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SOME NEW FUNCTIONAL SPACES

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The present paper is devoted to the study of two new types of Banach spaces $\Lambda(\alpha)$, $\mathrm{M}(\alpha)$, $0 < \alpha < 1$, whose elements are integrable functions over an interval (or rather classes of functions; as usual, we identify functions equal almost everywhere). To a certain extent these spaces are akin to the F. Riesz's spaces L_p; there is a correspondence between $\Lambda(\alpha)$ and L_p for $p = \alpha^{-1}$ as well as between $\mathrm{M}(\alpha)$ and L_p, $p = (1 - \alpha)^{-1}$. The conjugate to $\Lambda(\alpha)$ is $\mathrm{M}(\alpha)$; but the conjugate to $\mathrm{M}(\alpha)$ proves to be not $\Lambda(\alpha)$, and so our spaces are not reflexive. §1 is devoted to main properties of the spaces $\Lambda(\alpha)$, $\mathrm{M}(\alpha)$ and to their relations to each other and to the spaces L_p. Some more general spaces are defined. In §2 we investigate linear functionals in $\Lambda(\alpha)$, $\mathrm{M}(\alpha)$ and find their general form in $\Lambda(\alpha)$; the inequality 2.1(1), proved here, plays a fundamental rôle in our theory. In the last three sections, §§3–5, we are concerned with applications to Fourier series, to the integration of fractional order and to the moment problem for a finite interval.

§1. The spaces $\Lambda(\alpha)$, $\mathrm{M}(\alpha)$ and their properties

1.1. For simplicity, we confine our attention to the interval (0, 1); our results will be valid for any finite interval $(0, a)$ as well (and, with some precautions, described in 4.1, also for infinite range). A real measurable function $f(x)$, $0 < x < 1$, belongs to $\Lambda(\alpha)$, $0 < \alpha < 1$, if and only if

$$\text{1.1(1)} \qquad \|f\|_{\Lambda(\alpha)} = \|f\| = \alpha \int_0^1 x^{\alpha-1} f^*(x)\, dx$$

is finite. Here and in the sequel we write $f^*(x)$ for the rearrangement of $|f(x)|$ in decreasing order, that is for the function $f^*(x)$ in $0 < x < 1$, which is decreasing (in the wider sense) and equimeasurable with $|f(x)|$. (For these notions see Hardy, Littlewood, Pólya [8, pp. 276–279], Zygmund [17] and Lorentz [14]).

A measurable function $f(x)$ belongs to $\mathrm{M}(\alpha)$, $0 < \alpha < 1$, if

$$\text{1.1(2)} \qquad \|f\|_{\mathrm{M}(\alpha)} = \|f\| = \sup_e \left\{ (me)^{-\alpha} \int_e |f(x)|\, dx \right\} < +\infty;$$

here e signifies an arbitrary measurable subset of (0, 1). For the excluded value $\alpha = 1$ we would obtain for $\Lambda(1)$ and $\mathrm{M}(1)$ the spaces L_1 and M (Banach [1]). We define the sum $f + g$ of two elements of $\Lambda(\alpha)$ or $\mathrm{M}(\alpha)$ and the product af of a real number a and an element in the natural way, and the norm $\|f\|$ by 1.1(1) or 1.1(2). Then we have

Theorem 1. *$\Lambda(\alpha)$ and $\mathrm{M}(\alpha)$ for $0 < \alpha < 1$ are Banach spaces.*

We first show that $\Lambda(\alpha)$ and $\mathrm{M}(\alpha)$ are linear normed spaces; for this purpose it will be sufficient to prove, that with every pair f, g the sum $f + g$ also belongs to $\Lambda(\alpha)$ and to $\mathrm{M}(\alpha)$ respectively and that

$$1.1(3) \qquad \| f + g \| \leqq \| f \| + \| g \| .$$

By 1.1(2), this is immediate for $\mathrm{M}(\alpha)$; and for $\Lambda(\alpha)$ we use the inequality

$$1.1(4) \qquad \int_0^1 h(x)[f(x) + g(x)]^* \, dx \leqq \int_0^1 h(x)f^*(x) \, dx + \int_0^1 h(x)g^*(x) \, dx.$$

Here f, g are arbitrary and h is decreasing and positive (in the wider sense). Putting $h(x) = \alpha x^{\alpha-1}$, we obtain 1.1(3) for $\Lambda(\alpha)$.

To prove 1.1(4), observe first that for a step function $h_c(x)$ of the type $h_c(x) = 1$ for $0 < x < c$, $h_c(x) = 0$ for $c \leqq x < 1$, $0 \leqq c \leqq 1$, 1.1(4) is

$$\int_0^c (f + g)^* \, dx \leqq \int_0^c f^* \, dx + \int_0^c g^* \, dx;$$

and this is true, since, by definition of the rearrangement $(f + g)^*$,

$$\int_0^c (f + g)^* \, dx = \int_e | f + g | \, dx$$

for some measurable set $e \subset (0, 1)$ with $me = c$. Returning to 1.1(4) we see that this inequality is likewise true for any decreasing positive step function h, every such h being a linear combination with positive coefficients of functions h_c. Finally, if h is any decreasing function, we can choose an increasing sequence of decreasing step functions h_n, and making $n \to \infty$ we obtain 1.1(4) by Lebesgue's theorem.

We shall now prove that $\Lambda(\alpha)$ is complete. Every $f \in \Lambda(\alpha)$ is an integrable function, $f \in L_1$. Write $\| f \| = \int_0^1 | f | \, dx$ for $f \in L_1$, and $\phi_\delta(f) = \alpha \int_\delta^1 x^{\alpha-1} f^* dx$. $\phi_\delta(f)$ is a continuous function in the space L_1, increasing for $\delta \to 0+$, and therefore $\phi(f) = \alpha \int_0^1 x^{\alpha-1} f^* dx = \lim_{\delta \to 0} \phi_\delta(f)$ is a lower semicontinuous function in L_1, with values $0 \leqq \phi \leqq +\infty$. Clearly, $\| f \| \leqq \alpha^{-1} \phi(f)$.

Suppose now that $f_n \in \Lambda(\alpha)$ and that $\phi(f_n - f_m) \to 0$ for $n, m \to \infty$. Then $\| f_n - f_m \| \to 0$, and so a function $f \in L_1$ exists such that $\| f_n - f \| \to 0$. For every $\varepsilon > 0$ there is a p for which $\phi(f_n - f_m) < \varepsilon$ if $n, m \geqq p$. By the lower semi-continuity of ϕ,

$$\phi(f_n - f) \leqq \varliminf_{m \to \infty} \phi(f_n - f_m) \leqq \varepsilon, \qquad n \geqq p.$$

We therefore have $f_n \to f$ in the metric of $\Lambda(\alpha)$, which proves the result. The proof for $\mathrm{M}(\alpha)$ is similar, and our theorem is complete.

1.2. Theorem 2. *The spaces* $\Lambda(\alpha)$ *are separable, while the spaces* $\mathrm{M}(\alpha)$ *are not.*

For $f \in \Lambda(\alpha)$ we put $f_n(x) = f(x)$ if $| f(x) | \leqq n$, $= n$ if $f(x) > n$ and $= -n$ if $f(x) < -n$. Then $f_n(x)$ is bounded and we have $\| f_n - f \|_{\Lambda(\alpha)} \to 0$. We thus may confine our attention on bounded $f \in \Lambda(\alpha)$. For uniformly bounded functions

$\|f_n - f\|_{\Lambda(\alpha)} \to 0$ is equivalent to $\|f_n - f\|_{L_1} \to 0$. From the known facts about L_1 we deduce that the two subsets of $\Lambda(\alpha)$: (a) the set of all linear combinations of characteristic functions $f_e(x) = 1$ for $x \in e$, $f_e(x) = 0$ for $x \bar{\epsilon}\, e$ and (b) the set of all polynomials with rational coefficients are everywhere dense in $\Lambda(\alpha)$. Thus $\Lambda(\alpha)$ is separable.

