J. Math. Anal. Appl. 458 (2018) 112–122

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Remarks on Stampacchia Lemma

Hongya Gao[a], Francesco Leonetti[b,*], Lianhong Wang[a]

[a] College of Mathematics and Information, Hebei University, Baoding, 071002, China
[b] DISIM, L'Aquila University, L'Aquila, 67100, Italy

ARTICLE INFO

Article history:
Received 26 May 2017
Submitted by A. Cianchi

Keywords:
Stampacchia Lemma
Regularity
Solution

ABSTRACT

We compare two versions of Stampacchia Lemma and we give an application to the regularity of entropy solutions to some anisotropic boundary value problems.
© 2017 Elsevier Inc. All rights reserved.

In dealing with regularity issues of weak solutions of elliptic partial differential equations as well as minima of variational integrals, an efficient tool is the well-known Stampacchia Lemma (see Lemma 4.1 in [14]).

Lemma 1. *Let c_1, α, β be positive constants. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be decreasing and such that*

$$\varphi(h) \leq \frac{c_1}{(h-k)^\alpha}[\varphi(k)]^\beta \tag{1}$$

for every h, k with $h > k \geq k_0$. It results that:
(i) if $\beta > 1$ then we have

$$\varphi(k_0 + d) = 0, \tag{2}$$

where

$$d = \left\{c_1[\varphi(k_0)]^{\beta-1}2^{\alpha\beta/(\beta-1)}\right\}^{1/\alpha}; \tag{3}$$

(ii) if $\beta = 1$ then for any $k \geq k_0$ we have

* Corresponding author.
E-mail address: leonetti@univaq.it (F. Leonetti).

http://dx.doi.org/10.1016/j.jmaa.2017.08.056
0022-247X/© 2017 Elsevier Inc. All rights reserved.

$$\varphi(k) \leq \varphi(k_0) e^{1-(c_1 e)^{-1/\alpha}(k-k_0)}; \tag{4}$$

(iii) if $\beta < 1$ and $k_0 > 0$ then for any $k \geq k_0$ we have

$$\varphi(k) \leq 2^{\frac{\alpha}{(1-\beta)^2}} \left[c_1^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} \varphi(k_0) \right] \left(\frac{1}{k} \right)^{\frac{\alpha}{1-\beta}}. \tag{5}$$

In [10] Kovalevsky and Voitovich quote Stampacchia Lemma and give more general Lemmas that they need in order to deal with more difficult situations. Among such Lemmas, there is the following one (see Lemma 4 in [10])

Lemma 2. *Let c_2, α, β be positive constants. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be decreasing and such that*

$$\varphi(2k) \leq \frac{c_2}{k^\alpha} [\varphi(k)]^\beta \tag{6}$$

for every $k \geq k_0$.
If $\beta < 1$ and $k_0 > 0$ then for any $k \geq k_0$ we have

$$\varphi(k) \leq 2^{\frac{\alpha}{(1-\beta)^2}} \left[c_2^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} \varphi(k_0) \right] \left(\frac{1}{k} \right)^{\frac{\alpha}{1-\beta}}. \tag{7}$$

Let us compare Lemma 1 (iii) with Lemma 2: the statement is the same; is assumption (6) weaker than (1)? The answer is No: the two assumptions are equivalent! Indeed we have the following

Remark 1. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be decreasing. Let $\alpha \in (0, +\infty)$ and $\beta \in (0, 1)$ be constants. Then

$$(1) \iff (6). \tag{8}$$

Proof. "$\Longrightarrow$". Assume (1). We take $h = 2k$ and we get (6) with $c_2 = c_1$.
"$\Longleftarrow$". Assume (6). Let us consider $h > k \geq k_0$. We split the proof into two cases: $2^{n+1}k \geq h > 2^n k$ for some integer $n \geq 1$ and $2k \geq h > k$.
Case $2^{n+1}k \geq h > 2^n k$ for some integer $n \geq 1$. Since φ decreases, we have $\varphi(h) \leq \varphi(2^n k) = \varphi(2(2^{n-1}k))$; we keep in mind that $n \geq 1$ so $2^{n-1}k \geq k \geq k_0$ and we can use (6) with $2^{n-1}k$ in place of k: we have

$$\varphi(2(2^{n-1}k)) \leq \frac{c_2}{(2^{n-1}k)^\alpha} [\varphi(2^{n-1}k)]^\beta.$$

