

Regularity results for the gradient of solutions of linear elliptic systems with *VMO*-coefficients and $L^{1,\lambda}$ data

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Abstract. We prove that if a vector-function f belongs to the Morrey space $L^{1,\lambda}(\Omega, \mathbb{R}^N)$, with $\Omega \subset \mathbb{R}^n$, $n \geq 3$, $N \geq 2$, $\lambda \in [0, n-2]$, then there exists a very weak solution u of the system

$$\begin{cases} -D_i(A_{ij}(x)D_j u) = f & \text{in } \Omega \\ u \in W_0^{1,1}(\Omega, \mathbb{R}^N) \end{cases}$$

such that Du belongs to the space $L_{\text{loc}}^{q,n-q(n-\lambda-1)}(\Omega, \mathbb{R}^{nN})$ for any $q \in [1, \frac{n}{n-1}]$, provided the matrix of coefficients (A_{ij}) has $L^\infty \cap VMO$ entries.

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1 Introduction

The paper is devoted to the study of existence and regularity of solutions to the Dirichlet problem associated to the system¹

$$(1) \quad \begin{cases} A(u) \equiv -D_i(A_{ij}(x)D_j u) = f \\ u \in W_0^{1,1}(\Omega, \mathbb{R}^N) \end{cases}$$

where Ω is an open bounded subset of \mathbb{R}^n ($n \geq 3$) with C^1 -boundary, A is an elliptic operator with *VMO*-coefficients and f belongs to the Morrey space $L^{1,\lambda}(\Omega, \mathbb{R}^N)$, $\lambda \in [0, n-2]$.

The study of linear elliptic equations ($N = 1$) with L^1 (or measure) right-hand side and bounded coefficients was started by G. Stampacchia (see [33, 39]) and was treated later by many authors and by different approaches (see [8, 19, 34, 4]); while, for elliptic systems ($N \geq 2$), several existence results were obtained under additional structural conditions.

As a matter of fact in [20] the authors consider nonlinear elliptic systems

$$\begin{aligned} -\operatorname{div} \sigma(x, u(x), Du(x)) &= \mu & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{aligned}$$

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¹ Einstein's convention will be used throughout the paper.

with $p - q$ growth and a structural condition

$$(2) \quad \sigma(x, u, F) : MF \geq 0$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$, $F \in \mathbb{R}^{nN}$ and all matrices $M \in \mathcal{M}^{N \times N}$ of the form

$$M = Id - a \otimes a$$

with $|a| \leq 1$ (which is satisfied for p -Laplacean).

In this paper we consider linear systems with VMO coefficients without further structural conditions (as for instance condition (2)) and prove existence and regularity results for (suitably defined) weak solutions. A main feature of the paper is that we show that the regularity of solutions improves when the right-hand side function belongs to the Morrey space $L^{1,\lambda}$. The details of the proof show that the results are valid also for BMO coefficients with sufficiently small BMO seminorm.

Here we also recall that the VMO space of functions with “vanishing mean oscillations” (not only bounded), introduced by Sarason in [38], turns out to be very useful in the study of smoothness of weak solutions to elliptic equations or systems (see [13] for a survey and [1]). In fact the VMO condition provides the natural integral-type generalization of continuity allowing for extending several classical results for constant coefficients problems to those with variable ones.

An important ingredient in our approach is the so called A-harmonic approximation Lemma of Duzaar & Steffen [24], a new method allowing for a rapid and elegant implementation of certain comparison procedures. The technique has already been employed in the analysis of partial regularity of solutions to non-linear elliptic systems, while suitable analogs have been obtained for degenerate and parabolic problems [22, 23]. Here we shall use an extension of the A-harmonic approximation lemma, in the version which can be found in the appendix of [21], suitable for applications to problems with a right-hand side exhibiting certain decay properties; see Lemma 3.1 below.

We remark that recent regularity results in Morrey spaces of the type $L^{1,\lambda}$ are in the papers [15, 16, 17, 35].

The paper is organized as follows: we start with notations and a few auxiliary results in Section 2. In Section 3 we recall a known (see [41]) Morrey spaces regularity result saying that for $f \equiv D_i g_i$, with $g_i \in L^{2,\lambda}(\Omega, \mathbb{R}^N)$, the gradient Du of the solution u to the problem (1) belongs to the same space $L^{2,\lambda}(\Omega, \mathbb{R}^{nN})$.

This assertion allows us to prove the existence of the very weak (briefly Stampacchia) solution to (1) for any $f \in L^1(\Omega, \mathbb{R}^N)$ by duality method in Section 4.

An analog of Saint-Venant’s principle for solutions of (1) with $f \equiv 0$ is given in Section 5. In the last Section 6 these results, combined with a Campanato-type approach, yield the local regularity of Du in a suitable Morrey space.

2 Some notations and auxiliary results

In \mathbb{R}^n ($n \geq 3$), with generic point $x = (x_1, x_2, \dots, x_n)$, we shall denote by Ω a bounded open nonempty set with diameter d_Ω and C^1 -boundary $\partial\Omega$.