To prove the second part of the theorem, we let correspond to every real $0 < \theta \leqq 1$ the dyadic expansion with an infinity of 1's $\theta = 0, \theta_1 \theta_2 \cdots$ and the function

$$f_\theta(x) = 2^{\nu(1-\alpha)}\theta_\nu, \qquad 2^{-\nu} \leqq x < 2^{-(\nu-1)}, \qquad \nu = 1, 2, \cdots.$$

Then $f_\theta(x) \leqq x^{\alpha-1}$ and if $e \subset (0, 1)$,

$$\int_e f_\theta(x)\, dx \leqq \int_0^{me} x^{\alpha-1}\, dx = \alpha^{-1}(me)^\alpha.$$

Thus all of the f_θ belong to $\mathrm{M}(\alpha)$. On the other hand, they are at a distance $\geqq 1$ from each other, for if $\theta' \neq \theta$, then $|\theta'_n - \theta_n| = 1$ for some n and therefore

$$\|f_{\theta'} - f_\theta\| \geqq 2^{n\alpha} \int_{2^{-n}}^{2^{-(n-1)}} |f_{\theta'} - f_\theta|\, dx = 2^{n\alpha} 2^{-n} 2^{n(1-\alpha)} = 1.$$

The $f_\theta(x)$ forming a non enumerable set, the space $\mathrm{M}(\alpha)$ is not separable.

1.3. We investigate into the relations among the spaces $\Lambda(\alpha)$, $\mathrm{M}(\alpha)$, L_p. Consider two linear normed spaces X, Y, whose elements are measurable functions on (0, 1). It is convenient to write

1.3(1) $$X \prec Y,$$

if every function $f \in X$ also belongs to Y and if, for some constant K,

$$\|f\|_Y \leqq K\|f\|_X$$

for all $f \in X$. We have for instance,

1.3(2) $$\Lambda(\alpha) \prec \Lambda(\alpha'), \qquad \mathrm{M}(\alpha') \prec \mathrm{M}(\alpha) \quad \text{for} \quad \alpha < \alpha'.$$

THEOREM 3. *For every* $0 < \alpha < 1$,

1.3(3) $$\Lambda(\alpha) \prec L_{\alpha^{-1}},$$

1.3(4) $$L_{\alpha^{-1}} \prec \mathrm{M}(1 - \alpha).$$

1.3(4) is easily deduced from Hölder's inequality. For the proof of 1.3(3) we need the inequality

1.3(5) $$\int_0^1 x^{p-1} f(x)^p\, dx \leqq K_p \left\{\int_0^1 f(x)\, dx\right\}^p, \qquad p \geqq 1,$$

where $f(x)$ is positive and decreasing, the constant K_p depending only on p. For sums instead of integrals and with $K_p = p^{-1}$ this has been proved by Hardy, Littlewood and Pólya [9], [8, p. 100]; for convenience of the reader we insert a simple proof for integrals which does not claim to obtain the best value of K_p.

Let p be first an integer; then, with the aid of polar coordinates in the p-dimensional space,

$$\left\{\int_0^1 f(x)\,dx\right\}^p = \int_0^1 dx_1 \cdots \int_0^1 f(x_1) \cdots f(x_p)\,dx_p$$

$$\geqq \int_0^{\pi/2} d\vartheta_1 \cdots \int_0^{\pi/2} d\vartheta_{p-1} \int_0^1 f(r \sin\vartheta_1 \cdots \sin\vartheta_{p-1}) f(r\cos\vartheta_1 \sin\vartheta_2 \cdots \sin\vartheta_{p-1})$$

$$\cdots f(r\cos\vartheta_{p-1}) r^{p-1} \sin\vartheta_2 \cdots \sin^{p-2}\vartheta_{p-1}\,dr$$

$$\geqq \int_0^{\pi/2} \cdots \int_0^{\pi/2} \sin\vartheta_2 \cdots \sin^{p-2}\vartheta_{p-1}\,d\vartheta_1 \cdots d\vartheta_{p-1} \int_0^1 r^{p-1} f(r)^p\,dr$$

$$= K_p' \int_0^1 x^{p-1} f(x)^p\,dx,$$

which is equivalent to 1.3(5). For an arbitrary $p \geqq 1$ choose an integer $p_0 > p$ and apply Hölder's inequality, with the exponents $r = (p_0 - 1)/(p - 1)$ and $s = (p_0 - 1)/(p_0 - p)$, to the right hand side of

$$x^{p-1} f(x)^p = (x^{p_0-1} f^{p_0})^{r^{-1}} f^{s^{-1}}.$$

To prove 1.3(3), put $p = \alpha^{-1}$. For any $f \in \Lambda(\alpha)$, the function $g(x) = x^{\alpha-1} f^*(x)$ is decreasing. We therefore have

$$\int_0^1 |f|^p\,dx = \int_0^1 f^{*p}\,dx = \int_0^1 x^{(1-\alpha)p} g^p\,dx = \int_0^1 x^{p-1} g^p\,dx$$

$$\leqq K_p \left\{\int_0^1 g\,dx\right\}^p = K_{\alpha^{-1}} \alpha^{-p} \, \| x \|_{\Lambda(\alpha)}^p,$$

which completes the proof.

Counter-examples for Theorem 3 are provided by the functions $f_1 = x^{-\alpha} \log^{-1}\frac{1}{x}$, $f_2 = x^{-\alpha}$. We have, indeed, $f_1 \bar\in \Lambda(\alpha)$, $f_1 \in L_{\alpha^{-1}}$ and $f_2 \bar\in L_{\alpha^{-1}}$, $f_2 \in \mathrm{M}(1 - \alpha)$.

From the theorem we see that

1.3(6) $$\Lambda(\alpha) \prec \mathrm{M}(1 - \alpha)$$

(this is easily proved immediately). A counterpart to this inequality is contained in the theorem:

THEOREM 4. $\mathrm{M}(1 - \alpha') \prec \Lambda(\alpha)$ *if* $\alpha' < \alpha$.

Suppose $f \in \mathrm{M}(1 - \alpha')$; then

$$g(x) = \int_0^x f^*(t)\,dt \leqq \| f \|_{\mathrm{M}(1-\alpha')} x^{1-\alpha'},$$

$$\int_0^1 x^{\alpha-1} f^*(x)\,dx = \int_0^1 x^{\alpha-1}\,dg(x) = x^{\alpha-1} g(x)\Big|_0^1 + (1 - \alpha)\int_0^1 x^{\alpha-2} g(x)\,dx$$

$$\leqq \| f \| + (1 - \alpha) \| f \| \int_0^1 x^{\alpha-\alpha'-1}\,dx = K_1 \| f \|_{\mathrm{M}},$$

that is $f \in \Lambda(\alpha)$ and $\| f \|_\Lambda \leqq K \| f \|_{\mathrm{M}}$.

1.4. There is another mode of introducing the space $M(\alpha)$. For indefinite integrals $F(x) = \int_0^x f(t)dt$ we introduce, with Carathéodory [3, p. 511] the modulus of total continuity of $F(x)$:

$$\tau(\delta) = \sup \sum_{\nu=1}^{n} | F(b_\nu) - F(a_\nu) | , \qquad \delta > 0;$$

the least upper bound being taken over all finite groups of intervals (a_ν , b_ν) without common points, such that $\sum(b_\nu - a_\nu) < \delta$. Then $M(\alpha)$ can be considered as the totality of all $f \in L_1$ such that $\tau(\delta) \leqq K\delta^\alpha$, $\delta > 0$, and the least possible K in this inequality is $|| f ||_{M(\alpha)}$. For this least K is

$$K = \sup_{\delta>0} \delta^{-\alpha} \tau(\delta) = \sup \{ (\sum(b_\nu - a_\nu))^{-\alpha} \sum | F(b_\nu) - F(a_\nu) | \}$$

and the supremum is identical with 1.1(2).

