiently small so that (b) also holds. □

Lemma 4.8 *Let m, ρ, M_1 be three positive constants. Then there exists $\epsilon > 0$ such that if $(x_0, x_1) \subset [a, b]$, $0 < x_1 - x_0 < \epsilon$, $|u_0| \le m$, and $|(u_1 - u_0)/(x_1 - x_0)| < M_1$, then there exists a unique solution $\tilde{u} \in C^2([x_0, x_1])$ of* (4.12) *satisfying $\tilde{u}(x_0) = u(x_0)$, $\tilde{u}(x_1) = u_1$, and $\max_{[x_0, x_1]} |\tilde{u}(x) - u_0| < \rho$. Moreover, $\tilde{u}$ is the unique minimizer of*

$$\mathcal{F}(u; (x_0, x_1)) := \int_{x_0}^{x_1} F(x, u, u')\, dx$$

over the set

$$\mathcal{A} := \{u \in H^{1,1}(x_0, x_1) : u(x_0) = u_0, u(x_1) = u_1, \max_{[x_0, x_1]} |\tilde{u}(x) - u_0| < \rho\}.$$

Proof Set $\sigma := m + \rho$, $A = (a, b) \times (-\sigma, \sigma)$, and choose $M > \max(M_1, 2\rho)$, $0 < \delta < M - M_1$. Accordingly, let $\epsilon > 0$ be as in Lemma 4.7, and suppose in addition that $3M\epsilon < \rho$. By integrating (4.14) we find that for $x \in [x_0, x_1]$

$$\left|u(x; \alpha, \beta) - u_0 - \alpha - \beta(x - x_0)\right| \leq \delta(x - x_0). \tag{4.16}$$

Therefore, observing that by assumption

$$u_0 - M_1(x - x_0) \leq u_1 \leq u_0 + M_1(x - x_0),$$

we get

$$u(x_1; 0, M) \geq u_0 + M_1(x_1 - x_0) + (M - M_1 - \delta)(x_1 - x_0) > u_1$$
$$u(x_1; 0, -M) \leq u_0 - M_1(x_1 - x_0) - (M - M_1 - \delta)(x_1 - x_0) < u_1.$$

Since $\partial u/\partial\beta(x_1; 0, \beta) > 0$ for $\beta \in [-M, M]$, we infer that there is a unique $\beta_0 \in [-M, M]$ such that $u(x_1; 0, \beta_0) = u_1$. We now define

$$\tilde{u}(x) := u(x; 0, \beta_0).$$

Setting $x = x_1, \alpha = 0, \beta = \beta_0$ in (4.16), we obtain

$$\left|u_1 - u_0 - \beta_0(x_1 - x_0)\right| \leq \delta(x_1 - x_0);$$

thus

$$\left|\beta_0\right| \leq \delta + M_1. \tag{4.17}$$

Therefore, again by (4.16), for $x \in [x_0, x_1]$:

$$\left|\tilde{u}(x) - u_0\right| \leq (\delta + \left|\beta_0\right|)(x - x_0) \leq (2\delta + M_1)\epsilon < \rho.$$

Now suppose that $v \in C^2([x_0, x_1])$ is also a solution of (4.12) satisfying $v(x_0) = u_0$, $v(x_1) = u_1$, and $\max_{[x_1, x_2]} \left|v(x) - u_0\right| < \rho$. Then for some $\overline{x} \in (x_0, x_1)$

$$v'(\overline{x}) = \frac{u_1 - u_0}{x_1 - x_0}$$

and $(\overline{x}, v(\overline{x})) \in \mathcal{A}$; thus, applying Lemma 4.7 and in particular (4.14) with $(\overline{x}, v(\overline{x}))$ replacing (x_0, u_0) and $v'(\overline{x})$ replacing β, we deduce for $x \in [x_0, x_1]$

$$\left|v'(x) - \frac{u_1 - u_0}{x_1 - x_0}\right| \leq \delta.$$

In particular

$$\left|v'(x_0)\right| \leq M_1 + \delta < M.$$

Since, as we have seen, there exists a unique $\beta_0 \in [-M, M]$ such that the solution of (4.12) with initial values $u(x_0) = u_0$, $u'(x_0) = \beta_0$ has values u_1 at x_1, we deduce that $v'(x_0) = \beta$, and thus $v = \tilde{u}$.

To show that $\tilde{u}$ minimizes $\mathcal{F}(u; (x_0, x_1))$ in $\mathcal{A}$, we consider the one-parameter family of solutions $\{u(\cdot; \alpha, \beta_0) : |\alpha| \le M\}$. By (4.16), (4.17) we have for $x \in [x_0, x_1]$

$$u(x; M, \beta_0) - u_0 \ge M + (\beta_0 - \delta)(x - x_0) \ge M - (2\delta + M_1)\epsilon > \rho$$

and

$$u(x; -M, \beta_0) - u_0 \le -M + (\beta_0 + \delta)(x - x_0) \le -M + (2\delta + M_1)\epsilon < -\rho.$$

Since $\partial u/\partial\alpha(x; \alpha, \beta_0) > 0$, it follows that $\tilde{u}$ is embedded in a field of extremals that simply cover the region $[x_0, x_1] \times [u_0 - \rho, u_0 + \rho]$. Since $F_{pp} > 0$, it follows from the Weierstrass formula (cf. Section 1.3) that

$$\mathcal{F}(u; (x_0, x_1)) > \mathcal{F}(\tilde{u}; (x_0, x_1))$$

for all $u \in \mathcal{A}$, with equality if and only if $u = \tilde{u}$. □

Proof of Theorem 4.6 Denote by $C(a, b)$ the class of absolutely continuous functions on (a, b) with the same boundary values as u. Then there exists a constant $\delta_1 > 0$ such that $\mathcal{F}(u) \le \mathcal{F}(v)$ for all $v \in C(a, b)$ with $\max_{[a,b]} |u(x) - v(x)| \le \delta_1$. Let $\overline{x}$ be a point in $[a, b]$ where the difference quotients of u are equibounded, or simply where

$$M(\overline{x}) := \liminf_{\substack{\to\overline{x} \\ x\in[a,b]}} \left| \frac{u(x) - u(\overline{x})}{x - \overline{x}} \right| < +\infty. \tag{4.18}$$

Suppose that $\overline{x} \neq b$ and take $\overline{x}_1 > \overline{x}$ with $\overline{x}_1 - \overline{x}$ sufficiently small so that $\max_{x\in[\overline{x},\overline{x}_1]} |u(x) - u(\overline{x})| < \delta_1/2$. Choose $M_1 > M(\overline{x})$. By (4.18) we can apply Lemma 4.8 with $x_0 = \overline{x}, u_0 = u(\overline{x}), \rho = \delta_1/2, u_1 = u(x_1)$, where $x_1 \in (\overline{x}, \overline{x}_1)$ satisfies

$$x_1 - \overline{x} < \epsilon, \qquad \left| \frac{u(x_1) - u(\overline{x})}{x_1 - \overline{x}} \right| < M_1.$$

Let $\tilde{u}$ be the corresponding solution of the Euler equation, and let $\hat{u} \in (a, b)$ be defined as

$$\hat{u}(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in [\overline{x}, x_1] \\ u(x) & \text{otherwise.} \end{cases}$$

Then $\max_{[a,b]} |\hat{u}(x) - u(x)| \le \delta_1$, and so

$$\mathcal{F}(\hat{u}) - \mathcal{F}(u) = \mathcal{F}(\tilde{u}; (\overline{x}_1, x_1)) - \mathcal{F}(u; (\overline{x}_1, x_1)) \ge 0.$$

Since $\tilde{u}$ is the unique minimizer of $\mathcal{F}(\cdot; (\overline{x}_1, x_1))$ with $\tilde{u}(\overline{x}_1) = u(\overline{x}_1), \tilde{u}(x_1)$, and $\max |\tilde{u}(x) - \tilde{u}(\overline{x}_1)| < \delta_1/2$, it follows that $\tilde{u} = u$ in $[\overline{x}, x_1]$ and hence that $u \in C^2([\overline{x}, x_1])$. Similarly, if $\overline{x} \neq a$ then $u \in C^2([x_0, \overline{x}])$ for some $x_0 < \overline{x}$. In particular u is Lipschitz continuous in a neighbourhood of any $\overline{x} \in [a, b]$ for which $M(\overline{x}) < \infty$, and thus by Proposition 4.4 of Section 4.1 u is smooth in a neighbourhood of any such $\bar{x}$. Since u is differentiable almost everywhere in $[a, b]$ it follows that

$$\Omega_0 := \{x \in [a, b] \colon M(x) < \infty\}$$

is an open subset of $[a, b]$ of full measure, and that $u \in C^\infty(\Omega_0)$.

It remains to show that the classical derivative $[u'(x)]$ of $u(x)$ exists everywhere and is a continuous function with values in $\mathbb{R} \cup \{\pm\infty\}$. In order to prove this, it suffices to consider points x_0 in $E := [a, b] \setminus \Omega_0$. We first assume that $x_0 \in (a, b)$; then, by an appropriate reflection of the variables x and (or) u we can assume without loss of generality that there exist points $y_j \to x_0$, $y_j < x_0$ such that

$$\lim_{j \to \infty} \frac{u(x_0) - u(x_j)}{x_0 - y_j} = +\infty.$$

In this case, the existence of $[u'(x_0)] = +\infty$ and the continuity of $[u'(x)]$ at x_0 follows if we show that for $x_j \to x_0$, $z_j \to x_0$, $x_j \geq z_j$ we have

$$\lim_{j \to \infty} \frac{u(x_j) - u(z_j)}{x_j - z_j} = +\infty. \tag{4.19}$$

Let M, δ be arbitrary and apply Lemma 4.7 with $u_0 = u(x_0)$. The solutions $\{u(\cdot; \alpha, M) \colon |\alpha| \leq M\}$ of the Euler equation form a field of extremals simply covering some neighbourhood of (x_0, u_0) in $\mathbb{R}^2$. Thus for $|x - x_0|$ sufficiently small there exists a unique $\alpha(x)$ with $|\alpha(x)| \leq M$ such that $u(x) = u(x; \alpha(x), M)$, and by the implicit function theorem and (4.15) α depends continuously on x. Clearly $\alpha(x_0) = 0$. We claim that $\alpha(x)$ is non-decreasing near x_0. Then we would obtain

$$\begin{aligned}
\frac{u(x_j) - u(z_j)}{x_j - z_j} &= \frac{u(x_j; \alpha(x_j), M) - u(z_j; \alpha(z_j), M)}{x_j - z_j} \\
&\geq \frac{u(x_j; \alpha(z_j), M) - u(z_j; \alpha(z_j), M)}{x_j - z_j} \\
&= u'(w_j; \alpha(z_j), M) \geq M - \delta
\end{aligned}$$

where $x_j \geq w_j \geq z_j$ and we have used (4.14). Since M, δ are arbitrary we obtain relation (4.19).

Let us finally show that $\alpha(x)$ is monotone near x_0. Suppose this is not true; then there exist sequences $a_j \to x_0$, $b_j \to x_0$, $c_j \to x_0$ with $a_j < b_j < c_j$ and $\alpha(a_j) = \alpha(c_j) \neq \alpha(b_j)$. For large j the solution $v_j(x) := u(x; \alpha(a_j), M)$, $a_j \leq x \leq c_j$, satisfies

$$v_j(a_j) = u(a_j), \quad v_j(b_j) \neq u(b_j), \quad v_j(c_j) = u(c_j), \quad \max_{[a_j, c_j]} |u(x) - v_j(x)| \leq \delta_1.$$

Since v_j is embedded in a field of extremals, Weierstrass's formula yields

$$\int_{a_j}^{c_j} F(x, u, u')\, dx > \int_{a_j}^{c_j} F(x, v_j, v_j')\, dx$$

contradicting our hypothesis that u is a strong relative minimizer. Thus α is either non-decreasing or non-increasing near x_0; the latter possibility is excluded by noticing that by integrating (4.14), compare (4.16), we obtain

$$\frac{\alpha(y_j)}{x_0 - y_j} \le \delta + M - \frac{u(x_0) - u(y_j)}{x_0 - y_j}$$

so that $\alpha(y_j) < 0$ for j sufficiently large. By observing that a similar argument applies if $x_0 = a$ or $x_0 = b$, we complete the proof of Theorem 4.6. □

As a consequence of Theorem 4.6 we now have

Theorem 4.9 *Suppose that $F(x, u, p)$ is a smooth Lagrangian with superlinear growth, i.e.*

$$\lim_{|p|\to\infty} \frac{F(x, u, p)}{|p|} = \infty$$

whose Hessian F_{pp} satisfies $F_{pp} > 0$. Let $u \in AC(a, b)$ be a strong local minimizer of the functional

$$\mathcal{F}(u) = \int_a^b F(x, u, u')\,dx$$

with respect to its own boundary values, and suppose either that

$$F_z(\cdot, u, u') \in L^1(a, b) \tag{4.20}$$

or that

$$F_x(\cdot, u, u') \in L^1(a, b). \tag{4.21}$$

Then u is smooth and satisfies both the Euler equation

$$-\frac{d}{dx} F_p(x, u, u') + F_z(x, u, u') = 0 \tag{4.22}$$

and the DuBois-Reymond equation

$$\frac{d}{dx}[F(x, u, u') - u'(x) F_p(x, u, u')] = F_x(x, u, u'). \tag{4.23}$$

Proof Let Ω_0 be a maximal interval in $\Omega^* := [a, b]$ minus the singular set E of u. By Theorem 4.6, u is smooth and satisfies (4.22) and thus (4.23) on Ω_0. If (4.20) holds, by integrating (4.22) we find that

$$\left|F_p(x, u(x), u'(x))\right| \le \text{const} \quad \text{for } x \in \Omega_0; \tag{4.24}$$

if (4.21) holds, we deduce from (4.23) that

$$\left|u'(x)F_p(x,u(x),u'(x)) - F(x,u(x),u'(x))\right| \leq \text{const} \quad \text{for } x \in \Omega_0. \quad (4.25)$$

The result then follows since superlinear growth and convexity imply

$$\left|F_p(x,u,p)\right| \to \infty \qquad pF_p(x,u,p) - F(x,u,p) \to \infty \quad (4.26)$$

as $|p| \to \infty$, uniformly in $x \in [a,b]$ and u in a compact set of $\mathbb{R}$. Therefore from (4.24) or (4.25) we see that u' is bounded in Ω_0, and thus $\Omega_0 = [a,b]$. Let us prove (4.26). By the convexity of $F(x,u,p)$ in p we have

$$F(x,u,0) \geq F(x,u,p) - pF_p(x,u,p);$$

hence, for $p \neq 0$,

$$\frac{p}{|p|}F_p(x,u,p) \geq \frac{F(x,u,p)}{|p|} - \frac{F(x,u,0)}{|p|}.$$

Therefore for fixed x, u we deduce that

$$\lim_{p\to+\infty} F_p(x,u,p) = +\infty \qquad \lim_{p\to-\infty} F_p(x,u,p) = -\infty.$$

Since F_p is increasing in p, we also have, for example, for $p \geq M$ that

$$F(x,u,p) \geq F(x,u,M).$$

From this we deduce that the first limit in (4.26) is uniform in (x,u) in a compact set; otherwise there would exist a convergent sequence (x_j, u_j) and a sequence $p_j \to +\infty$ such that $\liminf_{j\to\infty} F_p(x_j,u_j,p_j) < \infty$, in contradiction to

$$\liminf_{j\to\infty} F_p(x_j,u_j,p_j) \geq \liminf_{j\to\infty} F_p(x_j,u_j,M) \geq F_p(x,u,M)$$

for all M. Similarly one proceeds if $p \to -\infty$.

To prove the second claim in (4.26) we note that

$$F(x,u,1) \geq F(x,u,p) - (p-1)F_p(x,u,p),$$

and hence

$$pF_p(x,u,p) - F(x,u,p) \geq \frac{F(x,u,p)}{p}\frac{p}{p-1} - F(x,u,1)\frac{p}{p-1},$$

provided $p > 1$. Therefore, for fixed x, u

$$\lim_{p\to\infty}[pF_p(x,u,p) - F(x,u,p)] = \infty.$$

Uniformity in x, u follows as before by observing that $pF_p - F$ is increasing in p. $\square$

An immediate consequence of Theorem 4.9 is the following regularity result.

Theorem 4.10 *Let $F(x, u, p)$ be a smooth Lagrangian which has superlinear growth and satisfies $F_{pp} > 0$. Suppose either that F depends only on u and p, $F = F(u, p)$, or that F depends only on x and p, $F = F(x, p)$. Then any minimizer $u \in AC(a, b)$ of $\mathcal{F}(u)$ with respect to its own boundary values is of class $C^\infty([a, b])$.*

Remark 1 Finally we remark that if $1 < m < \infty$ then the previous theorems still hold (with the same proof) if we replace AC by $H^{1,m}$ both in the statements and in the definition of strong relative minimizers.

4.3 The Lavrentiev phenomenon and the singular set

In this section we shall show that the strict inequality

$$\begin{aligned}&\inf\{\mathcal{F}(u)\colon u \in AC(a, b), u(a) = \alpha, u(b) = \beta\}\\ &\quad < \inf\{\mathcal{F}(u)\colon u \in C^1([a, b]), u(a) = \alpha, u(b) = \beta\}\end{aligned}$$

may even hold for variational integrals $\mathcal{F}$ whose Lagrangians $F(x, z, p)$ are superlinear. This is the so-called *Lavrentiev phenomenon*, discovered by Lavrentiev in 1926 (see [161]), and investigated later by the Tonelli's school in Pisa (cf. [271]), and in particular by Manià [178], who presented a polynomial example of a Lagrangian $F(x, u, p)$ exhibiting the Lavrentiev phenomenon.

We shall now discuss the Manià example in detail, together with some further extensions.

Consider the variational integral

$$\mathcal{F}(u) := \int_0^1 (u^3 - x)^2 u'^6 \, dx. \tag{4.27}$$

Obviously $u(x) := x^{1/3}$ is a minimizer of $\mathcal{F}(u)$ in the class

$$\mathcal{C}(0, 1) := \{u \in H^{1,1}(0, 1)\colon u(0) = 0, u(1) = 1\}$$

as we have $\mathcal{F}(u) \geq 0$ for all $u \in \mathcal{C}(0, 1)$ and $\mathcal{F}(x^{1/3}) = 0$. The function $x^{1/3}$ is also a minimizer of $\mathcal{F}$ in the class $\mathcal{C}(0, 1) \cap C^1(0, 1)$.

We shall now show that *there exists a positive constant η such that $\mathcal{F}(u) > \eta$ for any function in* $\mathcal{C}(0, 1)$ *with bounded derivatives in* $(0, 1)$. Moreover, for any sequence of Lipschitz functions $\{u_k\}$ in $\mathcal{C}(0, 1)$ which converges uniformly to $x^{1/3}$ we have

$$\mathcal{F}(u_k) \to \infty. \tag{4.28}$$

In particular we obtain

$$0 = \inf_{\mathcal{C}(0,1)} \mathcal{F} < \inf_{\mathcal{C}(0,1) \cap Lip(0,1)} \mathcal{F}. \tag{4.29}$$

Formulae (4.28) and (4.29) show that $x^{1/3}$ cannot be approximated *in energy* by functions in $\mathcal{C}(0, 1) \cap Lip(0, 1)$, and also that, in extending the functional in (4.27) from the class

$C(0,1) \cap Lip(0,1)$ to all of $C(0,1)$ by means of the Lebesgue integral, we have picked some semicontinuous extension of $\mathcal{F}$ which is not the *best extension*, i.e. not the largest semicontinuous extension $\overline{\mathcal{F}}$ of $\mathcal{F}$. Recall that this largest extension $\overline{\mathcal{F}}$ is given by

$$\overline{\mathcal{F}}(u) = \inf\left\{\liminf_{n\to\infty} \mathcal{F}(u_n) : u_n \in C(0,1) \cap Lip(0,1), u_n \rightrightarrows u \text{ on } [0,1]\right\}.$$

(Notation: '$u_n \rightrightarrows u$ on $[0,1]$' means that the functions u_n converge uniformly in $[0,1]$ to u.)

The situation changes completely if we do not require that the approximating sequence $\{u_n\}$ satisfies the boundary conditions $u_n(0) = 0, u_n(1) = 1$. Then $u(x) := x^{1/3}, 0 \le x \le 1$, can be approximated by functions $u_n \in Lip(0,1)$ such that $u_n \rightrightarrows u$ on $[0,1]$ and $\mathcal{F}(u_n) \to \mathcal{F}(u)$, e.g. by

$$u_n(x) := \begin{cases} x^{1/3} & \text{if } 1/n \le x \le 1 \\ n^{-1/3} & \text{if } 0 \le x \le 1/n. \end{cases}$$

The reader may verify that one can even choose a sequence of function $u_n \in C^1([0,1])$ such that $u_n \rightrightarrows u$ in $[0,1]$ and $\mathcal{F}(u_n) \to \mathcal{F}(u)$. Moreover, we can approximate $u(x) = x^{1/3}$ by $u_n \in C(0,1) \cap C^1(0,1)$ such that $u_n \rightrightarrows u$ and $\mathcal{F}(u_n) \to \mathcal{F}(u)$, simply by taking $u_n = u$. These considerations show that the most reasonable generalization of the problem

$$\min\{\mathcal{F}(u) : u \in C(0,1) \cap Lip(0,1)\} \tag{$\mathcal{P}$}$$

is in general not the problem

$$\min\{\mathcal{F}(u) : u \in C(0,1)\}, \tag{$\mathcal{P}_1$}$$

but the problem

$$\min\{\overline{\mathcal{F}}(u) : u \in C(0,1)\}. \tag{$\overline{\mathcal{P}}$}$$

The functional $\overline{\mathcal{F}}$ is called the *relaxed functional* associated with $\mathcal{F}$, and problem $(\overline{\mathcal{P}})$ is the *relaxed minimum problem* associated with $(\mathcal{P})$.

In general we have $\mathcal{F}(u) \le \overline{\mathcal{F}}(u)$, and the class $\{u \in C(0,1) : \overline{\mathcal{F}}(u) < \infty\}$ may be considerably smaller than $C(0,1)$, although every $H^{1,1}(0,1)$-function can be approximated in the $H^{1,1}$-norm by functions of class $Lip(0,1)$ or even of class $C^\infty([0,1])$.

It is an interesting question whether $\overline{\mathcal{F}}$ has a Lagrangian, i.e. whether $\overline{\mathcal{F}}$ can be represented as an integral of the form

$$\overline{\mathcal{F}}(u) = \int_0^1 \overline{F}(x, u(x), u'(x))\, dx.$$

We do not treat this question but refer the interested reader to Buttazzo–Mizel [51] and to the survey paper of Buttazzo–Belloni [48] where a complete discussion of this topic is given. Some remarks concerning this question can also be found in the scholia.

A final remark is appropriate. One might think that all the trouble with the functional $\mathcal{F}$ stems from the fact that the Lagrangian $F(x, z, p) = (z^3 - x)^2|p|^6$ is not genuinely superlinear and degenerates for $u(x) = x^{1/3}$. This, however, is not the case.

In fact, consider any real number σ with $1 < \sigma < 3/2$; then $x^{1/3} \in H^{1,\sigma}(0, 1)$. Therefore we can find a positive constant ϵ such that

$$\epsilon \int_0^1 |Dx^{1/3}|^\sigma \, dx < \eta.$$

Now set

$$\mathcal{F}_1(u) := \int_0^1 [(u^3 - x)^2 \dot{u}^6 + \epsilon|\dot{u}|^\sigma] \, dx.$$

Obviously

$$\mathcal{F}_1(u) > \eta$$

for all u in $C(0, 1)$ with bounded derivatives in $(0, 1)$, whereas

$$\mathcal{F}_1(u_0) < \eta \quad \text{for } u_0(x) = x^{1/3},$$

i.e. (4.29) holds with $\mathcal{F}$ replaced by $\mathcal{F}_1$. The integrand $F_1(x, u, p)$ of $\mathcal{F}_1$ is now non-degenerate, convex in p, and satisfies

$$\epsilon|p|^\sigma \le F_1(x, u, p) \le c_1|p|^6 + c_2$$

where c_1 is a positive constant depending on the upper bound of u. We shall see later that similar phenomena may also occur for integrands $F(x, u, p)$ which are smooth, convex in p, and satisfy growth conditions of the type

$$|p|^2 \le F(x, u, p) \le c_1|p|^m + c_2,$$

with $m > 2$.

Let us now prove our claims.

Denote by C_0 the curve $(x, x^{1/3})$, $0 \le x \le 1$, in the x, y-plane, and by Γ_1 and Γ_2 respectively the curves $(x, \frac{1}{2}x^{1/3})$, $(x, \frac{1}{4}x^{1/3})$, $0 \le x \le 1$ (see Fig. 4.1).

Moreover, for any $\xi \in \left(0, \frac{1}{2}\right)$ we denote by R_ξ the region in the x, y-plane which is bounded by Γ_1, Γ_2, and the straight lines $x = \xi$ and $x = 2\xi$. It is easy to see that in R_ξ the expression $(y^3 - x)^2$ has an absolute minimum given by $\epsilon(\xi) = \frac{49}{64}\xi^2$. Let $(x, u(x))$, $x_1 \le x \le x_2$, be an absolute continuous arc contained in R_ξ such that $(x_1, u(x_1)) = (\xi, \frac{1}{4}\xi^{1/3})$. Then

$$\mathcal{F}(u, (x_1, x_2)) = \int_{x_1}^{x_2} [u^3(x) - x]^2 |u'|^6 \, dx \ge \frac{49}{64}\xi^2 \int_{x_1}^{x_2} |u'|^6 \, dx.$$

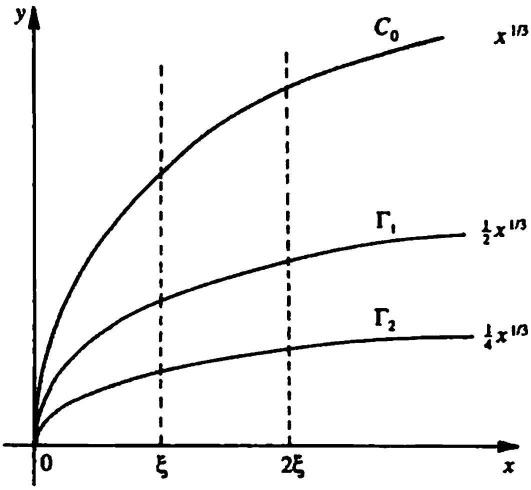

FIG. 4.1.

By a simple convexity argument one can easily see that the integral $\int_{x_1}^{x_2} |u'|^6\,dx$ takes its minimum value for a linear function u in (x_1, x_2). Hence we have

$$\mathcal{F}(u, (x_1, x_2)) \geq \frac{49}{64}\xi^2[u(x_2) - u(x_1)]^6(x_2 - x_1)^{-5}.$$

Therefore, if $(x_1, u(x_1)) = \left(\xi, \frac{1}{4}\xi^{1/3}\right)$ and $(x_2, u(x_2)) = (2\xi, y_2)$ with $\frac{1}{4}(2\xi)^{1/3} \leq y_2 \leq \frac{1}{2}(2\xi)^{1/3}$, then

$$\begin{aligned}\mathcal{F}(u, (x_1, x_2)) &\geq \frac{49}{64}\xi^2 \cdot \xi^{-5}\left(\frac{1}{4}(2\xi)^{1/3} - \frac{1}{4}\xi^{1/3}\right)^6\\ &= 4^{-6}\frac{49}{64}(2^{1/3} - 1)^6\xi^{-1}.\end{aligned}$$

The last expression is positive and tends to $+\infty$ as ξ goes to zero; hence there is a constant η_1 such that $\mathcal{F}(u, (x_1, x_2)) \geq \eta_1$ if we take $\xi \in (0, 1/2)$ into account.

If again $(x_1, u(x_1)) = (\xi, \frac{1}{4}\xi^{1/3})$ as stated, but $(x_2, u(x_2)) = (x_2, \frac{1}{2}x_2^{1/3})$ with $\xi \leq x_2 \leq 2\xi$, then

$$\mathcal{F}(u, (x_1, x_2)) \geq \frac{49}{64}\xi^2\Lambda(x_2, \xi),$$

$$\Lambda(x_2, \xi) := (x_2, \xi)^{-5}(x_2 - \xi)^{-5}\left(\frac{1}{2}x_2^{1/3} - \frac{1}{4}\xi^{1/3}\right)^6.$$

Thus, for any fixed $\xi \in \left(0, \frac{1}{2}\right)$ we see that $\Lambda(x_2, \xi) \to +\infty$ as $x_2 \to \xi$, whence the minimum of Λ must be either at $x_2 = 2\xi$ or at some $x_2 = \alpha\xi$ with $1 < \alpha < 2$. In the first case we have again $\mathcal{F}(u, (x_1, x_2)) \geq \eta_1$. In the second case we obtain

$$\frac{d}{dx_2}\Lambda(x_2, \xi)\Big|_{x_2=\alpha\xi} = 0,$$

and α must satisfy the equation

$$(\alpha - 1) - 5\alpha^{2/3}\left(\frac{1}{2}\alpha^{1/3} - \frac{1}{4}\right) = 0$$

which is independent of ξ and does not admit any solution $\alpha \in (1, 2)$. Thus $\mathcal{F}(u, (x_1, x_2)) \geq \eta_2$ where η_2 is a positive constant.

Finally if $(x, u(x))$, $\frac{1}{2} \leq x_1 \leq x \leq x_2 \leq 1$, is an arc lying between Γ_1 and Γ_2 with end points on Γ_2 and Γ_1 respectively, then

$$\tfrac{1}{4}x^{1/3} \leq u(x) \leq \tfrac{1}{2}x^{1/3}, \quad x_1 \leq x \leq x_2, \quad u(x_1) = \tfrac{1}{4}x_1^{1/3}, \quad u(x_2) = \tfrac{1}{2}x_2^{1/3},$$

whence

$$\mathcal{F}(u, (x_1, x_2)) \geq \tfrac{49}{64}2^{-2}(x_2 - x_1)^{-5}\left(\tfrac{1}{2}x_2^{1/3} - \tfrac{1}{4}x_1^{1/3}\right)^6,$$

and again $\mathcal{F}(u, (x_1, x_2)) \geq \eta_3$ where η_3 is a positive constant. We set $\eta := \min(\eta_1, \eta_2, \eta_3)$.

Now, if $(x, u(x))$, $0 \leq x \leq 1$, is any smooth arc C with $u'(x)$ bounded in $(0, 1)$, then C is below Γ_1 in some right-neighbourhood of the origin. Therefore C has a maximal arc $(x, u(x))$, $x_1 \leq x \leq x_2$, lying between Γ_1 and Γ_2 and with end points on Γ_2 and Γ_1 respectively. If $0 < x_1 < \frac{1}{2}$, then $\mathcal{F}(u) \geq \min(\eta_1, \eta_2)$; if $\frac{1}{2} \leq x_1 < 1$, then $\mathcal{F}(u) \geq \eta_3$. Thus we obtain

$$\int_0^1 (u^3 - x)^2 |u'|^6 \, dx \geq \eta$$

for all Lipschitz functions u in $(0, 1)$ with $u(0) = 0$, $u(1) = 1$.

From the previous discussions one readily sees that if $\{u_k\}$ is a sequence of Lipschitz functions with $u_k(0) = 0$, $u_k(1) = 1$, converging weakly in $H^{1,1}$ to $x^{1/3}$, then $\mathcal{F}(u_k)$ tends to $+\infty$.

We conclude this section by proving that any closed subset E of $[a, b]$ of measure zero can occur as the singular set of a minimizer of a variational integral whose Lagrangian $F(x, u, p)$ has a superlinear growth and actually satisfies

$$F(x, u, p) \geq c_0|p|^2 \quad \text{for every } p$$

where c_0 is a positive constant. We have

Proposition 4.11 *Let E be a closed subset of $[a, b]$ having measure zero, and set*

$$C(0, 1) := \{u \in AC(a, b) : \ u(a) = 0, \ u(b) = 1\}.$$

Then we can find functions $v \in C(0, 1)$ and $\varphi, \psi \in C^\infty(\mathbf{R})$, as well as a number $\epsilon > 0$ such that

$$\psi \geq 0, \psi'' \geq 0 \quad on\ \mathbf{R}, \ \psi \circ v \in C^\infty(\mathbf{R}).$$

and that for

$$F(x, u, p) := [\varphi(u) - \varphi(v(x))]^2 \psi(p) + \epsilon p^2$$

we have

(i) *The functional $\mathcal{F}(u) := \int_a^b F(x, u, u')\, dx$ attains its infimum on $C(0, 1)$.*
(ii) *If u is any minimizer of $\mathcal{F}$ in $C(0, 1)$, then the singular set of u is exactly E.*
(iii) *Lavrentiev's phenomenon occurs:*

$$\inf_{C(0,1)} \mathcal{F} < \inf_{C(0,1) \cap Lip} \mathcal{F}.$$

Proof To construct the functions v, φ, ψ, we choose a sequence $\{U_k\}$ of open sets of $\mathbf{R}$ such that

$$\text{meas } U_k < 2^{-k}, \qquad \overline{U}_{k+1} \subset U_k, \qquad \cap_k U_k = E,$$

and a sequence of functions $\{g_k\} \subset C^\infty(\mathbb{R})$ such that

$$0 \leq g_k \leq 1, \ g_k = 0 \quad \text{on } \mathbb{R}^n - U_k, \qquad g_k = 1 \quad \text{on } \overline{U}_{k+1}.$$

Now we set

$$g := 1 + \sum_{k=1}^\infty g_k, \qquad \alpha := \int_a^b g\, dx.$$

Clearly we have

$$g - 1 \in L^2(\mathbf{R}), \ g \in C^\infty(\mathbb{R} - E), \ g \geq 1 \quad \text{on } \mathbf{R}, \qquad g \geq k \quad \text{on } U_{k+1}.$$

Finally we set

$$v(t) := \alpha^{-1} \int_a^t g(x)\, dx \quad \text{for } t \in [a, b].$$

One can easily see that

$$\begin{aligned} &v \in C(0, 1), \ v' \in L^2(a, b), \ v \in C^\infty([a, b] - E) \\ &v' \geq \alpha^{-1} \quad \text{on } [a, b] \\ &v' \geq k\alpha^{-1} \quad \text{on } U_{k+1}. \end{aligned} \tag{4.30}$$

□

Let $F := v(E)$; then F is a closed and nowhere dense subset of $[0, 1]$. Setting $V := (0, 1) - F$ we can write

$$V = \bigcup_{n=1}^{\infty} G_n$$

where $\{G_n\}$ is a sequence of compact sets such that $G_n \subseteq G_{n+1}$. Let h_n be of class $C^\infty(\mathbf{R})$, satisfy $h_n = 0$ outside G_n, $h_n \geq 0$ on $\mathbf{R}$, and $h_n > 0$ on G_{n-1}, and define

$$\varphi_n(t) := \int_0^t h_n(x)\,dx;$$

then $\varphi_n \in C^\infty(\mathbf{R})$ and $\varphi_n \circ v \in C^\infty([a, b])$. For each n we choose $\delta_n > 0$ such that, for $k = 0, \ldots, n$,

$$\delta_n \left| D^k \varphi_n(t) \right| \leq 2^{-n}, \quad \delta_n \left| D^k (\varphi_n \circ v)(t) \right| \leq 2^{-n} \qquad \text{for } t \in [a, b],$$

and we define

$$\varphi = \sum_n \delta_n \varphi_n;$$

this series and all its derivatives converge uniformly on $\mathbf{R}$. Since $\varphi' = \sum \delta_n h_n$ and therefore $\varphi' > 0$, we conclude that

$$\varphi \in C^\infty(\mathbf{R}), \quad \varphi \circ v \in C^\infty([a, b]), \quad \varphi \text{ is strictly increasing on } [0, 1].$$

For every $t \in [0, 2\alpha]$ we define

$$\eta(t) := \inf\{[\varphi(x) - \varphi(y)]^2 : x, y \in [0, 1], |x - y| \geq t/2\alpha\}. \tag{4.31}$$

Since φ is continuous and strictly increasing on $[0, 1]$, η is continuous and increasing on $[0, 2\alpha]$, $\eta(0) = 0$, and $\eta(t) > 0$ for $0 < t < 2\alpha$.