Since $2^{n-1}k \geq k$, we use the monotonicity of φ to have $\varphi(2^{n-1}k) \leq \varphi(k)$; then $[\varphi(2^{n-1}k)]^\beta \leq [\varphi(k)]^\beta$. Since $2^{n+1}k \geq h$, we have $(2^{n+1} - 1)k \geq h - k$, then

$$2^{n-1}k = \frac{2^{n+1}k}{4} \geq \frac{(2^{n+1} - 1)k}{4} \geq \frac{h-k}{4},$$

thus

$$\varphi(h) \leq \varphi(2^n k) = \varphi(2(2^{n-1}k)) \leq \frac{c_2}{(2^{n-1}k)^\alpha} [\varphi(2^{n-1}k)]^\beta \leq \frac{4^\alpha c_2}{(h-k)^\alpha} [\varphi(k)]^\beta.$$

Case $2k \geq h > k$. Since φ decreases we have $\varphi(h) \leq \varphi(k) = [\varphi(k)]^\beta [\varphi(k)]^{1-\beta}$. We use Lemma 2 and we get (7):

$$\varphi(k) \le c_3 \left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}},$$

where

$$c_3 = 2^{\frac{\alpha}{(1-\beta)^2}} \left[c_2^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} \varphi(k_0)\right].$$

Then

$$\varphi(h) \le \varphi(k) = [\varphi(k)]^\beta [\varphi(k)]^{1-\beta} \le [\varphi(k)]^\beta \left[c_3 \left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}}\right]^{1-\beta} = [\varphi(k)]^\beta c_3^{1-\beta} \left(\frac{1}{k}\right)^\alpha.$$

Since $2k \ge h$ we get $k > h-k$ and $\left(\frac{1}{k}\right)^\alpha < \left(\frac{1}{h-k}\right)^\alpha$, then

$$\varphi(h) \le [\varphi(k)]^\beta c_3^{1-\beta} \left(\frac{1}{k}\right)^\alpha \le [\varphi(k)]^\beta c_3^{1-\beta} \left(\frac{1}{h-k}\right)^\alpha.$$

In both cases we have obtained (1) with $c_1 = \max\{4^\alpha c_2; c_3^{1-\beta}\}$. $\square$

The previous Remark 1 suggests the following question: when $\beta = 1$ are (1) and (6) equivalent? The answer is No, as the following Remark says.

Remark 2. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be decreasing. Let $\alpha \in (0, +\infty)$ and $\beta = 1$ be constants. Then

$$(1) \not\Leftarrow (6). \tag{9}$$

More precisely, the function

$$\varphi(k) = e^{-[\ln(k)]^2}, \qquad k \in [1, +\infty) \tag{10}$$

verifies (6) with $\beta = 1$, $\alpha = 2\ln(2)$, $c_2 = 2^{-\ln(2)}$ but it does not satisfy (1) with $\beta = 1$, for any choice of the two constants $\alpha > 0$ and $c_1 > 0$.