1.5. We shall only mention, without going into details, definitions of some more general Banach spaces, which we denote by $\Lambda(\alpha, p)$, $\alpha > 0$, $p \geqq 1$ and $M(\alpha, p)$, $0 \leqq \alpha \leqq 1$, $p \geqq 1$. They are obtained by putting

$$\Lambda(\alpha, p): \qquad || f || = \left\{ \alpha \int_0^1 x^{\alpha-1} f^*(x)^p \, dx \right\}^{1/p},$$

$$M(\alpha, p): \qquad || f || = \sup_{e \subset (0,1)} (me)^{-\alpha} \left\{ \int_e | f(x) |^p \, dx \right\}^{1/p}.$$

Spaces of this type had been considered indirectly formerly, and the well known theorems of Hardy and Littlewood, and Paley (compare Zygmund [17, pp. 208, 214]) may be regarded as to refer to the space $\Lambda(p - 1, p)$, $p > 1$. Without proof, we mention the inclusion

$$1.5(1) \qquad \Lambda(\alpha, p) \prec M\left(\frac{1}{p'} - \frac{\alpha}{p}, p'\right), \qquad 1 \leqq p' \leqq p.$$

A special case is $\Lambda(\alpha, p) \prec M((1 - \alpha)/p, p)$, which is more general than 1.3(6).

The inclusion

$$1.5(2) \qquad \Lambda(\alpha, p) \prec \Lambda(\alpha', p')$$

holds if and only if either $\alpha p^{-1} < \alpha' p'^{-1}$ or $\alpha p^{-1} = \alpha' p'^{-1}$ and $p \leqq p'$. Especially, $\Lambda(\alpha, p) \prec \Lambda(\alpha p' p^{-1}, p')$ if $p' \geqq p$, and for $p' = p\alpha^{-1}$ we obtain $\Lambda(\alpha, p) \prec L_{p\alpha^{-1}}$, which contains 1.3(3).

§2. Linear functionals in the spaces $\Lambda(\alpha)$ and $M(\alpha)$

2.1. We shall make a repeated use of the following inequalities:

THEOREM 5. *If f, g are measurable over* (0, 1),

$$2.1(1) \qquad \left| \int_0^1 f(x)g(x) \, dx \right| \leqq \sup_e \left\{ (me)^{-\alpha} \int_e | f(x) | \, dx \right\} \cdot \alpha \int_0^1 x^{\alpha-1} g^*(x) \, dx;$$

and if f, g are positive and g is decreasing,

$$2.1(2) \qquad \int_0^1 f(x)g(x)\,dx \leqq \sup_{\delta>0}\left\{\delta^{-\alpha}\int_0^\delta f(x)\,dx\right\}\cdot\alpha\int_0^1 x^{\alpha-1}g(x)\,dx.$$

We have $\left|\int_0^1 fg\,dx\right| \leqq \int_0^1 f^*g^*\,dx$, hence it will be sufficient to prove 2.1(2). Suppose first that $g(x)$ is a step function $g_\Delta(x) = 1$ for $0 < x \leqq \Delta$, and $= 0$ for $\Delta < x < 1$. Then 2.1(2) reduces to

$$\int_0^\Delta f\,dx \leqq \sup_\delta\left\{\delta^{-\alpha}\int_0^\delta f\,dx\right\}\cdot\Delta^\alpha,$$

and is clearly true. Again, 2.1(2) is true for any positive decreasing step function $g(x)$ with a finite number of jumps, since such $g(x)$ is expressible as a linear combination with positive coefficients of the $g_\Delta(x)$. And by passing to the limit, as in Theorem 1, we obtain 2.1(2) for general $g(x)$.

2.2. The main result of this paragraph is:

THEOREM 6. *Every linear functional $\Phi(f)$ in $\Lambda(\alpha)$ is of the form*

$$2.2(1) \qquad \Phi(f) = \int_0^1 f(x)g(x)\,dx$$

with an appropriate $g \in \mathrm{M}(\alpha)$; this g is uniquely determined by Φ and

$$2.2(2) \qquad \|\Phi\| = \|g\|_{\mathrm{M}(\alpha)}$$

We need a lemma:

LEMMA. *Suppose that $g(x); f_*(x), g^*(x)$ are measurable in $(0, 1)$ ($g^*(x)$ being the rearrangement of $|g(x)|$ in decreasing order). Then there is a measurable $f(x)$ such that f and f_*, as well as fg and f_*g^* are equimeasurable.*

To prove the lemma, we use the fact (Lorentz [14]), that there are two decompositions $K + \sum_{\xi\in\Xi}K_\xi$ and $K^* + \sum_{\xi\in\Xi}K_\xi^*$ of the interval $(0, 1)$ in null sets such that (a) on the corresponding sets K_ξ and K_ξ^* the functions g and g^* take the same value; (b) if a sum $\sum_{\xi\in\mathrm{H}}K_\xi$ is measurable, then also $\sum_{\xi\in\mathrm{H}}K_\xi^*$ is so (and conversely), and both sets have the same measure; (c) every set K_ξ^* consists of a single point.

We now define $f(x)$ by $f(x) = f_*(y)$ for all $x \in K_\xi$, if y is the only point of K_ξ^*; $f(x)$ may be taken arbitrary on the set K. It is easily seen that $f(x)$ is measurable and that it possesses the properties required by the lemma.

PROOF OF THEOREM 6. If $g \in \mathrm{M}(\alpha)$, then 2.2(1) is a linear functional, and $\|\Phi\| \leqq \|g\|_{\mathrm{M}(\alpha)}$, according to 2.1(1). Conversely, suppose that $\Phi(f)$ is a linear functional in the space $\Lambda(\alpha)$. We wish to show that Φ can be represented in the form 2.2(1). Put $\varphi(e) = \Phi(f_e)$ for any measurable set $e \subset (0, 1)$, where f_e is the characteristic function of e; $\varphi(e)$ is an additive set function. We have $\|f_e\|_{\Lambda(\alpha)} = \|f_e^*\| = \alpha\int_0^{me} x^{\alpha-1}\,dx = (me)^\alpha$, and therefore $|\varphi(e)| \leqq \|\Phi\|\,(me)^\alpha$. Thus

$\varphi(e)$ is totally continuous and has a representation $\varphi(e) = \int_e g(x)\,dx$, with some $g \in L_1$; and g is uniquely determined by φ (or by Φ). We clearly have

$$\Phi(f) = \int_0^1 f_e\, g\, dx.$$

Choose $\delta > 0$ arbitrary and put $\bar{e} = (0, \delta)$. For the three functions $g(x); f_*(x) = f_{\bar{e}}(x)$, $g^*(x)$ we find, according to the lemma, the fourth $f(x)$, which is clearly an $f_e(x)$ with $me = \delta$. Since

$$\int_0^\delta g^*\, dx = \int_0^1 f_{\bar{e}} g^*\, dx = \int_0^1 f_e\, g\, dx = \Phi(f_e) \leqq \|\Phi\| \cdot \|f_e\| = \|\Phi\|\,\delta^\alpha,$$

we conclude that $g \in \mathrm{M}(\alpha)$; moreover, $\|g\|_{\mathrm{M}(\alpha)} \leqq \|\Phi\|$.

Now $\Phi(f)$ and $\Phi_1(f) = \int_0^1 fg\, dx$ are two linear functionals in $\Lambda(\alpha)$, and $\Phi(f) = \Phi_1(f)$ if $f = f_e$. By 1.2(a) these functionals are identical on the whole of $\Lambda(\alpha)$. This completes the proof.

For any fixed $f \in \Lambda(\alpha)$, 2.2(1) is a linear functional in $\mathrm{M}(\alpha)$ with the norm $= \|f\|_{\Lambda(\alpha)}$. But this is not the general form of a linear functional in $\mathrm{M}(\alpha)$, since $\mathrm{M}(\alpha)$ is not separable, and $\Lambda(\alpha)$ is (Banach [1, p. 189]).