For $k = 1, 2, \ldots$, let $d_k := \operatorname{dist}(E, \mathbf{R} - U_k)$ so that d_n decrease to zero. Define ρ on $(0, \infty)$ by

$$\rho(d_n) := (n - 2)\alpha^{-1}/8, \quad n = 3, 4, \ldots,$$

extending ρ to be constant on $[d_3, \infty)$ and linear on each interval $[d_{n+1}, d_n]$, $n = 3, 4, \ldots$. Then ρ is continuous and decreasing, and we have $\rho(t) \to +\infty$ as $t \to 0$. Taking (16) into account we obtain

$$v'(t) \geq 8\rho(\operatorname{dist}(t, E)) \quad \text{for every } t \in [a, b] - E. \tag{4.32}$$

Furthermore, ρ is strictly decreasing on $(0, d_3]$, and has a continuous, decreasing inverse function on $(\alpha^{-1}/8, +\infty)$

$$h(t) := \frac{t}{\rho^{-1}(t)\eta(\rho^{-1}(t))}.$$

Thereby we define a positive, continuous, increasing function h on $[\alpha^{-1}/8, +\infty]$. We extend h to be continuous and increasing on $[0, +\infty)$, with $h = 0$ on $[0, \beta]$ for some $\beta > 0$.

Finally, for $t \in \mathbb{R}$, we define

$$\psi(t) := h(1)\left(\frac{t}{\beta}\right)^2 + \sum_{n=1}^{\infty} h(n+1)\left(\frac{t}{n}\right)^{k_n}$$

where k_n are even positive integers chosen in such a way that the power series has an infinite radius of convergence. Then we have $\psi(t) \geq h(n+1)$ for $t \in [n, n+1]$, $n = 1, 2, \ldots$, whence

$$\psi(t) \geq h(t), \quad t \geq 0, \tag{4.33}$$

and $\psi \in C^\infty(\mathbb{R})$, $\psi \geq 0$, $\psi'' \geq 0$, $\psi(0) = 0$.

The key to the proof of Proposition 4.11 is in the following result:

Suppose that $u \in C(0, 1)$ and that for some $t_0 \in E$ the derivative $u'(t_0)$ exists and is finite. Then

$$\mathcal{F}_0(u) := \int_a^b [\varphi(u(x)) - \varphi(v(x))]^2 \varphi(u'(x))\, dx \geq \alpha^{-1}/8.$$

Let us prove this statement. We have either

$$u(t_0) \leq v(t_0), \quad t_0 < b,$$

or

$$u(t_0) \geq v(t_0), \quad t_0 > a.$$

Suppose now that $u(t_0) \leq v(t_0)$, $t_0 < b$; a similar argument works in the other case. Since $t_0 \in E$, it follows that $v'(t_0) = +\infty$ while $u'(t_0)$ is finite and $u(t_0) \leq v(t_0)$. Thus we can find a number r with $t_0 < r < b$ and

$$u(r) - v(t_0) \leq \tfrac{1}{4}[v(r) - v(t_0)]. \tag{4.34}$$

Moreover, since $v(b) = u(b) = 1$ we can find an s with $r < s < b$ such that

$$u(s) - v(t_0) = \tfrac{1}{2}[v(s) - v(t_0)] \tag{4.35}$$

and

$$u(t) - v(t_0) \leq \tfrac{1}{2}[v(t) - v(t_0)], \quad r \leq t \leq s.$$

Then for $r \leq t \leq s$ we have

$$v(t) - u(t) \geq \tfrac{1}{2}[v(t) - v(t_0)] \geq (t - t_0)/2\alpha$$

since $v' \geq \alpha^{-1}$ by (4.30). Using (4.31) we then obtain

$$[\varphi(u(t)) - \varphi(v(t))]^2 \geq \eta(t - t_0). \tag{4.36}$$

On account of (4.32), (4.34), (4.35), and the fact that $v(r) < v(s)$ for $r < s$, we arrive at

$$u(s) - u(r) \geq \frac{1}{4}[v(s) - v(t_0)] \geq 2\int_0^{s-t_0} \rho(t)\,dt. \tag{4.37}$$

On the other hand

$$u(s) - u(r) = \int_r^s u'(t)\,dt = \int_G u'(t)\,dt + \int_H u'(t)\,dt \tag{4.38}$$

where

$$G := \{t \in (r,s) :\ u'(t) \leq \rho(t - t_0)\},\quad H := (r,s) - G.$$

Then

$$\int_G u'\,dt \leq \int_0^{s-t_0} \rho(t)\,dt. \tag{4.39}$$

Now if $t \in H$, then by (4.33)

$$\psi(u'(t)) \geq h(u'(t)) \geq \frac{u'(t)}{(t-t_0)\eta(t-t_0)}$$

since $u'(t) \geq \rho(t - t_0)$ implies $\rho^{-1}(u'(t)) \leq t - t_0$. Hence

$$\begin{aligned}\int_H u'\,dt &\leq \int_r^s (t-t_0)\eta(t-t_0)\psi(u'(t))\,dt\\ &\leq \int_r^s (t-t_0)[\varphi(u(t)) - \varphi(v(t))]^2\psi(u'(t))\,dt\\ &\leq (s-t_0)\mathcal{F}_0(u)\end{aligned}$$

where we have used (4.36). Combining this with (4.37), (4.38), and (4.39) we obtain

$$\int_0^{s-t_0} \rho(t)\,dt \leq (s-t_0)\mathcal{F}_0(u),$$

and since $\rho(t) \geq \alpha^{-1}/8$ for all t we deduce that

$$\mathcal{F}_0(u) \geq \alpha^{-1}/8,$$

as required.

We are now ready to complete the proof of the proposition. Choose $\epsilon > 0$ in such a way that

$$8\alpha \int_a^b v'^2(t)\,dt < 1/\epsilon,$$

and observe that

$$\mathcal{F}(v) = \epsilon \int_a^b v'^2\,dt < \alpha^{-1}/8$$

since $\mathcal{F}_0(v) = 0$. Assertion (i) follows from Tonelli's existence theorem. Let u be a minimizer of I in $C(0,1)$. Then $\mathcal{F}(u) \leq \mathcal{F}(v) < \alpha^{-1}/8$, whence $\mathcal{F}_0(u) < \alpha^{-1}/8$, and the result above yields that the singular set E_0 of u contains E, and (iii) holds true. To complete the proof we have only to show that E_0 contains no points outside of E.

First we note that if u is a minimizer of $\mathcal{F}$, then u is monotonically increasing on $[a, b]$. Otherwise, we can find points t_0, t_1 with $a \le t_0 < t_1 \le b$ such that $u(t_0) = u(t_1)$ whereas u is not constant on $[t_0, t_1]$; then we can diminish $\mathcal{F}(u)$ by making u constant on $[t_0, t_1]$. Suppose now that $E_0 - E$ is non-void and let $t \in E_0 - E$. We can assume that t is an end point of an open interval $J \subset [a, b]$ on which u is smooth; we assume that t is the right end point of J. (If t is the left end point of J we can argue similarly.) The Euler equation

$$\frac{d}{dx}F_p = F_z$$

holds on J, and we have

$$F_z(\cdot, u, u') = 2\varphi'(u)[\varphi(u) - \varphi(v)]\psi(u'),$$
$$F_p(\cdot, u, u') = [\varphi(u) - \varphi(v)]^2\psi'(u') + 2\epsilon u'.$$

Now since u is monotonically increasing and t is in the singular set E_0 of u, we have $u'(s) \to \infty$ as $s \to t$, $s \in J$. Hence, as $s \to t$, $s \in J$,

$$F_p(s, u(s), u'(s)) \to +\infty. \tag{4.40}$$

If $u(t) \neq v(t)$ we have

$$\left|F_z(x, u(x), u'(x))\right| \le \text{const}\ F(x, u(x), u'(x))$$

for x close to t and hence $F_z \in L^1(s, t)$ for s close enough to t. In view of the Euler equation this contradicts (4.40). On the other hand if $u(t) = v(t)$, then we infer from $u'(t) = \infty$ and $v'(t) < \infty$ that we can find $s < t$ such that $u(x) < v(x)$ for $s < x < t$. Then $F_z(x, u(x), u'(x)) < 0$ for $s < x < t$, so that by the Euler equation $F_p(x, u(x), u'(x))$ is decreasing on this interval, again contradicting (4.40).

5

SOME APPLICATIONS

In this section we shall discuss a few applications of direct methods to some significant examples with the aim of illustrating the existence and regularity results of the previous sections.

Reversing the original idea that minimizers may be selected among solutions of Euler's equations satisfying given boundary conditions, the direct methods and the regularity theory, developed in Chaps 3 and 4, provide existence and regularity of solutions of *boundary value problems* associated with Euler's equations of one-dimensional variational integrals. A few examples will be discussed in Section 5.1.

In Sections 5.2 and 5.3 we shall discuss the variational approach to *eigenvalue problems* for Sturm–Liouville operators and the *vibrating string* problem. The important role of the vibrating string problem in the development of mathematical analysis and in particular in the process of definition of the concept of function, in the development of the theory of Fourier series and eventually of the theory of Hilbert spaces is well known. We shall also discuss briefly estimates for the first eigenvalue of linear operators.

In Section 5.4 we discuss the one-dimensional (non-parametric) *obstacle problem* as the easiest example of a series of constrained variational problems which lead to so-called *variational inequalities*.

In Sections 5.5 and 5.6 we shall discuss a few simple applications of direct methods to the existence of periodic solutions, and in particular to the existence of periodic solutions of Hamiltonian systems on energy-level sets.

Section 5.7 will deal with variational problems which do not satisfy the coerciveness assumptions; thus the direct methods cannot be applied, and some *compatibility conditions* have to be added in order to provide the existence of a solution.

In Section 5.8 we describe a general framework to treat *optimal control problems*. In the case of one-dimensional problems we present a general existence result for optimal pair state control.

Finally, in Section 5.9 we discuss existence and regularity questions for parametric variational problems.

5.1 Boundary value problems

Consider the variational integral

$$\mathcal{F}(v) := \frac{1}{2}\int_0^1 (\dot{v}^2 + v^2)\,dx - \int_0^1 f v\,dx \tag{5.1}$$

where f is a given function in $L^2(0, 1)$. Tonelli's existence theorem, or simply observing that

$$\int_0^1 (\dot{v}^2 + v^2)\, dx$$

is the square of the norm in $H_0^{1,2}(0, 1)$ and hence is lower semicontinuous with respect to the weak convergence in the Hilbert space $H_0^1(0, 1)$, while $\int_0^1 f v\, dx$ is continuous with respect to the weak convergence in $H_0^1(0, 1)$, yields at once the existence of a minimizer of (5.1) in H_0^1.

Indeed, for all $\epsilon > 0$

$$\begin{aligned}\mathcal{F}(u) &\geq \frac{1}{2}\int_0^1 (u'^2 + u^2)\, dx - \left(\int_0^1 f^2\, dx\right)^{1/2} \left(\int_0^1 u^2\, dx\right)^{1/2} \\ &\geq \frac{1}{2}\int_0^1 (u'^2 + u^2)\, dx - \frac{\epsilon}{2}\int_0^1 u^2\, dx - \frac{1}{2\epsilon}\int_0^1 f^2\, dx,\end{aligned}$$

and, for $\epsilon = 1/2$,

$$\mathcal{F}(u) \geq \frac{1}{4}\int_0^1 (u'^2 + u^2)\, dx - \int_0^1 f^2\, dx.$$

Since for all $\varphi \in H_0^1(0, 1)$ the function $\mathcal{F}(u + t\varphi)$ is differentiable with respect to t, the minimizer u (which is unique because of the strict convexity of $\mathcal{F}$) satisfies the Euler equation

$$\frac{d}{dt}\mathcal{F}(u + t\varphi)\Big|_{t=0} = 0 \quad \text{for all } \varphi \in H_0^1(0, 1)$$

which amounts to

$$\int_0^1 (u'\varphi' + u\varphi)\, dx = \int_0^1 f\varphi\, dx \quad \text{for all } \varphi \in H_0^1(0, 1). \tag{5.2}$$

Also, the regularity of u follows from the general results of Chap. 4, but in this case it follows actually more easily. In fact, from (5.2) we deduce that u solves (in the weak sense of distributions) the *boundary value problem*

$$\begin{cases} -u'' + u = f & \text{in } (0, 1) \\ u(0) = u(1) = 0. \end{cases} \tag{5.3}$$

In particular:

(i) Since $f \in L^2(0, 1)$ and $u \in H_0^1(0, 1)$, from (5.3) it follows that $u'' \in L^2(0, 1)$, i.e. $u \in H^2(0, 1)$; by induction we then find that if $f \in H^k(0, 1)$ with $k \in \mathbb{N}$, then $u \in H^{k+2}(0, 1)$.

(ii) If $f \in C([0,1])$, since every function in $H^1(0,1)$ belongs to $C([0,1])$, we have that $u'' \in C([0,1])$, i.e. $u \in C^2([0,1])$. As before we get, by induction, that $f \in C^k([0,1])$ gives $u \in C^{k+2}([0,1])$.

Let us now replace the *homogeneous* Dirichlet conditions $u(0) = u(1) = 0$ by non-homogeneous conditions, i.e. let us consider the problem

$$\begin{aligned} -u'' + u = f \quad & \text{in } (0,1) \\ u(0) = \alpha, \qquad & u(1) = \beta. \end{aligned} \tag{5.4}$$

By considering a smooth function u_0 such that $u_0(0) = \alpha$ and $u_0(1) = \beta$, for instance considering the linear affine function through the points $(0,\alpha)$, $(1,\beta)$, we see at once that (5.4) is equivalent to the homogeneous Dirichlet problem

$$\begin{aligned} -\tilde{u}'' + \tilde{u} &= f + u_0'' - u_0 \\ \tilde{u}(0) = \tilde{u}(1) &= 0, \end{aligned} \tag{5.5}$$

where we have set $u = \tilde{u} + u_0$. Equivalently, problem (4) can be treated by minimizing the variational integral $\mathcal{F}$ in (1) on the *closed convex set*, actually the linear affine subspace

$$\mathcal{C}(\alpha,\beta) := \{v \in H^1(0,1) : v(0) = \alpha, v(1) = \beta\}.$$

Both approaches of course yield similar results to those of the homogeneous case treated above.

Suppose now that f belongs to $C^0([0,1])$ and that $u \in H^1(0,1)$ is a *weak solution* of (5.4), i.e. $u \in \mathcal{C}(\alpha,\beta)$ and (5.2) holds.

As we have seen, u then belongs to $C^2([0,1])$. Let $x_0 \in [0,1]$ be a point where u attains its maximum. If $x_0 = 0$, we have $u(x) \leq \alpha$ for all $x \in [0,1]$; if $x_0 = 1$, we have $u(x) \leq \beta$ for all $x \in [0,1]$; otherwise $0 < x_0 < 1$, and then $u'(x_0) = 0$, $u''(x_0) \leq 0$, and hence

$$u(x_0) = f(x_0) + u''(x_0) \leq f(x_0).$$

Therefore we conclude that

$$u(x) \leq \max\left\{\alpha, \beta, \sup_{(0,1)} f(x)\right\} \quad \text{for all } x \in [0,1]. \tag{5.6}$$

Similarly we deduce that

$$\min\left\{\alpha, \beta, \inf_{(0,1)} f(x)\right\} \leq u(x) \quad \text{for all } x \in [0,1]. \tag{5.7}$$

These inequalities are known as *maximum principle*. They in fact hold for weak solutions of (5.4), under the weaker assumption that f belongs to $L^\infty(0,1)$ and therefore u is not necessarily of class C^2. The simplest way to see this is by means of the so-called *truncation method*. It consists in the following. Set

$$k = \max\left\{\alpha, \beta, \sup_{(0,1)} f\right\}.$$

As we have seen in Chap. 2 the function

$$u^{(k)}(x) := \max\{u - k, 0\}$$

belongs to $H^1(0, 1)$, and actually to $H_0^1(0, 1)$. Choosing $\varphi = u^{(k)}$ in (5.2) we get

$$\int_0^1 \dot{u}\dot{u}^{(k)}\,dx + \int_0^1 uu^{(k)}\,dx = \int_0^1 fu^{(k)}\,dx,$$

i.e.

$$\int_0^1 [(\dot{u}^{(k)})^2 + (u^{(k)})^2]\,dx = \int_0^1 (f - k)u^{(k)}\,dx,$$

and, since $f - k \leq 0$,

$$\int_0^1 [(\dot{u}^k)^2 + (u^{(k)})^2]\,dx \leq 0,$$

i.e. $u^{(k)} = 0$, which means $u \leq k$, namely (5.6). Similarly one proves (5.7) and we can state

Proposition 5.1 (Maximum principle) *Let $I = (0, 1)$ and let $u \in H^1(0, 1)$ be a weak solution of* (5.4) *with $f \in L^\infty(0, 1)$ and boundary data $u(0) = \alpha$, $u(1) = \beta$. Then*

$$\min\left\{\alpha, \beta, \inf_{(0,1)} f\right\} \leq u(x) \leq \max\left\{\alpha, \beta, \sup_{(0,1)} f\right\} \quad \textit{for all } x \in I.$$

In particular

(i) *if $u \geq 0$ on ∂I and $f \geq 0$ on I, then $u \geq 0$ in I;*
(ii) *if $u = 0$ on ∂I and $f \in L^\infty(I)$, then*

$$\|u\|_{L^\infty(I)} \leq \|f\|_{L^\infty(I)}$$

and more precisely

$$\inf_I f \leq u(x) \leq \sup_I f \quad \textit{for all } \; x \in \overline{I};$$

(iii) *if $f(x) \equiv 0$ on I, then*

$$\|u\|_{L^\infty(I)} \leq \|u\|_{L^\infty(\partial I)}.$$

We now consider the homogeneous Dirichlet problem for a second-order linear operator in 'divergence form', i.e.

$$\begin{aligned} -(pu')' + ru' + qu &= f \quad \text{in } I := (0, 1) \\ u(0) = 0, \qquad u(1) &= 0 \end{aligned} \tag{5.8}$$

where $p \in C^1(\overline{I}), r, q \in C^0(\overline{I}), f \in C^0(\overline{I})$ (or $f \in L^2(I)$) are given functions and we assume that

$$p(x) \geq \alpha > 0 \quad \text{for all } x \in \overline{I}. \tag{5.9}$$

Often the differential operator $-(pu')' + ru' + qu$ is referred to as the *Sturm–Liouville operator.*

In analogy with the above the *weak formulation of* (5.8) is:

Find $u \in H_0^1(0, 1)$ such that

$$\int_I (pu'\varphi' + ru'\varphi + qu\varphi)\,dx = \int_I f\varphi\,dx \quad \text{for all } \varphi \in H_0^1(I). \tag{5.10}$$

Notice that every second-order linear differential operator

$$au'' + bu' + cu, \quad a \in C^1(I), b \in C^0(\overline{I}),$$

can be written in the divergence form above, i.e. as a Sturm–Liouville operator, as

$$(au')' + (b - a')u' + cu.$$

Observe, however, that the weak formulation (5.10) makes sense even if p is just a continuous function on $\overline{I}$.

Here an extra difficulty appears, as the bilinear form

$$a(u, \varphi) := \int_I (pu'\varphi' + ru'\varphi + qu\varphi)\,dx$$

is *not symmetric*, and thus (5.10) cannot appear as the Euler equation of a variational integral. We shall see how in this case a simple trick allows us to reduce problem (5.8) to a variational problem. We introduce a primitive R of r/p, i.e. a function $R \in C^1(\overline{I})$ such that

$$R' = \frac{r}{p}$$

and set $\zeta = e^{-R}$. The equation in (5.8) can be equivalently written after a multiplication by ζ as

$$-\zeta pu'' - \zeta p'u' + \zeta ru' + \zeta qu = \zeta f$$

or else

$$-(\zeta p u')' + \zeta q u = \zeta f$$

since $\zeta' p + \zeta r = 0$.[12] Thus we are reduced to study problems of the type

$$\begin{aligned} -(pu')' + qu &= f \\ u(0) = u(1) &= 0 \end{aligned} \tag{5.11}$$

where p, q, f satisfy the same regularity conditions as above and the *coercivity* condition

$$p(x) \geq \nu > 0 \quad \text{for all } x \in \overline{I}. \tag{5.12}$$

Note that now the bilinear form

$$b(u, \varphi) := \int_I (pu'\varphi' + qu\varphi)\, dx$$

is symmetric, and arises as the Euler operator of

$$\frac{1}{2}\int_I (p\dot{v}^2 + qv^2)\, dx.$$

Suppose now that $q \geq 0$. Then, using Poincaré's inequality, we see that the variational integral

$$\mathcal{F}(v) := \frac{1}{2}\int_I (p\dot{v}^2 + qv^2)\, dx - \int_I fv\, dx$$

is *coercive* in $H_0^1(I)$ and precisely

$$\mathcal{F}(v) \geq \frac{\nu}{4}\|v\|^2_{H_0^1} - c\|f\|^2_{L^2(I)}. \tag{5.13}$$

In fact for any positive ϵ we have

$$\begin{aligned} \mathcal{F}(v) &\geq \frac{\nu}{2}\int_I \dot{v}^2\, dx - \|f\|_{L^2(I)}\|v\|_{L^2(I)} \\ &\geq \frac{\nu}{2}\int_I \dot{v}^2\, dx - c_0\|f\|_{L^2(I)}\|\dot{v}\|_{L^2(I)} \\ &\geq \frac{\nu}{2}\int_I \dot{v}^2\, dx - \frac{\epsilon}{2}\|\dot{v}\|^2_{L^2(I)} - \frac{c_0^2}{2\epsilon}\|f\|^2_{L^2(I)} \end{aligned}$$

and hence (5.13) follows by taking $\epsilon = \nu/2$.

[12] In the weak formulation (5.10) this amounts to replacing φ by $\zeta\varphi$.

The lower semicontinuity of $\mathcal{F}$ with respect to the weak convergence in H_0^1 and the coercivity now allow us to conclude at once with the existence of a minimizer of $\mathcal{F}(v)$ in $H_0^1(I)$. The minimizer u is then a weak solution of (5.11), i.e. it satisfies

$$b(u,\varphi) = \int_I f\varphi\, dx \quad \text{for all } \varphi \in H_0^1(I), \tag{5.14}$$

and one can easily verify that it is unique. Actually by means of the truncation method one can see that a maximum principle holds.[13] Finally, from (5.14), i.e. from

$$\int_I pu'\varphi'\, dx = \int (f - qu)\varphi\, dx \quad \text{for all } \varphi \in H_0^1(I),$$

we deduce that pu' belongs to $H^1(I)$; hence $u' = 1/p\, pu'$ belongs to $H^1(I)$. Consequently, we deduce that *u belongs to $H^2(I)$ if $f \in L^2(I)$*, and *u belongs to $C^2(\overline{I})$ if $f \in C^0(\overline{I})$, and it satisfies* (5.11) *in the classical sense.*

We notice that if $q < 0$, the method breaks down; for instance, the problem

$$-u'' - \pi^2 u = 0 \quad \text{in } (0,1)$$
$$u(0) = u(1) = 0$$

has the infinitely many solutions $c \sin \pi x$.

As we shall see in the next section, π^2 is an eigenvalue of the operator $-u''$ with zero boundary values on the interval $(0,1)$.

A similar approach also works for the *Neumann problem*. For instance, it is easily seen that for $f \in L^2(I)$ there is a unique minimizer of

$$\frac{1}{2}\int_0^1 (\dot{u}^2 + u^2) - \int_0^1 f u\, dx$$

in $H^1(I)$; moreover, it satisfies for all $\varphi \in H^1(I)$ the equality

$$\int_I (\dot{u}\dot{\varphi} + u\varphi) = \int_0^1 f\varphi\, dx. \tag{5.15}$$

In particular (5.15) holds for all φ in $H_0^1(I)$; thus $u \in H^2(I)$ and hence $u \in C^1(\overline{I})$, and $-u'' + u - f = 0$ on I. Integrating (5.15) by parts we find that for every $\varphi \in H^1(I)$

$$\int_I (-u'' + u - f)\varphi\, dx + \dot{u}(1)\varphi(1) - \dot{u}(0)\varphi(0) = 0.$$

Since the first integral is zero we then conclude that

$$\dot{u}(1)\varphi(1) - \dot{u}(0)\varphi(0) = 0 \quad \text{for all } \varphi \in H^1(I),$$

[13] Notice that the linearity of the Euler equation plus a maximum principle imply uniqueness.

i.e. $\dot{u}(1) = \dot{u}(0) = 0$. Therefore we have solved the problem

$$\begin{aligned} -u'' + u &= f \quad \text{on } (0,1) \\ \dot{u}(0) &= \dot{u}(1) = 0. \end{aligned} \tag{5.16}$$

Similarly, by minimizing the variational integral

$$\frac{1}{2}\int_0^1 (\dot{u}^2 + u^2)\,dx - \int_I fu\,dx + \alpha u(0) - \beta u(1)$$

one solves the problem

$$\begin{aligned} -u'' + u &= f \quad \text{on } (0,1) \\ \dot{u}(0) &= \alpha, \quad \dot{u}(1) = \beta. \end{aligned}$$

Analogously we can treat mixed problems, where in one end point we prescribe Dirichlet and in the other end point Neumann conditions, or periodic boundary value problems, provided the associated energy is lower semicontinuous and coercive, i.e. we avoid eigenvalues. But we shall not detail all these cases and we leave them to the reader. We only want to mention that for solutions of problem (5.16) the following maximum principle holds:

$$\inf_I f \le u(x) \le \sup_I f \quad \text{for all } x \in I$$

as the reader might easily verify.

5.2 The Sturm–Liouville eigenvalue problem

The problem we want to investigate in this section is the following. Given a Sturm–Liouville operator L on a bounded interval (a,b), i.e.

$$Lu := -(pu')' + qu$$

with $p \in C^1([a,b])$, $q \in C^0([a,b])$, where

$$p \ge \nu > 0 \quad \text{for all } x \in [a,b]$$

and, for the sake of simplicity, $q \ge 0$, we are concerned with the *eigenvalue problem*

$$\begin{aligned} -(pu')' + qu &= \lambda u \\ u(a) &= u(b) = 0. \end{aligned} \tag{5.17}$$

λ being a real number. Values λ for which problem (5.17) has a non-trivial solution are called *eigenvalues* of the differential operator $Au := -(pu')' + qu$ with Dirichlet's boundary conditions, and the non-trivial solutions corresponding to an eigenvalue λ are called *eigenfunctions* associated with λ.

Of course, in (5.17) we can replace the Dirichlet boundary conditions by Neumann or mixed conditions; we leave the analysis of those eigenvalue problems to the reader.

As we have seen in Section 5.1 every weak solution of (5.17) is in fact a classical C^2-solution; thus we should not worry about regularity, and we can work in the weak H_0^1-context. Moreover, for reasons that will become clear in the sequel of this section, it is convenient slightly to generalize problem (5.17) in the following form: find λ and $u, u \not\equiv 0$, such that

$$\begin{aligned} -(p(x)u')' + q(x)u &= \lambda\sigma(x)u \\ u(a) = u(b) &= 0, \end{aligned} \tag{5.18}$$

$\sigma(x)$ being a *continuous* and *strictly positive* function in $[a, b]$ while $p(x)$ and $q(x)$ satisfy the same assumptions as above. (Note that the assumption $q(x) \geq 0$ can be omitted since we may add a term $c\sigma(x)u$ with a positive constant c to both sides of the differential equation.)

The basic observation in approaching (5.18) is the following. Consider the Hilbert space $H_0^1(a, b)$ and any closed linear subspace V of $H_0^1(a, b)$. The general existence results of Chap. 3 in conjunction with the continuity of the functional $\int_a^b \sigma u^2\, dx$ in L^2 yield at once

Proposition 5.2 *Let V be a closed linear subspace of $H_0^1(a, b)$ different from the trivial space $\{0\}$. Then the functional*

$$\mathcal{F}(u) := \int_a^b (p\dot{u}^2 + qu^2)\, dx$$

assumes its infimum on the set

$$W := V \cap \left\{ u \in H_0^1(a, b) : \int_a^b \sigma u^2\, dx = 1 \right\}$$

in at least one element of W.

In fact Proposition 5.2 can be easily proved without appealing to the general results. Let $\{u_k\} \subset W$ be a minimizing sequence, i.e.

$$\mathcal{F}(u_k) \to \lambda = \inf_W \mathcal{F}(u).$$

Clearly $\{u_k\}$ is equibounded in $H_0^1(a, b)$. Now we observe that the quadratic form $\mathcal{F}(u)$ is derived from a symmetric bilinear form $\mathcal{A}(u, v)$ on $H_0^1(a, b)$ by

$$\mathcal{F}(u) = \mathcal{A}(u, u)$$

where

$$\mathcal{A}(u, v) := \int_a^b (pu'v' + quv)\, dx;$$

thus

$$\begin{aligned} \mathcal{F}(u_k) = \mathcal{A}(u_k, u_k) &= \mathcal{A}(u_k - u, u_k - u) + 2\mathcal{A}(u, u_k) - \mathcal{A}(u, u) \\ &\geq 2\mathcal{A}(u_k, u) - \mathcal{A}(u, u). \end{aligned}$$

From the last inequality we deduce that $\mathcal{F}$ is lower semicontinuous with respect to weak convergence in $H_0^1(a, b)$ and that a weakly converging minimizing sequence is in fact strongly converging in $H_0^1(a,b)$. Hence, passing to a subsequence, $u_k \to u$ weakly and strongly in $H_0^1(a,b)$ and $\int_a^b \sigma u_k^2\,dx \to \int_a^b \sigma u^2\,dx$, i.e. $\int_a^b \sigma u^2\,dx = 1$.

We now define inductively a sequence of closed linear subspaces of $H_0^1(a,b)$ as follows: set

$$V_1 := H_0^1(a,b), \quad W_1 = V_1 \cap \left\{u : \int_a^b \sigma u^2\,dx = 1\right\}.$$

Then Proposition 5.2 yields the existence of an element $u_1 \in W_1$ such that

$$\mathcal{F}(u_1) = \mathcal{A}(u_1, u_1) = \lambda_1 := \inf\{\mathcal{F}(u) \colon u \in W_1\}.$$

Next we define V_2 as the closed linear subspace of all vectors in $H_0^1(a,b)$ which are 'σ-orthogonal' to u_1, i.e.

$$V_2 := \left\{u \in H_0^1(a,b) \colon \int_a^b \sigma u u_1\,dx = 0\right\}$$

and

$$W_2 := V_2 \cap \left\{u : \int_a^b \sigma u^2\,dx = 1\right\}.$$

Again Proposition 5.2 yields an element $u_2 \in W_2$ with

$$\mathcal{F}(u_2) = \mathcal{A}(u_2, u_2) = \lambda_2 := \inf\{\mathcal{F}(u) : u \in W_2\}.$$

Thus we proceed by induction and at the nth step we have the subspace

$$V_n := \left\{u \in H_0^1(a,b) : \int_a^b \sigma u u_i\,dx = 0 \quad i = 1, 2, \dots, n-1\right\}$$
$$W_n := V_n \cap \left\{u : \int_a^b \sigma u^2\,dx = 1\right\}$$

and we find an element $u_n \in W_n$ with

$$\mathcal{F}(u_n) = \mathcal{A}(u_n, u_n) = \lambda_n := \inf\{\mathcal{F}(u) \colon u \in W_n\}.$$

Then V_{n+1} will be defined as

$$V_{n+1} := \left\{u \in V_n : \int_a^b \sigma u u_i\,dx = 0 \quad i = 1, 2, \dots, n\right\}.$$

Since

$$V_{n+1} = \left\{u \in V_n : \int_a^b \sigma u u_n\,dx = 0\right\}$$

we obtain

$$W_1 \supset W_2 \supset W_3 \supset \cdots \supset W_n \supset W_{n+1} \supset \cdots$$

and therefore

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \lambda_{n+1} \le \cdots.$$

Moreover, the sequence $\{u_n\}$ is 'σ-orthogonal', i.e.

$$\int_a^b \sigma u_n u_k \, dx = \delta_{nk}. \tag{5.19}$$

We now claim that λ_n, u_n are for every $n \in \mathbb{N}$ respectively the eigenvalue and eigenfunction of the Sturm–Liouville operator $Au = -(pu')' + qu$ with Dirichlet conditions. This is a consequence of the results of Section 1.2, as can be shown as follows. If $\varphi_i \in V_i, \epsilon \ne 0$, and $\int_a^b \sigma(u_i + \epsilon\varphi_i)^2 \, dx = 1$, we find that

$$\mathcal{F}(u_i + \epsilon\varphi_i) \ge \lambda_i$$

whence

$$\mathcal{A}(u_i + \epsilon\varphi_i, u_i + \epsilon\varphi_i) \ge \lambda_i \int_a^b \sigma(u_i + \epsilon\varphi_i)^2 \, dx \quad \text{for all } \varphi_i \in V_i.$$

Developing this in ϵ we get

$$2\epsilon \left[\mathcal{A}(u_i, \varphi_i) - \lambda_i \int_a^b \sigma u_i \varphi_i \, dx\right] + \epsilon^2 \left[\mathcal{A}(\varphi_i, \varphi_i) - \lambda_i \int_u^b \sigma \varphi_i^2 \, dx\right] \ge 0$$

for all $\epsilon \in \mathbf{R}$, and this implies

$$\mathcal{A}(u_i, \varphi_i) - \lambda_i \int_a^b \sigma u_i \varphi_i \, dx = 0 \quad \text{for all } \varphi_i \in V_i.$$

In conjunction with (5.19), we then arrive at

$$\mathcal{A}(u_i, u_k) = \lambda_i \delta_{ik} \tag{5.20}$$

and hence

$$\mathcal{A}(u_i, \varphi) - \lambda_i \int_a^b \sigma u_i \varphi \, dx = 0 \quad \text{for all } \varphi \in H_0^1(a, b), \tag{5.21}$$

which says that u_i is a weak solution of (5.18) when $\lambda = \lambda_i$.