For $\rho > 0$ and $x_o \in \mathbb{R}^n$ we define

$$\begin{aligned} B(x_o, \rho) &= \{x \in \mathbb{R}^n : |x - x_o| < \rho\}, \\ \Omega(x_o, \rho) &= \Omega \cap B(x_o, \rho), \\ d(x_o, \partial\Omega) &= \text{dist}(x_o, \partial\Omega). \end{aligned}$$

If $y_o = (y_{o1}, \dots, y_{on-1}, 0)$ we define

$$\begin{aligned} B^+(y_o, \rho) &= \{x \in B(y_o, \rho) : x_n > 0\}, \\ \Gamma(y_o, \rho) &= \{x \in B(y_o, \rho) : x_n = 0\}. \end{aligned}$$

Moreover, if $u \in L^1(\Omega(x_o, \rho), \mathbb{R}^N)$ we denote by

$$u_{\Omega(x_o, \rho)} = \frac{1}{|\Omega(x_o, \rho)|} \int_{\Omega(x_o, \rho)} u(x) dx$$

where $|\Omega(x_o, \rho)|$ is the n -dimensional Lebesgue measure of $\Omega(x_o, \rho)$.

Definition 2.1 (Morrey space). Let $q \geq 1$ and $0 \leq \lambda < n$. By $L^{q, \lambda}(\Omega, \mathbb{R}^N)$ we denote the linear space formed by the vector-functions $u \in L^q(\Omega, \mathbb{R}^N)$ for which

$$\|u\|_{L^{q, \lambda}(\Omega)} = \sup_{x_o \in \Omega, 0 < \rho \leq d_\Omega} \left\{ \rho^{-\lambda} \int_{\Omega(x_o, \rho)} |u(x)|^q dx \right\}^{1/q} < +\infty.$$

$L^{q, \lambda}(\Omega, \mathbb{R}^N)$ equipped with the above norm is a Banach space.

Definition 2.2 (Campanato space). Let $q \geq 1$ and $0 \leq \lambda < n + q$. By $\mathcal{L}^{q, \lambda}(\Omega, \mathbb{R}^N)$ we denote the space of all vector-functions $u \in L^q(\Omega, \mathbb{R}^N)$ such that

$$[u]_{\mathcal{L}^{q, \lambda}(\Omega)} = \sup_{x_o \in \Omega, 0 < \rho \leq d_\Omega} \left\{ \rho^{-\lambda} \int_{\Omega(x_o, \rho)} |u(x) - u_{\Omega(x_o, \rho)}|^q dx \right\}^{1/q} < +\infty.$$

Moreover we introduce the notion of *BMO* and *VMO* classes.

Definition 2.3 (John–Nirenberg space). Let Q be a cube in \mathbb{R}^n . By $BMO(Q)$ we denote the space of all functions $u \in L^1(Q, \mathbb{R}^{N^2})$ such that the seminorm defined by

$$[u]_{BMO(Q)} = \sup_{\tilde{Q} \subset Q} \frac{1}{|\tilde{Q}|} \int_{|\tilde{Q}|} |u - u_{\tilde{Q}}| dx$$

is finite, where the supremum is taken over all cubes with sides parallel to coordinate axes.

Let us recall that $\mathcal{L}^{q, n}(Q) \cong BMO(Q)$, $\forall q \geq 1$.

Definition 2.4 (Sarason space). For a matrix-function $w \in L^1(\Omega, \mathbb{R}^{N^2})$ and $r > 0$ we define

$$V(x, r) \equiv \sup_{0 < \rho \leq r} \frac{1}{|\Omega(x, \rho)|} \int_{\Omega(x, \rho)} |w(y) - w_{\Omega(x, \rho)}| dy$$

and we introduce the *VMO*-continuity modulus for w

$$V(r) \equiv \sup_{x \in \Omega} V(x, r).$$

By *VMO* we denote the space of all matrix-functions $w \in L^1(\Omega, \mathbb{R}^{N^2})$ such that

$$V(r) < +\infty \quad \text{for all } 0 < r \leq d_\Omega$$

and

$$\lim_{r \rightarrow 0} V(r) = 0.$$

In Section 3 we will need the following simple

Lemma 2.1. *Let $g_i \in L^\infty(\Omega) \cap VMO$ for $i = 1, 2$. Then $g_3 \equiv g_1 \cdot g_2 \in L^\infty(\Omega) \cap VMO$ and*

$$(3) \quad V_3(r) \leq \sqrt{2\|g_1\|_{L^\infty(\Omega)} \|g_2\|_{L^\infty(\Omega)}} \left(\sqrt{\|g_1\|_{L^\infty(\Omega)} V_2(r)} + \sqrt{\|g_2\|_{L^\infty(\Omega)} V_1(r)} \right),$$

where we denote, for $i = 1, 2, 3$,

$$V_i(r) \equiv \sup_{x \in \Omega, 0 < \rho \leq r} \frac{1}{|\Omega(x, \rho)|} \int_{\Omega(x, \rho)} |g_i(y) - (g_i)_{\Omega(x, \rho)}| dy.$$