Proof. Let us take φ as in (10), then

$$\begin{aligned}
\varphi(2k) &= e^{-[\ln(2k)]^2} = e^{-[\ln(2)+\ln(k)]^2} = e^{-[\ln(k)]^2 - 2\ln(k)\ln(2) - [\ln(2)]^2} \\
&= e^{-[\ln(k)]^2} e^{-2\ln(k)\ln(2) - [\ln(2)]^2} = \varphi(k) e^{-[\ln(2)][2\ln(k)+\ln(2)]} \\
&= \varphi(k) e^{-[\ln(2)][\ln(2k^2)]} = \varphi(k) e^{[\ln((2k^2)^{-\ln(2)})]} = \varphi(k)(2k^2)^{-\ln(2)} \\
&= \varphi(k) \left(\frac{1}{2k^2}\right)^{\ln(2)}.
\end{aligned}$$

This shows that (6) holds true with $\beta = 1$, $\alpha = 2\ln(2)$, $c_2 = 2^{-\ln(2)}$. Now we are going to show that (1) does not hold true with $\beta = 1$: by contradiction, if (1) would hold true with $\beta = 1$, then Stampacchia Lemma, part (ii), would guarantees (4), then

$$\varphi(k) \le c_4 e^{-\gamma k}, \qquad \forall k \in [1, +\infty)$$

for suitable constants $c_4, \gamma \in (0, +\infty)$. That is

$$e^{-[\ln(k)]^2} \le c_4 e^{-\gamma k},$$

this means that

$$\frac{1}{c_4} \le e^{-\gamma k+[\ln(k)]^2},$$

but this is false when $k \to +\infty$ since $e^{-\gamma k+[\ln(k)]^2} \to 0$. Then (1) cannot hold true with $\beta = 1$. □

Remark 1 suggests also the following question: when $\beta > 1$ are (1) and (6) equivalent? The answer is No, as the following Remark says.

Remark 3. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be decreasing. Let $\alpha \in (0, +\infty)$ and $\beta > 1$ be constants. Then

$$(1) \not\Leftarrow (6). \tag{11}$$

More precisely, the function

$$\varphi(k) = e^{-k^p}, \qquad p = \log_2(2\beta), \qquad k \in [1, +\infty) \tag{12}$$

verifies (6) with $\beta > 1$, $c_2 = 1$, any $\alpha > 0$ and a suitable $k_0 = k_0(\alpha, \beta) \ge 1$, but it does not satisfy (1) for any choice of the three constants $\beta > 1$, $\alpha > 0$ and $c_1 > 0$.

Proof. Let us take φ as in (12); we keep in mind that $2^p = 2\beta$ and we have

$$\varphi(2k) = e^{-(2k)^p} = e^{-2^p k^p} = e^{-2\beta k^p} = (e^{-k^p})^{2\beta} = (\varphi(k))^{2\beta} = (e^{-k^p})^{\beta}(\varphi(k))^{\beta}.$$

Note that there exists $k_0 = k_0(\alpha, \beta) \ge 1$ such that

$$(e^{-k^p})^{\beta} \le \left(\frac{1}{k}\right)^{\alpha}, \qquad \forall k \in [k_0, +\infty).$$

Then

$$\varphi(2k) = (e^{-k^p})^{\beta}(\varphi(k))^{\beta} \le \left(\frac{1}{k}\right)^{\alpha}(\varphi(k))^{\beta}, \quad \text{for every } k \ge k_0,$$

so that φ verifies (6) with the selected $\beta > 1$, with $c_2 = 1$, with any $\alpha > 0$ and with a suitable $k_0 = k_0(\alpha, \beta) \ge 1$. We claim that such a φ does not satisfy (1) for any choice of the constants $\beta > 1$, $\alpha > 0$, $c_1 > 0$, $k_0 \ge 1$. Indeed, if such a φ would satisfy (1) for some constants $\beta > 1$, $\alpha > 0$, $c_1 > 0$ and $k_0 \ge 1$, then part (i) of Stampacchia Lemma would imply (2):

$$\varphi(k_0 + d) = 0$$

for a suitable $d \ge 0$: this gives a contradiction since $\varphi(k) > 0$ for every $k \in [1, +\infty)$. □

In the sequel Ω will be a bounded open subset of $\mathbb{R}^N$ and $p_1, \cdots, p_N \in (1, +\infty)$. Moreover, let $\bar{p}$ be the harmonic mean of $p_1, \cdots, p_N$, that is, $\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$; let us assume that $\bar{p} < N$; let $\bar{p}^*$ be the Sobolev exponent of $\bar{p}$, that is, $\bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}$. The following proposition is a consequence of Stampacchia Lemma, which can be found, for example, in Proposition 2.1 of [9].