Thus we have proved that *the sequence of consecutive minimum problems*

$$\min\left\{\int_a^b (p\dot{u}^2 + qu^2)\, dx : u \in W_n\right\}$$

defines a sequence $\{(\lambda_n, u_n)\}$ *of eigenfunctions* u_n *and eigenvalues* λ_n.

Now we want to show that the system of the (λ_n, u_n) is *complete* in the sense that if (λ, u) *is a solution of* (5.18) *with* $u \ne 0$, *then* λ *must be one of the numbers* λ_n, *and* u *must be a multiple of* u_n. This is expressed by saying that λ is a simple eigenvalue.[14]

[14]Notice that eigenvalues corresponding to periodic boundary conditions are not simple: for instance, $\lambda = n^2$ is a double eigenvalue of $u'' + \lambda u = 0$, with eigenfunctions $\sin nx, \cos nx$ in $(0, 2\pi)$.

We first prove

$$\lim_{n\to\infty} \lambda_n = +\infty. \tag{5.22}$$

Otherwise the sequence $\{\lambda_n\}$ would be bounded and we could extract a subsequence $\{u_{n_i}\}$ converging weakly in $H_0^1(a, b)$ to some u. Then (passing to a subsequence) we would have

$$\int_a^b \sigma(u - u_{n_i})^2\, dx \to 0 \quad \text{as } i \to \infty$$

whereas (5.19) would imply that

$$\int_a^b \sigma(u_k - u_n)^2\, dx = 2 \quad \text{for } k \neq n$$

and therefore

$$\int_a^b \sigma(u - u_n)^2\, dx = 2 \quad \text{for all } n,$$

a contradiction. We now prove that every $v \in H_0^1(a, b)$ can be developed as a σ-Fourier series, i.e.

$$v = \sum_{i=1}^{\infty} c_i u_i$$

where c_i are the *σ-Fourier coefficients of u with respect to u_i*, i.e.

$$c_i := \int_a^b \sigma v u_i\, dx.$$

Set

$$v_n = \sum_{i=1}^{n} c_i u_i.$$

Clearly $v - v_n$ belongs to V_{n+1}, and we infer from the minimum property of λ_{n+1} that

$$\mathcal{A}(v - v_n, v - v_n) \geq \lambda_{n+1} \int_a^b \sigma(v - v_n)^2\, dx. \tag{5.23}$$

The relations (5.19) and (5.20) imply that

$$\int_a^b \sigma(v - v_n)^2\, dx = \int_a^b \sigma v^2\, dx - \sum_{i=1}^{n} c_i^2 = \int_a^b \sigma v^2\, dx - \int_a^b \sigma v_n^2\, dx \tag{5.24}$$

$$\mathcal{A}(v - v_n, v - v_n) = \mathcal{A}(v, v) - \sum_{i=1}^{n} \lambda_i c_i^2 = \mathcal{A}(v, v) - \mathcal{A}(v_n, v_n). \tag{5.25}$$

Because of (5.22) there is some N such that $\lambda_n > 0$ for all $n \geq N$. Hence we infer from (5.23) and (5.25) that

$$0 \leq \int_a^b \sigma(v - v_n)^2\, dx \leq \lambda_{n+1}^{-1} \mathcal{A}(v, v) \tag{5.26}$$

whence

$$\int_a^b \sigma(v - v_n)^2\, dx \to 0 \tag{5.27}$$

and (5.24) yields

$$\int_a^b \sigma v_n^2\, dx \to \int_a^b \sigma v^2\, dx$$

or equivalently

$$\int_a^b \sigma v^2\, dx = \sum_{i=1}^{\infty} c_i^2. \tag{5.28}$$

From (5.25) we also obtain

$$\mathcal{A}(v_n, v_n) = \sum_{i=1}^{n} \lambda_i c_i^2 \leq \mathcal{A}(v, v).$$

Consequently, the series $\sum_{i=1}^{\infty} \lambda_i c_i^2$ also is convergent. Thus we have

$$\int_a^b \sigma(v_n - v_m)^2\, dx = \sum_{i=m+1}^{n} c_i^2 \to 0$$

$$\mathcal{A}(v_n - v_m, v_n - v_m) = \sum_{i=m+1}^{n} \lambda_i c_i^2 \to 0$$

if $n > m$ and $n, m \to \infty$. On account of the coercivity of $\mathcal{F}$, we see that $\{v_n\}$ is a Cauchy sequence in $H_0^1(a, b)$ and in view of (5.27) we finally conclude that $v_n \to v$ in $H_0^1(a, b)$. Moreover,

$$\mathcal{A}(v, v) = \sum_{i=1}^{\infty} \lambda_i c_i^2.$$

Finally, consider an arbitrary solution (λ, u) of (5.18). Then we have

$$\lambda \int_a^b \sigma u u_i\, dx = \mathcal{A}(u, u_i) = \lambda_i \int_a^b \sigma u_i u\, dx$$

whence

$$(\lambda - \lambda_i) \int_a^b \sigma u u_i = 0 \quad \text{for all } i.$$

Consequently, if $\lambda \neq \lambda_i$ for all i, it follows from (5.28) that

$$\int_a^b \sigma u^2 \, dx = \sum_{i=1}^{\infty} \int_a^b \sigma u u_i \, dx = 0$$

whence $u = 0$. Hence $\lambda = \lambda_i$ for some i. To prove that all λ_i are simple eigenvalues, let us assume that λ agrees with two eigenvalues in the sequence $\{\lambda_n\}$. Then we can find two linearly independent functions $\overline{u}_1$ and $\overline{u}_2$ which are eigenfunctions associated with λ, and every linear combination

$$\mu \overline{u}_1 + \gamma \overline{u}_2 \neq 0$$

is also an eigenfunction corresponding to λ. Now we can obviously find numbers μ and γ such that

$$\begin{aligned} \mu \overline{u}_1(a) + \gamma \overline{u}_2(a) &= 0 \\ \mu \overline{u}_1'(a) + \gamma \overline{u}_2'(a) &= 0 \end{aligned}$$

but, by the unique solvability of the Cauchy problem, we would conclude that $\mu \overline{u}_1 + \gamma \overline{u}_2$ is identically zero, contradicting the independence of $\overline{u}_1$ and $\overline{u}_2$.

Therefore, the completeness of the system (λ_n, u_n) is proved.

We now can summarize the results just proved.

Theorem 5.3 *The eigenvalue problem (5.18) possesses infinitely many eigenvalues; they cluster exactly at ∞, and each eigenvalue is simple. A complete sequence $\{\lambda_n, u_n\}$ of eigenvalues λ_n and eigenfunctions u_n with $\lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ can be obtained by solving the recursive minimum problems*

$$\min \left\{ \mathcal{F}(u) := \int_a^b (p\dot{u}^2 + qu^2) \, dx : u \in W_n \right\}$$

where

$$W_n := V_n \cap \left\{ u : \int_a^b \sigma u^2 \, dx = 1 \right\}$$

and

$$\begin{aligned} V_1 &= H_0^1(a, b) \\ V_n &= \left\{ u \in H_0^1(a, b) : \int_a^b \sigma u u_i \, dx = 0, \quad i = 1, \ldots, n-1 \right\}. \end{aligned}$$

In other words,

$$\lambda_n = \mathcal{F}(u_n) = \min\left\{ \frac{\mathcal{F}(u)}{\int_a^b \sigma u^2\,dx} : u \neq 0,\ u \in V_n \right\}. \tag{5.29}$$

Moreover, the sequence $\{(\lambda_n, u_n)\}$ *satisfies the two completeness relations*

$$\int_a^b \sigma v w\,dx = \sum_{i=1}^{\infty} \int_a^b \sigma v u_i\,dx \int_a^b \sigma w u_i\,dx,$$

$$\mathcal{A}(v, w) = \sum_{i=1}^{n} \mathcal{A}(v, u_i)\mathcal{A}(w, u_i),$$

and each $v \in H_0^1(a, b)$ *can be written as a 'Fourier series' in the form*

$$v = \sum_{i=1}^{\infty} c_i u_i, \quad c_i = \int_a^b \sigma v u_i\,dx.$$

Moreover, notice that we have $\lambda_1 > 0$ *whenever* $q \geq 0$.

The quotient

$$\frac{\mathcal{F}(u)}{\int_a^b \sigma u^2\,dx}$$

is often referred to as the *Raleigh quotient*.

Remark 1 It is worth nothing that, actually, the Fourier series converges uniformly and even in suitable classes C^k if v and the coefficients p, q, σ are sufficiently smooth. We also mention that for general boundary value problems the eigenvalues are not necessarily simple, e.g. for periodic boundary values. The same is true for systems and for partial differential operators.

The variational characterization of the nth eigenvalue λ_n in Theorem 5.3 requires knowledge of the previous $n-1$ eigenfunctions $u_1, \ldots, u_{n-1}$; for this reason it is often denoted as the *recursive characterization of eigenvalues*. The next theorem provides an *independent characterization*.

Theorem 5.4 (Minimax characterization of eigenvalues) *Consider* $n-1$ *arbitrary functions* $v_1, \ldots, v_{n-1}$ *of class* $L^2(a, b)$ *and let* $d(v_1, \ldots, v_{n-1})$ *be the infimum of the Rayleigh quotient* $\mathcal{F}(u)/\int_a^b \sigma u^2\,dx$ *for all* $u \in H_0^1(a, b)$, $u \neq 0$, *satisfying*

$$\int_a^b \sigma u v_i\,dx = 0, \quad i = 1, \ldots, n-1.$$

Then the nth eigenvalue λ_n, *defined by* (5.29), *is given by*

$$\lambda_n = \max\{d(v_1, \ldots, v_{n-1})\}.$$

Proof Let $u_1, \dots, u_n$ be the first n eigenfunctions of (5.18) defined in Theorem 5.3. Set

$$u = \sum_{i=1}^{n} c_i u_i, \quad c_i \in \mathbb{R},$$

and consider the $n-1$ equations

$$\int_a^b \sigma u v_i \, dx = 0, \quad i = 1, \dots, n-1,$$

for the unknowns $c_1, \dots, c_n$. There is a non-trivial solution $c_1, \dots, c_n$ which can be normalized by the condition $\sum_{i=1}^n c_i^2 = 1$. Then we have

$$\int_a^b \sigma u^2 \, dx = 1 \quad \text{and} \quad \mathcal{F}(u) = \sum_{i=1}^{n} \lambda_i c_i^2 \le \lambda_n,$$

and therefore

$$d(v_1, \dots, v_{n-1}) \le \lambda_n$$

for any $v_1, \dots, v_{n-1} \in L^2(a, b)$, and $\lambda_n = d(u_1, \dots, u_{n-1})$. □

To conclude our discussion on the eigenvalue problem for Sturm–Liouville operators we notice that the eigenvalues depend on the boundary conditions and on the domain. In particular the variational characterization and the uniqueness of solution of the Cauchy problem yield that, if $\lambda_k(a, b)$ denotes the kth eigenvalue of problem (5.18) for the domain (a, b), we have

$$\lambda_k(c, d) \le \lambda_k(a, b) \quad \text{whenever } (a, b) \subset (c, d)$$

with strict inequality if (a, b) is a true subinterval of (c, d).

The weak monotonicity of the eigenvalues $\lambda_k(a, b)$ in dependence of the domain (a, b) follows at once from Theorem 5.4 since the set of competing functions for the larger interval contains that for the smaller interval. To prove strong monotonicity we suppose for instance that $c \le a < b < d$ and $\lambda_k(c, d) = \lambda_k(a, b)$. Then we choose a sequence $\{\beta_m\}$ of real numbers β_m satisfying

$$b < \beta_1 < \beta_2 < \cdots < \beta_m < \beta_{m+1} < \cdots < d.$$

Let $\lambda_k(c, \beta_m)$ be the eigenvalues for (c, β_m) and $u_k^m \in H_0^1(c, \beta_m)$ be the corresponding eigenfunctions. The weak monotonicity property of λ_k implies

$$\lambda_k(c, d) \le \lambda_k(c, \beta_m) \le \lambda_k(a, b)$$

whence

$$\lambda_k(c, d) = \lambda_k(c, \beta_m) = \lambda_k(a, b) \quad \text{for all } m \in \mathbb{N}.$$

Introduce $w_m \in H_0^1(c, d)$ by $w_m(x) := u_k^m(x)$ for $x \in [c, \beta_m]$ and $w_m(x) := 0$ otherwise. The unique solvability of the Cauchy problem implies that $u_k^m(x)$ cannot identically vanish in a subdomain of (c, β_m). Hence, for any $m \in \mathbb{N}$, the functions $w_1, \dots, w_m$ are linearly independent. Suppose now that $v_1, \dots, v_{m-1} \in H_0^1(c, d)$ are the eigenfunctions to $\lambda_1(c, d), \dots, \lambda_{m-1}(c, d)$ respectively for the interval (c, d), and set $w = c_1 w_1 + \cdots + c_m w_m \in H_0^1(c, d)$ with $c_1, \dots, c_m \in \mathbb{R}$. Then we can determine $(c_1, \dots, c_m) \neq (0, \dots, 0)$ such that

$$\int_c^d \sigma w v_j \, dx = 0 \quad \text{for } j = 1, \dots, m-1.$$

Since $w_1, \dots, w_m$ are linearly independent we obtain $w \neq 0$, and therefore we can also achieve that

$$\int_c^d \sigma w^2 \, dx = 1.$$

Then the minimum characterization of $\lambda_m(c, d)$, given in Theorem 5.3, implies that

$$\lambda_m(c, d) \leq \mathcal{F}(w) \quad \text{for all } m \in \mathbb{N},$$

where we have set

$$\mathcal{F}(\varphi) := \mathcal{A}(\varphi, \psi) \quad \text{and} \quad \mathcal{A}(\varphi, \psi) := \int_c^d (p\varphi'\psi' + q\varphi\psi) \, dx.$$

Moreover, we have

$$\int_c^{\beta_j} (p w_j' \varphi' + q w_j \varphi) \, dx = \lambda_k(c, \beta_j) \int_c^{\beta_j} \sigma w_j \varphi \, dx$$

for all $\varphi \in H_0^1(c, \beta_j)$. Thus it follows for $l \leq j$ that

$$\mathcal{A}(w_j, w_l) = \lambda_k(c, d) \int_c^d \sigma w_j w_l \, dx.$$

Since $\mathcal{A}(w_j, w_l) = \mathcal{A}(w_l, w_j)$, this relation holds for any pair (j, l), and therefore

$$\begin{aligned} \mathcal{F}(w) &= \sum_{j,l=1}^{m} c_j c_l \mathcal{A}(w_j, w_l) \\ &= \lambda_k(c, d) \sum_{j,l=1}^{m} c_j c_l \int_c^d \sigma w_j w_l \, dx = \lambda_k(c, d). \end{aligned}$$

Hence it follows that

$$\lambda_m(c, d) \leq \lambda_k(c, d) \quad \text{for all } m \in \mathbb{N}.$$

But this is impossible since $\lambda_m(c, d) \to \infty$ as $m \to \infty$, and thus we have

$$\lambda_k(c, d) < \lambda_k(a, b) \quad \text{if } (a, b) \subset (c, d), \ (a, b) \neq (c, d).$$

Finally, we notice that all the theory depends strongly on the fact that we considered problem (5.18) in a *bounded* interval. On unbounded intervals the *spectrum* of a Sturm–Liouville operator may be *continuous*.

Now we prove the following result for the number of zeros of any eigenfunction u. (Note that, by definition, $u(x) \not\equiv 0$.)

Theorem 5.5 *The nth eigenfunction* u_n *of problem* (5.18) has exactly $n - 1$ zeros in (a, b), and between two consecutive zeros of u_n in $[a, b]$ there is exactly one zero of u_{n+1}.

Proof We proceed in several steps.

(i) *Any eigenfunction u can have at most finitely many zeros in* $[a, b]$. Otherwise there would exist a cluster point c of zeros of u, whence $u(c) = u'(c) = 0$ and therefore $u(x) \equiv 0$, a contradiction.

(ii) *The eigenfunction* u_1 *to the smallest eigenvalue* λ *has no zero in* (a, b). Otherwise we consider the smallest zero ξ of $u_1(x)$ in (a, b) and set $v(x) := u_1(x)$ in $a \le x \le \xi$ and $v(x) := 0$ in $\xi < x \le b$. Then $v(x) \not\equiv 0$ in (a, b) and $v \in H_0^1(a, b)$. Furthermore from $\mathcal{A}(u, v) = \lambda_1 \int_a^b \sigma u v \, dx$ it follows that

$$\lambda_1 = \frac{\int_a^b (p\dot{v}^2 + qv^2)\, dx}{\int_a^b \sigma v^2 \, dx} .$$

Since λ_1 is the minimum of the Rayleigh quotient among all functions $u \ne 0$ in $H_0^1(a, b)$, it follows that v is an eigenfunction to the eigenvalue λ_1 and $v(x) = 0$ in $\xi < x < b$. This would contradict property (i).

(iii) *Suppose that u is an eigenfunction for* (a, b) *corresponding to the eigenvalue* λ, *and let* α, β *be two consecutive zeros of* $u(x)$ *with* $a \le \alpha < \beta \le b$. *Then* λ *is the smallest eigenvalue for the interval* (α, β), *and it has* $u_{|[\alpha,\beta]}$ *as a corresponding simple eigenfunction.* Clearly $u(x)$ restricted to $[\alpha, \beta]$ is an eigenfunction on this interval with the eigenvalue λ. Let v_1 be an eigenfunction for (α, β) corresponding to the smallest eigenvalue μ_1 on this interval. If $\lambda > \mu_1$ then $\int_\alpha^\beta \sigma u v_1 \, dx = 0$, and $v_1(x) \ne 0$ on (α, β) according to (ii). Since $\sigma(x) > 0$ it follows that $u(x)$ is changing its sign within (α, β), which contradicts the assumption on u.

(iv) *Suppose that u and v are eigenfunctions for* (a, b) *corresponding to the eigenvalues* λ *and* μ *respectively. Let* α, β *be two consecutive zeros of u in* $[a, b]$ *with* $\alpha < \beta$ *and assume that* $\lambda < \mu$. *Then there exists at least one zero* ξ *of v with* $\alpha < \xi < \beta$. Otherwise there would be two consecutive zeros γ and δ of v satisfying $a \le \gamma \le \alpha < \beta \le \delta \le b$. It is impossible that both $\alpha = \gamma$ and $\beta = \delta$ because in this case u and v were eigenfunctions for (α, β) corresponding to the smallest eigenvalue for (α, β). This is a contradiction to $\lambda < \mu$. Thus $[\alpha, \beta]$ is a true subinterval of $[\gamma, \delta]$, say, $\beta < \delta$, and $v_{|[\gamma,\delta]}$ is the eigenfunction for (γ, δ) corresponding to the eigenvalue μ, which is the smallest eigenvalue for (γ, δ) according to (iii). Therefore we have

$$\mu = \min \left\{ \frac{\int_\gamma^\delta (p\dot{w}^2 + qw^2)\, dx}{\int_\gamma^\delta \sigma w^2 \, dx} : w \in H_0^1(\gamma, \delta) , \ w \ne 0 \right\} .$$

On the other hand it follows that

$$\lambda = \frac{\int_\alpha^\beta (p\dot{u}^2 + qu^2)\,dx}{\int_\alpha^\beta \sigma u^2\,dx}$$

since $u(\alpha) = u(\beta) = 0$. Then the function $w(x)$ defined by $w(x) := u(x)$ for $\alpha \le x \le \beta$ and $w(x) := 0$ for $\gamma \le x \le \alpha$ and $\beta \le x \le \delta$ is of class $H_0^1(\gamma, \delta)$ and satisfies

$$\lambda = \frac{\int_\gamma^\delta (p\dot{w}^2 + qw^2)\,dx}{\int_\gamma^\delta \sigma w^2\,dx} \ge \mu.$$

This contradicts the assumption $\lambda < \mu$.

(v) *Between two consecutive zeros of u_n there is at least one zero of u_{n+1}.* This assertion follows readily from (iv) applied to $u = u_n$ and $v = u_{n+1}$, taking $\lambda_n < \lambda_{n+1}$ into account.

(vi) *The nth eigenfunction u_n for the interval (a, b) has at least $n - 1$ zeros in (a, b).* By induction this follows from (v).

(vii) *The eigenfunction u_n has at most $n-1$ zeros in (a, b).* Suppose otherwise that u_n has at least n zeros $\xi_1, \dots, \xi_n$ in (a, b), $a =: \xi_0 < \xi_1 < \xi_2 < \dots < \xi_n < \xi_{n+1} := b$, and set $I_j := [\xi_{j-1}, \xi_j]$, $j = 1, 2, \dots, n+1$, $\xi := \xi_n$, $I' := (a, \xi)$, i.e. $\overline{I'} = I_1 \cup I_2 \cup \dots \cup I_n$. Set $w_j(x) := k_j u_n(x)$ for $x \in I_j$ and $w_j(x) := 0$ for $x \in [a, b] - I$, $1 \le j \le n$, where $k_j \in \mathbb{R}$ is chosen in such a way that

$$\int_a^b \sigma w_j^2\,dx = 1.$$

Then the functions $w_1, \dots, w_n$ are of class $H_0^1(a, \xi)$ and satisfy

$$\int_a^\xi \sigma w_j w_k\,dx = \delta_{jk}$$

and

$$\int_a^\xi (pw_j' w_k' + qw_j w_k)\,dx = \lambda_n \delta_{jk}.$$

Choose arbitrary functions $v_1, \dots, v_{n-1} \in L^2(a, \xi)$. Then we can determine an n-tuple $(c_1, c_2, \dots, c_n) \in \mathbb{R}^n - \{0\}$ such that $w := c_1 w_1 + c_2 w_2 + \dots + c_n w_n \in H_0^1(a, \xi)$ satisfies

$$\int_a^\xi \sigma w v_l\,dx = 0, \quad 1 \le l \le n-1.$$

We can normalize $(c_1, \dots, c_n)$ in such a way that $c_1^2 + c_2^2 + \cdots + c_n^2 = 1$, and thus we also have

$$\int_a^\xi \sigma w^2 \, dx = 1.$$

Because

$$\int_a^\xi (p\dot{w}^2 + qw^2)\, dx = \sum_{j,k=1}^{n} \lambda_n \delta_{jk} c_j c_k = \lambda_n$$

we then infer from Courant's maximum–minimum principle (Theorem 5.4) that the nth eigenvalue λ_n' for the interval $\overline{I'} = [a, \xi]$ satisfies $\lambda_n' \le \lambda_n$. On the other hand this principle implies the strong monotonicity property $\lambda_n < \lambda_n'$ since $\overline{I'} = [a, \xi]$ is a true subinterval of $[a, b]$, and we have found a contradiction.

(viii) From (vi) and (vii) we conclude that *the nth eigenfunction u_n for (a, b) has exactly $n - 1$ zeros in (a, b)*. In conjunction with (v) we infer that *there is exactly one zero of u_{n+1} between two consecutive zeros of u_n, α, β with $a \le \alpha < \beta \le b$, and we have $u_{n+1}(\alpha) \ne 0$ if $\alpha > a$ and $u_{n+1}(\beta) \ne 0$ if $\beta < b$.* □

It is an interesting problem to give estimates for the eigenvalues, in particular for the first eigenvalue. This question is obviously connected with the problem of optimal inequalities such as the optimal constant in Poincaré's inequality. For this purpose, direct methods as well as field theory and Fourier series are very useful tools. We are not going to treat this kind of problem in general, but we want at least to mention some simple results concerning the operator $-u''$.

First let us consider the eigenvalue problem

$$\begin{aligned} -u'' &= \lambda u \quad \text{in } (a, b) \\ u(a) &= u(b) = 0. \end{aligned} \tag{5.30}$$

As we have seen, all eigenvalues of (5.30) are positive and the first eigenvalue λ_1 is given by the infimum of the Rayleigh–Ritz quotient

$$\lambda_1 = \inf \left\{ \frac{\int_a^b \dot{u}^2 \, dx}{\int_a^b u^2 \, dx} : \ u \in H_0^1(a, b), \ \ u \ne 0 \right\}.$$

Equivalently, the inverse of λ_1 can be seen as the best constant in Poincaré's inequality:

$$\int_a^b u^2 \, dx \le \frac{1}{\lambda_1} \int_a^b \dot{u}^2 \, dx \quad \text{for all } u \in H_0^1(a, b).$$

In this case we have

$$\lambda_1 = \left(\frac{\pi}{b - a} \right)^2$$

and more precisely

Proposition 5.6 *The sequence* $\{\lambda_n\}$ *of eigenvalues of* (5.30) *is given by*

$$\lambda_n = n^2\left(\frac{\pi}{b-a}\right)^2 \qquad n = 1, 2, \ldots$$

with eigenfunctions

$$u_n(x) = \sin\left(n\pi\frac{x-a}{b-a}\right).$$

In particular,

$$\int_a^b u^2\,dx \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b \dot{u}^2\,dx \quad \textit{for every } u \in H_0^1(a, b).$$

and equality holds if and only if u is proportional to the first eigenfunction,

$$u(x) = C\sin\left(\pi\frac{x-a}{b-a}\right).$$

Proof Since all eigenvalues of (5.30) are positive, the general solution of

$$u'' + \lambda u = 0$$

is given by

$$u(x) = C\sin[\sqrt{\lambda}(x - x_0)]$$

where C and x_0 are the integration constants. By imposing the boundary conditions $u(a) = u(b) = 0$ we obtain that $\sqrt{\lambda}(a - x_0)$ and $\sqrt{\lambda}(b - x_0)$ are both integer multiples of π, which yields the equality

$$\lambda = n^2\left(\frac{\pi}{b-a}\right)^2 \tag{5.31}$$

for some integer n. Therefore, every eigenvalue is of the form (5.31) and the corresponding eigenfunction is given by

$$u_n(x) = \sin\left(n\pi\frac{x-a}{b-a}\right). \tag{5.32}$$

On the other hand, it is easy to see that the functions u_n in (5.32) and the numbers $\lambda_n = n^2(\pi/(b-a))^2$ satisfy the equation

$$u_n'' + \lambda_n u_n = 0$$
$$u_n(a) = u_n(b) = 0$$

and so they are eigenfunctions and eigenvalues of (5.30). Taking $n = 1$ we obtain the first eigenvalue $\lambda_1 = (\pi/(b-a))^2$ whose inverse provides the best constant in Poincaré's inequality. □

Suppose now that on the admissible functions for $\mathcal{F}$ we require only $u(a) = 0$, while $u(b)$ is free to assume any value. By Section 1.2 we know that in this case the Euler equation has the *natural* condition $u'(b) = 0$, i.e. the equation for eigenvalues and eigenfunctions is

$$\begin{aligned} -u'' &= \lambda u \quad \text{in } (a, b) \\ u(a) &= 0, \quad u'(b) = 0. \end{aligned} \tag{5.33}$$

The general solution is given by

$$u(x) = C \sin[\sqrt{\lambda}(x - x_0)]$$

and, imposing the boundary conditions $u(a) = u'(b) = 0$, we obtain that $\sqrt{\lambda}(a - x_0)$ and $\sqrt{\lambda}(b - x_0) - \pi/2$ are both integer multiples of π, which yields the equality

$$\lambda = \left(\frac{n\pi - \pi/2}{b - a}\right)^2 \tag{5.34}$$

for some integer n. The corresponding eigenfunction is

$$u_n(x) = \sin\left[\left(n - \frac{1}{2}\right)\pi \frac{x - a}{b - a}\right]. \tag{5.35}$$

On the other hand, the functions u_n in (5.35) and the numbers λ_n given by (5.34) satisfy the equation (5.33). Therefore, the first eigenvalue is $\lambda_1 = \frac{1}{4}(\pi/(b-a))^2$, whose inverse provides the best constant in Poincaré's inequality where only $u(a) = 0$ is prescribed.

Summarizing, we can conclude with

Proposition 5.7 *The sequence $\{\lambda_n\}$ of eigenvalues of* (5.33) *is given by*

$$\lambda_n = \left(n - \frac{1}{2}\right)^2 \left(\frac{\pi}{b - a}\right)^2 \quad n = 1, 2, \dots$$

with eigenfunctions

$$u_n(x) = \sin\left[\left(n - \frac{1}{2}\right)\pi \frac{x - a}{b - a}\right].$$

In particular,

$$\int_a^b u^2\,dx \le 4\left(\frac{b - a}{\pi}\right)^2 \int_a^b \dot{u}^2\,dx \quad \text{for every } u \in H^1(a, b), \quad u(a) = 0,$$

and equality holds if and only if u is proportional to the first eigenfunction, i.e.

$$u(x) = C \sin\left(\frac{\pi}{2}\frac{x - a}{b - a}\right).$$

We now consider the case when both $u(a)$ and $u(b)$ are free to assume any value. This corresponds to the equation

$$\begin{aligned} -u'' &= \lambda u \quad \text{in } (a, b) \\ u'(a) &= u'(b) = 0 \end{aligned} \tag{5.36}$$

for eigenvalues and eigenfunctions. By a procedure similar to the previous ones we obtain that the eigenvalues are given by

$$\lambda_n = (n-1)^2 \left(\frac{\pi}{b-a}\right)^2 \qquad n = 1, 2, \ldots$$

and the corresponding eigenfunctions are

$$u_n(x) = \cos\left((n-1)\pi \frac{x-a}{b-a}\right).$$

Notice that for the first eigenvalue we have $\lambda_1 = 0$, and the eigenspace corresponding to $\lambda_1 = 0$ is the subspace of constant functions; its orthogonal in $L^2(a, b)$ is the subspace of functions with zero average in (a, b). The variational characterization of eigenvalues then yields at once that

$$\inf\left\{\frac{\int_a^b |u'|^2\,dx}{\int_a^b u^2\,dx} : u \in H^1(a, b), \int_a^b u\,dx = 0\right\} = \left(\frac{\pi}{b-a}\right)^2.$$

Thus we may conclude with

Proposition 5.8 *For every $u \in H^1(a, b)$ with zero average, i.e. such that*

$$\int_a^b u\,dx = 0,$$

we have

$$\int_a^b u^2\,dx \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b |u'|^2\,dx$$

and equality holds if and only if

$$u(x) = C\cos\left(\pi\frac{x-a}{b-a}\right).$$

Similarly, we obtain

Proposition 5.9 *For every $u \in H^1(a, b)$ periodic and with zero average we have*

$$\int_a^b u^2\,dx \le \left(\frac{b-a}{2\pi}\right)^2 \int_a^b |u'|^2\,dx$$

and equality holds if and only if

$$u(x) = A\cos\left(\frac{2\pi}{b-a}x\right) + B\sin\left(\frac{2\pi}{b-a}x\right).$$

More precisely, as a consequence of the Parseval identity, we find

Proposition 5.10 *For every periodic function $u \in H^1(a,b)$ we have*

$$\int_a^b u^2\,dx - \left(\frac{b-a}{2\pi}\right)^2 \int_a^b |u'|^2\,dx \le \frac{1}{b-a}\left(\int_a^b u\,dx\right)^2 \tag{5.37}$$

and equality holds if and only if

$$u(x) = A + B\cos\left(\frac{2\pi}{b-a}x\right) + C\sin\left(\frac{2\pi}{b-a}x\right).$$

Proof With the change of variable

$$x = a + \frac{b-a}{2\pi}\,\theta$$

and setting $f(\theta) = u(a + (b-a)/(2\pi)\theta)$, formula (5.37) becomes

$$\int_0^{2\pi} f^2\,d\theta - \int_0^{2\pi} |f'|^2 d\theta \le \frac{1}{2\pi}\left(\int_0^{2\pi} f\,d\theta\right)^2. \tag{5.38}$$

Expanding $f(\theta)$ as a Fourier series, and using the fact that, since $f \in H^1(0, 2\pi)$ the Fourier series converges in the H^1-norm, we find that

$$f(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta)$$

$$f'(\theta) = \sum_{k=1}^{\infty} k(-a_k \sin k\theta + b_k \cos k\theta).$$

Then, by the Parseval identity we get

$$\int_0^{2\pi} f^2\,d\theta = \pi\left[\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)\right]$$

$$\int_0^{2\pi} |f'|^2\,d\theta = \pi\sum_{k=1}^{\infty} k^2(a_k^2 + b_k^2)$$

and hence (5.38) follows. Moreover, we see that equality in (5.38) holds if and only if

$$f(\theta) = \frac{1}{2}a_0 + a_1\cos\theta + b_1\sin\theta,$$

i.e.

$$u(x) = A + B\cos\left(\frac{2\pi}{b-a}x\right) + C\sin\left(\frac{2\pi}{b-a}x\right). \qquad \square$$

As we have seen in Section 2.1 the inequalities

$$\left(\int_0^1 |u|^q\,dx\right)^{1/q} \le c(p,q)\left(\int_0^1 |u'|^p\,dx\right)^{1/p}$$

hold for all $u \in H_0^{1,p}(0,1)$ and all $p, q \ge 1$, or equivalently, by a scaling argument,

$$\left(\int_a^b |u|^q\,dx\right)^{1/q} \le c(p,q)(b-a)^{1-1/p+1/q}\left(\int_a^b |u'|^p\,dx\right)^{1/p} \tag{5.39}$$

for all $u \in H_0^{1,p}(a,b)$. We conclude this section by showing that the best constant $c(p,q)$ in (5.39) is given, when $p > 1$, by

$$c(p,q) = \frac{1}{2}\frac{q(1+p'/q)^{1/p}}{(1+q/p')^{1/q}B(1/q,1/p')} \tag{5.40}$$

where p' is the conjugate exponent of p, $1/p + 1/p' = 1$, and B is the so-called *beta function*. We recall that the beta function $B(x,y)$ is defined for $x, y > 0$ as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

or equivalently by

$$B(x,y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta\,\sin^{2y-1}\theta\,d\theta.$$

It is not difficult to show that, in terms of *Euler's gamma function*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad (x > 0),$$

we have

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We recall also that (for the proofs we refer for instance to Fleming [101])

$$\begin{aligned} \Gamma(x+1) &= x\Gamma(x) && \text{for all } x > 0\\ \Gamma(n+1) &= n! && \text{for all } n \in \mathbb{N}\\ B(1/2,1/2) &= \Gamma(1/2) = \sqrt{\pi}. \end{aligned}$$

The result we shall prove is the following.