Proof. For any $x \in \Omega$, $\rho \in]0, r]$ we have

$$\begin{aligned} V_3(x, \rho) &\leq \frac{1}{|\Omega(x, \rho)|^{1/2}} \left(\int_{\Omega(x, \rho)} |g_1(y) g_2(y) - (g_1)_{\Omega(x, \rho)} (g_2)_{\Omega(x, \rho)}|^2 dy \right)^{1/2} \\ &\leq \frac{1}{|\Omega(x, \rho)|^{1/2}} \left\{ \left(\int_{\Omega(x, \rho)} |g_1(y) g_2(y) - (g_1)_{\Omega(x, \rho)} g_2(y)|^2 dy \right)^{1/2} \right. \\ &\quad \left. + \left(\int_{\Omega(x, \rho)} |g_2(y) - (g_2)_{\Omega(x, \rho)}|^2 |(g_1)_{\Omega(x, \rho)}|^2 dy \right)^{1/2} \right\} \end{aligned}$$

where we have used Hölder inequality and the fact that

$$\int_{\Omega(x, \rho)} |g_3(y) - (g_3)_{\Omega(x, \rho)}|^2 dy = \inf_{c \in \mathbb{R}} \int_{\Omega(x, \rho)} |g_3(y) - c|^2 dy.$$

As, for $i = 1, 2$, it is $(g_i)_{\Omega(x, \rho)} \leq \|g_i\|_{L^\infty(\Omega)}$ we have

$$V_3(x, r) \leq \left\{ \|g_2\|_{L^\infty(\Omega)} \sqrt{2\|g_1\|_{L^\infty(\Omega)} V_1(r)} + \|g_1\|_{L^\infty(\Omega)} \sqrt{2\|g_2\|_{L^\infty(\Omega)} V_2(r)} \right\}$$

and the thesis follows at once.

3 $L^{2,\lambda}$ -regularity of Du on $\overline{\Omega}$

If $u : \Omega \rightarrow \mathbb{R}^N$, we set

$$D_i \equiv \frac{\partial}{\partial x_i}, \quad Du = (D_i u^r)_{\substack{i=1, \dots, n \\ r=1, \dots, N}}.$$

Let $A_{ij}(x)$, $i, j = 1, 2, \dots, n$, be matrix-functions for which the following conditions are satisfied:

$$(4) \quad \begin{aligned} A_{ij}(x) &= \left(A_{ij}^{rs}(x) \right)_{r,s=1, \dots, N} \in L^\infty(\Omega, \mathbb{R}^{N^2}) \cap VMO, \\ A_{ij}^{rs}(x) &= A_{ji}^{sr}(x) \quad \text{for a.a. } x \in \Omega, \end{aligned}$$

there exist two positive constants A_1 and A_2 such that

$$(5) \quad \begin{aligned} A_2 |\xi|^2 &\geq A_{ij}(x) \xi_i \xi_j \geq A_1 |\xi|^2 \\ \text{for a.a. } x \in \Omega, \forall \xi &= (\xi_i^r) \in \mathbb{R}^{nN}. \end{aligned}$$

For $x \in \overline{\Omega}$, $0 < r \leq d_\Omega$, we set

$$(6) \quad V(x, r) \equiv \sup_{\substack{0 < \rho \leq r \\ i, j=1, 2, \dots, n}} \frac{1}{|\Omega(x, \rho)|} \int_{\Omega(x, \rho)} |A_{ij}(y) - (A_{ij})_{\Omega(x, \rho)}| dy.$$

In this section we are concerned with regularity of a weak solution $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ of the problem

$$(7) \quad D_i(A_{ij}(x)D_j u) = D_i g_i \quad \text{in } \Omega$$

where Ω is an open bounded domain with C^1 boundary, $g_i \in L^{2,\lambda}(\Omega, \mathbb{R}^N)$ and the coefficients satisfy conditions (4) and (5).

The main result of this section, Theorem 3.1, is not new and can be found in [41]. However we present here a proof, for reader's convenience, based on Steffen, Duzaar, Grotowski method of A-harmonic approximation (see [24, 21, 30]).

Theorem 3.1. *Let $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ be the weak solution to the problem (7), let conditions (4) and (5) be satisfied. Assume that $g_i \in L^{2,\lambda}(\Omega, \mathbb{R}^N)$, for $\lambda \in [0, n[$.*

Then $Du \in L^{2,\lambda}(\Omega, \mathbb{R}^{nN})$ and there exists a positive constant $c = c_V(n, \lambda, A_1, A_2, \Omega)$ such that the inequality

$$(8) \quad \|Du\|_{L^{2,\lambda}(\Omega)} \leq c \|g\|_{L^{2,\lambda}(\Omega)}$$

holds.²

In particular, if

$$(9) \quad \lambda \in]n - 2, n[$$

then $u \in C^{0,\gamma}(\overline{\Omega}, \mathbb{R}^N)$, with $\gamma = 1 - \frac{n-\lambda}{2}$, and the inequality

$$(10) \quad [u]_{C^{0,\gamma}(\overline{\Omega})} \leq c \|g\|_{L^{2,\lambda}(\Omega)}$$

holds.