2.3. Theorem 7. (i) *If the integral*

$$\int_0^1 f(x)g(x)\, dx \tag{2.3(1)}$$

exists for a fixed measurable g and all $f \in \mathrm{M}(\alpha)$, then $g \in \Lambda(\alpha)$; (ii) *If* 2.3(1) *exists for a fixed g and all $f \in \Lambda(\alpha)$, then $g \in \mathrm{M}(\alpha)$.*

(i) For the functions $g(x); f_*(x) = \alpha x^{\alpha-1}$, $g^*(x)$ we find, by the lemma of 2.2, the corresponding $f(x)$. Since $f_* \in \mathrm{M}(\alpha)$, f also belongs to $\mathrm{M}(\alpha)$. Thus $\alpha \int_0^1 x^{\alpha-1} g^*\, dx = \int_0^1 fg\, dx$ is finite.

(ii) Suppose that $g \bar{\in} \mathrm{M}(\alpha)$, or that $\delta^{-\alpha} \int_0^\delta g^*\, dx$ is not bounded. Then there is a sequence $\delta_\nu > 0$, $(\delta_1 = 1)$, decreasing monotonously to 0, such that $\int_0^{\delta_\nu} g^*\, dx > \nu\delta_\nu^\alpha$ $(\nu = 2, 3, \cdots)$. We may assume, that even $\int_{\delta_{\nu+1}}^{\delta_\nu} g^*\, dx > \nu\delta_\nu^\alpha$ holds for all $\nu = 2, 3, \cdots$ and that $\sum \delta_\nu^\alpha < +\infty$. Choose a sequence of positive numbers a_ν such that

$$\sum a_\nu \delta_\nu^\alpha < +\infty, \qquad \sum \nu a_\nu \delta_\nu^\alpha = +\infty$$

and put

$$f_*(x) = a_\nu \quad \text{for} \quad \delta_{\nu+1} \leqq x < \delta_\nu.$$

For the three functions $g; f_*$, g^* we choose, by the lemma of 2.2, a fourth function f. Since

$$\alpha \int_0^1 x^{\alpha-1} f_*\, dx = \sum_1^\infty \alpha a_\nu \int_{\delta_{\nu+1}}^{\delta_\nu} x^{\alpha-1}\, dx \leqq \sum a_\nu \delta_\nu^\alpha < +\infty,$$

we have $f_* \in \Lambda(\alpha)$ and hence $f \in \Lambda(\alpha)$. Therefore, $\int_0^1 f_* g^*\, dx = \int_0^1 fg\, dx$ exists. On the other hand, however,

$$\int_0^1 f_* g^*\, dx = \sum \int_{\delta_{\nu+1}}^{\delta_\nu} \geqq \sum \nu a_\nu \delta_\nu^\alpha = +\infty.$$

This contradiction establishes our result.

§3. Applications to Fourier series

In this paragraph the range is $0 < x < 2\pi$.

3.1. Theorem 8. *A series*

3.1(1) $$a_0/2 + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a function $f(x)$ from $M(\alpha)$ *or* $\Lambda(\alpha)$, *if and only if the first arithmetical means $\sigma_n(x)$ of the partial sums of* 3.1(1) *have the property*

3.1(2) $$\| \sigma_n \| \leqq K$$

in the metric of $M(\alpha)$ *and* $\Lambda(\alpha)$ *respectively.*

First, if 3.1(1) is a Fourier series of some $f \in M(\alpha)$ (or $\Lambda(\alpha)$) then $\sigma_n(x) = \int_{-\pi}^{+\pi} K_n(t) f(x + t)\, dt$, $K_n(t)$ denoting the Fejér kernel. Suppose that $g \in \Lambda(\alpha)$ (or $M(\alpha)$) and that $\| g \| \leqq 1$. We have

$$\int_{-\pi}^{+\pi} \sigma_n g\, dx = \int_{-\pi}^{+\pi} K_n(t)\, dt \int_{-\pi}^{+\pi} f(x + t) g(t)\, dx \leqq \| f \| \int_{-\pi}^{+\pi} K_n\, dt = \| f \|.$$

Hence, by the end of 2.2, $\| \sigma_n \| \leqq \| f \|$.

Now suppose that 3.1(2) holds. By Theorem 4 and from 1.3(3) we deduce that $\| \sigma_n \|_{L_p}$ are also bounded for some $p > 1$. Hence, by a theorem of W. H. Young (Zygmund [17, p. 83]) the series 3.1(1) is the Fourier series of some $f \in L_p$. Since $\sigma_n(x) \to f(x)$ almost everywhere, we have also $\sigma_n^*(x) \to f^*(x)$ almost everywhere. Hence, by Fatou's lemma

$$\| f \| = \| f^* \| \leqq \underline{\lim} \| \sigma_n^* \| = \underline{\lim} \| \sigma_n \| \leqq K,$$

in the metric of $M(\alpha)$ (resp. $\Lambda(\alpha)$). Therefore, f belongs to $M(\alpha)$ (or $\Lambda(\alpha)$).

Theorem 9. *For all $f \in$* $M(\alpha)$ (or $\Lambda(\alpha)$)

3.1(3) $$\sigma_n \to f$$

in the metric of $M(\alpha)$ (*or* $\Lambda(\alpha)$).

By Theorem 8, the norms of the linear operators $\sigma_n = \sigma_n(f)$ are uniformly bounded. Since 3.1(3) is fulfilled for all continuous f, the conclusion follows.

THEOREM 10. *Suppose that* $f \in \Lambda(\alpha)$, $g \in \mathrm{M}(\alpha)$ *and that* a_n, b_n *are Fourier constants of* f *and* a_n', b_n' *are those of* g, *then*

$$\frac{a_0 a_0'}{2} + \sum_1^\infty (a_n a_n' + b_n b_n') = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x)g(x)\, dx$$

where the sum means the Cesàro C_k — *sum and* k *any positive number.*

The proof is the same as that in Zygmund [17, p. 88] and is omitted.

3.2. For the special case of the series

3.2(1) $$\sum_1^\infty a_n \cos nx, \qquad 3.2(2) \quad \sum_1^\infty a_n \sin nx$$

where the coefficients are monotonously decreasing to 0, much simpler criteria can be given in order that the sum belongs to $\Lambda(\alpha)$ or $\mathrm{M}(\alpha)$. For similar questions for other classes of functions, compare Hardy and Littlewood [7], Hardy and Rogosinski [10] and Lorentz [13].

THEOREM 11. *In order that the sum* $g(x)$ *of the series* 3.2(1) *or* 3.2(2) *should belong to* $\mathrm{M}(\alpha)$, *the condition*

3.2(3) $$a_n = O(n^{-\alpha})$$

is necessary and sufficient; and for $g \in \Lambda(\alpha)$ *the condition is*

3.2(4) $$\sum_1^\infty n^{-\alpha} a_n < +\infty.$$

We confine our attention on the cosine series 3.2(1); the series 3.2(2) may be treated on the same lines. We shall make use of the inequality (Hardy and Littlewood [7, p. 7], Zygmund [17, p. 213])

3.2(5) $$a_n \leqq C\, G(\pi/n),$$

where $G(x) = \int_0^x g(t)\, dt$ and C (and later on $C_1, C_2, \cdots$) stands for a constant, independent of g. If $g \in \mathrm{M}(\alpha)$, 3.2(3) follows. Suppose now $g \in \Lambda(\alpha)$, write $g^*(x)$ for the rearrangement of $| g(x) |$ in $0 < x < \pi$ in decreasing order and put $G^*(x) = \int_0^1 g^*\, dt$. Then

$$\sum_{n=2}^\infty n^{-\alpha} a_n \leqq C_1 \sum_{n=2}^\infty n^{-\alpha} G(\pi/n) \leqq C_1 \sum n^{-\alpha} G^*(\pi/n)$$
$$\leqq C_2 \sum \int_{\pi/n}^{\pi/(n-1)} x^{-2+\alpha} G^*(x)\, dx = C_2 \int_0^\pi x^{-2+\alpha} G^*(x)\, dx.$$