Proposition 1. *Let $v : \Omega \to \mathbb{R}$ be a measurable function, let c_5, α, k_0 be positive constants and $0 < \beta < 1$. If, for every $k, h \in \mathbb{R}$ such that $h > k \geq k_0$, we have*

$$|\{|v| > h\}| \leq \frac{c_5}{(h-k)^\alpha} |\{|v| > k\}|^\beta,$$

then $v \in L^{\frac{\alpha}{1-\beta}}_{weak}(\Omega)$.

About weak Lebesgue spaces $L^p_{weak}(\Omega)$, see [15]. With this proposition in hand, Kovalevsky proved in [9] the following proposition, which is useful for the description of integrability of solutions to anisotropic problems.

Proposition 2. *Let $v \in W^{1,(p_i)}_0(\Omega)$, and let c_6, k_0 be positive constants and $0 < \beta < 1$. If, for every $k \geq k_0$, we have*

$$\int_{\{|v|>k\}} \sum_{i=1}^N |D_i v|^{p_i} \leq c_6 |\{|v| > k\}|^{\frac{\beta \bar{p}}{\bar{p}^*}},$$

then $v \in L^{\frac{\bar{p}^*}{1-\beta}}_{weak}(\Omega)$.

Lemma 2 allows us to give some applications to regularity of functions.

Proposition 3. *Let $v : \Omega \to \mathbb{R}$ be a measurable function, let c_7, α, k_0 be positive constants and $0 < \beta < 1$. If, for every $k \geq k_0$, we have*

$$k^\alpha |\{|v| > 2k\}| \leq c_7 |\{|v| > k\}|^\beta, \tag{13}$$

then $v \in L^{\frac{\alpha}{1-\beta}}_{weak}(\Omega)$.

Proof. We let

$$\varphi(k) = |\{|v| > k\}|.$$

The assumption (13) yields for every $k \geq k_0$,

$$k^\alpha \varphi(2k) \leq c_7 [\varphi(k)]^\beta.$$

By Lemma 2,

$$|\{|v| > k\}| \leq c_8 \left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}}, \quad \forall k \geq k_0,$$

where

$$c_8 = 2^{\frac{\alpha}{(1-\beta)^2}} \left[c_7^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} |\Omega| \right].$$

Thus

$$\begin{aligned}&\sup_{k>0} k^{\frac{\alpha}{1-\beta}}|\{|v|>k\}|\\ &\le \left(\sup_{0<k<k_0}+\sup_{k\ge k_0}\right)k^{\frac{\alpha}{1-\beta}}|\{|v|>k\}|\\ &\le k_0^{\frac{\alpha}{1-\beta}}|\Omega|+c_8<\infty.\end{aligned}$$

This ends the proof. □

Proposition 4. *Let $v \in W_0^{1,(p_i)}(\Omega)$, and let c_9, k_0 be positive constants, $0 \le \theta < 1$ and $0 < \beta < 1$. If, for every $k \ge k_0$, we have*

$$\int_{\{|v|>k\}} \sum_{i=1}^{N} |D_i v|^{p_i} \le c_9 k^{\theta \bar{p}} |\{|v|>k\}|^{\frac{\beta\bar{p}}{\bar{p}^*}},$$

then $v \in L_{weak}^{\frac{\bar{p}^*(1-\theta)}{1-\beta}}(\Omega)$.