Corollary 3.1. *Let $\Omega = B(x_o, R)$, $0 < R \leq 1$, and assume the hypotheses of the theorem with $\lambda \in]n - 2, n[$. Then $u \in C^{0,\gamma}(\overline{\Omega}, \mathbb{R}^N)$ and there exists a positive constant $c = c_V(n, \lambda, A_1, A_2)$, which is independent of R , such that the inequality*

$$(11) \quad [u]_{C^{0,\gamma}(\overline{\Omega})} \leq c \|g\|_{L^{2,\lambda}(\Omega)}$$

holds.

Proof. The corollary is true for $\Omega = B(0, 1)$ by the previous theorem.

Let us perform the following change of variables

$$\tilde{u}(y) := R^{-1}u(x_o + Ry), \quad \tilde{A}_{ij}(y) := A_{ij}(x_o + Ry),$$

$$\tilde{g}(y) := g(x_o + Ry), \quad y \in B(0, 1),$$

and note that the transformed coefficients \tilde{A}_{ij} are still in the VMO class.

In fact it is not difficult to see that, denoting by \tilde{V} the VMO -continuity modulus for \tilde{A}_{ij} , it holds

$$\tilde{V}(r) = V(Rr) \leq V(r).$$

Moreover, \tilde{u} satisfies the system (7) with coefficients \tilde{A}_{ij} and right-hand side \tilde{g} and

$$\|\tilde{g}\|_{L^{2,\lambda}(B(0,1))} = R^{\lambda-n} \|g\|_{L^{2,\lambda}(B(x_o,R))},$$

$$[\tilde{u}]_{C^{0,\gamma}(\overline{B(0,1)})} R^{1-\gamma} = [u]_{C^{0,\gamma}(\overline{B(x_o,R)})}.$$

The aforementioned remarks prove the corollary.

² As a permanent convention we will denote by $c_V(\dots, \Omega)$ a constant which depends on various parameters, on the coefficients of the system through the smallness of their VMO -continuity modulus and on the geometrical properties of the involved domain Ω .

Before beginning the proof of the Theorem 3.1 we present some auxiliary results.

The first Lemma 3.1 has its origin in an A-harmonic approximation Lemma of F. Duzaar and J. F. Grotowski (Lemma 2.1, p. 292 of [21]) reformulated by M. Giaquinta (see Lemma A.1 of [21]) and slightly modified by M. Pošta (for parabolic boundary case see [37]). The second Lemma 3.2 can be deduced in a similar way from Lemma 2.1, p. 357 of [30].

Lemma 3.1. *Let $0 < \Lambda_1 < \Lambda_2$ and $n, N \in \mathbb{N}$, with $n \geq 2$, be fixed.*

Then for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that the following holds:

for any bilinear form \bar{A} on \mathbb{R}^{nN} , satisfying the conditions

$$(12) \quad \langle \bar{A} \xi, \xi \rangle \geq \Lambda_1 |\xi|^2,$$

$$(13) \quad |\bar{A}| \leq \Lambda_2,$$

and for any $u \in W^{1,2}(B(x_o, \rho), \mathbb{R}^N)$ there exists $h \in W^{1,2}(B(x_o, \rho), \mathbb{R}^N)$ such that

$$(14) \quad -\operatorname{div}(\bar{A} Dh) = 0 \quad \text{on } B(x_o, \rho),$$

$$(15) \quad \int_{B(x_o, \rho)} |Dh|^2 dx \leq \int_{B(x_o, \rho)} |Du|^2 dx,$$

there exists $\varphi \in C_0^1(B(x_o, \rho), \mathbb{R}^N)$ so that

$$(16) \quad \|D\varphi\|_{L^\infty(B(x_o, \rho))} \leq \frac{1}{\rho}$$

and

$$(17) \quad \int_{B(x_o, \rho)} |u - h|^2 dx \leq C(\varepsilon) \rho^{4-n} \left(\int_{B(x_o, \rho)} \bar{A} Du D\varphi dx \right)^2 + \varepsilon \rho^2 \int_{B(x_o, \rho)} |Du|^2 dx.$$

Lemma 3.2. *Let $0 < \Lambda_1 < \Lambda_2$ and $n, N \in \mathbb{N}$, with $n \geq 2$, be fixed.*

Then for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that the following holds:

For any bilinear form \bar{A} on \mathbb{R}^{nN} , satisfying the conditions (12) and (13), and for any $u \in W^{1,2}(B^+(y_o, \rho), \mathbb{R}^N)$, such that $u = 0$ on $\Gamma(y_o, \rho)$, there exists $h \in W^{1,2}(B^+(y_o, \rho), \mathbb{R}^N)$ such that:

$$(18) \quad \begin{aligned} -\operatorname{div}(\bar{A} Dh) &= 0 \quad \text{on } B^+(y_o, \rho), \\ h &= 0 \quad \text{on } \Gamma(y_o, \rho), \end{aligned}$$

$$(19) \quad \int_{B^+(y_o, \rho)} |Dh|^2 dx \leq \int_{B^+(y_o, \rho)} |Du|^2 dx,$$

there exists $\varphi \in C_0^1(B^+(y_o, \rho), \mathbb{R}^N)$ so that

$$(20) \quad \|D\varphi\|_{L^\infty(B^+(y_o, \rho))} \leq \frac{1}{\rho}$$

and

$$(21) \quad \int_{B^+(y_o, \rho)} |u - h|^2 dx \leq C(\varepsilon) \rho^{4-n} \left(\int_{B^+(y_o, \rho)} \bar{A} Du D\varphi dx \right)^2 \\ + \varepsilon \rho^2 \int_{B^+(y_o, \rho)} |Du|^2 dx.$$

Proof. Let us outline the proof of the lemma.

Firstly observe that it is enough to prove it for $\rho = 1$ since the full assertion follows by a standard homotopy argument. Lemma 2.1 of [30] in this case and in our notation reads as follows:

Let $0 < \Lambda_1 < \Lambda_2$ and $n, N \in \mathbb{N}$, with $n \geq 2$, be fixed.

Then for any $\varepsilon > 0$ there exists a constant $\delta(n, N, \Lambda_1, \Lambda_2, \varepsilon) \in]0, 1]$ with the following property:

For any bilinear form \bar{A} on \mathbb{R}^{nN} , satisfying the conditions (12) and (13), and for any $w \in W^{1,2}(B^+(y_o, 1), \mathbb{R}^N)$, such that $w = 0$ on $\Gamma(y_o, 1)$,

$$\int_{B^+(y_o, 1)} |Dw|^2 dx \leq 1$$

and

$$(22) \quad \left| \int_{B^+(y_o, 1)} \bar{A}(Dw, D\varphi) dx \right| \leq \delta \sup_{y \in B^+(y_o, 1)} |D\varphi(y)|$$

for all $\varphi \in C_0^1(B^+(y_o, 1), \mathbb{R}^N)$, there exists a function $v \in W^{1,2}(B^+(y_o, 1), \mathbb{R}^N)$ solving

$$-\operatorname{div}(\bar{A}Dv) = 0 \quad \text{on } B^+(y_o, 1)$$

such that

$$\int_{B^+(y_o, 1)} |Dv|^2 dx \leq 1, \quad v = 0 \quad \text{on } \Gamma(y_o, 1)$$

and

$$\int_{B^+(y_o, 1)} |v - w|^2 dx \leq \varepsilon.$$

Let us deduce now the assertion of Lemma 3.2.

Fix any $\varepsilon > 0$ and find a δ , as in the quoted lemma, so that (22) holds for $w := \frac{u}{\int_{B^+(y_o,1)} |Du|^2 dx}$.³ Then there is a function v satisfying the above requirements, i.e. the function $h := v \int_{B^+(y_o,1)} |Du|^2 dx$ fulfills (18), (19) and (21) choosing $\varphi \equiv 0$.

If, vice versa, there is a nonzero $\psi \in C_0^1(B^+(y_o, 1), \mathbb{R}^N)$ such that

$$(23) \quad \left| \int_{B^+(y_o,1)} A(Dw, D\psi) dx \right| > \delta \sup_{y \in B^+(y_o,1)} |D\psi(y)|,$$

we set

$$h = 0, \quad \varphi = \frac{\psi}{\sup_{y \in B^+(y_o,1)} |D\psi|}, \quad C(\varepsilon) = \frac{C_P}{\delta}$$

where C_P is the constant from Poincaré's inequality (see e.g. Theorem 2.2 p. 359 of [30]).

As $\|D\varphi\|_{L^\infty(B^+(y_o,1))} = 1$, inequality (23) and Poincaré's inequality imply

$$\int_{B^+(y_o,1)} |w - h|^2 dx \leq C_P \int_{B^+(y_o,1)} |Dw|^2 dx \leq C(\varepsilon) \left| \int_{B^+(y_o,1)} A(Dw, D\varphi) dx \right|.$$

Then multiplying the last inequality by $\int_{B^+(y_o,1)} |Du|^2 dx$ we have the assertion.

Next we prove local interior and boundary analogues of Theorem 3.1. The following lemma is the analog of Theorem 9.1 at p. 339 of [11].