To estimate the last integral observe that for $\delta > 0$

$$\int_\delta^\pi x^{-2+\alpha} G^*\, dx = -C_3 \int_\delta^\pi G^*\, dx^{\alpha-1} = -C_3 \left\{ x^{\alpha-1} G^* \Big|_\delta^\pi - \int_\delta^\pi x^{\alpha-1} g^*\, dx \right\}$$
$$\leqq C_3 \delta^{\alpha-1} G^*(\delta) + C_3 \int_\delta^\pi x^{\alpha-1} g^*\, dx.$$

The first term of the right hand side is $\leqq C_8 \|g\|_{M(1-\alpha)} \leqq C_4 \|g\|_{\Lambda(\alpha)}$, and the same holds for the second term. Thus

$$\sum_2^\infty n^{-\alpha} a_n \leqq C_5 \|g\|_{\Lambda(\alpha)} < +\infty.$$

To prove that the conditions are sufficient, we note that by Abel's lemma

$$|g(x)| \leqq \sum_{\nu=1}^n a_\nu + \left| \sum_{\nu=n+1}^\infty a_\nu \cos \nu x \right| \leqq \sum_1^n a_\nu + C_6 a_n/x.$$

Consequently 3.2(3) implies

$$|g(x)| \leqq C_7 \sum_1^n a_\nu \leqq C_8 \sum_1^n \nu^{-\alpha} \leqq C_9 n^{1-\alpha} \tag{3.2(6)}$$

$$\text{for} \quad \pi/(n+1) \leqq x < \pi/n.$$

Put $h(x) = C_9 n^{1-\alpha}$ for $\pi/(n+1) \leqq x < \pi/n$; clearly $|g(x)| \leqq h(x)$ and therefore also $g^*(x) \leqq h(x)$. Hence

$$\int_0^{1/n} g^* \, dx \leqq C_{10} \sum_{\nu=n}^\infty \nu^{1-\alpha} \frac{1}{\nu(\nu+1)} \leqq C_{11} n^{-\alpha},$$

so that $g \in M(\alpha)$, what we wished to show.

Again, if 3.2(4) holds, we deduce as in 3.2(6) that $g^*(x) \leqq h(x)$ where $h(x) = C_7 \sum_1^n a_n = A_n$ for $\pi/(n+1) \leqq x < \pi/n$. Hence

$$\int_0^\pi x^{\alpha-1} g^* \, dx \leqq \sum_{n=1}^\infty \int_{\pi/(n+1)}^{\pi/n} x^{\alpha-1} h \, dx \leqq C_{12} \sum_1^\infty A_n n^{-1-\alpha}$$

$$= C_{12} \sum_{\nu=1}^\infty a_\nu \sum_{n=\nu}^\infty n^{-1-\alpha} \leqq C_{13} \sum_{\nu=1}^\infty \nu^{-\alpha} a_\nu < +\infty.$$

§4. Applications to fractional integrals

4.1. We first derive some inequalities dealing with functions in the range $(-\infty, +\infty)$ or $(0, \infty)$. Here $f^*(x)$ need not to have a sense for an arbitrary measurable $f(x)$; $f^*(x)$ exists if, and only if, for every $a > 0$ the set $|f(x)| > a$ has a finite measure. If the spaces $\Lambda(\alpha)$, $M(\alpha)$ for an infinite range are restricted to contain only functions of this kind, the results of §§1–2 are still applicable.

THEOREM 12. *Suppose that $f(x)$, $g(x)$ are positive for $-\infty < x < +\infty$, $f \in \Lambda(\alpha)$, $g \in M(\beta)$, $0 < \alpha < 1$, $0 < \beta < 1$ and let $\mu = 2 - \alpha - \beta < 1$. Then*

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{f(x)g(y)}{|x-y|^\mu} \, dx \, dy \leqq K \|f\|_{\Lambda(\alpha)} \|g\|_{M(1-\beta)}, \tag{4.1(1)}$$

with a constant $K = K(\alpha_1, \beta)$ depending only upon α and β.

By a well known inequality of F. Riesz (Hardy, Littlewood and Pólya [8, p. 279]), the integral in 4.1(1) can become only larger if we replace f, g by their symmetrically decreasing rearrangements. Thus we may assume that f, g are decreasing in $(0, +\infty)$ and even, and therefore replace the range $(-\infty, +\infty)$ in 4.1(1) by $(0, +\infty)$. But

4.1(2)
$$\int_0^\infty \int_0^\infty fg\,|x-y|^{-\mu}\,dx\,dy = \int_0^\infty g(y)\,dy \int_0^y f(x)\,|x-y|^{-\mu}\,dx + \int_0^\infty f(x)\,dx \int_0^x g(y)\,|x-y|^{-\mu}\,dy.$$

Putting $F(y) = \int_0^y f(x)\,dx$, we obtain

$$\int_0^y f(x)(y-x)^{-\mu}\,dx = \int_0^{y/2} + \int_{y/2}^y \leqq (y/2)^{-\mu}F(y/2) + f(y/2)(y/2)^{1-\mu}(1-\mu)^{-1}$$
$$\leqq C_1 y^{-\mu}F(y) + C_2 y^{-\mu}F(y) = C_3 y^{-\mu}F(y);$$

hence the first integral in the right hand side of 4.1(2) is $\leqq C_3 \int_0^\infty y^{-\mu}F(y)g(y)\,dy$. A similar inequality holds for the second integral. Therefore, Theorem 12 is a consequence of

THEOREM 13. *Suppose that $f(x)$, $g(x)$ are positive and decreasing in $(0, +\infty)$ and that $F(y)$, $G(y)$ are their integrals over $(0, y)$. Let $0 < \alpha < 1$, $0 < \beta < 1$ and $\mu = 2 - \alpha - \beta < 1$. Then*

(i)
$$\int_0^\infty x^{-\mu}f(x)G(x)\,dx \leqq \frac{1}{1-\mu}\,\|f\|_{\Lambda(\alpha)}\,\|g\|_{M(1-\beta)},$$

(ii)
$$\int_0^\infty x^{-\mu}g(x)F(x)\,dx \leqq K\,\|f\|_{\Lambda(\alpha)}\,\|g\|_{M(1-\beta)},$$

where the constant K depends only upon α and β.

PROOF. The proofs of the two cases are different; and (i) is simpler. From the "fundamental inequality" 2.1(2) we see that the integral in (i) does not exceed $\|f\| \sup_{\delta>0} \delta^{-\alpha} \int_0^\delta x^{-\mu}G\,dx$; and since

$$\delta^{-\alpha}\int_0^\delta x^{-\mu}G(x)\,dx \leqq \delta^{-\alpha}G(\delta)\int_0^\delta x^{-\mu}\,dx = \delta^{\beta-1}G(\delta)/(1-\mu)$$
$$= (1-\mu)^{-1}\delta^{\beta-1}\int_0^\delta g\,dx \leqq (1-\mu)^{-1}\|g\|,$$

(i) follows.

To prove (ii), we again observe, that we need only prove the inequality for a step function $f(x) = 1$ for $0 < x \leqq \delta$, $f(x) = 0$ for $x > \delta$. For this f (ii) becomes

4.1(3)
$$\int_0^\delta x^{1-\mu}g\,dx + \delta\int_\delta^\infty x^{-\mu}g\,dx \leqq K\delta^\alpha\,\|g\|.$$

The first integral is $\leqq \delta^{1-\mu}\int_0^\delta g\,dx \leqq \delta^{-\alpha}\,\|g\|_{M(1-\beta)}$. Moreover, if $f_1(x)$ is the function $= 0$ for $0 < x < \delta$ and $= x^{-\mu}$ for $x \geqq \delta$, then $f_1^*(x) = (x + \delta)^{-\mu}$ and by 2.1(1),

$$\delta \int_\delta^\infty x^{-\mu} g\, dx \leqq \delta \,||\, g \,||_{\mathrm{M}(1-\beta)} \,||\, f_1 \,||_{\Lambda(1-\beta)}$$

$$= \delta \,||\, g \,||\, (1 - \beta) \int_0^\infty (x + \delta)^{-\mu} x^{-\beta}\, dx = K\delta^\alpha \,||\, g \,||,$$

where $K = (1 - \beta) \int_0^\infty (v + 1)^{-\mu} v^{-\beta}\, dv$. This proves 4.1(3) and therefore (ii).