Proof. Define, for s in $\mathbb{R}$ and for $k>0$, the truncation T_k at level k as follows: $T_k(s)=s$ if $-k \le s \le k$, $T_k(s)=k$ if $k<s$, $T_k(s)=-k$ if $s<-k$. Let us consider the function $G_k(s)=s-T_k(s)$. Note that $G_k(s)=0$ for $-k \le s \le k$. Applying the anisotropic Sobolev inequality (see [7,11,12,16])

$$\|u\|_{L^{\bar{p}^*}(\Omega)} \le C \prod_{j=1}^{N} \|D_j u\|_{L^{p_j}(\Omega)}^{\frac{1}{N}}, \quad \forall u \in W_0^{1,(p_i)}(\Omega) \tag{14}$$

to $u = G_k(v) \in W_0^{1,(p_i)}(\Omega)$, we get for every $k \ge k_0$,

$$\begin{aligned}&\left(\int_{\{|v|>k\}} |G_k(v)|^{\bar{p}^*}\right)^{\frac{\bar{p}}{\bar{p}^*}}\\ &= \left(\int_{\Omega} |G_k(v)|^{\bar{p}^*}\right)^{\frac{\bar{p}}{\bar{p}^*}}\\ &\le C\prod_{j=1}^{N}\left(\int_{\Omega}|D_j G_k(v)|^{p_j}\right)^{\frac{\bar{p}}{p_j N}}\\ &\le C\prod_{j=1}^{N}\left(\sum_{i=1}^{N}\int_{\Omega}|D_i G_k(v)|^{p_i}\right)^{\frac{\bar{p}}{p_j N}}\\ &= C\sum_{i=1}^{N}\int_{\Omega}|D_i G_k(v)|^{p_i}\\ &= C\int_{\{|v|>k\}}\sum_{i=1}^{N}|D_i v|^{p_i}\\ &\le Cc_9 k^{\theta\bar{p}}|\{|v|>k\}|^{\frac{\beta\bar{p}}{\bar{p}^*}}.\end{aligned}$$

We observe that $G_k(v) \ge k$ on $\{|v|>2k\}$ and we thus have

$$k^{\bar{p}^*}|\{|v|>2k\}| \le \int_{\{|v|>2k\}} |G_k(v)|^{\bar{p}^*} \le \int_{\{|v|>k\}} |G_k(v)|^{\bar{p}^*}.$$

Then

$$k^{\bar{p}^*(1-\theta)}|\{|v|>2k\}| \le c_{10}|\{|v|>k\}|^{\beta}, \quad k \ge k_0,$$

where $c_{10} = (Cc_9)^{\frac{\bar{p}^*}{\bar{p}}}$. The assumption of Proposition 3 holds with

$$\alpha = \bar{p}^*(1-\theta).$$

This ends the proof. □

Previous Proposition 4 can be weakened as follows.

Proposition 5. *Let $v : \Omega \to \mathbb{R}$ be measurable and let c_{11}, k_0 be positive constants, $0 \le \theta < 1$ and $0 < \beta < 1$. If, for every $k \ge k_0$, we have $T_k(v) \in W_0^{1,1}(\Omega)$ with*

$$\int_\Omega \sum_{i=1}^N |D_i[T_k(G_k(v))|^{p_i} \le c_{11}k^{\theta\bar{p}}|\{|v|>k\}|^{\frac{\beta\bar{p}}{\bar{p}^*}}, \tag{15}$$

then $v \in L_{weak}^{\frac{\bar{p}^*(1-\theta)}{1-\beta}}(\Omega)$.

Remark 4. Under the assumptions of Proposition 5 we have $T_k(G_k(v)) = T_{2k}(v) - T_k(v) \in W_0^{1,1}(\Omega)$ for every $k \ge k_0$.