Lemma 3.3. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a solution to the equation (7) where $g_i \in L^{2,\lambda}(\Omega, \mathbb{R}^N)$, with $\lambda \in [0, n]$, and let conditions (4) and (5) be satisfied.*

Then there exist two positive constants $c = c(n, \lambda, A_1, A_2)$ and $\rho_o = \rho_V(n, \lambda, A_1, A_2)$ such that

$$(24) \quad \|Du\|_{L^{2,\lambda}(B(z_o, R/4))} \leq c \left(R^{-\lambda/2} \|Du\|_{L^2(B(z_o, R))} + \|g\|_{L^{2,\lambda}(\Omega)} \right)$$

for any $z_o \in \Omega$ and $0 < R < \min\{d(z_o, \partial\Omega), \rho_o\}$.

Proof. Let $x_o \in \Omega$ and fix σ and ρ such that $0 < \sigma < \rho < d(x_o, \partial\Omega)$.

By Caccioppoli's inequality (see e.g. Theorem 1.V at p. 46 of [12] or Lemma 5.II at p. 328 of [11]) we have

$$(25) \quad \int_{B(x_o, \sigma/2)} |Du|^2 dx \leq c_1(A_1, A_2) \left[\frac{1}{\sigma^2} \int_{B(x_o, \sigma)} |u - u_{B(x_o, \sigma)}|^2 dx + \int_{B(x_o, \sigma)} |g|^2 dx \right].$$

³ Note that the assertion is trivial for $\int_{B^+(y_o,1)} |Du|^2 dx = 0$.

Set $\bar{A} = (A_{ij})_{B(x_o, \rho)}$ and recall that Lemma 3.1 ensures the existence of a function $h \in W^{1,2}(B(x_o, \rho), \mathbb{R}^N)$ satisfying system (14) and inequalities (15) and (17). Thus, from (25) we deduce

$$(26) \quad \int_{B(x_o, \sigma/2)} |Du|^2 dx \leq 2c_1 \frac{1}{\sigma^2} \left(\int_{B(x_o, \sigma)} |u - u_{B(x_o, \sigma)} - (h - h_{B(x_o, \sigma)})|^2 dx \right. \\ \left. + \int_{B(x_o, \sigma)} |h - h_{B(x_o, \sigma)}|^2 dx \right) + c_1 \sigma^\lambda \|g\|_{L^{2,\lambda}(\Omega)}.$$

As the function h is a solution to (14), by Poincaré and the well-known Campanato's inequality (see Theorem 3.1 at p. 54 of [12]), we obtain

$$I_1 \equiv \int_{B(x_o, \sigma)} |h - h_{B(x_o, \sigma)}|^2 dx \leq c_2(n) \sigma^2 \int_{B(x_o, \sigma)} |Dh|^2 dx \\ \leq c_3(n, \Lambda_1, \Lambda_2) \sigma^2 \left(\frac{\sigma}{\rho} \right)^n \int_{B(x_o, \rho)} |Dh|^2 dx$$

whence, by (15),

$$(27) \quad I_1 \leq c_3 \sigma^2 \left(\frac{\sigma}{\rho} \right)^n \int_{B(x_o, \rho)} |Du|^2 dx.$$

On the other hand, (17) yields

$$(28) \quad I_2 \equiv \int_{B(x_o, \sigma)} |u - u_{B(x_o, \sigma)} - (h - h_{B(x_o, \sigma)})|^2 dx \\ \leq \int_{B(x_o, \rho)} |u - h|^2 dx \\ \leq \varepsilon \rho^2 \int_{B(x_o, \rho)} |Du|^2 dx + C(\varepsilon) \rho^{4-n} \left(\int_{B(x_o, \rho)} (A_{ij})_{B(x_o, \rho)} D_i u D_j \varphi dx \right)^2$$

for a function $\varphi \in C_0^1(B(x_o, \rho), \mathbb{R}^N)$ with $\|D\varphi\|_{L^\infty(B(x_o, \rho))} \leq \frac{1}{\rho}$.

Using the fact that u is solution to (7) we have

$$\int_{B(x_o, \rho)} (A_{ij})_{B(x_o, \rho)} D_i u D_j \varphi dx = \int_{B(x_o, \rho)} [(A_{ij})_{B(x_o, \rho)} - A_{ij}] D_i u D_j \varphi dx \\ + \int_{B(x_o, \rho)} g_i D_i \varphi dx$$

whence, with $\|D\varphi\|_{L^\infty(B(x_o, \rho))} \leq \frac{1}{\rho}$ and Hölder's inequality, we get

$$(29) \quad \left| \int_{B(x_o, \rho)} (A_{ij})_{B(x_o, \rho)} D_i u D_j \varphi dx \right|^2 \\ \leq \frac{2}{\rho^2} \left\{ \int_{B(x_o, \rho)} |A_{ij} - (A_{ij})_{B(x_o, \rho)}|^2 dx \int_{B(x_o, \rho)} |Du|^2 dx + \rho^n \int_{B(x_o, \rho)} |g|^2 dx \right\}.$$

Inserting (29) into (28) and using the properties of the matrix $A = (A_{ij})$, we achieve

$$(30) \quad I_2 \leq \varepsilon \rho^2 \int_{B(x_o, \rho)} |Du|^2 dx \\ + C(\varepsilon) \Lambda_2 V(x_o, \rho) \rho^2 \int_{B(x_o, \rho)} |Du|^2 dx \\ + C(\varepsilon) \rho^2 \int_{B(x_o, \rho)} |g|^2 dx.$$