4.2. For an integrable function $f(x)$ with the period 2π and mean value 0 the fractional integral $f_\rho(x)$ of Weyl of the order $0 < \rho < 1$ is defined by

$$f_\rho(x) = \frac{1}{\Gamma(\rho)} \int_{-\infty}^x f(t)(x - t)^{\rho-1}\, dt; \tag{4.2(1)}$$

$f_\rho(x)$ exists almost everywhere. The main result of Hardy and Littlewood [6] is embodied in the fact that $f \in L_{1/\alpha}$ implies $f_\rho \in L_{1/(\alpha-\rho)}$, if $0 < \rho < \alpha$. A similar result is true for our spaces.

THEOREM 14. *Let* $0 < \rho < \alpha < 1$. *Then* $f \in \Lambda(\alpha)$ *implies* $f_\rho \in \Lambda(\alpha - \rho)$ *and* $f \in \mathrm{M}(1 - \alpha)$ *implies* $f_\rho \in \mathrm{M}(1 - \alpha + \rho)$.

The proof follows the same lines as that of Hardy and Littlewood [6, p. 575] (see also Zygmund [17, p. 232]); it depends upon the inequality 4.1(1) and is left to the reader.

4.3. We note also the following inequality

THEOREM 15. *If* $f(x)$, $g(x)$ *are positive in* $0 < x < +\infty$, *if* $f \in \Lambda(\alpha)$, $g \in \mathrm{M}(\alpha)$ *and* $0 < \alpha < 1$, *then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y}\, dx\, dy \leqq K(\alpha) \,||\, f \,||_{\Lambda(\alpha)} \,||\, g \,||_{\mathrm{M}(\alpha)}\, . \tag{4.3(1)}$$

We apply Schur's argument in his proof of Hilbert's inequality (I. Schur [15], Hardy, Littlewood and Pólya [8, p. 230]). The integral in 4.3(1) is

$$\begin{aligned} &= \int_0^\infty g(y)\, dy \int_0^\infty (x + y)^{-1} f(x)\, dx = \int_0^\infty g\, dy \int_0^\infty (1 + t)^{-1} f(yt)\, dt \\ &= \int_0^\infty (1 + t)^{-1}\, dt \int_0^\infty f(xt) g(x)\, dx. \end{aligned} \tag{4.3(2)}$$

For any fixed t,

$$||\, f(xt) \,||_{\Lambda(\alpha)} = \alpha \int_0^\infty x^{\alpha-1} f(xt)^*\, dx$$

$$= \alpha t^{-\alpha} \int_0^\infty u^{\alpha-1} f^*(u)\, du = t^{-\alpha} \,||\, f \,||,$$

and the result follows from 4.3(2) by means of 2.1(2).

§5. Applications to moment problems

5.1. In this paragraph we deal with the moment problem for the interval (0, 1), in the case when the generating function belongs to $\Lambda(\alpha)$ or $M(\alpha)$. The problem consists in describing those real sequences μ_n, for which the system

5.1(1) $$\mu_n = \int_0^1 x^n f(x)\,dx, \qquad n = 0, 1, 2, \cdots$$

possesses a solution $f \in \Lambda(\alpha)$ or $f \in M(\alpha)$. We first give necessary conditions on μ_n.

THEOREM 16. *If $f \in \Lambda(\alpha)$, then*

5.1(2) $$\sum_1^\infty n^{-\alpha} |\mu_n| < +\infty;$$

and if $f(x)$ is positive (that is, $\geqq 0$) and decreasing, then 5.1(2) *is necessary and sufficient in order that $f \in \Lambda(\alpha)$.*

It is sometimes convenient to make use of the binomial coefficients $A_n^\beta = \binom{n+\beta}{n} \cong n^\beta/\Gamma(\beta+1)$ rather than of the powers n^β. With their aid we have, if $f \in \Lambda(\alpha)$,

5.1(3) $$\begin{aligned}\sum_1^\infty n^{-\alpha} |\mu_n| &\leqq C_1 \sum_0^\infty A_n^{-\alpha} |\mu_n| \leqq C_1 \int_0^1 |f(x)| \sum_0^\infty A_n^{-\alpha} x^n\,dx \\ &= C_1 \int_0^1 |f(x)|\,(1-x)^{\alpha-1}\,dx \leqq C_1\alpha^{-1} \|f\|.\end{aligned}$$

On the other hand, if f is positive and decreasing, from 5.1(2) we derive $\sum A_n^{-\alpha}\mu_n < +\infty$, and, by 5.1(3), $\int_0^1 f^* x^{\alpha-1} dx < +\infty$.

THEOREM 17. *If $f \in M(\alpha)$, then*

5.1(4) $$\mu_n = O(n^{-\alpha});$$

and for positive decreasing $f(x)$ 5.1(4) *is equivalent to $f \in M(\alpha)$.*

Suppose that $f \in M(\alpha)$, then, by 2.1(2),

$$\|\mu_n\| \leqq \|f\|\,\alpha \int_0^1 (1-x)^{\alpha-1} x^n\,dx = \alpha B(n+1, \alpha)\,\|f\| = O(n^{-\alpha}).$$

Conversely, lel 5.1(4) be fulfilled and suppose $f(x) \geqq 0$ and decreasing. If we define the positive integer n by $(n+1)^{-1} < \delta \leqq n^{-1}$, then for $0 < \delta < \frac{1}{2}$,

$$\int_{1-\delta}^1 f(x)\,dx \leqq C_1 \int_{1-\delta}^1 x^n f(x)\,dx \leqq C_2 n^{-\alpha} \leqq C_2 \delta^\alpha$$

and therefore $f \in M(\alpha)$.

For the sake of completeness we mention the theorem:

THEOREM 18. *If* $f \in L_p$, $p > 1$ *then* $\sum_1^\infty n^{p-2} \mid \mu_n \mid^p < +\infty$; *and if this condition is fulfilled and* $f(x)$ *is positive and decreasing, then* $f \in L_p$.

The first part of the theorem was given by Hardy and Littlewood [5]; and the second part is proved on the same lines as in Theorem 17.

5.2. We now proceed to solve the moment problems for general μ_n and prove the Theorems 19 and 20 below. In the proof of the necessity of our conditions the main difficulty consists in the estimating of certain sums involving the "Newton probabilities"

$$5.2(1) \qquad p_\nu(x) = p_{n\nu}(x) = \binom{n}{\nu} x^\nu(1-x)^{n-\nu}; \qquad 0 \leqq \nu \leqq n, \quad n = 0, 1, \cdots.$$

In the sufficiency part we use an idea of T. H. Hildebrandt [12], applied by him to a solution of Hausdorff's moment problem. We write

$$\mu_{n\nu} = \binom{n}{\nu}\Delta^{n-\nu}\mu_\nu = \binom{n}{\nu}\left\{\mu_\nu - \frac{n-\nu}{1}\mu_{\nu+1} + \cdots + (-1)^{n-\nu}\mu_n\right\};$$
$$0 \leqq \nu \leqq n, \quad n = 0, 1, \cdots.$$