Proposition 5.11 *Let $1 < p < +\infty$ and $1 \le q < +\infty$. For every $u \in H_0^{1,p}(a,b)$ we have*

$$\left(\int_a^b |u|^q\,dx\right)^{1/q} \le \frac{1}{2}(b-a)^{1-1/p+1/q}\frac{q(1+p'/q)^{1/p}}{(1+q/p')^{1/q}B(1/q,1/p')}\left(\int_a^b |u'|^p\,dx\right)^{1/p}.$$

Proof By the change of variable

$$x = a + (b-a)t$$

we may assume that $a = 0$ and $b = 1$. The best constant $c(p,q)$ in (5.39) is then obviously given by the inverse of

$$\inf\left\{\left(\int_0^1 |u'|^p\,dx\right)^{1/p} : u \in H_0^{1,p}(0,1),\ \int_0^1 |u|^q\,dx = 1\right\} \tag{5.41}$$

or equivalently by

$$\sup\left\{\frac{(\int_0^1 |u|^q\,dx)^{1/q}}{(\int_0^1 |u'|^p\,dx)^{1/p}} : u \in H_0^{1,p}(0,1),\ u \not\equiv 0\right\}. \tag{5.42}$$

By the direct methods of the calculus of variations and also taking into account the compactness of the embedding $H_0^{1,p} \hookrightarrow L^q$, we see at once that the infimum in (5.41) or the supremum in (5.42) are attained. By replacing each element u_n of a minimizing sequence by $|u_n|$ we also see that the infimum in (5.41) (or the supremum in (5.42)) is attained on a non-negative function u. From the general theory of Section 1.2 we also know that u is a weak solution of

$$\begin{cases}(|u'|^{p-2}u')' + \lambda u^{q-1} = 0\\ u(0) = u(1) = 0, \quad u \ge 0\end{cases} \tag{5.43}$$

in the sense that

$$\int_0^1 |u'|^{p-2}u'\varphi'\,dx - \lambda\int_0^1 u^{q-1}\varphi\,dx = 0 \quad \text{for all } \varphi \in C_c^1(0,1). \tag{5.44}$$

Here the Lagrange multiplier λ is given by

$$\lambda = \int_0^1 |u'|^p\,dx \bigg/ \int_0^1 |u|^q dx.$$

From (5.43) we deduce in particular that $|u'|^{p-2}u'$ is an absolutely continuous function.

Now fix $x_1 < x_2$ in $(0, 1)$ and choose in (5.44) the sequence φ_n defined by (it is clear that (5.44) also holds for all Lipschitz-continuous functions φ with compact support in $(0, 1)$)

$$\varphi_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le x_1 - 1/n \\ n(x - x_1 + 1/n) & \text{if } x_1 - 1/n < x \le x_1 \\ 1 & \text{if } x_1 < x < x_2 \\ -n(x - x_2 - 1/n) & \text{if } x_2 \le x < x_2 + 1/n \\ 0 & \text{if } x_2 + 1/n \le x \le 1. \end{cases}$$

We find that

$$\begin{aligned} 0 \le \lambda \int_{x_1}^{x_2} u^{q-1}\, dx &= \lim_{n \to +\infty} \lambda \int_0^1 u^{q-1} \varphi_n \, dx \\ &= \lim_{n \to +\infty} \int_0^1 |u'|^{p-2} u' \varphi_n' \, dx \\ &= |u'(x_1)|^{p-2} u'(x_1) - |u'(x_2)|^{p-2} u'(x_2). \end{aligned}$$

Moreover, the inequality is strict as soon as u does not vanish in (x_1, x_2). Hence $|u'|^{p-2}u'$ is decreasing in $(0, 1)$, and consequently u' is decreasing in $(0, 1)$.

By possibly modifying the value of λ we may normalize the function u so that

$$\max_{[0,1]} u = 1.$$

Multiplying (5.43) by u' and observing that

$$u'(|u'|^{p-2}u')' = \frac{1}{p'}(|u'|^p)'$$

we deduce that

$$\frac{1}{p'}|u'|^p + \frac{\lambda}{q} u^q = \text{const.}$$

Writing this equation at the point $\overline{x}$ where u attains its maximum, $u(\overline{x}) = 1$, we then find that

$$\frac{1}{p'}|u'|^p + \frac{\lambda}{q} u^q = \frac{1}{p'}|u'(\overline{x})|^p + \frac{\lambda}{q} u^q(\overline{x}) = \frac{\lambda}{q},$$

and hence

$$|u'|^p = \frac{p'\lambda}{q}(1 - u^q).$$

It is easy to see that the point $\bar{x}$ is also the unique zero of u'; then we have

$$u'(1-u^q)^{-1/p} = \begin{cases} (p'\lambda/q)^{1/p} & \text{on } (0, \bar{x}) \\ -(p'\lambda/q)^{1/p} & \text{on } (\bar{x}, 1). \end{cases}$$

Consequently, since

$$\int_0^1 (1-u^q)^{-1/p} du = \int_0^{\bar{x}} u'(1-u^q)^{-1/p}\, dx = \bar{x}(p'\lambda/q)^{1/p}$$
$$\int_1^0 (1-u^q)^{-1/p} du = \int_{\bar{x}}^1 u'(1-u^q)^{-1/p}\, dx = (\bar{x}-1)(p'\lambda/q)^{1/p},$$

we find $\bar{x} = 1/2$. Therefore,

$$(p'\lambda/q)^{1/p} = 2\int_0^1 (1-u^q)^{-1/p} du = \frac{2}{q} B(1/q, 1/p')$$

and we deduce for u

$$u'(1-u^q)^{-1/p} = \frac{2}{q} B(1/q, 1/p') \quad \text{on } (0, 1/2)$$
$$u(0) = 0, \quad u(1/2) = 1.$$

Thus u is defined implicitly by

$$\int_0^{u(x)} (1-s^q)^{-1/p}\, dx = \frac{2x}{q} B(1/q, 1/p') \quad \text{for } 0 \le x \le 1/2$$

and by

$$u(x) = u(1-x) \quad \text{for } 1/2 \le x \le 1.$$

After some simple calculations we find that

$$\int_0^1 u^q\, dx = 2\int_0^{1/2} u^q\, dx = 2\int_0^1 \frac{u^q}{u'} du = \frac{p'}{q+p'}$$
$$\int_0^1 |u'|^p\, dx = 2\int_0^{1/2} |u'|^p\, dx = 2\int_0^1 |u'|^{p-1} du = \frac{(2B(1/q, 1/p'))^p}{q^{p-1}(q+p')}$$

so that the conclusion follows at once by formula (5.42). □

Remark 2 In the limit case $p = 1$ it is easy to see that the supremum in formula (5.42) is not attained on $H_0^{1,1}(0, 1)$ but on the function $u \in BV(\mathbf{R})$ given by

$$u(x) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } x \ge 1 \\ 1 & \text{if } x \in (0, 1) \end{cases}$$

so that

$$c(1, q) = 1/2 \quad \text{for all } q \geq 1.$$

Analogously, when $q = +\infty$ and $p \geq 1$, it is easy to see that the supremum

$$\sup \left\{ \frac{\|u\|_{L^\infty}}{\|u'\|_{L^p}} : u \in H_0^{1,p}(0, 1), u \not\equiv 0 \right\}$$

is attained on the function $u(x) = 1 - |2x - 1|$, so that

$$c(p, \infty) = 1/2 \quad \text{for all } p \geq 1.$$

5.3 The vibrating string

Consider a perfectly elastic string stretched under constant tension τ along the x-axis with its end points fixed at $x = 0$ and $x = L$; in this configuration the string is in *equilibrium*. The string is permitted to vibrate freely in a plane containing the x-axis so that each particle of the string moves in a straight line perpendicular to the x-axis. The amplitude of vibration is supposed so small that the slope of the string at any point is small at any instant; the elongation consequently is small so that we can assume that the tension remains constant. We finally assume that the system is conservative, in particular there is no friction. The transverse displacement at time t of the point x, $0 \leq x \leq L$, is denoted by $w(x, t)$; thus $w(x, t)$ describes the shape of the string at time t and, since the string is fixed at its end points, we have $w(0, t) = w(L, t) = 0$ for all t.

Since the string is perfectly elastic, the work necessary to reach a distorted configuration is merely employed to increase the length of the string compared with its equilibrium length. Thus the potential energy at a given instant is given by

$$V = \tau \int_0^L \left(\sqrt{1 + w_x^2} - 1 \right) dx.$$

With the assumption that $|w_x|$ is small, we may expand $\sqrt{1 + |w_x|^2}$ as

$$\sqrt{1 + |w_x|^2} = 1 + \frac{1}{2}|w_x|^2 + o(|w_x|^2),$$

and, neglecting the higher-order terms in $|w_x|^2$, we may assume that

$$V = \frac{1}{2}\tau \int_0^L |w_x|^2 \, dx.$$

If $\sigma(x)$ denotes the density of the distribution of mass along the string, the kinetic energy is given by

$$T = \frac{1}{2} \int_0^L \sigma |\dot{w}|^2 \, dx$$

and, applying Hamilton's principle, the actual motion of the string is the one which makes the integral

$$\int_{t_1}^{t_2} (T - V)\,dt = \frac{1}{2}\int_{t_1}^{t_2}\int_0^L (\sigma \dot{w}^2 - \tau w_x^2)\,dx\,dt$$

stationary. The motion is therefore described by d'Alembert's equation

$$\frac{\partial^2 w}{\partial x^2} = \frac{\sigma(x)}{\tau}\frac{\partial^2 w}{\partial t^2}, \tag{5.45}$$

which is the Euler equation of the preceding integral $\int_{t_1}^{t_2} (T - V)\,dt$. This Euler equation is derived in a similar way as in the one-dimensional case. Equation (5.45) together with the boundary conditions

$$w(0, t) = w(L, t) = 0$$

and the initial conditions on the position and the velocity

$$\begin{aligned} w(x, 0) &= \Phi(x) \\ \dot{w}(x, 0) &= \Psi(x) \end{aligned} \tag{5.46}$$

determine the actual motion of the string.

Daniel Bernoulli proposed to attack the problem by searching for solutions in which every particle executes a simple harmonic motion differing only in amplitude from the motion of the other particles. Such a motion, which is called an *eigenvibration* of the string, is represented by a function of the form

$$w(x, t) = u(x) f(t) \tag{5.47}$$

where $u(x)$ represents the *shape factor* and $f(t)$ the *magnification factor*. Inserting (5.47) into (5.45) we get

$$\frac{\tau}{\sigma(x)}\frac{u''(x)}{u(x)} = \frac{f''(t)}{f(t)}. \tag{5.48}$$

Since the left-hand side depends upon x alone and the right-hand side upon t alone, the only possibility is that both sides be equal to a constant that we denote by $-\lambda$. Thus (5.48) implies the two ordinary differential equations

$$\tau\frac{d^2u}{dx^2} + \lambda\sigma(x)u = 0 \qquad \frac{d^2 f}{dt^2} + \lambda f = 0$$

while the boundary condition yields

$$u(0) = u(L) = 0.$$

This argument is often called the *method of separation of variables*. The first problem

$$\begin{cases} \tau \dfrac{d^2u}{dx^2} + \lambda\sigma(x)u = 0 \\ u(0) = u(L) = 0 \end{cases}$$

is a Sturm–Liouville eigenvalue problem, and, as we have seen in Section 5.2, it has non-trivial solutions only for a countable set of eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \qquad \lambda_n \to \infty,$$

with corresponding eigenfunctions u_n. For each λ_n the general solution of

$$\frac{d^2f}{dt^2} + \lambda_n f = 0$$

is given by

$$a_n \cos\sqrt{\lambda_n}t + b_n \sin\sqrt{\lambda_n}t$$

where a_n and b_n are arbitrary constants. Thus we conclude that for each n the functions

$$w_n(x,t) := u_n(x)(a_n \cos\sqrt{\lambda_n}t + b_n \sin\sqrt{\lambda_n}t) \tag{5.49}$$

are solutions of (5.45) satisfying the boundary conditions. Since every smooth function can be expanded as a Fourier series with respect to u_n as

$$v = \sum_{i=1}^{\infty} c_n u_n \qquad c_n := \int_0^L \sigma(x) v u_n\, dx,$$

we deduce that we can choose the constants a_n, b_n in such a way that

$$w(x,t) := \sum_{n=0}^{\infty} w_n(x,t) = \sum_{n=1}^{\infty} u_n(x)(a_n \cos\sqrt{\lambda_n}t + b_n \sin\sqrt{\lambda_n}t) \tag{5.50}$$

fits the initial conditions (5.46). In conclusion the actual motion is obtained as the *superposition* of eigenvibrations.

Clearly the solution (5.49) is periodic in time with *period* $2\pi/\sqrt{\lambda_n}$ and *frequency* $\sqrt{\lambda_n}/2\pi$ and the string has the possibility of vibrating with any of the discrete set of frequencies which are determined by the eigenvalues λ_n. The lowest eigenvalue λ_1 provides the *fundamental frequency* $\sqrt{\lambda_1}/2\pi$ of the string and the general motion of a vibrating string is a linear superposition of the various frequency modes of vibrations represented by (5.49).

5.4 Variational problems with obstacles

Consider the variational integral

$$\mathcal{F}(u) = \int_a^b F(x, u, u')\, dx$$

with a smooth Lagrangian $F(x, z, p)$ that is convex with respect to p. For simplicity we also assume that $F(x, z, p)$ has a polynomial growth $m > 1$ (see Section 3.2). By means of direct methods we have proved in Section 3.2 that $\mathcal{F}(u)$ attains its absolute minimum on each affine subspace of $H^{1,m}(a, b)$

$$C(a, b) = \{u \in H^{1,m}(a, b) : u(a) = \alpha,\ u(b) = \beta\}.$$

In exactly the same way one can show that $\mathcal{F}(u)$ attains its minimum on any non-empty *closed convex set* K of $C(a, b)$.

Suppose now that u is a minimizer of $\mathcal{F}$ on a closed convex set K of $C(a, b)$. In general u will not be an 'inner point' of K; hence the first variation of $\mathcal{F}$ at u need not vanish in all smooth directions. However, we have $\lambda u + (1 - \lambda)v \in K$ for all $\lambda \in [0, 1]$ provided that $v \in K$, and as in Section 1.1 it is not difficult to see that the function $\phi : [0, 1] \to \mathbf{R}$, defined by

$$\phi(\lambda) := \mathcal{F}[\lambda u + (1 - \lambda)v],$$

is differentiable. Then we conclude that $\phi'(1) \leq 0$, which leads to the inequality

$$\int_a^b [F_p(x, u, u') \cdot (u' - v') + F_z(x, u, u') \cdot (u - v)]\, dx \leq 0 \tag{5.51}$$

for all $v \in K$.

Relation (5.51) is called a *variational inequality*. It has to be satisfied for all $v \in K$ if u is a minimizer for $\mathcal{F}$ on K. Such an inequality is a substitute for the vanishing of the first variation of $\mathcal{F}$ at minimizers of $\mathcal{F}$ on the affine subspaces $C(a, b)$.

An interesting example of this kind of problem is the so-called *obstacle problem*. We choose a continuous function ψ on $[a, b]$ satisfying $\psi(a) < \alpha$ and $\psi(b) < \beta$, which is called the *obstacle*, and consider the subset K_ψ of functions $u \in C(a, b)$ satisfying $u(x) \geq \psi(x)$ for $a \leq x \leq b$. It is easily seen that K_ψ is closed, convex, and non-empty. Hence the minimum problem

$$\min \left\{ \frac{1}{2} \int_a^b |u'|^2\, dx :\ u \in K_\psi \right\} \tag{5.52}$$

has a solution which can be interpreted as the equilibrium position of an elastic string (under small perturbations) which is forced to stay above the obstacle ψ and is fixed at its two end points which lie 'above ψ' (i.e. above the graph of ψ). The minimizer u of problem (5.52) is uniquely determined since the functional $\frac{1}{2} \int_a^b |u'|^2\, dx$ is strictly convex on K_ψ. By the minimization procedure we obtain a minimizer u of class $H^{1,2}(a, b)$, and

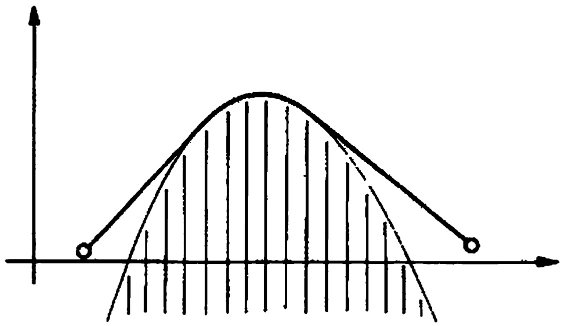

FIG. 5.1.

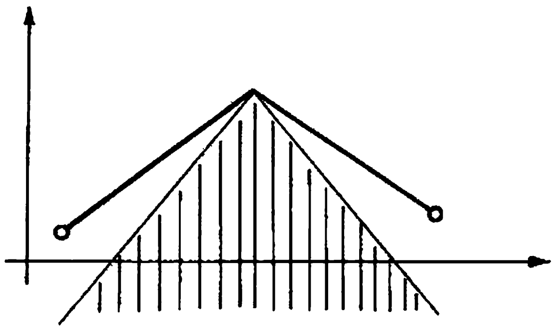

FIG. 5.2.

therefore $u \in C^{0,1/2}([a,b])$. Figure 5.1 suggests that u cannot be better than $C^{1,1}$, even if the obstacle ψ is real analytic, and if ψ is less regular than C^1 we should even expect less regularity than C^1, as shown by Fig. 5.2.

We shall now briefly discuss the regularity question for the minimizer u. To this end we introduce the *coincidence set* $I := \{x \in [a,b] : u(x) = \psi(x)\}$ where the minimizer u of problem (5.52) coincides with the obstacle ψ. This set is closed (since u and ψ are both continuous), possibly empty, and certainly $I \subset (a,b)$ since $\psi(a) < \alpha = u(a)$ and $\psi(b) < \beta = u(b)$. The complement I' of I,

$$I' := \{x \in [a,b] : u(x) > \psi(x)\},$$

is open in $[a,b]$.

We claim that $u''(x) = 0$ on I'. In fact, if $x_0 \in I' \cap (a,b)$ then there is a neighbourhood U of x_0 in (a,b) and a number $m > 0$ such that $u(x) \geq m$ for all $x \in U$. Hence for any $\varphi \in C_c^\infty(U)$ and $|\epsilon| \ll 1$ we have $u + \epsilon\varphi \in K_\psi$, whence

$$\int_a^b u'(x)\varphi'(x)\,dx = 0$$

and therefore $u \in C^\infty(I' \cap (a,b))$ and $u''(x) = 0$ for all $x \in I' \cap (a,b)$. Since $[a, a+\delta] \cup [b-\delta, b] \subset I'$ for $0 < \delta \ll 1$, we infer that $u \in C^\infty(I')$ and $u''(x) = 0$ on I', whereas $u(x) = \psi(x)$ for any $x \in I$. Hence $u(x)$ can be non-regular only at points of coincidence.

Now we apply variational inequality (5.51) to the Lagrangian $F(p) = \frac{1}{2}|p|^2$, the confining set $K = K_\psi$ and to test function $v = u + \varphi$ where φ is an arbitrary non-negative function of class $C_c^\infty(a,b)$. We obtain

$$\int_a^b u'(x)\varphi'(x)\,dx \geq 0 \quad \text{for all } \varphi \in C_c^\infty(a,b) \text{ with } \varphi(x) \geq 0. \tag{5.53}$$

Let $\{\varphi_j\}$ be a sequence of functions of class $C_c^\infty(a,b)$ with support in a fixed compact set A in (a,b) which converge uniformly to zero. We choose a function $\Phi \in C_c^\infty(a,b)$ with $\Phi(x) \geq 0$ on (a,b) and $\Phi(x) \geq 1$ on A. Then we obtain

$$-\epsilon_j \Phi(x) \leq \varphi_j(x) \leq \epsilon_j \Phi(x) \quad \text{for } a \leq x \leq b$$

where $\{\epsilon_j\}$ is a suitable sequence of positive real numbers with $\epsilon_j \to 0$. Applying (5.53) to $\varphi = \epsilon_j \Phi \pm \varphi_j$ we obtain

$$-\epsilon_j \int_a^b u'(x)\Phi'(x)\,dx \leq \int_a^b u'(x)\varphi_j'(x)\,dx \leq \epsilon_j \int_a^b u'(x)\Phi'(x)\,dx,$$

and this implies that

$$\lim_{j\to\infty} \int_a^b u'(x)\varphi_j'(x)\,dx = 0.$$

Therefore the mapping $T(\varphi) := \int_a^b u'(x)\varphi'(x)\,dx$, $\varphi \in C_c^\infty(a,b)$, is a continuous positive linear functional on $C_c^\infty(a,b)$ equipped with the sup convergence. By Riesz's representation theorem this implies the existence of a positive measure μ such that

$$\int_a^b u'\varphi'\,dx = \int_a^b \varphi\,d\mu \quad \text{for all } \varphi \in C_c^\infty(a,b).$$

This relation can also be written as

$$u'' = -\mu \tag{5.54}$$

in the sense of distributions on (a,b). Moreover, from what we have seen above, we infer that the support of the measure μ is contained in the coincidence set I. In particular, it follows from (5.54) that the solution u is a concave function; hence locally Lipschitz on (a,b). Therefore, as a consequence of the concavity, we have

$$u'(x^-) \geq u'(x^+) \quad \text{for every } x \in (a,b). \tag{5.55}$$

Suppose now that ξ is a point in the coincidence set I, so that

$$u(x) - u(\xi) \geq \psi(x) - \psi(\xi) \quad \text{for every } x \in (a, b).$$

If $x < \xi$ we therefore have

$$\frac{u(x) - u(\xi)}{x - \xi} \leq \frac{\psi(x) - \psi(\xi)}{x - \xi};$$

hence

$$u'(\xi^-) \leq \psi'(\xi^-)$$

provided $\psi'(\xi^-)$ exists, while, if $x > \xi$,

$$\frac{u(x) - u(\xi)}{x - \xi} \geq \frac{\psi(x) - \psi(\xi)}{x - \xi};$$

hence

$$u'(\xi^+) \geq \psi'(\xi^+)$$

provided that $\psi'(\xi^+)$ exists.

Suppose now that ψ' has only discontinuities $x \in (a, b)$ satisfying

$$\psi'(x^-) \leq \psi'(x^+).$$

For instance, this is the case if ψ is of class C^1. Then we find that

$$u'(\xi^-) \leq \psi'(\xi^-) \leq \psi'(\xi^+) \leq u'(\xi^+),$$

and by virtue of (5.55) it follows that

$$u'(\xi^-) = u'(\xi^+).$$

Hence $u'(x)$ is continuous on I and therefore also on (a, b). Thus we obtain the following result: *If $\psi \in C^1(a, b)$ then also $u \in C^1([a, b])$, and $u'(x) = \psi(x)$ for all $x \in I$.* Actually we can show that the minimizer u coincides with the concave envelope $\bar{\psi}$ of the function γ_ψ defined by

$$\gamma_\psi(x) := \begin{cases} \psi(x) & \text{for } x \in (a, b) \\ \alpha & \text{if } x = a \\ \beta & \text{if } x = b, \end{cases}$$

i.e. with the lowest concave function which is larger than or equal to γ_ψ. Indeed, since u is concave and $u \geq \gamma_\psi$ we also have $u \geq \bar{\psi}$. On the other hand we have $u(x) = \psi(x) \leq \bar{\psi}(x)$ on I, while the function u is affine on every maximal subinterval $\mathcal{I}$ of I' and coincides with γ_ψ at the boundary of $\mathcal{I}$, which implies $u(x) \leq \bar{\psi}(x)$ on $\mathcal{I}$.

Now we are going to prove that $u \in C^{1,1}(a,b)$ if we assume that $\psi \in C^{1,1}(a,b)$. First we show that $u \in H^{2,2}_{loc}(a,b)$. For this purpose we introduce the difference–quotient operator Δ_h by

$$\Delta_h u(x) := \frac{1}{h}[u(x+h) - u(x)]$$

for $h \in \mathbb{R}$ with $0 < |h| < \delta$ and $x \in (a+\delta, b-\delta)$, $0 < \delta \ll 1$. It follows that

$$\begin{aligned}&\Delta_{-h}[\eta^2(x)\Delta_h u(x)]\\&\quad = h^{-2}\{\eta^2(x)u(x+h) + \eta^2(x-h)u(x-h) - [\eta^2(x) + \eta^2(x-h)]u(x)\}\end{aligned} \tag{5.56}$$

for any function $\eta(x)$, and the analogous identity holds for $\Delta_{-h}[\eta^2(x)\Delta_h\psi(x)]$. For any δ with $0 < 2\delta < b-a$ we denote by $\mathcal{I}_\delta$ the interval $(a+\delta, b-\delta)$. Choose some $r > 0$ with $6r < b-a$ and some $\eta \in C_c^\infty(\mathcal{I}_{2r})$ with $\eta(x) = 1$ on $\mathcal{I}_{3r}$ and $0 \le \eta(x) \le 1$. Then we fix some h with $0 < |h| < r$ and some ϵ with $0 < 2\epsilon < r^2$ and set

$$v := u + \varphi, \quad \varphi := \epsilon\Delta_{-h}[\eta^2\Delta_h(u-\psi)]. \tag{5.57}$$

Then $\varphi(x) = 0$ for $x - a \ll 1$ or $b - x \ll 1$, $x \in [a,b]$, and therefore $v \in \mathcal{C}(a,b)$. On account of (5.56) and (5.57) we have

$$\begin{aligned}v(x) &= \psi(x) + [u(x) - \psi(x) + \varphi(x)]\\&= \psi(x) + \{1 - \epsilon h^{-2}[\eta^2(x) + \eta^2(x-h)]\}[u(x) - \psi(x)]\\&\quad + \epsilon h^{-2}\{\eta^2(x)[u(x+h) - \psi(x+h)] + \eta^2(x-h)[u(x-h) - \psi(x-h)]\},\end{aligned}$$

and this implies $v(x) \ge \psi(x)$ for $a \le x \le b$, whence $u \in K_\psi$. By virtue of (5.51) it follows that

$$-\epsilon\int_a^b u'\Delta_{-h}[\eta^2\Delta_h(u' - \psi')]\,dx \le 0.$$

Dividing by ϵ we arrive at

$$\int_a^b u'(-\Delta_{-h})[\eta^2\Delta_h(u' - \psi')]\,dx \le 0.$$

This inequality can be transformed into

$$\int_a^b \eta^2(\Delta_h u')\Delta_h(u' - \psi')\,dx \le 0,$$

and by Schwarz's inequality we obtain

$$\int_a^b \eta^2 |\Delta_h u'|^2\,dx \le \int_a^b \eta^2 (\Delta_h u')(\Delta_h \psi')\,dx$$
$$\le \left[\int_a^b \eta^2 |\Delta_h u'|^2\,dx \int_a^b \eta^2 |\Delta_h \psi'|^2\,dx\right]^{1/2}.$$

This leads to

$$\int_a^b \eta^2 |\Delta_h u'|^2\,dx \le \int_a^b \eta^2 |\Delta_h \psi'|^2\,dx$$

whence

$$\int_{a-3r}^{b-3r} |\Delta_h u'|^2\,dx \le \int_{a-2r}^{b-2r} |\Delta_h \psi'|^2\,dx.$$

Since $\psi \in C^{1,1}(a,b)$ there is a constant $c(r) > 0$ such that $|\psi'(y)-\psi'(x)| \le c(r)|y-x|$ for all $x, y \in \mathcal{I}_r$, and thus it follows that

$$\int_{a-2r}^{b-2r} |\Delta_h \psi'|^2\,dx \le c^2(r)(b-a) =: c^*(r).$$

Therefore we find that

$$\int_{a-3r}^{b-3r} |\Delta_h u'|^2\,dx \le c^*(r) \tag{5.58}$$

for any h satisfying $0 < |h| < r$. In particular we have

$$\int_{a-3r}^{b-3r} |\Delta_{1/n} u'|^2\,dx \le c^*(r)$$

for all sufficiently large integers n. Hence there is a sequence $\{h_j\}$ of positive real numbers with $h_j \to 0$ such that $\Delta_{h_j} u'$ tends weakly in $L^2(\mathcal{I}_{3r})$ to some function $w \in L^2(\mathcal{I}_{3r})$. From

$$\int_{\mathcal{I}_{3r}} \varphi \Delta_{h_j} u'\,dx = -\int_{\mathcal{I}_{3r}} u' \Delta_{-h_j} \varphi\,dx$$

we infer that

$$\int_{\mathcal{I}_{3r}} \varphi w\,dx = -\int_{\mathcal{I}_{3r}} u'\varphi'\,dx.$$

Thus w is the distributional derivative of u' on $\mathcal{I}_{3r}$ and we have $u'' \in L^2(\mathcal{I}_{3r})$; that is, $u \in H^{2,2}_{loc}(a,b)$ and therefore $u \in C^{1,1/2}(a,b)$ and $u' \in AC(\mathcal{I})$ for any subinterval $\mathcal{I}$ of (a,b). Since $u(x) = \psi(x)$ on the coincidence set I we obtain $u'(x) = \psi'(x)$ a.e. on I, and therefore $u''(x) = \psi''(x)$ a.e. on I, whereas $u''(x) = 0$ for all $x \in I'$. This implies $u'' \in L^\infty([a, b])$ and $u \in C^{1,1}([a, b])$, as we have claimed.

In essentially the same way we can prove analogous regularity results for solutions of obstacle problems posed for the general variational integral $\mathcal{F}(u) = \int_a^b F(x, u, u')\,dx$ with an elliptic Lagrangian F, i.e. for Lagrangians $F(x, z, p)$ with a positive definite Hessian F_{pp}.

5.5 Periodic solutions of variational problems

Periodicity is an important feature of many motions in nature and especially in celestial mechanics. Therefore it is not surprising that there is a large and very interesting literature on the existence of periodic solutions of non-linear systems both in Lagrangian and Hamiltonian formalism. Fortunately there are excellent recent books dealing with this subject and in particular with applications of variational methods to this topic, such as for instance Ekeland [93], Mawhin–Willem [181], Rabinowitz [215], Struwe [247], so that we could be relieved from entering this area. But while we refer the interested reader to the literature, we would nevertheless like to discuss some very simple examples in this section, where we look for periodic solutions with a specified period. In the next section we shall be interested in periodic solutions of autonomous systems whose period will not be specified in advance.

For simplicity, we restrict our attention to the problem of the existence of periodic solutions of the non-autonomous system

$$\ddot{u}(t) = \nabla V(t, u(t)) \quad \text{in } [0, T], \tag{5.59}$$

i.e. to the existence of functions $u \in C^2([0, T]; \mathbb{R}^N)$ satisfying eqn (5.59) and the periodicity boundary conditions

$$u(0) = u(T), \qquad \dot{u}(0) = \dot{u}(T). \tag{5.60}$$

We shall always assume that the potential $V(t, x)$ is a smooth function, say of class C^1, from $[0, T] \times \mathbb{R}^N$ into $\mathbb{R}^N$ which is periodic with period T in the variable t. (This periodicity assumption is not really necessary, but otherwise seeking a solution with period T seems quite artificial.) Note that here ∇V means taking the gradient of $V(t, u)$ with respect to the u-variable only.

Formally (5.59) is the Euler equation of the variational integral

$$\mathcal{E}(u) := \int_0^T \left[\frac{1}{2}|\dot{u}|^2 + V(t, u(t))\right] dt, \tag{5.61}$$

and we would like to discuss a few elementary conditions on V which ensure the existence of a minimizer of (5.61) in the class of smooth T-periodic functions. For the sake of simplicity we shall denote by H_T^1 the subclass of $H^{1,2}(0, T; \mathbb{R}^N)$ of periodic functions in the sense that $u(0) = u(T)$, i.e.

$$H_T^1 := \{u \in H^{1,2}(0, T; \mathbb{R}^N) : u(0) = u(T)\}.$$

Notice that we do not require the relation $\dot{u}(0) = \dot{u}(T)$, a kind of natural boundary condition, which in principle is meaningless in $H^{1,2}$. It is worth mentioning that even for the very special case

$$\ddot{u} = f(t)$$

with $V(t, u) = f(t)u$ there is an obvious necessary condition, $\int_0^T f(t)\,dt = 0$. More generally we observe that a necessary condition to solve (5.59) is that there is a function $u : [0, T] \to \mathbf{R}^N$ satisfying (5.60) and $\int_0^T \nabla V(t, u(t))\,dt = 0$.