Remark 5. If $v \in W_0^{1,1}(\Omega)$, then $T_k(v) \in W_0^{1,1}(\Omega)$ for every $k \ge 0$. Moreover $D_i[T_k(G_k(v))] = 1_{\{k<|v|<2k\}}D_iv$ and (15) reads as follows

$$\int_{\{k<|v|<2k\}} \sum_{i=1}^N |D_iv|^{p_i} \le c_{11}k^{\theta\bar{p}}|\{|v|>k\}|^{\frac{\beta\bar{p}}{\bar{p}^*}}.$$

Proof. Let us set $A_k = \{|v|>k\}$; we consider the function $T_k(G_k(v)) = T_{2k}(v) - T_k(v) \in W_0^{1,1}(\Omega)$; the assumption (15) tells us that $T_k(G_k(v)) \in W_0^{1,(p_i)}(\Omega)$ and we can use the anisotropic Sobolev inequality (14),

$$\begin{aligned}
&k|A_{2k}|^{\frac{1}{\bar{p}^*}} \\
&= \left(\int_{A_{2k}} |T_k(G_k(v))|^{\bar{p}^*}\right)^{\frac{1}{\bar{p}^*}} \\
&\le \left(\int_\Omega |T_k(G_k(v))|^{\bar{p}^*}\right)^{\frac{1}{\bar{p}^*}} \\
&\le C\prod_{j=1}^N \left(\int_\Omega |D_jT_k(G_k(v))|^{p_j}\right)^{\frac{1}{Np_j}} \\
&\le C\prod_{j=1}^N \left(\int_\Omega \sum_{i=1}^N |D_iT_k(G_k(v))|^{p_i}\right)^{\frac{1}{Np_j}} \\
&= C\left(\int_\Omega \sum_{i=1}^N |D_iT_k(G_k(v))|^{p_i}\right)^{\frac{1}{\bar{p}}} \\
&\le c_{12}k^{\theta}|A_k|^{\frac{\beta}{\bar{p}^*}}
\end{aligned} \tag{16}$$

where $c_{12} = Cc_{11}^{\frac{1}{\bar{p}}}$. (16) gives

$$k^{\bar{p}^*(1-\theta)}|A_{2k}| \leq c_{12}^{\bar{p}^*}|A_k|^{\beta}. \tag{17}$$

Then (6) holds true with $\varphi(k) = |A_k|$ and $\alpha = \bar{p}^*(1-\theta)$; we note that

$$\frac{\alpha}{1-\beta} = \frac{\bar{p}^*(1-\theta)}{1-\beta},$$

and Proposition 3 yields the desired result. □

We now consider boundary value problems of the form

$$\begin{cases} -\sum_{i=1}^{N} D_i(a_i(x, Du(x))) = f(x), & \text{in } \Omega, \\ u(x) = u_*(x), & \text{on } \partial\Omega, \end{cases} \tag{18}$$

where $a_i(x,z) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ with $x \mapsto a_i(x,z)$ measurable and $z \mapsto a_i(x,z)$ continuous. We assume that

$$\sum_{i=1}^{N} |z_i - \tilde{z}_i|^{p_i} \leq \sum_{i=1}^{N} (a_i(x,z) - a_i(x,\tilde{z}))(z_i - \tilde{z}_i) \tag{19}$$

and

$$\sum_{i=1}^{N} |a_i(x,z)|^{p_i'} \leq b\left(1 + \sum_{i=1}^{N} |z_i|^{p_i}\right), \tag{20}$$

for almost all $x \in \Omega$ and all $z, \tilde{z} \in \mathbb{R}^N$, where $b \geq 1$ is a constant and p_i' is the Hölder conjugate of p_i. For $m \geq 1$ we let

$$f \in L^m_{weak}(\Omega) \tag{21}$$

and

$$u_* \in W^{1,1}(\Omega) \ \text{ with } \ D_i u_* \in L^{mp_i}_{weak}(\Omega), \ i = 1, \cdots, N. \tag{22}$$

We introduce the following definition.

Definition 1. $\mathcal{T}_0^{1,(p_i)}(\Omega)$ is the set of measurable functions $v : \Omega \to \mathbb{R}$ such that for any $k > 0$ the truncated function $T_k(v)$ belongs to $W_0^{1,(p_i)}(\Omega)$.

Under the assumptions (21) and (22) with $m \geq 1$, it is reasonable to work with entropy solutions, which need less regularity than the usual weak solutions.