Combining together (26), (27) and (30) we obtain, for all $0 < \sigma < \rho < d(x_o, \partial\Omega)$ and for all $\varepsilon > 0$,

$$(31) \quad \int_{B(x_o, \sigma/2)} |Du|^2 dx \\ \leq c_4(n, \Lambda_1, \Lambda_2) \left[\varepsilon \left(\frac{\rho}{\sigma} \right)^2 + C(\varepsilon) \Lambda_2 V(x_o, \rho) \left(\frac{\rho}{\sigma} \right)^2 + \left(\frac{\sigma}{\rho} \right)^n \right] \int_{B(x_o, \rho)} |Du|^2 dx \\ + C_1(\varepsilon) \left(\frac{\rho}{\sigma} \right)^2 \rho^\lambda \|g\|_{L^{2, \lambda}(\Omega)}^2.$$

The above inequality can be rewritten as

$$(32) \quad \int_{B(x_o, \tau\rho)} |Du|^2 dx \\ \leq c_4 \tau^\alpha \left[\varepsilon \tau^{-2-\alpha} + C(\varepsilon) \Lambda_2 V(x_o, \rho) \tau^{-2-\alpha} + \tau^{n-\alpha} \right] \int_{B(x_o, \rho)} |Du|^2 dx \\ + C_1(\varepsilon) \tau^{-2} \rho^\lambda \|g\|_{L^{2, \lambda}(\Omega)}^2$$

for any $x_o \in \Omega$, $\tau \in]0, 1[$, $\rho \in]0, d(x_o, \partial\Omega)[$, $\varepsilon > 0$ and $\alpha \in]\lambda, n[$.

Let us fix $\alpha = \frac{n+\lambda}{2}$ and choose $\tau_o \in]0, \min\{1, (\frac{1}{4c_4})^{\frac{1}{n-\alpha}}\}[$ and $\varepsilon_o \in]0, \frac{1}{4c_4} \tau_o^{2+\alpha}[$. Thus, inequality (32) becomes

$$(33) \quad \int_{B(x_o, \tau_o\rho)} |Du|^2 dx \\ \leq \tau_o^\alpha \left[1/2 + c_4 C(\varepsilon_o) \Lambda_2 V(x_o, \rho) \tau_o^{-2-\alpha} \right] \int_{B(x_o, \rho)} |Du|^2 dx \\ + C_1(\varepsilon_o) \tau_o^{-2} \rho^\lambda \|g\|_{L^{2, \lambda}(\Omega)}^2, \quad \forall \rho \in]0, d(x_o, \partial\Omega)[.$$

Finally, observe that there exists $\rho_o = \rho_V(n, \lambda, \Lambda_1, \Lambda_2) > 0$ such that $c_4 C(\varepsilon_o) \cdot \Lambda_2 \tau_o^{-2-\alpha} V(x_o, \rho) \leq 1/2$ for all $x_o \in \Omega$ and for all $\rho \in]0, \min\{\rho_o, d_\Omega\}]$.⁴

Hence, for all $\rho \in]0, \min\{\rho_o, d(x_o, \partial\Omega)\}]$, inequality (33) becomes

$$(34) \quad \int_{B(x_o, \tau_o \rho)} |Du|^2 dx \leq \tau_o^{\frac{\lambda+n}{2}} \int_{B(x_o, \rho)} |Du|^2 dx + B \rho^\lambda$$

where $B = C_1(\varepsilon_o) \tau_o^{-2} \|g\|_{L^{2,\lambda}(\Omega)}^2$.

Then Lemma 7.3 at p. 229 of [29] gives, for all $0 < \theta < \rho < \min\{\rho_o, d(x_o, \partial\Omega)\}$,

$$(35) \quad \int_{B(x_o, \theta)} |Du|^2 dx \leq C(\tau_o, \varepsilon_o, \lambda, n) \theta^\lambda \left(\frac{1}{\rho^\lambda} \int_{B(x_o, \rho)} |Du|^2 dx + \|g\|_{L^{2,\lambda}(\Omega)}^2 \right).$$

Let us fix now $z_o \in \Omega$, $R \in]0, \min\{\rho_o, d(z_o, \partial\Omega)\}]$, $x_o \in B(z_o, R/4)$ and $\theta \in]0, R/2]$.

Then, due to the fact that $B(x_o, R/2) \subset B(z_o, 3/4R) \subset \subset \Omega$, we observe that $R/2 < \min\{\rho_o, d(x_o, \partial\Omega)\}$.