Our first result is the following.

Proposition 5.12 *Suppose that the potential $V(t, x)$ is periodic in x, i.e. that there exist linearly independent vectors $\tau_1, \dots, \tau_N \in \mathbf{R}^N$ such that*

$$V(t, x + \tau_j) = V(t, x) \quad \textit{for all } t \in [0, T] \textit{ and all } x \in \mathbf{R}^N, \quad j = 1, \dots, N.$$

Then the functional $\mathcal{E}(u)$ in (5.61) has at least one minimizer u in H^1_T. Moreover, u is of class C^2 in $[0, T]$ and solves (5.59), (5.60).

Proof Since V is smooth and periodic we deduce that $V(t, x) \geq h(t)$ for some function $h \in L^1(0, T)$, actually for some constant function $h(t)$. Consequently there is a constant c_1 such that

$$\mathcal{E}(u) \geq \frac{1}{2}\int_0^T |\dot{u}|^2\,dt - c_1 \quad \text{for all } u \in H^1_T.$$

It follows that there exists a constant c_2 such that any minimizing sequence $\{u_k\}$ satisfies

$$\int_0^T |\dot{u}|^2\,dt \leq c_2 \quad \text{for all } k \in \mathbb{N}. \tag{5.62}$$

We now write $u_k = \overline{u}_k + \tilde{u}_k$ where

$$\overline{u}_k := \frac{1}{T}\int_0^T u_k(t)\,dt,$$

and therefore

$$\frac{1}{T}\int_0^T \tilde{u}\,dt = 0.$$

It follows from (5.62) and Poincaré's inequality (Section 2.1) that

$$\|\tilde{u}_k\|_{H^{1,2}} \leq c_3 \quad \text{for all } k \in \mathbb{N}$$

and some constant $c_3 > 0$. On the other hand the periodicity of V yields

$$\mathcal{E}(u + \tau_j) = \mathcal{E}(u) \quad \text{for all } u \in H^1_T, \; j = 1, \dots, N;$$

hence also the sequence $\{v_k\}$ defined by

$$v_k := u_k + \sum_{j=1}^{N} \lambda_{jk} \tau_j, \quad \lambda_{jk} \in \mathbb{Z},$$

is a minimizing sequence for $\mathcal{E}$, and we can therefore assume that

$$|\overline{u}_k| \le \sum_{j=1}^{N} |\tau_j|.$$

Consequently we have found a minimizing sequence which is equibounded in $H^{1,2}$, and, since $\mathcal{E}$ is lower semicontinuous with respect to weak convergence in $H^{1,2}$, the existence of a minimizer of $\mathcal{E}$ in H^1_T follows at once.

By the results of Section 4.1 we readily deduce that the minimizer u belongs to $C^2([0,T], \mathbb{R}^N)$ and satisfies (5.59) and obviously $u(0) = u(T)$. Thus it remains to prove that $\dot{u}(0) = \dot{u}(T)$. Multiplying (5.59) by any smooth and T-periodic function φ and integrating by parts we find that

$$0 = \int_0^T [-\dot{u}\dot{\varphi} + \nabla V(t,u)\varphi]\, dt + \dot{u}(T)\dot{\varphi}(T) - u(0)\dot{\varphi}(0) = [\dot{u}(T) - \dot{u}(0)] \cdot \dot{\varphi}(0)$$

which immediately yields $\dot{u}(T) - \dot{u}(0) = 0$ since φ is arbitrary. □

Problem (5.59), (5.60) becomes more complicated if we want to deal with non-periodic potentials V. For this case we shall state two results. The first one deals with the scalar case $N = 1$ and illustrates an early result by Lichtenstein [172] and Tonelli [265], while the apparently more recent second result (cf. Mawhin–Willem [181]) deals with the case $N > 1$.

Proposition 5.13 *Suppose that $N = 1$, and that for $|x|$ larger than some fixed value l and for all $t \in [0,T]$ the inequality*

$$xV_x(t,x) \ge 0 \tag{5.63}$$

holds true. Then the functional $\mathcal{E}(u)$ in (5.61) *has at least one minimizer u on H^1_T. Moreover, u is of class C^2 in $[0,T]$ and solves* (5.59), (5.60).

Proof We first observe that (5.63) yields

$$V(t,u) \ge \min\{V(\tau,x) \colon \tau \in [0,T], x \in [-l,l]\};$$

hence with a suitable constant c_1 we have

$$\mathcal{E}(u) \ge \frac{1}{2}\int_0^T |\dot{u}|^2\, dt - c_1 \quad \text{for all } u \in H^1_T.$$

It follows that there exists a constant c_2 such that any minimizing sequence $\{u_k\}$ in H^1_T satisfies

$$\int_0^T |\dot{u}_k|^2\,dt \le c_2 \quad \text{for all } k \in \mathbb{N}.$$

Now we claim that we can assume that each u_k takes at least one value in the interval $[-l, l]$. Otherwise we would have $u_k(t) > l$ or $u_k(t) < -l$ in $[0, T]$ since u_k is continuous. Suppose for instance that $u_k(t) > l$ in $[0, T]$, and let m_k be the minimum of u_k in $[0, T]$. Instead of u_k we then consider the new function

$$\hat{u}_k(t) := u_k(t) - (m_k - l).$$

Then the minimum of $\hat{u}_k$ is l and, because of (5.63), we have

$$\mathcal{E}(\hat{u}_k) \le \mathcal{E}(u_k).$$

In a similar way we can treat the case $u_k(t) < -l$ in $[0, T]$. Since now u_k assumes a value in $[-l, l]$ at some point t_0, we deduce that

$$|u_k(t)| = \left|\int_{t_0}^t \dot{u}_k(\tau)d\tau + u_k(t_0)\right| \le l + \int_0^T |\dot{u}_k|\,dt.$$

Hence $\{u_k\}$ is equibounded in H^1_T, and we can complete the proof similarly to the proof of Proposition 5.12. □

Proposition 5.14 *Suppose that $V(t, x), x \in \mathbb{R}^N$, is convex in x for all $t \in [0, T]$ and that*

$$\int_0^T V(t, x)\,dt \to \infty \quad \text{if } |x| \to \infty. \tag{5.64}$$

Then problem (5.59), (5.60) has a solution which minimizes $\mathcal{E}$ in H^1_T.

Proof As above it suffices to show that a minimizing sequence in H^1_T is bounded in $H^{1,2}$. By assumption (5.64) the convex function

$$x \mapsto \int_0^T V(t, x)\,dt$$

has a minimum at some point $\bar{x}$, and

$$\int_0^T V_x(t, \bar{x})\,dt = 0.$$

Also, by convexity,

$$V(t, x) \ge V(t, \bar{x}) + V_x(t, \bar{x})(x - \bar{x}).$$

Therefore, if $\{u_k\}$ is a minimizing sequence for $\mathcal{E}$, we deduce that

$$\begin{aligned}\mathcal{E}(u_k) &\geq \frac{1}{2}\int_0^T |\dot{u}_k|^2\,dt + \int_0^T V(t,\overline{x})\,dt + \int_0^T V_x(t,\overline{x})[u_k(t)-\overline{x}]\,dt \\ &= \frac{1}{2}\int_0^T |\dot{u}_k|^2\,dt + \int_0^T V(t,\overline{x})\,dt + \int_0^T V_x(t,\overline{x})\tilde{u}_k(t)\,dt\end{aligned}$$

where

$$\tilde{u}_k(t) := u_k - \frac{1}{T}\int_0^T u_k(t)\,dt.$$

On the other hand, since $\tilde{u}_k$ has mean value zero, we find

$$|\tilde{u}_k(t)| \leq \int_0^T |\dot{u}_k|\,dt \quad \text{for all } t \in [0,T],$$

i.e.

$$\|\tilde{u}_k\|_\infty^2 \leq \left(\int_0^T |\dot{u}_k|\,dt\right)^2 \leq T\int_0^T |\dot{u}_k|^2\,dt. \tag{5.65}$$

Hence we conclude that

$$\begin{aligned}\mathcal{E}(u_k) &\geq \frac{1}{2}\int_0^T |\dot{u}_k|^2\,dt + \int_0^T V(t,\overline{x})\,dt - \left(\int_0^T |V_x(t,\overline{x})|\,dt\right)\|\tilde{u}_k\|_\infty \\ &\geq \frac{1}{2}\int_0^T |\dot{u}_k|^2\,dt - c_1 - c_2\left(\int_0^T |\dot{u}_k|^2\,dt\right)^{1/2}\end{aligned}$$

for some constants c_1 and c_2. Hence there exists a constant c_3 such that

$$\int_0^T |\dot{u}_k|^2\,dt \leq c_3 \quad \text{for all } k \in \mathbb{N} \tag{5.66}$$

and, consequently, by (5.65)

$$\|\tilde{u}_k\|_\infty \leq c_4 \quad \text{for all } k \in \mathbb{N}. \tag{5.67}$$

It remains to show that the sequence $\{\overline{u}_k\}$ of mean values

$$\overline{u}_k := \frac{1}{T}\int_0^T u_k\,dt$$

is bounded. By convexity

$$V\left(t,\frac{\overline{u}_k}{2}\right) = V\left(t,\frac{1}{2}\left[u_k(t)-\tilde{u}_k(t)\right]\right) \leq \frac{1}{2}V(t,u_k(t)) + \frac{1}{2}V(t,-\tilde{u}_k(t));$$

hence

$$\mathcal{E}(u_k) \geq \frac{1}{2}\int_0^T |\dot{u}_k|^2\,dt + 2\int_0^T V\left(t, \frac{\overline{u}_k}{2}\right) dt - \int_0^T V(t, -\tilde{u}_k(t))\,dt,$$

and this implies by (5.67) that

$$\mathcal{E}(u_k) \geq 2\int_0^T V\left(t, \frac{\overline{u}_k}{2}\right) dt - c_5$$

for some constant c_5. Since $\{u_k\}$ is a minimizing sequence, we obtain by assumptions (5.64) that $\{\overline{u}_k\}$ is bounded. □

Remark We note that if $V(t, x)$ is strictly convex in x, then (5.64) is equivalent to the existence of some $\overline{x} \in \mathbf{R}^N$ such that

$$\int_0^T V_x(t, \overline{x})\,dt = 0. \tag{5.68}$$

In fact, if (5.68) holds, $\overline{x}$ is then the unique minimal point for

$$x \mapsto \int_0^T V(t, x)\,dt;$$

hence

$$\delta := \min_{|x|=1} \int_0^T [V(t, \overline{x} + x) - V(t, \overline{x})]\,dt > 0.$$

Thus we obtain for $|x| > 1$ that

$$\begin{aligned}
\delta &\leq \int_0^T V\left(t, \frac{1}{|x|}(x + \overline{x}) + \left(1 - \frac{1}{|x|}\right)\overline{x}\right) dt - \int_0^T V(t, \overline{x})\,dt \\
&\leq \frac{1}{|x|}\int_0^T V(t, \overline{x} + x)\,dt + \left(1 - \frac{1}{|x|}\right)\int_0^T V(t, \overline{x})\,dt - \int_0^T V(t, \overline{x})\,dt \\
&= \frac{1}{|x|}\left(\int_0^T V(t, x + \overline{x})\,dt - \int_0^T V(t, \overline{x})\right).
\end{aligned}$$

i.e.

$$\int_0^T V(t, x + \overline{x})\,dt \geq \delta|x| + \int_0^T V(t, \overline{x})\,dt,$$

and we obtain relation (5.64).

5.6 Periodic solutions of Hamiltonian systems

In Hamiltonian mechanics one is naturally led to look for periodic solutions of Hamiltonian systems

$$\dot{x}(t) = \mathcal{I}H_x(x(t)) \tag{5.69}$$

where H is a C^2-Hamiltonian in $\mathbf{R}^{2N}$, and $\mathcal{I}$ is the standard symplectic matrix operating on $\mathbf{R}^{2N}$, i.e.

$$\mathcal{I} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

where E denotes the unit matrix operating on $\mathbf{R}^N$. In fact one is interested in periodic solutions with a prescribed period. Since we obtain

$$\frac{d}{dt}H(x,(t)) = \nabla H(x) \cdot \dot{x} = -(\mathcal{I}\dot{x}, \dot{x}) = 0$$

along any solution $x(t)$ of (5.69), we have $H(x(t)) \equiv$ const on solutions of (5.69). Thus we are looking for periodic solutions on a given constant-energy surface. More generally, one is interested in understanding the structure of all periodic solutions as this could give information on the structure of all trajectories.

Equation (5.69) is the Euler equation of the associated Hamiltonian action

$$\mathcal{H}(x; (0, T)) = \int_0^T \left(\frac{1}{2}(x, \mathcal{I}\dot{x}) + H(x(t))\right) dt,$$

and solutions with prescribed period T may be regarded as critical points of $\mathcal{H}(x)$ in the set of T-periodic trajectories of class C^1. Also, periodic solutions on a given energy surface $\{H = \alpha\}$ may be regarded as stationary points of $\mathcal{H}(x; (0, 1))$ in the class

$$\mathcal{S}_\alpha := \left\{ x \in C^1(\mathbf{R}, \mathbf{R}^{2N}) : x(t+1) = x(t), \int_0^1 H(x(t))\, dt = \alpha \right\}.$$

Indeed we may hope to show by means of the Lagrange multiplier rule that, under suitable assumptions, at any critical point x of $\mathcal{H}$ in $\mathcal{S}_\alpha$ there exists a constant $T \neq 0$ such that

$$\dot{x} = T\mathcal{I}H_x(x).$$

Scaling time by a factor T we then obtain a T-periodic solution of (5.69) on the energy surface $\{x : H(x) = \alpha\}$. However, the integral

$$\frac{1}{2}\int_0^1 (x, \mathcal{I}\dot{x})\, dt$$

is bounded neither from above nor from below, as we easily see by substituting

$$x_k(t) := (\cos\lambda_k t)x_0 - (\sin\lambda_k t)\mathcal{J}x_0, \quad \lambda_k = 2\pi k, \ x_0 \in \mathbf{R}^{2N}, \ |x_0| = 1$$

so that $|x_k| = 1$, $(\mathcal{J}\dot{x}_k, x_k) = \lambda_k$, and consequently

$$\frac{1}{2}\int_0^1 (x_k, \mathcal{J}\dot{x}_k)\,dt = \frac{1}{2}\lambda_k \to \pm\infty \quad \text{as } k \to \pm\infty.$$

Because of this difficulty it was thought for a long time to be hopeless to approach the existence problem for periodic solutions of Hamiltonian systems by variational methods. The turning point was Rabinowitz's paper [213] where the existence of periodic trajectories on a strictly convex hypersurface was established by minimax methods. A short time afterwards it was proved that such trajectories could in fact also be obtained by minimizing a suitable *dual functional*; see Clarke [63], [62] and Clarke–Ekeland [66]. Since then several variational methods have been developed and many interesting results are available. This area is still very fruitful, and challenging problems remain open. We cannot survey this interesting but vast field of research; instead we refer the reader to the well-written books by Ekeland [93], Mawhin–Willems [181], Rabinowitz [215], and Struwe [247] that were quoted in the previous section. Here we content ourselves with presenting the 'dual minimization proof' of the following basic result of Rabinowitz [213] and Weinstein [292], which extends the earlier work of Seifert [235]; compare also Moser [196].

Theorem 5.15 *Suppose that $H \in C^1(\mathbb{R}^{2N})$ is strictly convex, non-negative, and coercive, i.e. $H(x) \to \infty$ as $|x| \to \infty$, with $H(0) = 0$. Then for any $\alpha > 0$ there is a periodic solution $x \in C^1(\mathbb{R}, \mathbb{R}^{2N})$ of the Hamiltonian system*

$$\dot{x} = \mathcal{I} H_x(x) \tag{5.70}$$

with $H(x(t)) = \alpha$ for all t.

Proof We first show that whether or not a level surface $H =$ const carries a periodic solution of (5.70) is a question concerning the *surface* and the symplectic structure $\mathcal{I}$ and not the particular Hamiltonian H.

Obviously the Hamiltonian $H_\alpha := 1/\alpha H$ satisfies the same assumptions as H; moreover, if the level surface $H_\alpha = 1$ carries a periodic solution of the Hamiltonian system

$$\dot{x} = \mathcal{I}\nabla H_\alpha(x),$$

then by scaling the time by a factor of α we obviously find a periodic solution of (5.70) on the surface $H = \alpha$. Thus we can assume $\alpha = 1$.

By strict convexity of H the level surface $S = H^{-1}(1)$ is the boundary of the convex set

$$C := \{x \in \mathbb{R}^{2N} : H(x) < 1\}.$$

Consider now the distance function F_C of the convex set C, which is defined as follows. For any ξ on the unit sphere S^{2N-1} of $\mathbb{R}^{2N}$ there exists a unique number $r(\xi) > 0$ such that $r(\xi)\xi \in \mathcal{S}$; then we set for $\rho \geq 0$ and $\xi \in S^{2N-1}$

$$F_C(\rho\xi) := \begin{cases} \rho/r(\xi) & \text{if } \rho > 0 \\ 0 & \text{if } \rho = 0. \end{cases}$$

Now for a fixed number q, $1 < q < 2$, we define[15]

$$\tilde{H}(x) := F_C^q(x),$$

i.e.

$$\tilde{H}(\rho\xi) = \begin{cases} (\rho/r(\xi))^q & \text{if } \rho > 0,\ \xi \in S^{2N-1} \\ 0 & \text{if } \rho = 0. \end{cases}$$

By the strict convexity of the Hamiltonian H, its differential map dH is strongly monotone, and hence one-to-one. Then by the implicit function theorem $r \in C^1(S^{2N-1})$, and we deduce that $\tilde{H}$ belongs to $C^1(\mathbf{R}^{2N})$; also, $\tilde{H}$ is positively homogeneous of degree q. Moreover, if we set $\tilde{\mathcal{S}} := \{x \in \mathbf{R}^{2N} : \tilde{H}(x) = 1\}$, we have $\tilde{\mathcal{S}} = \mathcal{S}$; hence $\nabla\tilde{H}(x)$ is proportional to $\nabla H(x)$ for $x \in \mathcal{S}$, say

$$\nabla H(x) = \lambda(x)\nabla\tilde{H}(x) \quad \text{at any } x \in \mathcal{S}.$$

After a parameter transformation

$$x(t) := \tilde{x}(s(t)), \quad \dot{s}(t) = \lambda(\tilde{x}(s)),$$

a periodic solution $\tilde{x}$ on $\tilde{\mathcal{S}}$ of

$$\frac{d\tilde{x}}{dt} = \mathcal{I}\nabla\tilde{H}(\tilde{x})$$

will yield a periodic solution on H of the original Hamiltonian system (5.70). Finally, one can easily verify that $\tilde{H}$ is strictly convex. In conclusion we can assume from now on that H is equal to $\tilde{H}$.

Let H^* be the Legendre–Fenchel transform[16] of H. Since H is positively homogeneous of degree $q > 1$, the function H^* is everywhere finite. Moreover, $H^*(0) = 0$ and $H^* \geq 0$. Since H has a strongly monotone gradient, we deduce that $H^* \in C^1$; compare for instance Giaquinta–Hildebrandt [113], Vol. II, Chap. 7, Section 3.3. Finally, denoting by $p = \frac{q}{q-1} > 2$ the conjugate exponent of q, we have

$$\begin{aligned} \frac{H^*(y)}{|y|^p} &= \sup\left\{\left(\frac{x}{|y|^{p-1}}, \frac{y}{|y|}\right) - \frac{H(x)}{|y|^p} : x \in \mathbf{R}^{2N}\right\} \\ &= \sup\left\{\left(\frac{x}{|y|^{p-1}}, \frac{y}{|y|}\right) - H\left(\frac{x}{|y|^{p-1}}\right) : x \in \mathbf{R}^{2N}\right\} = H^*\left(\frac{y}{|y|}\right), \end{aligned} \tag{5.71}$$

i.e. H^* is positively homogeneous of degree $p > 2$.

[15]The reason for such a choice will become clear in the sequel.

[16]Note that x includes both position and momentum variables. Thus the conjugate H^* of H differs from the usual Legendre transform of H which customarily only involves the momentum variables. We recall that $H^*(y) := \sup\{xy - H(x) : x \in \mathbf{R}^{2N}\}$.

We are now ready to give an equivalent dual formulation of (5.70). We introduce the space

$$X := \left\{ y \in L^p((0,1), \mathbb{R}^{2N}) : \int_0^1 y\,dt = 0 \right\}.$$

If $x \in C^1([0,1], \mathbb{R}^{2N})$ is a 1-periodic solution of (5.70), the function

$$y := -\mathcal{I}\dot{x}$$

belongs to X and solves the system of equations

$$y = -\mathcal{I}\dot{x} \tag{5.72}$$
$$y = \nabla H(x). \tag{5.73}$$

Introducing the compact integral operator $K : X \to H^{1,p}((0,1), \mathbb{R}^{2N})$ defined by

$$(Ky)(t) := \int_0^t \mathcal{I}y\,dt,$$

we can invert eqn (5.72) up to an integration constant $x_0 \in \mathbb{R}^{2N}$. Since relation (5.73) is equivalent to the relation $x = \nabla H^*(y)$ (see for instance Giaquinta–Hildebrandt [113], Vol. II, Proposition 6, p. 91), we obtain that (5.72), (5.73) are equivalent to

$$x = Ky + x_0 \tag{5.74}$$
$$x = \nabla H^*(y) \tag{5.75}$$

for some $x_0 \in \mathbb{R}^{2N}$. This leads to

$$\int_0^1 [\nabla H^*(y) - Ky]\eta\,dt = 0 \quad \text{for all } \eta \in X. \tag{5.76}$$

In turn, this equation implies (5.74), (5.75). In fact, if $y \in X$ solves (5.76), it follows (cf. Section 1.1) that

$$\nabla H^*(y) - K(y) = \text{const } = x_0.$$

Hence y solves (5.74), (5.75) for some $x \in H^{1,p}((0,1), \mathbb{R}^{2N})$. Transforming back to (5.72), (5.73), we infer from (5.73) that $y \in C^0([0,1], \mathbb{R}^{2N})$. Therefore x is of class $C^1([0,1], \mathbb{R}^{2N})$, and it is a 1-periodic solution of eqn (5.70) which is equivalent to (5.76).

Now (5.76) is the Euler equation of the functional

$$\mathcal{H}^*(y) := \int_0^1 \left[H^*(y) - \frac{1}{2}(y, Ky) \right] dt.$$

By (5.71) the functional $\mathcal{H}^*$ is coercive on X. Moreover, H^* is convex and K is a compact operator; hence $\mathcal{H}^*$ is lower semicontinuous with respect to weak convergence of L^p. Thus we easily deduce the existence of a minimizer y^* in X which solves (5.76). By (5.71) the quadratic term $-\int_0^1 (y, Ky)\,dt$ dominates near $y = 0$; thus, since K also possesses positive eigenvalues, as we may easily verify, we deduce that

$$\inf_X \mathcal{H}^* < 0;$$

consequently $y^* \neq 0$. The discussion above shows that there is a constant x_0 such that $x = Ky^* + x_0$ solves (5.70), and, since $y^* \neq 0$, x is also non-zero; hence $H(x(t)) = \beta$ for some $\beta > 0$. But $H = \tilde{H}$ is positively homogeneous; thus a suitable multiple $\tilde{x}$ of x will satisfy (5.70) with $H(\tilde{x}(t)) = 1$, as desired. □

5.7 Non-coercive variational problems

In Section 5.5 we have already encountered minimum problems of the form

$$\min\left\{\int_0^T \left[\frac{1}{2}|u'|^2 + V(t,u)\right] dt : u \in H^1(0,T;\mathbf{R}^N),\, u(0) = u(T)\right\} \tag{5.77}$$

which are not coercive in the sense that there exist sequences of functions $\{u_n\}_{n\in\mathbf{N}}$ with equibounded energies that are not weakly compact. For instance, if in problem (5.77) the potential $V(t,\cdot)$ is periodic in the sense of Proposition 5.12 of Section 5.5, then the sequence $u_n(t) := n\tau$ is not weakly compact in $H^1(0,T;\mathbf{R}^N)$, whereas the corresponding energies $\int_0^T [\frac{1}{2}|u_n'|^2 + V(t,u_n)]\,dt$ are equibounded.

In this kind of situation the direct methods of the calculus of variations cannot be applied immediately, and further assumptions on the potential V have to be added in order to guarantee the existence of minimizers (see Propositions 5.12, 5.13, 5.14 of Section 5.5).

In this section we present a general scheme to attack non-coercive minimum problems, and we give some applications of it to minimum problems of the form (5.77).

From now on we denote by X a reflexive Banach space, and $\mathcal{F} : X \to \mathbf{R} \cup \{\infty\}$ will be a given functional which is always supposed to be sequentially weakly lower semicontinuous on X. We are interested in finding conditions on the functional $\mathcal{F}$ which are weaker than the usual coercivity but still imply the existence of a solution to the minimum problem

$$\min\{\mathcal{F}(u) : u \in X\}. \tag{5.78}$$

To this end we introduce the so-called *recession functional* $\mathcal{F}_\infty$ associated with $\mathcal{F}$, which is defined by

$$\mathcal{F}_\infty(u) = \inf\left\{\liminf_{n\to+\infty} \frac{\mathcal{F}(t_n u_n)}{t_n} :\ t_n \to +\infty\ ,\ u \text{ weakly in } X\right\}$$

for every $u \in X$. For the existence of a solution to problem (5.78) we obtain the following necessary condition in terms of the functional $\mathcal{F}_\infty$.

Proposition 5.16 *If the minimum problem* (5.78) *admits a solution, then*

$$\mathcal{F}_\infty(u) \geq 0 \quad \textit{for every } u \in X.$$

Proof Assume that the minimum problem (5.78) admits a solution, or even less that

$$\inf\{\mathcal{F}(u) : u \in X\} = m > -\infty.$$

Then, for every $u \in X$

$$\mathcal{F}_\infty(u) \geq \inf\left\{\liminf_{n\to\infty} \frac{m}{t_n} : t_n \to \infty\right\} \geq 0.$$

□

In general, non-negativity of $\mathcal{F}_\infty$ does not guarantee the existence of a minimizer, as we can easily see by taking $X = \mathbb{R}$ and $\mathcal{F}(u) = e^u$. Therefore we add some further assumptions on $\mathcal{F}$ which, together with the non-negativity of $\mathcal{F}_\infty$, will imply the existence of a solution of problem (5.78).

Theorem 5.17 *Let $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ be a sequentially weakly lower semicontinuous functional. Assume that*

(i) *the following compactness condition is fulfilled: for every $t_n \to \infty$ and any weakly converging sequence $\{u_n\}$ the boundedness of $\mathcal{F}(t_n u_n)$ from above implies that $\{u_n\}$ converges strongly;*
(ii) *the necessary condition $\mathcal{F}_\infty \geq 0$ on X is satisfied;*
(iii) *the following compatibility condition is verified: if $\mathcal{F}_\infty(w) = 0$, then for some suitable $\mu > 0$ we have*

$$\mathcal{F}(u - \mu w) \leq F(u) \quad \textit{for all } u \in X.$$

Then the minimum problem (5.78) *possesses at least one solution.*

Proof Since $\mathcal{F}$ is lower semicontinuous with respect to weak convergence, the minimum problem

$$\min\{\mathcal{F}(u) : u \in X,\ \|u\| \leq n\} \tag{$\mathcal{P}_n$}$$

has a solution u_n for any $n \in \mathbb{N}$. Moreover, again by the lower semicontinuity of $\mathcal{F}$, we may choose u_n in such a way that

$$\|u_n\| = \min\{\|u\| : u \text{ solves } (\mathcal{P}_n)\}. \tag{5.79}$$

If such a sequence $\{u_n\}$ of minimizers u_n is bounded in norm, then a suitable subsequence (still denoted by $\{u_n\}$) converges weakly to some element u of X which is a solution of problem (5.78) because

$$\mathcal{F}(u) \leq \liminf_{n\to\infty} \mathcal{F}(u_n) = \inf\{\mathcal{F}(u) : u \in X\}.$$

Now we show that such a sequence $\{u_n\}$ cannot be unbounded, which will complete the proof. We argue by contradiction and assume that a subsequence of $\{\|u_n\|\}$ (which we still index by n) tends to ∞. Since the normalized vectors $w_n = u_n/\|u_n\|$ are bounded, by the reflexivity of X we may assume (again extracting a subsequence) that $\{w_n\}$ converges weakly to some $w \in X$. Since u_n is a solution of problem $(\mathcal{P}_n)$ we have

$$\mathcal{F}(u_{n+1}) \leq \mathcal{F}(u_n) \quad \text{for every } n \in \mathbb{N};$$

thus the values $\mathcal{F}(u_n)$ are bounded from above (as the case of $\mathcal{F}$ being constantly equal to ∞ is trivial). By definition of $\mathcal{F}_\infty$ we then obtain

$$\mathcal{F}_\infty(w) \leq \liminf_{n\to\infty} \frac{\mathcal{F}(\|u_n\|w_n)}{\|u_n\|} = \liminf_{n\to\infty} \frac{\mathcal{F}(u_n)}{\|u_n\|} \leq 0.$$

On account of the necessary condition (ii) we arrive at

$$\mathcal{F}_\infty(w) = 0.$$

Recall now that $\|u_n\| \to \infty$, $w_n \to w$ weakly in X, and that the values $\mathcal{F}(\|u_n\|w_n) = F(u_n)$ are bounded from above. Assumption (i) then implies that $w_n \to w$ strongly in X, and this implies that $\|w\| = 1$ because of $\|w_n\| = 1$ for all $n \in \mathbb{N}$. Furthermore, since $\mathcal{F}_\infty(w) = 0$, compatibility condition (iii) implies that there is some $\mu > 0$ such that

$$\mathcal{F}(u_n - \mu w) \leq F(u_n) \quad \text{for every } n \in \mathbb{N}. \tag{5.80}$$

Finally we have

$$\begin{aligned}
\|u_n - \mu w\| &= \left\| \left(1 - \frac{\mu}{\|u_n\|}\right) u_n + \mu(w_n - w) \right\| \\
&\leq \left(1 - \frac{\mu}{\|u_n\|}\right) \|u_n\| + \mu \|w_n - w\| \\
&= \|u_n\| + \mu(\|w_n - w\| - 1),
\end{aligned}$$

and the right-hand side of the last equality is strictly less than $\|u_n\|$ for $n \gg 1$ since $\|w - w_n\| \to 0$. Therefore, by (5.80), $u_n - \mu w$ is a solution of $(\mathcal{P}_n)$ whose norm is strictly less than $\|u_n\|$ for $n \gg 1$. But this is a contradiction to (5.79). □

Remark 1 The usual coercive case of the classical direct method of the calculus of variations is covered by Theorem 5.17. In fact, consider a sequentially weakly lower semicontinuous functional $\mathcal{F}$ on X which fulfils the standard coercivity condition: for every $t \in \mathbf{R}$ the set $\{u \in X : \mathcal{F}(u) \leq t\}$ is bounded in X. This condition can be rephrased by saying that there exists a function $\phi : \mathbf{R} \to \mathbf{R}$ (which can be assumed to be continuous, positive, and strictly increasing) such that the inequality $\mathcal{F}(u) \leq t$ implies $\|u\| \leq \phi(t)$. This property turns out to be equivalent to the inequality

$$\|u\| \leq \phi(\mathcal{F}(u)) \quad \text{for every } u \in X. \tag{5.81}$$

Now the existence of a minimizer for $\mathcal{F}$ is clearly equivalent to the existence of a minimizer for the functional $\Phi(u) := \phi(\mathcal{F}(u))$. Therefore, to establish the existence of a minimizer for $\mathcal{F}$ it is enough to show that the functional Φ satisfies all the assumptions of Theorem 5.17 above. The lower semicontinuity of Φ follows immediately from that of $\mathcal{F}$, and assumptions (i), (ii), (iii) are consequences of (5.81), as one can easily verify. In particular, condition (iii) is trivial because $\Phi_\infty(w) = 0$ implies that $w = 0$.

Let us particularly consider the case of convex (and sequentially weakly lower semicontinuous) functionals $\mathcal{F}$; here the definition of the recession functional $\mathcal{F}_\infty$ actually reduces to the more common *convex recession* $\mathcal{F}^\infty$ defined by

$$\mathcal{F}^\infty(u) := \lim_{t\to\infty} \frac{\mathcal{F}(u_0 + tu)}{t} \tag{5.82}$$

(cf. for instance Rockafellar [228]) where u_0 is any element of X such that $\mathcal{F}(u_0) < \infty$. Note that the limit on the right-hand side of (5.82) exists because of the convexity of the function $t \mapsto \mathcal{F}(u_0 + tu)/t$. The equivalence between $\mathcal{F}_\infty$ and $\mathcal{F}^\infty$ is shown in the following proposition.