Definition 2. A function $u \in u_* + \mathcal{T}_0^{1,(p_i)}(\Omega)$ is an entropy solution of (18) if

$$\int_\Omega \sum_{i=1}^N a_i(x, Du) D_i T_k(u - \varphi) \le \int_\Omega f T_k(u - \varphi) \tag{23}$$

holds true for all $k > 0$ and all $\varphi \in u_* + [W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)]$.

The concept of entropy solution was first introduced in [2] for nonlinear elliptic problems in 1995. It was then adapted to the study of some nonlinear elliptic and parabolic problems. We refer to [1–6,8,13] for some results on entropy solutions. We now use Proposition 5 to prove the following:

Theorem 1. *Let $1 < m < \frac{N}{\bar{p}}$. Let $u \in u_* + \mathcal{T}_0^{1,(p_i)}(\Omega)$ be an entropy solution of (18) under (19), (20), (21) and (22). Then*

$$u \in u_* + L^\gamma_{weak}(\Omega)$$

where

$$\gamma = \frac{Nm(\bar{p} - 1)}{N - m\bar{p}}.$$

Proof. Denote

$$A_k = \{|u - u_*| > k\} \quad \text{and} \quad B_k = \{k < |u - u_*| < 2k\}.$$

Let u be an entropy solution of (18). Our nearest goal is to show that $v = u - u_*$ satisfies (15). To this end, we use

$$\varphi = u_* + T_k(u - u_*) \in u_* + [W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)]$$

as a test function in (23); please, note that $u - \varphi = u - u_* - T_k(u - u_*) = G_k(u - u_*)$, so that

$$T_k(G_k(u - u_*)) = T_{2k}(u - u_*) - T_k(u - u_*) \in W_0^{1,(p_i)}(\Omega)$$

and

$$D_i[T_k(G_k(u - u_*))] = 1_{B_k}(D_i u - D_i u_*);$$

then we have

$$\begin{aligned} &\int_{B_k} \sum_{i=1}^N a_i(x, Du)(D_i u - D_i u_*) \\ &= \int_\Omega \sum_{i=1}^N a_i(x, Du) D_i[T_k(G_k(u - u_*))] \\ &\le \int_\Omega f[T_k(G_k(u - u_*))] \\ &\le k \int_{A_k} |f|. \end{aligned} \tag{24}$$

We let $k_0 = 1$ and we assume $k \geq k_0 = 1$. Using (19), (24), (20) and Young inequality, we thus have

$$\begin{aligned}
&\int_{B_k} \sum_{i=1}^{N} |D_i u - D_i u_*|^{p_i} \\
&\leq \int_{B_k} \sum_{i=1}^{N} (a_i(x, Du) - a_i(x, Du_*))(D_i u - D_i u_*) \\
&\leq k \int_{A_k} |f| + \int_{B_k} \sum_{i=1}^{N} |a_i(x, Du_*)||D_i u - D_i u_*| \\
&\leq k \int_{A_k} |f| + C(\varepsilon) \int_{B_k} \sum_{i=1}^{N} |a_i(x, Du_*)|^{p_i'} + \varepsilon \int_{B_k} \sum_{i=1}^{N} |D_i u - D_i u_*|^{p_i} \\
&\leq k \int_{A_k} |f| + C(\varepsilon) b \int_{B_k} \sum_{i=1}^{N} (1 + |D_i u_*|^{p_i}) + \varepsilon \int_{B_k} \sum_{i=1}^{N} |D_i u - D_i u_*|^{p_i}, \\
&\leq [1 + C(\varepsilon) b] k \int_{A_k} g + \varepsilon \int_{B_k} \sum_{i=1}^{N} |D_i u - D_i u_*|^{p_i},
\end{aligned}$$

here

$$g = |f| + \sum_{i=1}^{N} (1 + |D_i u_*|^{p_i}) \in L^m_{weak}(\Omega)$$

by (21) and (22), and we have used the facts $k \geq 1$ and $B_k \subset A_k$. Take $\varepsilon = \frac{1}{2}$, then