In the same manner $p_{n\nu}$ is expressed by the powers x^ν. Therefore 5.1(1) implies

$$5.2(2) \qquad \mu_{n\nu} = \int_0^1 p_{n\nu}(x)f(x)\,dx.$$

We also write $\mu_{n\nu}^*$ for the $\mid \mu_{n\nu} \mid$, $\nu = 0, \cdots, n$, arranged in decreasing order. Finally, we state some inequalities, concerning finite sums, and analogous to the inequalities for integrals of §§1–2. We have

$$5.2(3) \qquad \sum_{\nu=0}^{n} (\nu+1)^{p-1} a_\nu^p \leqq K_p\left(\sum_{\nu=0}^{n} a_\nu\right)^p, \qquad p > 1,$$

if a_ν are $\geqq 0$ and decreasing,

$$5.2(4) \qquad \sum_{\nu=0}^{n}\left(\frac{\nu+1}{n+1}\right)^{\beta-1} a_\nu^* \leqq K_{\alpha,\beta} \max_{k,\nu_0,\cdots,\nu_k}\left\{\left(\frac{k+1}{n+1}\right)^{-\alpha}\sum_{i=0}^{k} \mid a_{\nu_i} \mid\right\},$$
$$1-\alpha < \beta < 1;$$

whenever $0 < \alpha < 1$, $0 < \beta < 1$ and $1 - \alpha < \beta$; here the maximum runs over all integers $0 \leqq k \leqq n$ and $0 \leqq \nu_0 < \nu_1 < \cdots < \nu_k \leqq n$.

$$5.2(5) \qquad \left|\sum_{\nu=0}^{n} a_\nu b_\nu\right| \leqq \alpha \max_{k,\nu_0,\cdots,\nu_k}\left\{\left(\frac{k+1}{n+1}\right)^{-\alpha}\sum_{i=0}^{k}\frac{\mid a_{\nu_i}\mid}{n+1}\right\}\sum_{\nu=0}^{n}\left(\frac{\nu+1}{n+1}\right)^{\alpha-1} b_\nu^*,$$
$$0 < \alpha < 1.$$

Of these inequalities, 5.2(3) is the Hardy, Littlewood and Pólya inequality mentioned in 1.3; 5.2(4) corresponds to Theorem 4 and 5.2(5) to the inequality 2.1(1). They may be proved by arguments similar to those used for the integral inequalities, or derived from these; we leave the details to the reader.

5.3. THEOREM 19. *A sequence μ_n is the moment sequence of a function $f \in \Lambda(\alpha)$ if and only if for some constant K*

5.3(1) $$\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1}\mu_{n\nu}^{*} \leqq K(n+1)^{\alpha-1}, \qquad n = 1, 2, \cdots.$$

a) *The condition is necessary.* If $\rho = \rho(\nu)$ is an appropriate rearrangement of the $\nu = 0, 1, \cdots, n$, then, according to 5.2(2) and by 2.1(1) for the sum 5.3(1) we have

$$\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1}\mu_{n\nu}^{*} \leqq \int_0^1 |f(x)| \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} p_{n,\rho(\nu)}(x)\, dx$$
$$\leqq \|f\|_{\Lambda(\alpha)} \sup_{\delta>0}\left\{\delta^{-\alpha}\int_e \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} p_{n\rho}(x)\, dx\right\}; \quad (\delta = me).$$

The last integral is

$$= \int_e \sum_{\nu\leqq n\delta} dx + \int_e \sum_{\nu>n\delta} dx$$
$$\leqq \sum_{\nu\leqq n\delta} (n+1)^{-1}(\nu+1)^{\alpha-1} + (n\delta)^{\alpha-1}\int_e \sum_{\nu>n\delta} p_{n\rho}\, dx$$
$$\leqq C_1(n+1)^{-1}(n\delta)^{\alpha} + \delta(n\delta)^{\alpha-1} \leqq C_2(n+1)^{\alpha-1}\delta^{\alpha}.$$

Hence our sum is $\leqq C_2 \|f\| (n+1)^{\alpha-1}$, and 5.3(1) follows.

b) *The condition is sufficient.* Let 5.3(1) be fulfilled; applying 5.2(3) to $a_\nu = (\nu+1)^{\alpha-1}\mu_{n\nu}^{*}$, $\nu = 0, 1, \cdots, n$ and $p = \alpha^{-1}$, we obtain

$$\sum_{\nu=0}^{n} |\mu_{n\nu}|^{p} = \sum_{\nu=0}^{n} \mu_{n\nu}^{*p} \leqq C_1(n+1)^{1-p}, \qquad n = 1, 2, \cdots.$$

But according to Hausdorff [11] (see also Widder [16, p. 109]) this is sufficient to guarantee that the moment problem 5.1(1) is solvable by a function $f \in L_p$. It remains to show that this f belongs to $\Lambda(\alpha)$.

Let $P(x) = a_0 + a_1x + \cdots + a_nx^m$ be an arbitrary polynomial; we denote by

5.3(2) $$B_n^P(x) = \sum_{\nu=0}^{n} P(\nu/n)p_{n\nu}(x) = a_0^{(n)} + a_1^{(n)}x + \cdots + a_m^{(n)}x^m$$

its n^{th} Bernstein Polynomial; it is known that $a_i^{(n)} \to a_i$ when $n \to \infty$. Hence

$$\int_0^1 P(x)f(x)\, dx = \lim_{n\to\infty}\int_0^1 B_n^P(x)f(x)\, dx;$$

on the other hand, by 5.3(2), 5.2(2), 5.2(5) and 5.3(1),

$$\left|\int_0^1 B_n^P f\, dx\right| = \left|\sum_{\nu=0}^{n} P(\nu/n)\mu_{n\nu}\right|$$
$$\leqq K_\alpha K \max_{k,\nu_0,\cdots,\nu_k}\left\{\left(\frac{k+1}{n+1}\right)^{-\alpha}\sum_{i=0}^{k} |P(\nu_i/n)| \frac{1}{n+1}\right\}.$$

Write $\Pi_n(x)$ for the function, equal to $|P(\nu/n)|$ in $\nu/(n+1) \leqq x < (\nu+1)/(n+1)$; then the last sum is $= \int_e \Pi_n \, dx$ over some set e, whose measure is equal to $(k+1)/(n+1)$. Therefore,

$$\left| \int_0^1 B_n^P f \, dx \right| \leqq C_2 \| \Pi_n \|_{M(\alpha)} .$$

Since $\Pi_n(x) \to |P(x)|$ uniformly and since $\| \Pi_n \| \to \| P \|$, as $n \to \infty$, this obtains

$$\left| \int_0^1 P(x) f(x) \, dx \right| \leqq C_2 \| P \|_{M(\alpha)} .$$

For an arbitrary bounded measurable function $\varphi(x)$, $0 < x < 1$, there is a sequence of polynomials $P_n(x)$, uniformly bounded on $0 < x < 1$ and converging almost everywhere to $\varphi(x)$. It is easy to see that $\| P_n \| \to \| \varphi \|$. We thus obtain $\left| \int_0^1 \varphi f \, dx \right| \leqq C_2 \| \varphi \|$; and replacing φ by $\varphi \operatorname{sign} f$, $\int_0^1 \varphi |f| \, dx \leqq C_2 \| \varphi \|$. Using the lemma of 2.2, we deduce $\int_0^1 \psi f^* \, dx \leqq C_2 \| \psi \|$ for any bounded measurable function $\psi(x)$. Putting $\psi(x) = \alpha x^{\alpha - 1}$ for $\delta \leqq x < 1$, $\psi(x) = 0$ for $0 < x < \delta$, this becomes

$$\alpha \int_\delta^1 x^{\alpha-1} f^*(x) \, dx \leqq C_2 , \qquad (\delta > 0).$$

Thus in fact $f(x)$ belongs to $\Lambda(\alpha)$.