Proposition 5.18 *Let $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ be a convex and sequentially weakly lower semicontinuous functional. Then*

$$\mathcal{F}_\infty(u) = \mathcal{F}^\infty(u) \quad \textit{for every } u \in X.$$

In particular, the definition (5.82) *of $\mathcal{F}^\infty$ does not depend on the choice of u_0.*

Proof Let $u \in X$, and let u_0 be any point with $\mathcal{F}(u_0) < \infty$. By choosing a sequence of numbers t_n with $t_n \to \infty$ and

$$\mathcal{F}^\infty(u) = \lim_{n\to\infty} \frac{\mathcal{F}(u_0 + t_n u)}{t_n}$$

and setting $u_n := u + u_0/t_n$, we obtain

$$\mathcal{F}_\infty(u) \le \liminf_{n\to\infty} \frac{\mathcal{F}(t_n u_n)}{t_n} = \lim_{n\to\infty} \frac{\mathcal{F}(u_0 + t_n u)}{t_n} = \mathcal{F}^\infty(u).$$

To prove the opposite inequality we use the convexity and the lower semicontinuity of $\mathcal{F}$. For every $t > 0$ and for arbitrary sequences $t_n \to \infty$ and $u_n \to u$ weakly in X we have

$$\begin{aligned}
\mathcal{F}(u_0 + tu) &\le \liminf_{n\to\infty} \mathcal{F}\left(\left(1 - \frac{t}{t_n}\right) u_0 + \frac{t}{t_n} t_n u_n\right) \\
&\le \liminf_{n\to\infty} \left[\left(1 - \frac{t}{t_n}\right) \mathcal{F}(u_0) + \frac{t}{t_n} \mathcal{F}(t_n u_n)\right] \\
&= F(u_0) + \liminf_{n\to\infty} \frac{\mathcal{F}(t_n u_n)}{t_n}.
\end{aligned}$$

Since $t_n \to \infty$ and $u_n \to u$ were arbitrary sequences, we arrive at

$$\mathcal{F}(u_0 + tu) \le \mathcal{F}(u_0) + t\mathcal{F}_\infty(u).$$

Therefore,

$$\frac{\mathcal{F}(u_0+tu)-\mathcal{F}(u_0)}{t} \le \mathcal{F}_\infty(u)$$

for every $t > 0$, and passing to the limit as $t \to \infty$, we obtain

$$\mathcal{F}^\infty(u) \le \mathcal{F}_\infty(u). \qquad \square$$

Remark 2 If $\mathcal{F}$ is convex, a much simpler assumption implying the compatibility condition (iii) of Theorem 5.17 above is the following:

(iii$'$) *the set* $\ker \mathcal{F}^\infty = \{u \in X : \mathcal{F}^\infty(u) = 0\}$ *is a linear subspace of* X.

In fact, assume that (iii$'$) is fulfilled and let $w \in X$ be such that $\mathcal{F}^\infty(w) = 0$. Then, since $\ker \mathcal{F}^\infty$ is a linear subspace, we also have $\mathcal{F}^\infty(-w) = 0$. Therefore, if u_0 is a point where $\mathcal{F}(u_0) < \infty$ we have for every $u \in X$ that

$$\begin{aligned}
\mathcal{F}(u-w) &\le \liminf_{t\to\infty} \mathcal{F}\left(\left(1-\frac{1}{t}\right)u + \frac{1}{t}(u_0 - tw)\right) \\
&\le \liminf_{t\to\infty} \left(1-\frac{1}{t}\right)\mathcal{F}(u) + \frac{1}{t}\mathcal{F}(u_0 - tw) \\
&= \mathcal{F}(u) + \mathcal{F}^\infty(-w) = \mathcal{F}(u)
\end{aligned}$$

which is the compatibility condition (iii) with $\mu = 1$.

Remark 3 An inspection of the proof of Theorem 5.17 shows that in the case of a product space $X_1 \times \cdots \times X_N$ the compatibility condition (iii) can be replaced by the following weaker condition:

(iii$''$) *If* $\mathcal{F}_\infty(w_1, \ldots, w_N) = 0$, *then for suitable positive numbers* $\mu_1, \ldots, \mu_N$ *we have*

$$\mathcal{F}(u_1 - \mu_1 w_1, \ldots, u_N - \mu_N w_N) \le F(u_1, \ldots, u_N) \quad \text{for all } (u_1, \ldots, u_N).$$

We now specialize our results to the case when X is the space $H^1_T = \{u \in H^1(0, T; \mathbb{R}^N) : u(0) = u(T)\}$ and the functional $\mathcal{F}$ is of the form

$$\mathcal{F}(u) = \int_0^T \left[\frac{1}{2}|u'|^2 + V(t,u)\right] dt$$

where V is a Borel function with $V(t,\cdot)$ lower semicontinuous on $\mathbb{R}^N$ and such that, for suitable $q < 2$ and $a(t), b(t)$ in $L^1(0,T)$, we have

$$V(t,u) \ge -a(t) - b(t)|u|^q \quad \text{for } t \text{ a.e. in } (0,T) \text{ and for every } u \in \mathbb{R}^N. \tag{5.83}$$

The sequential weak lower semicontinuity of $\mathcal{F}$ is straightforward (cf. Section 3.1); let us prove the compactness property (i) of Theorem 5.17. If $t_n \to \infty$ and $u_n \to u$ weakly in H^1_T with $\mathcal{F}(t_n u_n)$ bounded from above, then we deduce by (5.83) that

$$\lim_{n\to\infty} \int_0^T |u'|^2\,dt = 0,$$

and this immediately implies the strong convergence of $\{u_n\}$ to a constant. Furthermore, we note that the necessary condition (ii) $\mathcal{F}_\infty(u) \geq 0$ for every $u \in H^1_T$ is trivially fulfilled if u is a non-constant function, because in this case we have $\mathcal{F}_\infty(u) = \infty$. Indeed, if $t_n \to \infty$ and $u_n \to u$ weakly in H^1_T with u being non-constant, we have

$$\liminf_{n\to\infty} \int_0^T |u_n'|^2\,dt \geq \int_0^T |u'|^2\,dt > 0.$$

Therefore, by (5.83) and using the fact that $q < 2$, we obtain

$$\liminf_{n\to\infty} \frac{\mathcal{F}(t_n u_n)}{t_n} \geq \liminf_{n\to\infty} \int_0^T \left[\frac{t_n}{2}|u_n'|^2 - b(t)t_n^{q-1}|u_n|^q\right] dt = \infty.$$

The necessary condition (ii) is then reduced to

$$\mathcal{F}_\infty(c) \geq 0 \quad \text{for every constant vector } c \in \mathbf{R}^N. \tag{5.84}$$

The simplest case of the existence of minimizers for problem (5.77) is that the potential V satisfies a Lipschitz condition of the form

$$|V(t,u) - V(t,v)| \leq k(t)|u-v| \quad \text{for } t \text{ a.e. in } (0,T) \text{ and for every} u, v \in \mathbf{R}^N, \tag{5.85}$$

with $k \in L^1(0,T)$ as well as the coercivity condition

$$\lim_{|c|\to\infty} \int_0^T V(t,c)\,dt = \infty. \tag{5.86}$$

In this case problem (5.77) is actually coercive, and the existence result follows immediately from the direct methods of the calculus of variations. We need only to show that we have $\mathcal{F}(u) \to \infty$ as soon as $\|u\|_{H^1_T} \to \infty$. By using the Lipschitz condition (5.85) we obtain

$$\mathcal{F}(u) \geq \int_0^T \left[\frac{1}{2}|u'|^2 + V(t,u(0))\right] dt - \int_0^T k(t)|u(t) - u(0)|\,dt$$

for every $u \in H^1_T$. Moreover, we have

$$|u(t) - u(0)| = \left|\int_0^t u'(s)ds\right| \leq \left(T\int_0^T |u'|^2\,dt\right)^{1/2}.$$

and therefore

$$\mathcal{F}(u) \geq \int_0^T \left[\frac{1}{2}|u'|^2 + V(t, u(0)) \right] dt - C \left(\int_0^T |u'|^2 dt \right)^{1/2}$$

for a suitable constant C. Since on H_T^1 the norm is equivalent to

$$\left(\int_0^T |u'|^2 dt + |u(0)|^2 \right)^{1/2}$$

we immediately obtain that $\|u\|_{H_T^1} \to \infty$ implies $\mathcal{F}(u) \to \infty$.

We now consider the case when $V(t, \cdot)$ is convex and $V(t, u_0(t))$ is integrable for some function $u_0 \in H_T^1$. Then the functional $\mathcal{F}$ turns out to be convex, and Theorem 5.17 together with Proposition 5.18 and Remark 2 can be applied. Taking (5.84) into account we have that the conditions

$$\lim_{\lambda \to \infty} \int_0^T \frac{V(t, u_0(t) + \lambda c)}{\lambda} dt \geq 0 \text{ for every } \quad c \in \mathbf{R}^N, \tag{5.87}$$

and $\lim_{\lambda \to \infty} \int_0^T V(t, u_0(t) + \lambda c)/\lambda \, dt = 0$ implies

$$\lim_{\lambda \to \infty} \int_0^T \frac{V(t, u_0(t) - \lambda c)}{\lambda} dt = 0, \tag{5.88}$$

yield the existence of a minimizer for $\mathcal{F}$ on H_T^1. In other words, (5.87) and (5.88) are equivalent to the property that for every $c \in \mathbf{R}^N$ the function Φ_c defined by

$$\Phi_c(\lambda) := \int_0^T V(t, u_0(t) + \lambda c) \, dt, \quad \lambda \in \mathbf{R},$$

either is constant or satisfies

$$\lim_{|\lambda| \to \infty} \Phi_c(\lambda) = \infty.$$

For instance, in the case of Proposition 5.14 of Section 5.5 we have

$$\mathcal{F}^\infty(c) > 0 \quad \text{for every } c \in \mathbf{R}^N - \{0\},$$

and then (5.87) and (5.88) are trivially satisfied.

Consider now the special case in which $V(t, x) := V(x) - h(t) \cdot x$, where $V : \mathbf{R}^N \to \mathbf{R} \cup \{\infty\}$ is a convex lower semicontinuous function (finite in at least a point x_0) and $h \in L^1(0, T)$. If V is differentiable, then the Euler–Lagrange equation associated with the minimum problem for $\mathcal{F}$ on H_T^1 is

$$\begin{gathered} -u'' + \nabla V(u) = h(t), \\ u(0) = u(T), \qquad u'(0) = u'(T). \end{gathered}$$

By (5.84) and (5.87), setting

$$\bar{h} := \frac{1}{T} \int_0^T h(t)\, dt,$$

we infer that the necessary condition for existence reads as

$$\bar{h} \cdot c \leq V^{\infty}(c) \quad \text{for every } c \in \mathbf{R}^N, \tag{5.89}$$

while, by (5.88), the compatibility condition, sufficient for the existence of a minimizer, states:

$$\bar{h} \cdot c = V^{\infty}(c) \text{ implies } V^{\infty}(-c) = V^{\infty}(c). \tag{5.90}$$

In the case when V is non-negative and positively homogeneous of degree $q > 1$, (5.89) and (5.90) above become

$$\bar{h} \cdot c \leq 0 \quad \text{whenever } V(c) = 0, \tag{i}$$

and

$$\{c \in \mathbf{R}^N : V(c) = 0,\ \bar{h} \cdot c = 0\} \text{ is a subspace} \tag{ii}$$

respectively since

$$V^{\infty}(x) = \begin{cases} 0 & \text{if } V(x) = 0 \\ \infty & \text{if } V(x) \neq 0. \end{cases}$$

The gap between the necessary condition (i) and the sufficient condition (ii) above contains some interesting examples like the following one. Take $N = 1$ and consider the differential equation

$$\begin{aligned} -u'' + u^+ &= h(t), \\ u(0) = u(T), &\qquad u'(0) = u'(T). \end{aligned} \tag{5.91}$$

The functional related to (5.91) is

$$\mathcal{F}(u) = \int_0^T \left\{ \frac{1}{2}[|u'|^2 + |u^+|^2] - h(t)u \right\} dt. \tag{5.92}$$

Because of the convexity of $\mathcal{F}$, solving (5.91) is equivalent to minimizing (5.92) on H_T^1. By (5.89) and (5.90) above we obtain that:

- if $\bar{h} < 0$ a solution always exists;
- if $\bar{h} > 0$ no solution may exist.

It remains to analyse the case $\bar{h} = 0$. Here we obtain the following results:

(a) *A solution of problem* (5.91) *always exists.*

(b) *Every solution of* (5.91) *is non-positive.*

To prove these claims we consider the linearized problem

$$\begin{gathered} -u'' = h(t), \\ u(0) = u(T), \qquad u'(0) = u'(T). \end{gathered} \tag{5.93}$$

Since $\overline{h} = 0$, problem (5.93) admits a solution $w \in H^1(0, T)$ that, up to a suitable addition of a constant, can be supposed to be non-positive. Then w solves problem (5.91) too, and claim (a) is proved. Assume now that u is a solution of (5.91) with $u^+ \not\equiv 0$; then for every $c > 0$ we have

$$\mathcal{F}(u - c) < \mathcal{F}(u)$$

which contradicts the fact that u is a minimum point for $\mathcal{F}$. Hence $u \leq 0$ and (b) is proved too.

Another interesting case in which the general scheme above applies is when $V(t, \cdot)$ is periodic (cf. Proposition 5.12 of Section 5.5); that is, for suitable independent vectors $\tau_i \in \mathbf{R}^N$ we have for almost all $t \in (0, T)$ and for every $u \in \mathbf{R}^N$ that

$$V(t, u + \tau_i) = V(t, u), \quad i = 1, \dots, N.$$

We also assume that there exists a function $a(t)$ in $L^1(0, T)$ such that

$$V(t, u) \geq a(t) \quad \text{for almost all } t \in (0, T) \text{ and for every } u \in \mathbf{R}^N.$$

In this case the necessary condition (5.84) is clearly fulfilled. Hence, to obtain the existence of a minimizer, it is enough to show that the compatibility condition (iii$''$) of Remark 3 holds too. This means that, for every $c \in \mathbf{R}^N$, we have to find a vector $\mu \in \mathbf{R}^N$ whose components μ_i are positive and satisfy

$$\mathcal{F}(u_1 - \mu_1 c_1, \dots, u_N - \mu_N c_N) \leq \mathcal{F}(u_1, \dots, u_N) \quad \text{for all} \quad u_1, \dots, u_N.$$

This can be achieved by choosing the vector $(\mu_1 c_1, \dots, \mu_N c_N)$ as one of the elements of the lattice $\{h_i \tau_i : i = 1, \dots, N,\ h_i \in \mathbb{Z}\}$, a choice which is clearly always possible.

5.8 An existence result in optimal control theory

Optimal control problems are minimum problems which describe the behaviour of systems that can be modified by the action of an operator; therefore two kinds of variables are involved: one of them describes the state of the system and cannot be modified by a direct action of the operator, and is called the *state variable*; the second one, on the contrary, is under the direct control of the operator who can choose his strategy among the admissible ones, and is called the *control variable*.

The state of the system (i.e. the state variable) can be modified by the operator in an indirect way, by acting on the control variable through a link control state, which is usually given by a differential equation, the so-called *state equation*. Finally, acting directly on the control variables, the operator has to achieve a certain goal. This aim is usually assumed to be the minimization of a functional which depends on control and states. The functional to be minimized is called the *cost functional*.

Driving a car leads in many ways to an optimal control problem. For instance, the cost functional could be the amount of fuel consumed, with position and velocity of the car as state variables, while the acceleration and the steering wheel's angle are the control variables. The state equation is given as the balance of forces. Other typical examples of optimal control problems are the guiding of a rocket to a moving target in the shortest possible time, or the management of an enterprise in order to maximize profits.

To fix ideas, we denote by Y the space of states, by U the set of controls, by A the set of *admissible pairs*, i.e. the subset of all pairs $(u, y) \in U \times Y$ such that y is linked to u through the state equation, and by I the cost functional defined on $U \times Y$. The optimal control problem is assumed to be a minimization problem of the form

$$\min\{I(u, y) : (u, y) \in A\}. \tag{5.94}$$

Note that constraints can be incorporated by defining $I = +\infty$ whenever the constraints are violated.

Here we do not want to develop a general theory of optimal control problems since many good books on the subject are available, for instance Berkovitz [31], Cesari [59], or Warga [291]. As an application of the results presented in the previous chapters we restrict our attention to the following very special class of problems:

$$\min\left\{\int_0^T f(t, y(t), u(t))\,dt : \quad y' = a(t, y) + b(t, y)u, \; y(0) = y_0\right\}. \tag{5.95}$$

Here we take the space of states Y equal to the space $H^{1,1}(0, T; \mathbb{R}^k)$ of absolutely continuous functions on $(0, T)$, introduced in Chapter 2, the set of controls U equal to the space $L^1(0, T; \mathbb{R}^m)$, the set A of admissible pairs equal to all pairs

$$(u, y) \in L^1(0, T; \mathbb{R}^m) \times H^{1,1}(0, T; \mathbb{R}^k)$$

such that

$$\begin{aligned} y' &= a(t, y) + b(t, y)u, \\ y(0) &= y_0, \end{aligned}$$

and the cost functional I as given by the integral

$$I(u, y) = \int_0^T f(t, y(t), u(t))\,dt.$$

We shall prove that, under suitable assumptions on the functions a, b, f, the minimum problem (5.95) admits a solution.

The following lemma deals with the behaviour of solutions of sequences of ordinary differential equations.

Lemma 5.19 *Let $g_n : (0, T) \times \mathbb{R}^N \to \mathbb{R}^N$ be a sequence of Carathéodory functions (i.e. $g_n(t, y)$ are measurable in t and continuous in y) such that*

(i) *the Lipschitz condition*

$$|g_n(t, y_1) - g_n(t, y_2)| \le L_n(t)|y_1 - y_2|$$

holds, with $\{L_n(t)\}_n$ weakly compact in $L^1(0, T)$;

(ii) *$g_n(\cdot, y) \to g_\infty(\cdot, y)$ weakly in $L^1(0, T; \mathbb{R}^N)$ for every $y \in \mathbb{R}^N$.*

Then the solutions $y_n \in H^{1,1}(0, T; \mathbb{R}^N)$ of the Cauchy problems

$$\begin{aligned} y' &= g_n(t, y) \quad \text{in } (0, T) \\ y(0) &= y_0 \end{aligned}$$

converge uniformly on $[0, T]$ to the solution $y_\infty \in H^{1,1}(0, T; \mathbb{R}^N)$ of the Cauchy problem

$$\begin{aligned} y' &= g_\infty(t, y) \quad \text{in } (0, T) \\ y(0) &= y_0 \end{aligned}$$

and $y_n' \to y_\infty'$ weakly in $L^1(0, T; \mathbb{R}^N)$.

Proof We show first that the function g_∞ satisfies a Lipschitz condition similar to (i). The sequence $\{L_n\}$ has a subsequence (which we still denote by the same indices) converging weakly in $L^1(0, T)$ to some integrable function $L(t)$. Moreover, the Lipschitz condition for g_n yields

$$\int_0^T (g_n(t, y_1) - g_n(t, y_2)) \cdot \eta(t)\, dt \le \int_0^T L_n(t)|y_1 - y_2||\eta(t)|\, dt$$

for every function $\eta \in L^\infty(0, T; \mathbb{R}^N)$. Passing to the limit as $n \to \infty$ we obtain by (ii) that

$$\int_0^T (g_\infty(t, y_1) - g_\infty(t, y_2)) \cdot \eta(t)\, dt \le \int_0^T L(t)|y_1 - y_2||\eta(t)|\, dt,$$

and defining η by $\eta(t) := \frac{1}{\tau - t_0} v$ if $t \in (t_0, \tau)$ with $v \in \mathbb{R}^N$, and by $\eta(t) := 0$ if $t \notin (t_0, \tau)$, we have

$$\frac{1}{\tau - t_0} \int_{t_0}^{\tau} (g_\infty(t, y_1) - g_\infty(t, y_2)) \cdot v\, dt \le \frac{1}{\tau - t_0} \int_{t_0}^{\tau} L(t)|y_1 - y_2||v|\, dt,$$

which readily implies the Lipschitz condition (i) for g_∞ with $L(t)$. Using Gronwall's lemma, we deduce from (i) that $\{y_n\}$ is bounded in $L^\infty(0, T; \mathbb{R}^N)$ and, by the equations, $\{y_n'\}$ is bounded in $L^1(0, T; \mathbb{R}^N)$. Since $L_n(t)$ and $g_n(\cdot, 0)$ are weakly compact in L^1, we easily obtain that the y_n' are weakly compact in L^1 too; hence a subsequence of $\{y_n\}$

tends to some function $y \in H^{1,1}(0,T;\mathbf{R}^N)$. Since the Cauchy problem with g_∞ has a unique solution, we only have to show that $y' = g_\infty(t,y)$, or equivalently that for every $t \in [0,T]$

$$\lim_{n\to\infty} \int_0^t g_n(s,y_n(s))\,ds = \int_0^t g_\infty(s,y(s))\,ds. \tag{5.96}$$

For every $\epsilon > 0$ let y_ϵ be a piecewise constant function such that $\|y - y_\epsilon\|_\infty \le \epsilon$; then, by using the Lipschitz assumption (i), we obtain

$$\begin{aligned}
&\left|\int_0^t g_n(s,y_n(s))\,ds - \int_0^t g_\infty(s,y(s))\,ds\right| \\
&\le \int_0^t |g_n(s,y_n(s)) - g_n(s,y_\epsilon(s))|\,ds \\
&\quad + \int_0^t |g_\infty(s,y(s)) - g_\infty(s,y_\epsilon(s))|ds \\
&\quad + \left|\int_0^t g_n(s,y_\epsilon(s))ds - \int_0^t g_\infty(s,y_\epsilon(s))ds\right| \\
&\le \|y_n - y_\epsilon\| \int_0^T L_n(s)ds + \|y - y_\epsilon\|_\infty \int_0^T L(s)ds \\
&\quad + \left|\int_0^t g_n(s,y_\epsilon(s))ds - \int_0^t g_\infty(s,y_\epsilon(s))ds\right|.
\end{aligned}$$

By assumption (ii) and by taking into account that the y_ϵ are piecewise constant, the last term tends to zero as $n \to \infty$; hence

$$\limsup_{n\to\infty}\left|\int_0^t g_n(s,y_n(s))ds - \int_0^t g_\infty(s,y(s))ds\right| \le C\epsilon$$

for a suitable constant C. Thus we obtain (5.96) by letting $\epsilon \to 0$. □

Now we consider the minimum problem (5.95) with the following set of assumptions on the data a, b, f. The functions $a : (0,T)\times\mathbf{R}^k \to \mathbf{R}^k$ and $b : (0,T)\times\mathbf{R}^m \to \mathbf{R}^{km}$ are of Carathéodory type and

$$|a(t,y_1) - a(t,y_2)| \le \alpha(t)|y_1 - y_2| \quad \text{with } \alpha \in L^1(0,T), \tag{5.97}$$

$$a(t,0) \in L^1(0,T;\mathbf{R}^k), \tag{5.98}$$

$$|b(t,y_1) - b(t,y_2)| \le \beta(t)|y_1 - y_2| \quad \text{with } \beta \in L^{p'}(0,T), \tag{5.99}$$

$$b(t,0) \in L^{p'}(0,T;\mathbf{R}^{km}), \tag{5.100}$$

where $p \in [1,\infty]$ is given, and p' is its conjugate exponent. The integrand $f : (0,T)\times\mathbf{R}^k\times\mathbf{R}^m \to [0,+\infty]$ is supposed to be a Borel function (or at least $\mathcal{L}\otimes\mathcal{B}_k\otimes\mathcal{B}_m$-measurable, as in Theorem 3.6 of Section 3.1) such that

$$f(t,\cdot,\cdot) \text{ is lower semicontinuous on } \mathbf{R}^k\times\mathbf{R}^m, \tag{5.101}$$

$$f(t,y,\cdot) \text{ is convex on } \mathbf{R}^m. \tag{5.102}$$

and it satisfies a coercivity condition of the form: if $p \in (1, +\infty)$ there exist $c > 0$ and $\gamma \in L^1(0, T)$ such that

$$f(t, y, u) \geq c|u|^p - \gamma(t), \tag{5.103}$$

if $p = 1$ there exist a superlinear function $\theta : \mathbb{R} \to \mathbb{R}$ and $\gamma \in L^1(0, T)$ such that

$$f(t, y, u) \geq \theta(|u|) - \gamma(t), \tag{5.104}$$

and if $p = +\infty$ there exists an $R > 0$ such that

$$f(t, y, u) = \infty \quad \text{whenever } |u| \geq R. \tag{5.105}$$

Proposition 5.20 *Under the assumptions* (5.97)–(5.102) *the functional*

$$\mathcal{F}(u, y) = \begin{cases} \int_0^T f(t, y, u)\, dt & \text{if } y' = a(t, y) + b(t, y)u, \ y(0) = y_0 \\ +\infty & \text{otherwise} \end{cases}$$

is sequentially lower semicontinuous with respect to the (weak $\mathbb{R}^p$) $\times$ (weak $H^{1,1}$)-convergence ((weak L^∞) $\times$ (weak $H^{1,1}$) in the case $p = +\infty$).*

Proof For the sake of simplicity we only consider the case $p < \infty$. Let $u_n \to u$ weakly in $L^p(0, T; \mathbb{R}^m)$ and $y_n \to y$ weakly in $H^{1,1}(0, T; \mathbb{R}^k)$. Furthermore we may assume that

$$\liminf_{n \to \infty} \mathcal{F}(u_n, y_n) < \infty,$$

since otherwise the assertion is trivially correct. By possibly passing to subsequences we may then assume that $\mathcal{F}(u_n, y_n) < \infty$ for every $n \in \mathbb{N}$. As a consequence, the differential equations

$$\begin{aligned} y_n' &= a(t, y_n) + b(t, y_n)u_n \\ y_n(0) &= y_0 \end{aligned}$$

are fulfilled. Setting

$$\begin{aligned} g_n(t, y) &= a(t, y) + b(t, y)u_n(t) \\ g_\infty(t, y) &= a(t, y) + b(t, y)u(t) \end{aligned}$$

we are in the framework of Lemma 5.19. Therefore the limit function $y(t)$ satisfies the differential equation

$$\begin{aligned} y' &= a(t, y) + b(t, y)u \\ y(0) &= y_0. \end{aligned}$$

On the other hand, by the lower semicontinuity theorem 3.6 of Section 3.1 we have

$$\int_0^T f(t, y, u)\, dt \leq \liminf_{n \to \infty} \int_0^T f(t, y_n, u_n)\, dt,$$

whence

$$\mathcal{F}(u, y) \leq \liminf_{n \to \infty} \mathcal{F}(u_n, y_n). \qquad \square$$

Now we obtain our main existence result.

Theorem 5.21 *Under the assumptions* (5.97)–(5.105) *above the minimum problem* (5.95) *has at least one solution.*

Proof By Proposition 5.20 above only the coercivity of the functional $\mathcal{F}$ remains to be proved. Again, for simplicity, we consider only the case $p < \infty$. Let $u_n \in L^p(0, T; \mathbf{R}^m)$ and $y_n \in H^{1,1}(0, T; \mathbf{R}^k)$ be two sequences such that $\mathcal{F}(u_n, y_n) \le C$ for a suitable constant C; we have to show that, by possibly passing to subsequences, $\{u_n\}$ converges weakly in $L^p(0, T; \mathbf{R}^m)$ and $\{y_n\}$ converges weakly in $H^{1,1}(0, T; \mathbf{R}^k)$. By assumptions (5.103)–(5.105) we may assume that $u_n \to u$ weakly in $L^p(0, T; \mathbf{R}^m)$ for some u, and that

$$\begin{aligned} y_n' &= a(t, y_n) + b(t, y_n)u_n \\ y(0) &= y_0. \end{aligned}$$

Now it follows by Lemma 5.19 that $y_n \to y$ weakly in $H^{1,1}(0, T; \mathbf{R}^k)$ where y is the solution of

$$\begin{aligned} y' &= a(t, y) + b(t, y)u \\ y(0) &= y_0. \end{aligned}$$

□

5.9 Parametric variational problems

In this final section we deal with variational integrals

$$\mathcal{F}(x) := \int_{t_1}^{t_2} F(x(t), \dot{x}(t))\, dt \tag{5.106}$$

whose Lagrangian $F(x, v)$ is positively homogeneous of first order with respect to v. Such integrals $\mathcal{F}(x)$ are defined on curves $x : [t_1, t_2] \to \mathbf{R}^N$ in $\mathbf{R}^N$, and on account of the homogeneity condition

$$F(x, \lambda v) = \lambda F(x, v) \text{ for } \lambda > 0 \tag{5.107}$$

we see that $\mathcal{F}$ is invariant under reparametrization of curves. That is, if

$$\sigma : [\tau_1, \tau_2] \to [t_1, t_2]$$

is an arbitrary C^1-diffeomorphism of $[\tau_1, \tau_2]$ onto $[t_1, t_2]$ with $d\sigma/d\tau > 0$, and if we set $z := x \circ \sigma$, i.e.

$$z(\tau) = x(\sigma(\tau)), \quad \tau_1 \le \tau \le \tau_2,$$

then we obtain

$$\int_{t_1}^{t_2} F(x(t), \dot{x}(t))\, dt = \int_{\tau_1}^{\tau_2} F(z(\tau), \dot{z}(\tau))\, d\tau, \tag{5.108}$$

or in other words,

$$\mathcal{F}(x) = \mathcal{F}(x \circ \sigma).$$

It is not difficult to see that, conversely, the invariance property (5.108) implies that the Lagrangian $F(x, v)$ of (5.106) has to satisfy the homogeneity condition (5.107). The most prominent examples are the *Euclidean length*

$$\mathcal{L}(x) = \int_{t_1}^{t_2} |\dot{x}(t)|\, dt \tag{5.109}$$

of a curve in $\mathbb{R}^N$ and its *Riemannian length*

$$\mathcal{F}(x) = \int_{t_1}^{t_2} \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}\; dt \tag{5.110}$$

with respect to a Riemannian line element given by

$$ds^2 = g_{ik}(x)\, dx^i\, dx^k.$$

In the following discussion *we assume throughout that* $F(x, v)$ *is a Lagrangian of class* $C^0(K \times \mathbb{R}^N)$ *satisfying* (5.107) *for all* $(x, v) \in K \times \mathbb{R}^N$. *Here* K *denotes a closed connected set in* $\mathbb{R}^N$. Clearly, $F(x, 0) = 0$. The integrands $F(v) = |v|$ and $F(x, v) = [g_{ik}(x)v^i v^k]^{1/2}$ of the length functionals (5.109) and (5.110) show that we may not assume F to be smooth at $v = 0$. Thus we shall assume $F \in C^2(K \times (\mathbb{R}^N - \{0\}))$ whenever we want to state the Euler equations

$$F_x(x, \dot{x}) - \frac{d}{dt} F_v(x, \dot{x}) = 0 \tag{5.111}$$

of the parametric integral (5.106). Since $F_v(x, v)$, $F_{vv}(x, v)$, etc., need not be defined for $v = 0$, we have to confine ourselves to the *regular curve* $x(t)$, $t_1 \le t \le t_2$, if we want to avoid a special discussion of $x(t)$ at singularities, i.e. we have to assume that

$$\dot{x}(t) \neq 0.$$

For a smooth curve we can then suppose that

$$|\dot{x}(t)| = 1 \quad \text{on } [t_1, t_2]. \tag{5.112}$$

A parameter curve $x(t)$ with this property is called a *normal curve* or a *normal representation*. For every regular C^2-curve $x : [t_1, t_2] \to K$ we can define the Eulerian covector field $e(t) = (e_1(t), \ldots, e_N(t))$ by

$$e := L_F(x) = F_x(x, \dot{x}) - \frac{d}{dt} F_v(x, \dot{x}). \tag{5.113}$$

As an immediate consequence of (5.107) we obtain

$$L_F(x) \cdot \dot{x} = 0, \tag{5.114}$$

i.e. the Eulerian covector field $e(t)$ along any regular C^2-motion $x(t)$ is perpendicular to the velocity field $\dot{x}(t)$.

In the sequel we assume that all curves $x : \overline{I} \to \mathbb{R}^N$ are parametrized on the fixed interval $\overline{I} = [0, 1]$ where we have set $I := (0, 1)$. Then we define the functional $\mathcal{F}(x)$ by

$$\mathcal{F}(x) := \int_0^1 F(x, \dot{x})\, dt. \tag{5.115}$$

We want to minimize $\mathcal{F}(x)$ in a suitable class of curves $x : \overline{I} \to K \subset \mathbb{R}^N$ connecting two given points P_1 and P_2 of K. There are various difficulties in working with this functional. For example, we could insert one or several 'constancy intervals' into a given curve $x : \overline{I} \to K$ and reparametrize to $\overline{I}$. For the resulting curve $z : I \to K$ we would have $\mathcal{F}(z) = \mathcal{F}(x)$, but $\dot{z}(t) = 0$ on a subset of positive measure in I. Since the derivatives of $F(x, v)$ are not defined for $v = 0$, we therefore cannot conclude that minimizers of $\mathcal{F}$ contained in the interior of the set K are necessarily $\mathcal{F}$-extremals. One way out would be to restrict the minimization process to normal curves. But then we have to impose the subsidiary condition $|\dot{x}(t)| = 1$ a.e. on $\overline{I}$, which is not closed with respect to weak convergence in the Sobolev space $H^{1,1}(I, \mathbb{R}^N)$ which is the natural space to use if we assume that

$$m_1|v| \le F(x, v) \le m_2|v| \quad \text{for } (x, v) \in K \times \mathbb{R}^N \tag{5.116}$$

where m_1, m_2 denote constants satisfying $0 < m_1 \le m_2$. In order to circumvent this and related difficulties we shall, instead of $\mathcal{F}(\overline{x})$, minimize the functional

$$\mathcal{Q}(x) := \int_0^1 Q(x, \dot{x})\, dt \tag{5.117}$$

with the Lagrangian

$$Q(x, v) := \frac{1}{2} F^2(x, v) \quad \text{for } (x, v) \in K \times \mathbb{R}^N, \tag{5.118}$$

and then we shall make use of Proposition 1.16 and Remark 3 of Section 1.1, thereby ensuring that any minimizer x of $\mathcal{Q}$ satisfies

$$Q(x(t), \dot{x}(t)) \equiv h > 0 \quad \text{a.e. on } I. \tag{5.119}$$

In conjunction with (5.116) we then conclude that $|\dot{x}|$ is of class $L^\infty(\overline{I})$, whence $x \in Lip(\overline{I}, \mathbb{R}^N)$. We can apply the results of Section 1.1.1 leading to

$$\partial \mathcal{Q}(x, \lambda) = 0 \quad \text{for all } \lambda \in C_c^\infty(I)$$

and therefore to relation (5.119) since $Q(x, v)$ is positively homogeneous of second order with respect to v and consequently $Q \in C^1(K \times \mathbb{R}^N)$ if we assume that F is of class C^1 on $K \times (\mathbb{R}^N - \{0\})$.