$$\int_{\Omega} \sum_{i=1}^{N} |D_i[T_k(G_k(u - u_*))]|^{p_i} = \int_{B_k} \sum_{i=1}^{N} |D_i u - D_i u_*|^{p_i} \leq c_{13} k \int_{A_k} g \leq c_{13} k |||g|||_m |A_k|^{\frac{1}{m'}},$$

where c_{13} is a positive constant depending only on $p_1, ..., p_N, b$; moreover

$$|||g|||_m = \sup_{E \subset \Omega, |E| > 0} \frac{1}{|E|^{\frac{1}{m'}}} \int_E |g|.$$

Thus (15) holds with $v = u - u_*$, $c_{11} = c_{13} |||g|||_m$, $\theta = \frac{1}{\bar{p}}$ and $\beta = \frac{\bar{p}^*}{m' \bar{p}}$, here

$$0 < \theta < 1 \Leftrightarrow 1 < \bar{p} < \infty$$

and

$$0 < \beta < 1 \Leftrightarrow 1 < m < \frac{N}{\bar{p}}.$$

Note that

$$\frac{\bar{p}^*(1 - \theta)}{1 - \beta} = \gamma.$$

This ends the proof. $\square$

Acknowledgments

The first author thanks NSFC (10371050) and NSF of Hebei Province (A2015201149) for the support; the second author thanks GNAMPA, INdAM, MIUR and UNIVAQ for the support. This paper was finished while the first author was visiting L'Aquila University, he would like to thank DISIM for warm hospitality.

References

[1] A. Alvino, L. Boccardo, V. Ferone, L. Orsina, G. Trombetti, Existence results for nonlinear elliptic equations with degenerate coercivity, Ann. Mat. Pura Appl. 182 (2003) 53–79.
[2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez, An L^1-theory of existence and uniqueness of nonlinear elliptic equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. 22 (1995) 241–273.
[3] L. Boccardo, G.R. Cirmi, Existence and uniqueness of solution of unilateral problems with L^1 data, J. Convex Anal. 6 (1999) 195–206.
[4] L. Boccardo, G. Croce, Elliptic Partial Differential Equations, De Gruyter Studies in Math., vol. 55, 2014.
[5] L. Boccardo, T. Gallouët, L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996) 539–553.
[6] A. Di Castro, Elliptic Problems for Some Anisotropic Operators, Ph.D. Thesis, University "Sapienza", Rome, 2008/2009.
[7] I. Fragalá, F. Gazzola, B. Kawohl, Existence and nonexistence results for anisotropic quasilinear elliptic equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004) 715–734.
[8] T. Kilpeläinen, G. Li, Estimates for p-Poisson equations, Differential Integral Equations 13 (4–6) (2000) 791–800.
[9] A.A. Kovalevsky, Integrability and boundedness of solutions to some anisotropic problems, J. Math. Anal. Appl. 432 (2015) 820–843.
[10] A.A. Kovalevsky, M.V. Voitovich, On the improvement of summability of generalized solutions of the Dirichlet problem for nonlinear equations of the fourth order with strengthened ellipticity, Ukrainian Math. J. 58 (2006) 1717–1733.
[11] S.N. Kruhkov, I.M. Kolodii, On the theory of embedding of anisotropic Sobolev spaces, Russian Math. Surveys 38 (1983) 188–189.
[12] S.M. Nikolsky, Imbedding theorems for functions with partial derivatives considered in various metrics, Izd. Akad. Nauk SSSR 22 (1958) 321–336.
[13] A. Prignet, Existence and uniqueness of entropy solutions of parabolic problems with L^1 data, Nonlinear Anal. 28 (12) (1997) 1943–1954.
[14] G. Stampacchia, Équations elliptiques du second ordre á coefficients discontinus, in: Séminaire de mathématiques supérieures, no. 16, Été, 1965, Les Presses de l'Université de Montréal, 1966.
[15] E. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, 1971.
[16] M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, Ric. Mat. 18 (1969) 3–24.