5.4. Theorem 20. *A sequence μ_n is the moment sequence of a function $f \in M(\alpha)$ if and only if there is a constant K such that*

5.4(1) $$\sum_{i=0}^{k} |\mu_{n\nu_i}| \leqq K \left(\frac{k+1}{n+1} \right)^\alpha, \qquad 0 \leqq \nu_0 < \cdots < \nu_k \leqq n; \;\; n = 1, 2, \cdots$$

a) *The condition is necessary.* For brevity, we write

5.4(2) $$s(x) = \sum_{i=0}^{k} p_{n\nu_i}(x), \qquad \sigma(x) = \sum_{\nu=0}^{k} p_{n\nu}(x)$$

Using 5.2(2), we deduce for the sum of our condition 5.4(1),

$$\sum_{i=0}^{k} |\mu_{n\nu_i}| \leqq \int_0^1 |f(x)| \sum_{i=0}^{k} p_{n\nu_i}(x) \, dx \leqq \| f \|_{M(\alpha)} \| s \|_{\Lambda(\alpha)}.$$

Thus our proof will be complete if we can show that for the function $s(x)$,

5.4(3) $$\| s \|_{\Lambda(\alpha)} \leqq C \left(\frac{k+1}{n+1} \right)^\alpha,$$

where the constant C is independent of the choice of the integers $n, k, \nu_0, \cdots, \nu_k$. This is easy for the function $\sigma(x)$, for $\sigma(x)$ is readily seen to decrease, and so

$$\begin{aligned} \|\sigma\|_{\Lambda(\alpha)} &= \alpha \sum_{\nu=0}^{k} \binom{n}{\nu} \int_0^1 x^{\nu+\alpha-1}(1-x)^{n-\nu}\,dx \\ &= \alpha \sum_{0}^{k} \binom{n}{\nu} \mathrm{B}(\nu+\alpha, n-\nu+1) = \alpha \sum_0^k \frac{\Gamma(\nu+\alpha)\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(\nu+1)} \\ &\leqq C_1 \sum_0^k \frac{(\nu+1)^{\alpha-1}}{(n+1)^\alpha} \leqq C_2 \left(\frac{k+1}{n+1}\right)^\alpha. \end{aligned}$$

Thus we only need to show that

$$\|s\| \leqq C_3 \|\sigma\|. \tag{5.4(4)}$$

We need the following lemma:

Lemma. *Let $g(x)$, $h(x)$ be positive on* $(0, 1)$ *and suppose that $h(x)$ is decreasing. If for every measurable set $e \subset$* $(0, 1)$

$$\int_e g(x)\,dx \leqq \int_0^{me} h(x)\,dx,$$

then, in the metric of $\Lambda(\alpha)$,

$$\|g\| \leqq \|h\|. \tag{5.4(5)}$$

For the function g^* we have $\int_0^\xi g^*\,dx \leqq \int_0^\xi h\,dx$, and hence $\int_0^\xi f\,dx \geqq 0$ for every $0 \leqq \xi \leqq 1$, with $f = h - g^*$. By the second law of mean, for every bounded decreasing positive function $\varphi(x)$ we have

$$\begin{aligned} \int_0^1 \varphi f\,dx &= \varphi(0) \int_0^\xi f\,dx + \varphi(1) \int_\xi^1 f\,dx \\ &= [\varphi(0) - \varphi(1)] \int_0^\xi f\,dx + \varphi(1) \int_0^1 f\,dx \geqq 0. \end{aligned} \tag{5.4(6)}$$

Putting $\varphi(x) = x^{\alpha-1}$ in $\delta \leqq x \leqq 1$ and $\varphi(x) = \delta^{\alpha-1}$ in $0 \leqq x \leqq \delta$ and making $\delta \to 0$ we obtain

$$\int_0^1 x^{\alpha-1} g^*\,dx \leqq \int_0^1 x^{\alpha-1} h\,dx,$$

which is equivalent to 5.4(5). This establishes our result.

By the lemma, and since $\sigma(x)$ is decreasing, it is sufficient to show that for some constant C_3

$$\int_e s(x)\,dx \leqq C_3 \int_0^\delta \sigma(x)\,dx, \qquad \delta = me. \tag{5.4(7)}$$

We have

$$\int_e s(x)\,dx \leqq \delta \quad \text{and} \quad \leqq \sum_{i=0}^{k} \int_0^1 p_{n\nu_i}(x)\,dx = \frac{k+1}{n+1}.$$

First suppose $k = 0$. Putting $\delta_1 = \min(\delta, (n+1)^{-1})$, we obtain

$$\int_0^\delta \sigma\,dx \geqq \int_0^{\delta_1} (1-x)^n\,dx \geqq \int_0^{\delta_1} \left(1 - \frac{1}{n+1}\right)^n dx \geqq C\delta_1, \qquad C > 0,$$

which proves 5.4(7). Next suppose $k \geqq 1$. 5.4(7) will follow, if we can show that

$$\int_0^\delta \sigma(x)\,dx \geqq C_4 \min(\delta, k/n), \qquad C_4 > 0,$$

Let $\delta_1 = 2^{-1} \min(\delta, k/n)$, then

$$\text{5.4(8)} \qquad \int_0^\delta \sigma\,dx \geqq \int_0^{\delta_1} \sum_{\nu \leqq 2n\delta_1} p_\nu\,dx \geqq \int_0^{\delta_1} \sum_{\nu \leqq 2nx} p_\nu\,dx = \int_0^{\delta_1} \sigma_1(x)\,dx,$$

say. We wish to show that the sum $\sigma_1(x)$ has a positive lower bound for all $0 \leqq x \leqq \delta_1$ and all $n = 1, 2, \cdots$. (This fact has a simple probabilistic meaning.) By an inequality of S. Bernstein [2] (see also Fréchet [4, p. 130])

$$\text{5.4(9)} \qquad \sum_{|\nu - nx| \geqq 2\alpha(nx(1-x))^{\frac{1}{2}}} p_{n\nu}(x) \leqq 2e^{-\alpha^2},$$

if

$$\text{5.4(10)} \qquad 0 < x < 1 \quad \text{and} \quad 0 \leqq \alpha \leqq \tfrac{3}{2}(nx(1-x))^{\frac{1}{2}}.$$

If we put $\alpha = \dfrac{1}{2}\left(\dfrac{nx}{1-x}\right)^{\frac{1}{2}}$, 5.4(10) is fulfilled for $0 \leqq x \leqq \delta_1 \leqq \frac{1}{2}$ and by 5.4(9),

$$\text{5.4(11)} \qquad \sum_{\nu > 2nx} p_{n\nu}(x) = \sum_{|\nu - nx| > nx} p_{n\nu}(x) < 2e^{-nx/4}.$$

Let A be so large, that $2\exp(-A/4) < \frac{1}{2}$; for $x > An^{-1}$ the sum 5.4(11) is $<\frac{1}{2}$, and thus

$$\sigma_1(x) = \sum_{\nu \leqq 2nx} p_{n\nu}(x) \geqq \tfrac{1}{2}.$$

On the other hand, if $0 \leqq x \leqq An^{-1}$,

$$\sigma_1(x) \geqq p_{n,0}(x) \geqq (1 - An^{-1})^n \geqq C_5,$$

the constant $C_5 > 0$ being independent of n. Therefore, putting $C_6 = \min(C_5, \frac{1}{2})$, we have $\sigma_1(x) \geqq C_6$ for all $0 \leqq x \leqq \delta_1$ and from 5.4(8) we deduce

$$\int_0^\delta \sigma(x)\,dx \geqq C_6\delta_1,$$

which completes the proof.

b). *The condition is sufficient.* The proof runs parallel to that of the corresponding part of Theorem 19. Instead of the inequality 5.2(3) we use 5.2(4) to show that 5.4(1) implies

$$\sum_{\nu=0}^{n} \left(\frac{\nu+1}{n+1}\right)^{\beta-1} \mu_{n\nu}^{*} \leqq C_1, \qquad (\beta > 1 - \alpha).$$

Thus 5.2(5) is fulfilled with some $0 < \beta < 1$ instead of α, and the moment problem possesses a solution $f \in \Lambda(\beta)$. We then show that $f \in M(\alpha)$ in the same manner as in Theorem 19.

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