A curve $x \in H^{1,1}(I, \mathbb{R}^N)$ satisfying (5.119) for some constant $h > 0$ is said to be *quasinormal*. On account of (5.116) every quasinormal curve is of class $H^{1,\infty}(I, \mathbb{R}^N)$ and therefore Lipschitz continuous on $\overline{I}$, and we infer from (5.116) and (5.119) that

$$\sqrt{2h}/m_2 \leq |\dot{x}(t)| \quad \text{a.e. on } I.$$

This allows us to derive the Euler equations $L_Q(x) = 0$ for any minimizer x of Q satisfying $x(I) \subset \operatorname{int} K$ provided that $F(x, v)$ is assumed to be of class C^2 on $K \times (\mathbb{R}^N - \{0\})$, whence $Q \in C^2(K \times (\mathbb{R}^N - \{0\}))$. Note that (5.119) is equivalent to

$$F(x(t), \dot{x}(t)) \equiv \sqrt{2h} > 0 \quad \text{a.e. on } I,$$

since $F \geq 0$. Moreover, we have

$$Q_v = FF_v \quad \text{and} \quad Q_x = FF_x$$

and therefore

$$L_Q(x) = \sqrt{2h} L_F(x)$$

with some constant $h > 0$ for any quasinormal curve $x \in C^2(I, \mathbb{R}^N)$. Hence *every quasinormal F-extremal is a Q-extremal, and vice versa.* Furthermore we shall see that every minimizer of Q is a minimizer of $\mathcal{F}$. Therefore we replace the minimization of $\mathcal{F}$ by that of Q thereby obtaining 'well-behaved' (i.e. quasinormal) minimizers and extremals. This idea is well known from Riemannian geometry where one replaces the length functional

$$\mathcal{L}(x) = \int_0^1 \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k}\, dt$$

by the Dirichlet integral

$$\mathcal{D}(x) = \frac{1}{2} \int_0^1 g_{ik}(x)\dot{x}^i\dot{x}^k\, dt.$$

Let us now fix our assumptions on the parametric Lagrangian $F(x, v)$ which is defined on $K \times \mathbb{R}^N$.

Condition A. The Lagrangian $F(x, v)$ is of class C^1 on $K \times (\mathbb{R}^N - \{0\})$ and has the following properties:

(i) There are numbers m_1 and m_2 with $0 < m_1 \leq m_2$ such that relations (5.116) are satisfied.

(ii) $F(x, v)$ is convex with respect to v.

Now we fix two points P_1 and P_2 in K with $P_1 \neq P_2$ and suppose that P_1 and P_2 can be connected in K by some Lipschitz arc. Then the set

$$C = C(P_1, P_2, K) := \{x \in H^{1,2}(I, \mathbb{R}^N),\ x(\bar{I}) \subset K,\ x(0) = P_1, x(1) = P_2\}$$

is not empty. Consider the variational problem

$$\mathcal{F} \to \min \text{ in } C. \tag{$\mathcal{P}$}$$

We have the following existence result.

Theorem 5.22 *Let K be a closed set in $\mathbb{R}^N$ and let $F(x, v)$ be a parametric Lagrangian on $K \times \mathbb{R}^N$ satisfying Condition A. Suppose also that $C = C(P_1, P_2, K)$ is non-void. Then there is a quasinormal (and therefore Lipschitz-continuous) curve $x \in C$ which minimizes Q in the set C. Moreover, x minimizes $\mathcal{F}$ among all quasinormal curves in C and even among all curves of C.*

Proof First we note that

$$(m_1^2/2)|v|^2 \le Q(x, v) \le (m_2^2/2)|v|^2 \quad \text{for all } (x, v) \in K \times \mathbb{R}^N.$$

Moreover, for any $x \in K$, the function $Q(x, v)$ is convex with respect to $v \in \mathbb{R}^N$. Then by Theorem 3.9 of Section 3.3.2 there is some $x \in C$ such that

$$Q(x) = \inf_C Q.$$

It follows as in Section 1.1 that the inner variation $\partial Q(x, \lambda)$ vanishes for all $\lambda \in C_c^\infty(I)$, and therefore

$$Q(x(t), \dot{x}(t)) \equiv h \quad \text{a.e. on } I$$

according to Proposition 1.14 and Remark 3 of Section 1.1. On account of (5.118) we have $h \ge 0$, and then we conclude that $h > 0$, since $h = 0$ would imply that $|\dot{x}(t)| = 0$ a.e. on I, whence $x(t) \equiv$ const on $\bar{I}$ and therefore $P_1 = P_2$, a contradiction. Thus x is quasinormal, and (5.116) yields that

$$|\dot{x}(t)| \le m_1^{-1}\sqrt{2h} \quad \text{a.e. on } \bar{I}$$

whence $\dot{x} \in L^\infty(I, \mathbb{R}^N)$. Thus $x : \bar{I} \to \mathbb{R}^N$ is Lipschitz continuous.

Finally Schwarz's inequality implies that

$$\mathcal{F}^2(z) \le 2Q(z) \quad \text{for all } z \in H^{1,2}(I, \mathbb{R}^N) \text{ with } z(\bar{I}) \subset K,$$

and the equality sign holds if and only if $Q(z(t), \dot{z}(t)) =$ const a.e. on I. Introducing $C^* = C^*(P_1, P_2, K)$ by

$$C^* := \{z \in C : z \text{ is quasinormal}\},$$

we then obtain

$$\mathcal{Q}(x) = \inf_{C} \mathcal{Q} = \inf_{C^*} \mathcal{Q} = \frac{1}{2}\left(\inf_{C^*} \mathcal{F}\right)^2$$

whence

$$\mathcal{F}(x) = \inf_{C^*} \mathcal{F}.$$

In other words, the minimizer x of the functional $\mathcal{Q}$ in C minimizes $\mathcal{F}$ in the class C^* and therefore also in the set of all Lipschitz curves in C which by Lipschitz-continuous parameter transformations $\tau : \overline{I} \to \overline{I}$ with $\tau'(u) > 0$ a.e. can be transformed to a quasinormal curve in C. This result suffices for all geometric purposes, but by the following lemma we even obtain

$$\inf_{C^*} \mathcal{F} = \inf_{C} \mathcal{F}$$

and therefore

$$\mathcal{F}(x) = \inf_{C} \mathcal{F}$$

as we have claimed. □

Lemma 5.23 *For any $x \in C \cap Lip(\overline{I}, \mathbb{R}^N)$ we can find a quasinormal $\xi \in C$ such that $\mathcal{F}(\xi) = \mathcal{F}(x)$.*

Proof The continuous increasing function $\sigma(t) := \int_0^t |\dot{x}|\,dt$ has at most denumerably many intervals of constancy; they agree with the constancy intervals of x. Removing the interior parts and pulling the holes together we obtain, after a linear reparametrization, some function $z \in C$ with $\mathcal{F}(z) = \mathcal{F}(x)$ which has no intervals of constancy. Thus we can assume that the original curve x has no constancy intervals and that $\sigma(t)$ is strictly increasing. Then σ defines a homeomorphism of $\overline{I}$ onto $[0, l]$ where l is the arc length of x. Set $\xi := x \circ \tau$ where τ is the inverse of σ. Set $s_1 = \sigma(t_1)$ and $s_2 = \sigma(t_2)$ for some $t_1, t_2 \in \overline{I},\ t_1 \le t_2$. Then

$$s_2 - s_1 = \int_{t_1}^{t_2} |\dot{x}|\,dt = \int_{t_1}^{t_2} |\,dx| = \int_{s_1}^{s_2} |d\xi|$$

and in particular

$$|\xi(s_2) - \xi(s_1)| \le |s_2 - s_1| \quad \text{for any } s_1, s_2 \in [0, l].$$

Therefore $\xi(s)$ is Lipschitz continuous on $[0, l]$ and

$$\int_{s_1}^{s_2} |\dot{\xi}(s)|ds = s_2 - s_1 \quad \text{for } 0 \le s_1 \le s_2 \le l$$

whence $|\dot{\xi}(s)| = 1$ a.e. on $[0, l]$ and

$$\mathcal{F}(x) = \int_0^l F(\xi, \dot{\xi})ds.$$

Thus we can assume that the original curve x is of class $\mathcal{C} \cap Lip(\overline{I}, \mathbb{R}^N)$ and satisfies $|\dot{x}(t)| = l > 0$ a.e. on $\overline{I}$. Set

$$c := \int_0^1 F(x, \dot{x})\,dt \text{ and } \sigma(t) := c^{-1} \int_0^t F(x, \dot{x})\,dt.$$

By (5.116) we have $c > 0$, and σ is a bi-Lipschitz-map of $\overline{I}$ onto itself. Then $\xi := x \circ \sigma^{-1}$ satisfies $\mathcal{F}(x) = \mathcal{F}(\xi)$, $\xi \in \mathcal{C}$, and $F(\xi(t), \dot{\xi}(t)) = c$ as well as $Q(\xi(t), \dot{\xi}(t)) = \frac{1}{2}c^2 > 0$ a.e. on $\overline{I}$. □

Remark 1 It suffices to assume that

$$m_1|v| \le F(x, v) \quad \text{for all } (x, v) \in K \times \mathbb{R}^N$$

instead of (5.116).

If $\mathcal{F}$ is chosen as the length functional and if we choose K as a connected Riemannian manifold that is isometrically embedded in some $\mathbb{R}^N$, we obtain that any two points P_1 and P_2 on K can be connected by a shortest line contained in K. The same is true if we choose K as the complement $\mathbb{R}^n - \Omega$ of some open set $\Omega \subset \mathbb{R}^N$, provided that $\mathbb{R}^n - \Omega$ is non-void, and that any two points of K can be connected in K by some Lipschitz arc. Thus we have solved the *obstacle problem* for the length functional and, more generally, for a fairly extended class of parametric variational integrals.

As we have seen earlier we cannot expect that a minimizer of $\mathcal{F}$ in $\mathcal{C}$ is an extremal. In fact there might even be only one Lipschitz curve in K connecting P_1 with P_2 since we have not imposed any regularity assumptions on K. However, we have

Proposition 5.24 *Suppose that $F(x, v)$ is of class C^1 on $K \times (\mathbb{R}^N - \{0\})$ and let $x \in \mathcal{C}(P_1, P_2, K)$ be a quasinormal minimizer of $\mathcal{F}$ among all curves in $\mathcal{C}(P_1, P_2, K)$, $P_1 \neq P_2$. Assume also that $x(I) \subset \operatorname{int} K$. Then x is a weak Lipschitz extremal of $\mathcal{F}$.*

Proof Let $\varphi \in C_c^\infty(I, \mathbb{R}^N)$ and consider the one-parameter family of curves

$$z(t, \epsilon) := x(t) + \epsilon\varphi(t), \quad t \in I, |\epsilon| < \epsilon_0.$$

For sufficiently small $\epsilon_0 > 0$ and $\delta > 0$ we obtain that $z(t, \epsilon) \in K$ and $|\dot{z}(t, \epsilon)| > \delta$ a.e. on I for all $\epsilon \in [-\epsilon_0, \epsilon_0]$. Hence $f(\epsilon) := \mathcal{F}(z(\cdot, \epsilon))$ is differentiable and $f(\epsilon) \ge f(0)$ for $|\epsilon| < \epsilon_0 \ll 1$. Then the reasoning of Section 1.1 yields $f'(0) = 0$, i.e.

$$\delta\mathcal{F}(x, \varphi) = \int_0^1 [F_x(x, \dot{x}) \cdot \varphi + F_v(x, \dot{x}) \cdot \dot{\varphi}]\,dt = 0. \qquad \square$$

Next we shall prove a *regularity theorem* for weak Lipschitz extremals which can be applied to minimizers x of $\mathcal{F}$ in $\mathcal{C}$ satisfying $x(I) \subset \operatorname{int} K$.

We say that $F(x,v)$ is *elliptic* if the coefficients $g_{ik}(x,v) := Q_{v^i v^k}(x,v)$ satisfy

$$g_{ik}(x,v)\xi^i\xi^k > 0 \quad \text{for all } \xi \in \mathbb{R}^N - \{0\} \text{ and } x \in K,\ v \in \mathbb{R}^N - \{0\}. \tag{5.120}$$

Note that

$$g_{ik} = F_{v^i} F_{v^k} + F F_{v^i v^k}$$

and

$$F(x,v) = v^i F_{v^i}(x,v),\quad v^k F_{v^i v^k}(x,v) = 0.$$

Thus $(F_{v^i v^k})$ can never be a positive definite matrix since v is in the kernel of F_{vv}. Therefore the strongest possible assumption on F_{vv} is that $F_{vv} \geq 0$ and that the null space of $F_{vv}(x,v)$ is one dimensional. It is not difficult to see that this assumption agrees with the ellipticity assumption formulated above.

Proposition 5.25 *Suppose that $F(x,v)$ is an elliptic Lagrangian of class C^2 which satisfies the assumptions of Theorem 5.22. Moreover, let x be a quasinormal curve in K which is a weak Lipschitz extremal of $\mathcal{F}$. Then x is an extremal of $\mathcal{F}$, i.e. $x \in C^2(I,\mathbb{R}^N)$, $\dot{x}(t) \neq 0$, and $L_F(x) = 0$.*

Proof There is a constant $c > 0$ such that $F(x,\dot{x}) = c$, whence

$$0 < c/m_2 \leq |\dot{x}(t)| \leq c/m_1 \quad \text{for almost all } t \in I. \tag{5.121}$$

Then there is a constant vector $\lambda \in \mathbb{R}^N$ such that

$$F_v(x(t),\dot{x}(t)) = \lambda + \int_0^t F_x(x(s),\dot{x}(s))ds. \tag{5.122}$$

If we multiply (5.122) by c and set $Q := \frac{1}{2}F^2$, it follows that

$$Q_v(x(t),\dot{x}(t)) = \lambda c + \int_0^t Q_x(x(s),\dot{x}(s))ds \ \text{ a.e. on } I.$$

Introducing the Hamilton function $\Phi(x,y)$ corresponding to $Q(x,v)$ which is also of class C^2 for $y \neq 0$, we obtain for the momentum $y(t) := Q_v(x(t),\dot{x}(t))$ the equation

$$y(t) = \lambda c - \int_0^t \Phi_x(x(s),y(s))\,dx \quad \text{a.e. on } I. \tag{5.123}$$

Our assumptions imply that the integrand $\Phi_x(x(t),y(t))$ is of class $L^\infty(I,\mathbb{R}^N)$, whence (5.123) yields that $y(t)$ is Lipschitz continuous on I. Thus $\Phi_x(x(t),y(t))$ is continuous on I, and (5.123) now implies that $y(t)$ is of class C^1 on I. From

$$\dot{x}(t) = \Phi_y(x(t),y(t))$$

and $\Phi \in C^2$ we then infer that $\dot{x} \in C^1(I,\mathbb{R}^N)$, i.e. $x \in C^2(I,\mathbb{R}^N)$. Differentiating (5.122), we obtain the Euler equation $L_F(x) = 0$ on I. □

Corollary 5.26 *In particular any solution x of the minimum problem $(\mathcal{P})$ is a C^2-extremal of the functional $\mathcal{F}$ provided that $x(I) \subset \operatorname{int} K$.*

In a similar way it follows that x is of class C^2 if K is a smooth Riemannian manifold without boundary which is embedded in $\mathbb{R}^N$. This is proved by locally flattening K which leads to a transformation of x and $\mathcal{F}$ by the flattening diffeomorphism u, and thereby a local Euler equation for u is derived. For a detailed discussion we refer to Giaquinta–Hildebrandt [113] Chapter 2, Section 2, and Chapter 8, Section 4.4.

6

SCHOLIA

6.1 Additional remarks on the calculus of variations

Variational problems such as the isoperimetric problem were already discussed in antiquity. However, the beginning of the calculus of variations is usually set in the year 1696 when Johann Bernoulli formulated the brachistochrone problem in the *Acta Eruditorum Lipsiae*. Actually Newton had already posed a true variational problem in 1686: his celebrated problem of determining the shape of a rotationally symmetric body of least resistance. The first textbook on the variational calculus was Leonhard Euler's *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes* which appeared in 1744. Here Euler treated notoriously difficult questions such as variational problems with differential equations as subsidiary conditions. In two appendices he studied elastic lines and gave the first satisfactory mathematical treatment of the *least action principle*. In 1755 Lagrange developed the so-called δ-calculus which he viewed as a kind of 'higher' infinitesimal calculus, while Euler showed in 1770 that, in fact, the δ-calculus can be reduced to the ordinary infinitesimal calculus. For this purpose he 'embedded' a given minimizing or maximizing curve in a one-parameter family of curves and differentiated the variational integral evaluated on the curves of the family with respect to the parameter, thus obtaining the value zero for the first variation of the integral at the extremum. This is the classical approach to the Euler–Lagrange equations which is still used today.

However, no *sufficient conditions* guaranteeing the minimum property for solutions of Euler's equations were known at the time of Euler and Lagrange. It seems that only Johann Bernoulli studied this question in a special case; his paper from 1718 remained unnoticed for two centuries. The first time the 'question of sufficiency' was systematically investigated was by Legendre in 1788. His paper was not correct as Lagrange pointed out in 1797, but Legendre's ideas were influential to Jacobi who resumed the study of sufficiency in 1837. In a very short paper he sketched his celebrated theory of conjugate points, essentially without proofs, which were supplied by other mathematicians in the following decades. About 50 years later L. Scheeffer and K. Weierstrass discovered that positivity of the second variation does not suffice to establish the (local) minimum property of an extremal, i.e. of a stationary curve. In 1879 Weierstrass discovered that the minimum property can be proved by means of a method which has become known as 'Weierstrass field theory'. This method was further developed by A. Mayer, A. Kneser, D. Hilbert, and C. Carathéodory. In Section 1.2 we have given a modified version of Carathéodory's ideas. A detailed treatment of these ideas can be found in the treatise of Giaquinta–Hildebrandt [113]. The scholia of that book can be used as an introduction to the long history of the calculus of variations. A systematic study of this history is presented in the monograph by H. H. Goldstine [124]. We also refer to the very interesting textbook by L. C. Young [294] which gives an original presentation both of the classical methods and of some ideas connected with the direct methods leading to the so-called Young measures. The study of these measures has in recent years become an important branch of the calculus of variations which, unfortunately, could not be discussed in our text. It is unavoidable that, in this book, the reader will miss many topics and many historical facts about which he or she would like to learn more. We should like to recommend Giaquinta–Hildebrandt [113] for further information and as a guide to the literature.

6.2 Semicontinuity and compactness

Notions of *general topology* can nowadays be found in undergraduate texts or in books about real functions or functional analysis, such as for instance Brézis [44], Rudin [230], Hewitt–Stromberg [139], Natanson [199], Yosida [293], Dunford–Schwartz [90]; specific treatises are, for example, Kelley [153], Dugundji [89], Bourbaki [42]. For the reader's convenience we collect here some basic definitions and results concerning the notions of semicontinuity and compactness. Usually, these notions are set up in the context of topological spaces.

Let (X, τ) be a *topological space*, i.e. a set X where a family τ, called topology, of *open sets* has been specified. A mapping

$$\mathcal{F} : X \to \mathbf{R} \cup \{\infty\}$$

is said to be τ-*lower semicontinuous*, τ-l.s.c. or l.s.c. for short, if for every $t \in \mathbf{R}$ the set

$$U_t^{\mathcal{F}} := \{x \in X : \mathcal{F}(x) > t\}$$

is open in X, or equivalently the set

$$V_t^{\mathcal{F}} := \{x \in X : \mathcal{F}(x) \leq t\}$$

is closed in X. One easily verifies

Proposition 6.1 *We have*

(i) $\mathcal{F} : X \to \mathbf{R} \cup \{\infty\}$ *is l.s.c. if and only if its epigraph*

$$epi(\mathcal{F}) := \{(x, t) \in X \times (\mathbf{R} \cup \{\infty\}) : \mathcal{F}(x) \leq t\}$$

is closed in $X \times (\mathbf{R} \cup \{\infty\})$.

(ii) *If* $\{\mathcal{F}_i\}_{i \in I}$ *is a family of l.s.c. functions, then the function*

$$\mathcal{F}(x) := \sup_{i \in I} \mathcal{F}_i(x)$$

is l.s.c.

(iii) *If* $\mathcal{F}$ *and* $\mathcal{G}$ *are l.s.c. and* $\lambda \geq 0$, *then* $\mathcal{F} + \mathcal{G}$ *and* $\lambda\mathcal{F}$ *are l.s.c.*

(iv) *If* $\{x_n\}$ *is a sequence which* τ-*converges to* x, *then*

$$\mathcal{F}(x) \leq \liminf_{n \to \infty} \mathcal{F}(x_n).$$

Condition (iv) in general does not imply lower semicontinuity of $\mathcal{F}$, but *if* (X, τ) *satisfies the first countability axiom*, i.e. *if every* $x \in X$ *has a countable fundamental system of neighbourhoods, then* (iv) *is equivalent to lower semicontinuity.*

Actually, from the point of view of the calculus of variations we need not work with a topological space, but with a space equipped with a notion of *convergence* and with functions which are *sequentially lower semicontinuous*, in short s.l.s.c.

Definition 6.2 *Let us assume that in* X *we have defined a notion of convergence. The mapping* $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ *is said to be s.l.s.c. if for every sequence* $\{x_k\}$ *converging to* $x \in X$ *we have*

$$\mathcal{F}(x) \leq \liminf_{k \to \infty} \mathcal{F}(x_k).$$

One can also prove that the notion of sequential semicontinuity is a topological concept; in fact we have (cf. e.g. Dolcher [88]):

Proposition 6.3 *Consider a topological space* (X, τ) *and denote by* τ_{seq} *the topology on* X *whose closed sets are the sequentially* τ*-closed subsets of* X. *Then*

(i) τ_{seq} *is the strongest topology on* X *for which the converging sequences are* τ*-converging.*
(ii) $\mathcal{F}$ *is sequentially* τ*-l.s.c. if and only if* $\mathcal{F}$ *is* τ_{seq}*-l.s.c.*
(iii) $\tau_{seq} = \tau$ *if* (X, t) *satisfies the first countability axiom.*

Notice that in the calculus of variations one often works with notions of *weak convergence*, for which $\tau_{seq} \neq \tau$. However, for convex functions in a Banach space the following result holds (see for instance Dunford–Schwartz [90] Vol. I, Chap. V).

Proposition 6.4 *Let* X *be a Banach space and let* $F : X \to (-\infty, \infty)$ *be a convex function. Then*

(i) F *is strongly l.s.c. if and only if* F *is weakly l.s.c.;*
(ii) *if* X^* *is separable, then* F *is weakly l.s.c. if and only if* F *is sequentially weakly l.s.c.;*
(iii) *if* $X = V^*$ *and* V *is a separable Banach space, then* F *is weakly* l.s.c. if and only if* F *is sequentially weakly* l.s.c.*

The other concept which is important here is that of *compactness.*

Definition 6.5 *A topological space* X *is called* compact *if every open covering of* X *has a finite subcovering.*

Even more important for us is the concept of *sequential compactness.*

Definition 6.6 *A space* X *equipped with a notion of convergence is called* sequentially compact *if every sequence in* X *has a converging subsequence.*

There exist compact spaces which are not sequentially compact and sequentially compact spaces which are not compact. But we have

Proposition 6.7 *In a metric space* X *the notions of compactness and sequential compactness coincide.*

Moreover, the following compactness criterion holds.

Proposition 6.8 *Let X be a metric space. Then X is (sequentially) compact if and only if X is totally bounded, i.e. if and only if for every positive ϵ, the space X can be covered by a finite number of balls of radius less than ϵ.*

One of the most important compactness theorems, which in fact is the first step in proving many other compactness results, is the following

Theorem 6.9 (Arzelà–Ascoli) *Let X be a compact metric space and let K be a subset of the space of continuous functions $C^0(X, \mathbb{R})$ equipped with the sup norm. Then K is (sequentially) compact if and only if the functions in K are equibounded and equicontinuous.*

In the case $X = [0, 1]$, Ascoli [16] pp. 545–549 showed the sufficiency of the condition for compactness, and Arzelà [14] proved the necessity of this condition. A clear presentation of this and related theorems was given by Arzelà [15]. The extension to a case in which the domain is a space with a notion of a limit (in particular a metric space) was carried out by Fréchet [102], [103], [104].

We emphasize again that our interest in semicontinuity and compactness comes from the following simple extension of Weierstrass's result on the existence of minimum points.

Theorem 6.10 (Weierstrass) *Let $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ be an s.l.s.c. function on X and let $C \subset X$ be a sequentially compact subset of X. Then $\mathcal{F}$ attains its minimum on C.*

The notions of *compactness* and *sequential compactness* coincide in a metric space, as we have already stated in Proposition 6.7 above. For a Banach space endowed with its weak topology we recall the Eberlein–Shmulian theorem (see for instance Dunford–Schwartz [90] Vol. I, p. 430):

Theorem 6.11 (Eberlein–Shmulian) *Let E be a subset of a Banach space X. Then the following conditions are equivalent:*

(i) *Every sequence in E has a weakly converging subsequence.*
(ii) *The weak closure of E is weakly compact.*

6.3 Absolutely continuous functions

One of the central problems in the work of Lebesgue concerns the correspondence between *integral* and *primitive*, i.e. the problem of determining a function from its derivative. This had been an important question at the end of the last century. In [164] Lebesgue showed that *if f is integrable* in (a, b) *then the function*

$$F(x) := \int_a^x f(t)\,dt$$

is differentiable almost everywhere with a derivative equal to $f(x)$. Conversely, if a function g is differentiable in (a, b) and its derivative $g' = f$ is bounded, then f is integrable, and one has

$$g(x) - g(a) = \int_a^x f(t)\,dt.$$

Much more complicated is the problem when g' is not bounded, since in this case g' is not necessarily defined everywhere and integrable. Assuming that g is continuous, g' exists almost everywhere, and that one of the *nombres dérivés* is everywhere finite, Lebesgue then proved that g is necessarily a function of bounded variation. The functions of bounded variation had been introduced by Jordan [150] in connection with the problem of the rectifiability of curves. Finally Lebesgue showed that *a function of bounded variation g has almost everywhere a derivative g' which is integrable, but that in general one has not*

$$g(x) - g(a) = \int_a^x g'(t)\,dt. \tag{6.1}$$

In a footnote on the last page of [164] Lebesgue claimed that for the validity of eqn (6.1) one has to assume that the total variation of g over a countable set of intervals of total length l tends to zero as l tends to zero.

It seems that Vitali discovered this condition independently of Lebesgue, and he gave a complete characterization of the functions which are primitive [283]. Vitali introduced the class of *absolutely continuous functions* in exactly the same way as we have done in Section 2.2, and he proved the following result: *A necessary and sufficient condition for* (6.1) *to hold is that g be absolutely continuous in* (a, b). He also exhibited an example of a continuous non-decreasing function of bounded variation which is not absolutely continuous. We mention that a similar example appears also in [168].

In Chap. 3 we saw the relevance of the class of absolutely continuous functions for the calculus of variations. Here we want to mention one more application connected with the problem of defining the length of a curve.

The length of a continuous curve C in $\mathbb{R}^3$ represented by $(x(t), y(t), z(t))$, $t \in (a, b)$, had been defined by Jordan [150] as the supremum of the lengths of all polygones with vertices on the curve. A curve of finite length is called *rectifiable*. Jordan also proved that *a necessary and sufficient condition for the curve $(x(t), y(t), z(t))$ to be rectifiable is that $x(t)$, $y(t)$, $z(t)$ be functions of bounded variation.*

As a consequence of his results Lebesgue showed in [163], [164] that if $\dot{x}$, $\dot{y}$, $\dot{z}$ exist everywhere, or if the functions x, y, z have bounded *nombres dérivés*, then

$$\text{length of } C = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt.$$

A complete answer was obtained by Tonelli [254]. Using Vitali's result, he proved the following:

Theorem 6.12 *The length l of a rectifiable curve satisfies*

$$l \geq \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt,$$

and equality holds if and only if $x(t)$, $y(t)$, and $z(t)$ are absolutely continuous.

Rather more complicated is the problem of defining the area of a surface, which we shall not touch. Concerning classical results dealing with this problem we refer to

[216], [232]. The question is related to the problem of minimal surfaces, one of the most important problems in the calculus of variations, which has been the source of many discoveries.

6.4 Sobolev spaces

The theory of Sobolev spaces was developed in connection with multidimensional variational problems, particularly in connection with Dirichlet's principle and with boundary value problems for partial differential equations. It proved to be useful in many other contexts such as, for example, approximation theory and real analysis. Consequently the literature on this topic is very large, and it extends far beyond the limits of this treatise. Many books on elliptic partial differential equations contain results about Sobolev spaces, for instance [2], [105], [121], [144], [145], [160], [173], [174], [193], [200]; a systematic study of Sobolev spaces is carried out in [1], [97], [158], [182], and Ziemer [295]. Some of these books also contain generalizations of Sobolev spaces such as Nikolskii, Slobodeskii, and Besov spaces; the reader may consult [245], [205], [33], [280], [279] for these topics.

It is nowadays customary to associate the name of S. L. Sobolev with these spaces although this is not very much justified from a historical point of view. In fact, these spaces were for quite some time called *Beppo Levi spaces*. In a systematic way Sobolev spaces were introduced independently by S. L. Sobolev [242], compare also [243], and by J. W. Calkin [53] and C. B. Morrey [191], compare also [192], but certainly these authors were not the first to consider functions with generalized derivatives in more than one variable.

Probably the first to use functions with generalized derivatives in the context of calculus of variations was Beppo Levi [171] in 1906. He considered continuous functions which are absolutely continuous in each variable for almost all values of the others and with first derivatives in L^2. Similar classes of functions were also considered by G. Fubini [111], and then by Tonelli [270], compare especially [275], and by Nikodym [204]. Continuous functions which are absolutely continuous in each variable for almost all values of the others and with derivatives in L^1 are often called *absolutely continuous in the sense of Tonelli*, and the set of such functions is denoted by ACT.

Functions of Sobolev type were also used by G. C. Evans [95], [96] in 1920 in his potential-theoretic studies. In 1930 F. Rellich [219] proved the L^2-compactness of bounded sets in $H^{1,2}$, and J. Leray [169] used $H^{1,2}$-spaces for his study of Navier–Stokes equations. Probably many authors have used functions with generalized derivatives, but an accurate historical account seems to be missing.

The study and the use of Sobolev spaces increased considerably during the 1940s and 1950s. To quote just a few of the most influential writers we mention Morrey [190], [192], Friedrichs [107], [108], [109], Kondrachov [157], Shiffman [237], Sigalov [240], [241], Deny [84], Deny–Lions [85], Ladyzhenskaya [159], [160], Aronszajn–Smith [12], [13], Nirenberg [206], John [148], [149], Lax [162], and Browder [46]. Starting in the 1950s the theory and its applications exploded, and it is extremely difficult to trace all the contributions in a historically correct way.

General inequalities such as *Poincaré's inequality* (often also referred to as *Wirtinger's inequality*) are of crucial importance in the calculus of variations and in the theory of elliptic partial differential equations. These inequalities go back to Poincaré [210]. For the early literature compare also [7], [261], [128], [136], [212].

The compact immersion theorem from $H^{1,p}$ into L^q is due to Rellich [219] for the case $p = 2$, and to Kondrachov [157] in general.

6.5 Non-convex functionals on measures and bounded variation functions

In Theorem 3.12 and Remark 1 of Section 3.3 we have seen that various integrands $F(x, p)$ which are convex with respect to p lead to weakly* lower semicontinuous functionals on the space of measures. These functionals are defined as

$$\mathcal{F}(\lambda) = \int_\Omega F\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_\Omega F^\infty(x, \lambda^s). \tag{6.2}$$

The indicated lower semicontinuity holds true, for example, for Lagrangians of the form $F = F(p)$ with F convex and lower semicontinuous.

Similarly we obtain from Theorem 3.6 of Section 3.1 that all functionals of the form

$$\mathcal{F}(u) = \int_\Omega F(x, u)\, dx \tag{6.3}$$

are weakly lower semicontinuous on $L^q(\Omega, \mathbf{R}^N)$ provided that $F(x, p)$ is a measurable function in (x, p) which is convex and lower semicontinuous with respect to p.

Vice versa, it is possible to prove (see Buttazzo–Dal Maso [49]) that functionals of the form (6.3) with a measurable Lagrangian F which is convex in p are the only ones that are weakly lower semicontinuous on $L^q(\Omega, \mathbf{R}^N)$ and local in the sense that

$$\mathcal{F}(u + v) = \mathcal{F}(u) + \mathcal{F}(v) \quad \text{whenever } u \cdot v \equiv 0 \quad \text{a.e. on } \Omega.$$

(Here we have assumed without loss of generality that $\mathcal{F}(0) = 0$.) In other words, for a functional on $L^q(\Omega, \mathbf{R}^N)$, weak lower semicontinuity and locality imply convexity. This is no longer true for functionals defined on the space $\mathcal{M}(\Omega, \mathbf{R}^N)$ of measures; in fact, it is possible to show (see Bouchitté–Buttazzo [38], [39], [40]) that if we define locality on $\mathcal{M}(\Omega, \mathbf{R}^N)$ by

$$\mathcal{F}(\lambda_1 + \lambda_2) = \mathcal{F}(\lambda_1) + \mathcal{F}(\lambda_2) \quad \text{whenever } \lambda_1, \lambda_2 \text{ are mutually singular on } \Omega,$$

we obtain that functionals of the form (6.2) are the only ones which are local, weakly* lower semicontinuous, and convex on $\mathcal{M}(\Omega, \mathbf{R}^N)$. On the other hand, if we do not impose convexity a priori we obtain the integral representation

$$\mathcal{F}(\lambda) = \int_\Omega F\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_\lambda} F^\infty(x, \lambda^s) + \int_{A_\lambda} G(x, \lambda^\#(x))\, d\#\tag{6.4}$$

for a suitable measure μ and integrands F and G, where A_λ is the set of *atoms* of λ, $\#$ is the counting measure, $\lambda^\#$ is the atomic part of λ, and $\lambda^\#(x)$ denotes the value $\lambda^\#(\{x\})$.

The Lagrangian $F(x, p)$ has to be convex in p, while $G(x, p)$ has to be *subadditive* in p, i.e.

$$G(x, p_1 + p_2) \leq G(x, p_1) + G(x, p_2) \quad \text{for all } x, p_1, p_2,$$

and F and G are related by the asymptotic condition

$$F^\infty(x, p) = \lim_{t \to +0} \frac{G(x, tp)}{t}. \tag{6.5}$$

For instance, taking $F(p) = |p|^2$ and $G(p) \equiv 1$, condition (6.5) is fulfilled; therefore the functional

$$\mathcal{F}(\lambda) = \begin{cases} \displaystyle\int_\Omega \left|\frac{d\lambda}{d\mu}\right|^2 d\mu + \sharp(A_\lambda) & \text{if } \lambda^s = 0 \\ \infty & \text{if } \lambda^s \neq 0 \end{cases} \tag{6.6}$$

turns out to be local and weakly* lower semicontinuous on $\mathcal{M}(\Omega, \mathbb{R}^N)$, but it is not convex.

An analogous discussion on $BV(I, \mathbb{R}^N)$, $I := (a, b)$, leads to functionals of the form

$$\mathcal{F}(\lambda) = \int_I F(x, \dot{u})\, dx + \int_{(a,b)\setminus S_u} F^\infty(x, u'_s) + \int_{S_u} G(x, [u])\, d\sharp$$

where $[u]$ denotes the jump of u and S_u is the set of discontinuity points of u. For instance, the functional in (6.6) becomes

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_a^b |\dot{u}|^2\, dx + \sharp(S_u) & \text{if } u'_s = 0, \\ \infty & \text{if } u'_s \neq 0, \end{cases}$$

which is the one-dimensional case of more general model problems in fracture mechanics or in image segmentation. We refer the reader to papers by De Giorgi [79], De Giorgi–Ambrosio [80], Ambrosio [9], De Giorgi *et al.* [83], Bouchitté *et al.* [41] where details and references can be found.

6.6 Direct methods

As was mentioned before, the origin of direct methods can be found in the work of Gauss, Thomson, Dirichlet, and Riemann in connection with Dirichlet's principle. The first justification of Dirichlet's principle was given by D. Hilbert in 1900, compare [140], [141], who showed that a direct approach can be used to solve one-dimensional regular problems concerning the existence of geodesics on a surface and to justify Dirichlet's principle. Inspired by these papers, further important contributions were given in [163], [55], [171], [111], [165]. Particularly relevant in our context are papers by Lebesgue [163], [165], where the importance of the concept of semicontinuity for the definition of the length of a curve and the area of a surface was pointed out.

Starting in 1911, in a series of papers Tonelli [256], [258], [260], [259] proved the existence of minimizers of one-dimensional variational problems (both in parametric and non-parametric form), and he applied his method to several specific questions such as the existence of periodic orbits [257], the stability of the equilibrium of a liquid mass under its molecular forces [262], and the minimum property of the sphere [264].[17] In his proofs Tonelli stressed the importance of the notion of semicontinuity. This notion also appeared in the papers of Lebesgue, but he and later authors established their existence results by using the fact that the Euler equation can be solved locally. As a consequence the essence of the direct methods and the reasons why they work remained hidden, and so the field of applications for these methods seemed to be rather limited. This fact was also observed by Hadamard [127], [126], who developed a steepest descent method for proving the existence of a minimizer, and by Goursat [125].

The very essence of the direct methods was first lucidly explained by Tonelli in his work on one-dimensional variational problems. Tonelli's fundamental paper [266] of 1915, anticipated by two short announcements [263] from 1914, is already written in the modern style, and it can be viewed as the second important step in the development of direct methods.

After the First World War Tonelli reviewed his theory [267], [268], and he gave a systematic exposition in his monograph [269]. He also described his ideas at several conferences, in particular at the International Congresses of Mathematicians in Toronto [272] and in Bologna [274], compare also [276], [277].

Later he tried to extend his results to two-dimensional problems [273], [275], still working with absolutely continuous functions, since he believed that this class of functions was the right tool to work with, and this probably was the main limitation of his theory. In fact, in order to have compactness he was forced to work with integrands having a growth larger than two with respect to the derivatives (because only in this case is continuity granted by Sobolev embedding theorems), although he could also handle certain limiting cases including Dirichlet's integral by means of so-called monotone function in the sense of Lebesgue, compare [165] and [273], [275].

An important turn took the development of the direct methods further in the work of C. B. Morrey [192], [190] where the space of absolute continuous functions was replaced by function spaces which are equivalent to Sobolev spaces. Morrey's work was the third decisive step in the development of direct methods. Using Tonelli's general ideas, Morrey formulated the modern *direct approach* to the existence and regularity of minimizers of multidimensional variational problems.

The semicontinuity and existence theorems of Chap. 3 appear in [269], see also [267]; in their presentation we have followed [276], [277], [192], and especially [193]. Manià's example appeared in [179], [178] following Lavrentiev's paper [161]; compare also the paper [59] which we have followed. Example $\boxed{2}$ is taken from Giaquinta *et al.* [114].

In [269] one also finds a proof of the Lipschitz regularity of minimizers of regular integrals. The step from Lipschitz (or actually from C^1) to C^2 is due to Hilbert.

[17]The most important contributions of Tonelli to the calculus of variations are collected in Vols 2 and 3 of his *Opere scelte* [278] and in his monograph [269].

The partial regularity theorem presented in Section 4.2 was discovered by Tonelli [269]. The proof we have given follows that of Ball–Mizel [21] which, apart from minor modifications and improvements, agrees with Tonelli's proof. Ball and Mizel were the first to show that the singular set E can be non-empty; further discussions of the singular set can be found in Clarke–Vinter [68], [70], [71], and in [76]; Proposition 4.11 of Section 4.3 is due to Davie [76]. An early result of a similar kind, but with a Lagrangian F satisfying only

$$|p| \le F(x, u, p) \le \text{const}(1 + |p|^2) \tag{6.7}$$

and

$$\frac{F(x, u, p)}{|p|} \to \infty \quad \text{as } |p| \to \infty, \tag{6.8}$$

was proved by Ball–Mizel [21].

We have already remarked that *under assumption* (6.7) *the class of absolutely continuous functions is by no means a suitable class to work with* since:

(a) limits of functions with equibounded gradients in L^1 are bounded variation functions, and derivatives of these functions are measures which in general have non-zero singular parts with respect to the Lebesgue measure;
(b) the existence of minimizers in AC may fail to be true. But even if a minimizer exists in AC, we would have to accept very pathological minimizers. For example, Davie [76] proved the following result.

Proposition 6.13 *Given a closed set $E \subset [a, b]$ of measure zero, a function $v \in C(0, 1)$ which is C^∞ outside E and satisfies $v' > 0$ and $v'(x) \to \infty$ as $dist(x, E) \to 0$, and a continuous function $\rho(x)$ on $[a, b]$ which is C^∞ outside E, we can find an integrand $F(u, p)$ such that*

$$F(u, p) \ge 0, \qquad F_{pp} > 0, \qquad c_0|p| \le F(u, p) \le c_1(1 + |p|^2)$$

$$F_p(v(x), v'(x)) = \rho(x) \quad \textit{for } x \in [a, b] - E$$

and v is a minimizer of $\mathcal{F}(u) := \int_a^b F(u, u')\, dx$.

In particular:

(a) If ρ is absolutely continuous, then Euler's equation will be satisfied in the sense that $F_z(v, v') \in L^1$ and $F_p(v, v')$ is its indefinite integral.
(b) If ρ is such that $\rho' \notin L^1$, then $F_z(v, v') \notin L^1$ and the Euler equation does not hold.
(c) If E is a Cantor set and ρ is the Cantor function then $F_z = 0$, but F_p is not constant.

For Lagrangians depending only on x, p one has the following; see Davie [76]. Consider the Lagrangian

$$F(x, p) := (1 + |p|^2)^{1/2} + x^2|p|^2, \quad x \in [-1, 1].$$

and the associated variational problem with the boundary conditions $u(1) = \alpha$, $u(-1) = -\alpha$ $(\alpha \geq 0)$, and notice that $F_{pp} = (1 + |p|^2)^{-3/2} + 2x^2 > 0$. Then there exists an $\alpha_1 > 0$ such that

(a) if $0 \leq \alpha \leq \alpha_1$, then $\mathcal{F}(u)$ has a unique minimizer; this minimizer is of class C^∞ if $\alpha < \alpha_1$, and it has a singularity at $x = 0$ if $\alpha = \alpha_1$;
(b) if $\alpha > \alpha_1$ there is no minimizer in AC with $u(1) = \alpha$, $u(-1) = -\alpha$.

Other examples can be found in [21], [76]. For further results we also refer to [22], [249], [250], [248].

6.7 Lavrentiev phenomenon

The first example of a gap between the infima of the values of a variational integral among smooth functions and absolutely continuous functions was given by Lavrentiev [161]; a simpler example was described later by Manià [178]. Conditions for excluding Lavrentiev's phenomenon were given by Angell [10], Loewen [176], and Clarke–Vinter [68]. Further investigations and examples can be found in Ball–Mizel [21], Davie [76], Heinricher–Mizel [138], [137].

We think that the following two facts are the reason why the gap phenomenon was found to be surprising and disturbing.

(i) Tonelli's strong belief that AC is the right space in which to minimize reasonable one-dimensional variational integrals; this conviction was and still is shared by many mathematicians.
(ii) The bad behaviour of many functionals with respect to weak convergence when they are extended (e.g. by means of Lebesgue's dominated convergence theorem).

More precisely, the occurrence of the gap phenomenon is caused by the fact that a reasonable functional defined for smooth functions might become a strange object if it is extended to absolutely continuous functions.

In order to have a complete picture of the Lavrentiev phenomenon, we want to present here some aspects of relaxation described by Buttazzo–Mizel [51], together with some further examples. In the abstract setting of relaxation we have two topological spaces X and Y, with Y dense in X, and a functional $\mathcal{F} : X \to (-\infty, \infty]$ which is lower semicontinuous with respect to the topology of X. Considering the restriction $\mathcal{F}|_Y$ of $\mathcal{F}$ to Y and its relaxation $\overline{\mathcal{F}}|_Y$ defined by

$$\overline{\mathcal{F}}|_Y := \max\{\mathcal{G} : X \to (-\infty, +\infty] : \mathcal{G} \text{ is } X\text{-lower semicontinuous}, \mathcal{G} \leq \mathcal{F} \text{ on } Y\}$$

we immediately have the inequality

$$\mathcal{F} \;\leq\; \overline{\mathcal{F}}|_Y \quad \text{on } X.$$

Thus we can write

$$\overline{\mathcal{F}}|_Y \;=\; \mathcal{F} + \mathcal{L}$$

where the functional $\mathcal{L} \geq 0$ is called the *Lavrentiev gap* associated with $\mathcal{F}$ and to X, Y. Note that $\mathcal{L}(u)$ makes sense only if $\mathcal{F}(u) < +\infty$, and we shall say that the Lavrentiev phenomenon is absent if the functional $\mathcal{L}$ vanishes identically. Since $\overline{\mathcal{F}}\big|_Y \le \mathcal{F}$ on Y we clearly have $\mathcal{L}(u) = 0$ for every $u \in Y$, but we may have $\mathcal{L}(u) > 0$ for some $u \in X \setminus Y$; in this case we say that $\mathcal{F}$ shows the Lavrentiev phenomenon between Y and X.

By the general theory of relaxed problems (see Buttazzo [47]) we have the relaxation equality

$$\begin{aligned}\inf\{\mathcal{F}(y) \colon y \in Y\} &= \inf\{\overline{\mathcal{F}}\big|_Y(x) \colon x \in X\}\\ &= \inf\{\mathcal{F}(x) + \mathcal{L}(x) \colon x \in X\}.\end{aligned}$$

Thus

$$\inf\{\mathcal{F}(y) \colon y \in Y\} = \inf\{\mathcal{F}(x) \colon x \in X\} \quad \text{if } \mathcal{L}(u) \equiv 0.$$

Consider functionals $\mathcal{F}$ of the form

$$\mathcal{F}(u) = \int_0^1 F(x, u, u')\,dx$$

defined on the space $H^{1,1}(0,1)$, where $F : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an integrand with the following properties:

(i) F is a Carathéodory function (i.e. $F(x, z, p)$ is measurable in x and continuous in (z, p));

(ii) $F(x, z, \cdot)$ is convex on $\mathbb{R}$ for every $(x, z) \in \Omega \times \mathbb{R}$;

(iii) $F(x, z, 0) = 0$ for every $(x, z) \in \Omega \times \mathbb{R}$;

(iv) there exists a function $\omega : \Omega \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ with $\omega(x, r, t)$ integrable in x and increasing in r and t such that

$$0 \le F(x, z, p) \le \omega(x, |z|, |p|) \qquad \text{for every } (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}.$$

To employ the abstract scheme above we denote by X the space of all functions $u \in H^{1,1}(0,1)$ with $u(0) = 0$, and by Y the space of all Lipschitz-continuous functions u with $u(0) = 0$. The following result has been obtained in Buttazzo–Mizel [51].

Theorem 6.14 *There exists a function $W : (0,1) \times \mathbb{R} \to \mathbb{R}$ such that the Lavrentiev gap functional $\mathcal{L}$ associated with $\mathcal{F}$ is given by the formula*

$$\mathcal{L}(u) = \liminf_{x \to 0} W(x, u(x))$$

for every $u \in H^{1,\infty}_{loc}((0,1]) \cap H^{1,1}(0,1)$ with $u(0) = 0$.

Remark 1 Here $H^{1,\infty}_{loc}((0,1])$ denotes the space of all functions which are Lipschitz continuous on every interval $[\delta, 1]$ with $\delta > 0$. In other words we deal with functions u which are singular only in one point (which for simplicity we choose to be the origin). The problem of representing $\mathcal{L}(u)$ for an arbitrary $u \in H^{1,1}(0,1)$ with $u(0) = 0$ is, as far as we know, still open.

The function W of Theorem 6.14 is given by

$$W(x,s) = \liminf_{t\to s} V(x,t)$$

where V is the *value function*

$$V(x,t) := \inf\left\{\int_0^x F(y,u,u')dy : u \in H^{1,\infty}(0,x),\, u(0)=0,\, u(x)=t\right\}.$$

As an example we now discuss Lagrangians of Manià type:

$$\text{(1)} \qquad F_{\alpha,\beta,m}(x,u,p) = (u|u|^{\alpha-1} - x^\beta)^2|p|^m$$

with $\alpha > \beta > 0$ and $m > 1$ (note that the original Manià integrand was $F_{3,1,6}$). According to the definition of Heinricher–Mizel [137] the integrand F is called homogeneous if

$$F(x,z,p) = tF(tx, t^\gamma z, t^{\gamma-1}p)$$

for every (x,z,p) and for every $t > 0$, where γ is a suitable number in $(0,1)$. Then case (1) is homogeneous if

$$m = \alpha\frac{1+2\beta}{\alpha-\beta}.$$

Therefore we distinguish in our discussion three cases: the *subhomogeneous*, the *homogeneous*, and the *superhomogeneous case*.

In the subhomogeneous case $m < \alpha(1+2\beta)/(\alpha-\beta)$ the gap functional $\mathcal{L}$ is identically zero and the Lavrentiev phenomenon does not occur. Indeed, every admissible function u can be approximated by the Lipschitz-continuous functions

$$u_\epsilon(x) = \begin{cases} u(x) & \text{if } x > \epsilon \\ u(\epsilon)x/\epsilon & \text{if } x \le \epsilon \end{cases}$$

for which we have

$$\lim_{\epsilon\to 0} \mathcal{F}(u_\epsilon) = \mathcal{F}(u).$$

In the homogeneous case $m = \alpha(1+2\beta)/(\alpha-\beta)$ we obtain

$$W(x,s) = K^{m-1}\left(\frac{|s|}{x^\gamma}\right)^m\left[\frac{m}{2\alpha+m}\left(\frac{|s|}{x^\gamma}\right)^{2\alpha} + 1 - \frac{2m}{\alpha+m}\left(\frac{|s|}{x^\gamma}\right)^\alpha\right]$$

where

$$K = 1 - \frac{2\beta}{m-1}, \qquad \gamma = 1 - \frac{1+2\beta}{m}.$$

For instance, on the function $u(x) = x^{\beta/\alpha}$ we get

$$\mathcal{L}(u) = K^{m-1}\frac{2\alpha^2}{(\alpha+m)(2\alpha+m)}.$$

Note that, even if the integrands $F_{q\alpha,q\beta,m}$ all vanish on the same function $u(x) = x^{\beta/\alpha}$, the corresponding gap functionals, evaluated on this function, are different.

In the superhomogeneous case $m > \alpha(1+2\beta)/(\alpha-\beta)$ we have

$$W(x,s) = K^{m-1}\left(\frac{|s|}{x^\gamma}\right)^m$$

where K and γ are as above. For instance, on the function $u(x) = x^{\beta/\alpha}$ we get $\mathcal{L}(u) = +\infty$.

Another class of integrands for which the explicit computation of the function W representing the gap $\mathcal{L}$ is possible is given by the functions

$$F_{q,m}(x,z,p) = |z - x^q||p|^m$$

with $0 < q < 1$. Here, the homogeneous case occurs when $m = (1+q)/(1-q)$, and we have the following situation (see Belloni [30]).

In the subhomogeneous case $m < (1+q)/(1-q)$ we infer by an argument similar to the previous one that the gap functional $\mathcal{L}$ is identically zero; hence the Lavrentiev phenomenon does not occur.

In the homogeneous case $m = (1+q)/(1-q)$ we have

$$W(x,s) = \begin{cases} \left(\dfrac{m}{m+1}\right)^m \left(\dfrac{|s|}{x^q}\right)^m \left[\dfrac{m+1}{m} - \dfrac{|s|}{x^\gamma}\right] & \text{if } s \le x^q \\[2ex] \left(\dfrac{m}{m+1}\right)^m \left[\dfrac{2}{m} + \left(\dfrac{|s|}{x^q}\right)^{m+1} - \dfrac{m+1}{m}\left(\dfrac{|s|}{x^q}\right)^m\right] & \text{if } s > x^q. \end{cases}$$

In the superhomogeneous case $m > (1+q)/(1-q)$ we have

$$W(x,s) = K^{m-1}\left(\frac{|s|}{x^\gamma}\right)^m$$

where

$$K = 1 - \frac{q}{m-1}, \qquad \gamma = 1 - \frac{q+1}{m}.$$

The problem of deciding when the approximation of a minimizer is possible by means of a minimizing sequence of Lipschitz functions was considered in the first paper of Lavrentiev [161] where he found that the following condition is sufficient for excluding the Lavrentiev phenomenon:

For every $r > 0$ there exists $c_r > 0$ such that for every $(x, z, p) \in (0, 1) \times [-r, r] \times \mathbb{R}$

$$\left|F_z(x, z, p)\right| \le c_r$$

(the convergence used was the uniform convergence). In the same year Tonelli [277] found a more general condition, still sufficient for $\mathcal{L} \equiv 0$:

$F(x, z, p) = G(x, z, p) + H(x, z, p)$ where G satisfies condition (α) below and H satisfies one of conditions (β) or (γ) below:

(α) *for every $r > 0$ there exists some $c_r > 0$ such that for every $(x, z, p) \in (0, 1) \times [-r, r] \times \mathbb{R}$*

$$\left|G(x, z, p)\right| \le c_r(1 + |p|);$$

(β) *for every $r > 0$ there exists some $c_r > 0$ such that for every $(x, z, p) \in (0, 1) \times [-r, r] \times \mathbb{R}$*

$$\left|H_z(x, z, p)\right| \le c_r(1 + |p|);$$

(γ) *for every $r > 0$ there exists some $c_r > 0$ such that for every $(x, z, w, p) \in (0, 1) \times [-r, r] \times [-r, r] \times \mathbb{R}$*

$$\left|H_z(x, z, p)\right| \le c_r(1 + \left|H_z(x, w, p)\right|).$$

In Clarke–Vinter [68] several cases have been considered where solutions of the problem

$$\min\left\{\int_0^1 F(x, u, u')\,dx : u \in H^{1,1}(0, 1), u(0) = a, u(1) = b\right\}$$

are Lipschitz continuous, under the following conditions:

(i) *$F(x, z, p)$ is locally bounded in (x, z, p) and measurable in x;*

(ii) *$F(x, z, p)$ is locally Lipschitz in (z, p) uniformly in x; that is, for every $r > 0$ there exists some $c_r > 0$ such that for every $x \in (0, 1)$ and every $|z_1|, |z_2|, |p_1|, |p_2| \le r$ we have*

$$\left|F(x, z_1, p_1) - F(x, z_2, p_2)\right| \le c_r(|z_1 - z_2| + |p_1 - p_2|);$$

(iii) *$F(x, z, p)$ is convex in p;*

(iv) *there exists a superlinear function θ (i.e. $\theta(r)/r \to +\infty$ as $r \to \infty$) such that*

$$F(x, z, p) \ge \theta(|p|).$$

We list some cases where minimizers are Lipschitz continuous.

(2) $F_x(x, u(x), u'(x)) \in L^1(0, 1)$;

(3) there exist $c > 0$ and $a \in L^1(0, 1)$ such that

$$\left|F_z(x, u(x), u'(x))\right| \le c\left|F_p(x, u(x), u'(x))\right| + a(x);$$

(4) Condition of Bernstein type: there exists $a \in L^1(0, 1)$ such that the expression

$$\Phi = \frac{F_x - F_{zp} - pF_{zp}}{F_{pp}},$$

satisfies

$$\left|\Phi(x, u(x), u'(x))\right| \le a(x)(1 + |u'(x)|).$$

One can prove (compare Clarke–Vinter [68]) that conditions (2), (3), (4) respectively are implied by the following assumptions:

(5) for every $r > 0$ there exist $c_r > 0$ and $a_r \in L^1(0, 1)$ such that for every $(x, z, p) \in (0, 1) \times [-r, r] \times \mathbf{R}$ we have

$$|F_x(x, z, p)| \le c_r|F(x, z, p)| + a_r(x);$$

(6) for every $r > 0$ there exist $c_r > 0, d_r > 0, a_r \in L^1(0, 1)$ such that for every $(x, z, p) \in (0, 1) \times [-r, r] \times \mathbf{R}$

$$|F_z(x, z, p)| \le c_r|F(x, z, p)| + d_r|F_p(x, z, p)| + a_r(x);$$

(7) there exist $\alpha \ge 1, k > 0$, $g : \mathbf{R} \to \mathbf{R}$ continuous, and for every $r > 0$ a positive constant c_r such that for every $(x, z, p) \in (0, 1) \times [-r, r] \times \mathbf{R}$ we have

$$F(x, z, p) \ge g(z) + k|p|^\alpha$$
$$|F(x, z, p)| \le c_r(1 + |p|^{1+\alpha}).$$

For integrands of the form $F(x, p)$ (which do not depend on z) we have the following proposition.

Proposition 6.15 *If $F : (0, 1) \to \mathbf{R} \times [0, \infty]$ is a Borel function such that*

(i) *$F(x, \cdot)$ is convex and lower semicontinuous for almost all $x \in (0, 1)$,*
(ii) *there exists $u_0 \in H^{1,\infty}$ such that $F(x, u_0') \in L^1$, then there is no Lavrentiev gap between $H^{1,1}$ and $H^{1,\infty}$.*

Proof We reduce our discussion to the case $u_0 = 0$ by considering the function

$$G(x, p) = F(x, p + u_0'(x)).$$

In order to show that the Lavrentiev gap functional $\mathcal{L}$ is identically zero, we have to prove that for every weakly lower semicontinuous functional $\mathcal{G} : H^{1,1}(0, 1) \to [0, +\infty]$

dominated by $\mathcal{F}$ from above on $H^{1,\infty}$ we have $\mathcal{G} \le \mathcal{F}$. To this end we fix an arbitrary $u \in H^{1,1}(0,1)$ and define for $n \in \mathbb{N}$

$$w_n(x) := \max\{-n, \min\{u'(x), n\}\},$$
$$u_n(x) := u(0) + \int_0^x w_n(t)\,dt.$$

We have that $u_n \in H^{1,\infty}(0,1)$, $u_n(0) = u(0)$, and $u_n \to u$ strongly in $H^{1,1}$; indeed, for a suitable constant $C > 0$

$$\|u_n - u\|_{H^{1,1}} \le C\|u_n' - u'\|_{L^1} \le C\int_{\{|u'|>n\}} (n + |u'|)\,dx$$
$$\le 2C\int_{\{|u'|>n\}} |u'|\,dx,$$

and the last integral tends to zero as $n \to \infty$ because of $u \in H^{1,1}$.

Since $F(x,\cdot)$ is convex we have

$$\mathcal{G}(u) \le \liminf_{n\to\infty} \mathcal{F}(u_n)$$
$$= \liminf_{n\to\infty}\left\{\int_{\{|u'|\le n\}} F(x,u')\,dx + \int_{\{u'>n\}} F(x,n)\,dx + \int_{\{u'<-n\}} F(x,-n)\,dx\right\}$$
$$\le \liminf_{n\to\infty}\left\{\int_{\{|u'|\le n\}} F(x,u')\,dx + \int_{\{|u'|>n\}}\left[\frac{n}{|u'|}F(x,u')\right.\right.$$
$$\left.\left.+\left(1-\frac{n}{|u'|}\right)F(x,0)\right]dx\right\}$$
$$\le \int_0^1 F(x,u')\,dx + \liminf_{n\to\infty}\int_{\{|u'|>n\}} F(x,0)\,dx.$$

By the integrability of $F(x,0)$ and the fact that $\operatorname{meas}\{|u'| > n\} \to 0$ as $n \to \infty$ we obtain $\mathcal{G}(u) \le \mathcal{F}(u)$ as required. □

In the autonomous case $F = F(z,p)$ the following general approximation result has been obtained by Alberti–Serra Cassano [6].

Theorem 6.16 *Let $F : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty]$ be a Borel function, and assume that for every $r > 0$ there exist $c_r > 0$ and $M_r > 0$ such that for $|z| \le r$ and $|p| \le c_r$ we have*

$$F(z,p) \le M_r.$$

Then, for every $m \in [1, \infty)$ and every $u \in H^{1,m}(I, \mathbb{R}^N)$, $I = (0, 1)$, there exists a sequence $u_n \in H^{1,\infty}(I, \mathbb{R}^N)$ which converges to u in the $H^{1,m}$-norm and approximates u in energy, i.e.

$$\int_0^1 F(u_n, u_n')\,dx \;\to\; \int_0^1 F(u, u')\,dx \quad as\, n \to +\infty.$$

Moreover, if the Lagrangian F is bounded on bounded sets, the approximating sequence can be taken in $C^1(\overline{I}, \mathbb{R}^N)$. Finally, if F is continuous, the approximating sequence can be taken in $C^\infty(\overline{I}, \mathbb{R}^N)$.

Theorem 6.16 can be generalized as follows (see Remark 2.9 of [6]). Let D be a subset of $\mathbb{R}^N$, let $\mathbf{V}$ be a family of mappings from $[0, 1]$ to $\mathbb{R}^N$, and let k, m be positive real numbers: we say that $\mathbf{V}$ (k, σ)-connects D when

– all functions $v \in \mathbf{V}$ are k-Lipschitz continuous;
– all functions $t \mapsto F(v(t), v'(t))$ with $v \in \mathbf{V}$ are uniformly integrable on $[0, 1]$;
– for every $y_1, y_2 \in D$ there exist $v \in \mathbf{V}$ and $x_1, x_2 \in [0, 1]$ such that $y_1 = v(x_1)$, $y_2 = v(x_2)$, and $|x_1 - x_2| \le \sigma |y_1 - y_2|$.

Then we have the following generalization of Theorem 6.16 (see Alberti–Serra Cassano [6]):

Let u be a function which belongs to $H^{1,m}(0, 1; \mathbb{R}^N)$ for some $m \in [1, \infty)$, and assume that there exists a family $\mathbf{V}$ which (k, σ)-connects the image of u for some k, σ. Then u can be approximated in energy by a sequence of Lipschitz functions which converge to u in the $H^{1,m}$-norm.

This statement may be applied to functionals of the form

$$\mathcal{F}(u) = \int_0^1 F(u, u')\,dx$$

with a constraint of the form $u \in T$, where

$$T := \{u \in H^{1,m}(I; \mathbb{R}^N) : u(t) \in M \quad \text{for all } t \in [0, 1]\},$$

and M is a closed Lipschitz submanifold of $\mathbb{R}^N$; then our autonomous functional assumes the value $+\infty$ if $u \notin T$, but the approximation result above still holds true. We want to point out that for Lagrangians of the kind $F(z, p)$ a constraint of the type $\{u \ge 0\}$ does not affect the approximation by regular functions, while for autonomous second-order integrands a constraint of this type may produce a gap (see Cheng–Mizel [61]).

6.8 The vibrating string problem

In 1747 D'Alembert found the general solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{6.9}$$

in the form

$$\varphi(x+at)+\psi(x-at).$$

Shortly afterwards, in 1753, Daniel Bernoulli treated the same problem by an entirely different method. Earlier, Taylor had observed that the functions

$$\sin\frac{n\pi x}{l}\cos\frac{n\pi a(t-\beta)}{l}\qquad n=1,2,\ldots$$

are solutions of the differential equation (6.9) which also satisfy the boundary conditions

$$u(0,t)=0\qquad u(l,t)=0.$$

This led Bernoulli to the conclusion that each solution of (6.9) could be written as the superposition of tones. Euler did not accept Bernoulli's solution as he could not see how such a sum of analytic functions could produce arbitrary initial data. This question was clarified by the work of Fourier and the subsequent work of Dirichlet, Riemann, and many other mathematicians at the beginning of this century.

The concepts and methods developed for the vibrating string problem proved to be of great importance for many other problems in physics, in particular the idea that the motion of a physical system near an equilibrium configuration can be obtained as a superposition of eigenvibrations. To describe this idea, following Lagrange we analyse a system with finitely many degrees of freedom.

In this case the potential energy with respect to generalized coordinates is given by a function $U(q)$, which we may assume to be zero at $q=0$, the equilibrium position. If this equilibrium position is stable, we can represent the potential energy up to higher-order terms as a *positive definite* quadratic form

$$U=\sum a_{ik}q_iq_k.$$

Similarly the kinetic energy is expressed by a positive definite quadratic form

$$T=\sum b_{ik}(q)\dot{q}_i\dot{q}_k,$$

and, up to higher-order terms, we can even think of the $b_{ik}(q)$ as constants. In this case the Euler equations

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_i}+\frac{\partial U}{\partial q_i}=0$$

take the form

$$B\ddot{q}+Aq=0\tag{6.10}$$

where A and B respectively denote the matrices (a_{ik}) and (b_{ik}). Replacing the coordinates q_j by suitable new Cartesian coordinates x_j we can transform T and U simultaneously to the diagonal forms

$$T = \dot{x}_1^2 + \cdots + \dot{x}_n^2,$$
$$U = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2.$$

Then eqns (6.10) reduce to the n harmonic oscillators

$$\ddot{x}_i + \lambda_i x_i = 0 \quad i = 1, \ldots, n.$$

Thus we see that the motion near a stable equilibrium point can be obtained as a finite sum of eigenvibrations. The systematic introduction of eigenvibrations into physics is described in the treatises of Thompson–Tait, Rayleigh, and Courant–Hilbert.

For a detailed historical account we refer to Truesdell [281] and to Cannon–Dostrovski [54].

The *variational approach to eigenvalue problems* goes back to Rayleigh and Fischer. Important applications of this ideas are due to W. Ritz, H. Weyl, and later to R. Courant. A systematic presentation of pertinent ideas can be found in the treatise of Courant–Hilbert [74]. For a modern approach with applications to differential geometry we refer to Chavel [60].

Best *constants in inequalities and estimates of eigenvalues* play an important role in physics as well as in geometry. The interested reader is referred for instance to [136], [212], [29], and [60].

6.9 Variational inequalities and the obstacle problem. Non-coercive problems

Variational problems with inequalities as constraints were investigated by Steiner, Weierstrass, Bolza, and Hadamard. The systematic study of variational inequalities began with the fundamental paper by Fichera [100] on Signorini's problem in 1964 and the work on monotone operators by G. J. Minty [187], [188] in 1962 and 1963. Further important contributions in the 1960s were made by G. Stampacchia, J. L. Lions–G. Stampacchia, F. E. Browder, H. Lewy, H. Lewy–G. Stampacchia, H. Brézis–G. Stampacchia, and J. C. C. Nitsche. In the following two decades the subject was thoroughly investigated. We refer the interested reader to the treatises of D. Kinderlehrer–G. Stampacchia [154] and A. Friedman [106] for further information.

A general theory of non-coercive problems, involving the recession functional associated with the total energy of the system, was developed by Baiocchi *et al.* in [19], together with several applications to problems from continuum mechanics. We refer the interested reader to this paper as well as to the related articles [18], [52], [253], [218] for details.

6.10 Periodic solutions

Our treatment of the important and fascinating topic of periodic solutions of Hamiltonian systems is rather inadequate, just as the treatment of the other items in Chap. 5 is fairly casual. In fact, Chap. 5 is no more than a collection of examples meant to illustrate the general theory presented in the preceding chapters. Nevertheless we should like to refer the reader also to the classical treatises of Poincaré [211], Birkhoff [34], Siegel–Moser [239], and to the encyclopedia article by Arnold *et al.* [11]. Despite its age the research on periodic solutions is very much alive and experiencing rapid progress. Problems involving singular potentials such as the Newtonian potential are of particular interest, and many questions still remain to be answered.

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One-dimensional Variational Problems
An Introduction

Giuseppe Buttazzo, Mariano Giaquinta and Stefan Hildebrandt

One-dimensional variational problems have been somewhat neglected in the literature on calculus of variations, as authors usually treat minimum problems for multiple integrals which in turn lead to partial differential equations which are considerably more difficult to handle. One-dimensional problems are connected with ordinary differential equations, and hence need many fewer technical prerequisites, but they exhibit the same kind of phenomena and surprises as variational problems for multiple integrals.

This book provides a modern introduction to this subject, placing special emphasis on direct methods. It combines the efforts of a distinguished team of authors who are all renowned mathematicians and expositors. Since there are fewer technical details, graduate students who want an overview of the modern approach to variational problems will be able to concentrate on the underlying theory and hence gain a good grounding in the subject. Except for results from the theory of measure and integration and from the theory of convex functions, the text develops all the required mathematical tools, including the basic results on one-dimensional Sobolev spaces, absolutely continuous functions, and functions of bounded variation.

Giuseppe Buttazzo and **Mariano Giaquinta** are both Professors of Mathematics at the University of Pisa. **Stefan Hildebrandt** is Professor of Mathematics at the University of Bonn.

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