The aforementioned remarks and (35) yield

$$(36) \quad \begin{aligned} \theta^{-\lambda} \int_{B(z_o, R/4) \cap B(x_o, \theta)} |Du|^2 dx &\leq \theta^{-\lambda} \int_{B(x_o, \theta)} |Du|^2 dx \\ &\leq C(\tau_o, \varepsilon_o, \lambda, n) \left(\frac{2^\lambda}{R^\lambda} \int_{B(x_o, R/2)} |Du|^2 dx + \|g\|_{L^{2,\lambda}(\Omega)}^2 \right) \\ &\leq C(\tau_o, \varepsilon_o, \lambda, n) \left(\frac{1}{R^\lambda} \int_{B(z_o, R)} |Du|^2 dx + \|g\|_{L^{2,\lambda}(\Omega)}^2 \right) \end{aligned}$$

which, taking supremum for $x_o \in B(z_o, R/4)$ and $\theta \in]0, R/2]$, concludes the proof.

Corollary 3.2. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a solution to the system (7) where $g_i \in L^{2,\lambda}(\Omega, \mathbb{R}^N)$ with $\lambda \in [0, n[$ and let conditions (4) and (5) be satisfied.*

Then $Du \in L_{\text{loc}}^{2,\lambda}(\Omega, \mathbb{R}^{nN})$ and for all $H \subset \subset \Omega$ there exists a positive constant $c = c(n, \lambda, \Lambda_1, \Lambda_2)$, such that

$$(37) \quad \|Du\|_{L^{2,\lambda}(H)} \leq c \left[(\min\{\rho_o, d(\overline{H}, \partial\Omega)\})^{-\lambda/2} \|Du\|_{L^2(\Omega)} + \|g\|_{L^{2,\lambda}(\Omega)} \right]$$

where ρ_o is as in the previous lemma.

Proof. Let $H \subset \subset \Omega$, $x_o \in H$ and set $\rho_1 = \min\{\rho_o, d(\overline{H}, \partial\Omega)\}$.

⁴ Note that ρ_o is independent of x_o because of the uniform convergence $V(x_o, \rho) \rightarrow 0$ for $\rho \rightarrow 0$ with respect $x_o \in \Omega$.

Since $\rho_1 \leq \min\{\rho_o, d(x_o, \partial\Omega)\}$, inequality (35) rewritten for $\rho = \rho_1$ gives

$$\int_{B(x_o, \theta) \cap H} |Du|^2 dx \leq c(\lambda, n, A_1, A_2) \theta^\lambda \left(\frac{1}{\rho_1^\lambda} \int_{\Omega} |Du|^2 dx + \|g\|_{L^{2,\lambda}(\Omega)}^2 \right), \quad \forall \theta \in]0, \rho_1[.$$

For $\theta \in [\rho_1, d_H]$ the proof is obvious.

Now, for fixed $y_o = (y_{o1}, y_{o2}, \dots, 0)$ and $R > 0$, let us take into account the system

$$(38) \quad \begin{aligned} D_i(B_{ij}(x)D_j u') &= D_i g_i && \text{in } B^+(y_o, R), \\ u' &= 0 && \text{on } \Gamma(y_o, R) \end{aligned}$$

under the following structural assumptions:

$$(39) \quad \begin{aligned} B_{ij}(x) &= \left(B_{ij}^{rs}(x) \right)_{r,s=1,\dots,N} \in L^\infty(B^+(y_o, R), \mathbb{R}^{N^2}) \cap VMO, \\ B_{ij}^{rs}(x) &= B_{ji}^{sr}(x) \quad \text{for a.a. } x \in B^+(y_o, R), \end{aligned}$$

there exist two positive constants A'_1 and A'_2 such that

$$(40) \quad \begin{aligned} A'_2 |\xi|^2 &\geq B_{ij}(x) \xi_i \xi_j \geq A'_1 |\xi|^2 \\ &\text{for a.a. } x \in B^+(y_o, R), \forall \xi = (\xi_i^r) \in \mathbb{R}^{nN}. \end{aligned}$$

We denote by V' the *VMO*-continuity modulus for the matrix B_{ij} .

Definition 3.1. A vector-function $u' \in W^{1,2}(B^+(y_o, R), \mathbb{R}^N)$ is a *weak solution* of the system (38) if

$$\begin{cases} \int_{B^+(y_o, R)} B_{ij}(x) D_i u' D_j \varphi, dx = \int_{B^+(y_o, R)} D_i g D_j \varphi dx, & \forall \varphi \in W_0^{1,2}(B^+(y_o, R), \mathbb{R}^N) \\ u' = 0 & \text{on } \Gamma(y_o, R). \end{cases}$$

The following lemma is the analog of Theorem 13.1 at p. 355 of [11].

Lemma 3.4. Let $u' \in W^{1,2}(B^+(y_o, R), \mathbb{R}^N)$ be a solution to the problem (38) where $g_i \in L^{2,\lambda}(B^+(y_o, R), \mathbb{R}^N)$, with $\lambda \in [0, n[$, and let conditions (39) and (40) be satisfied.

Then there exist two positive constants $c = c(n, \lambda, A'_1, A'_2)$ and $\bar{\rho} = \bar{\rho}_{V'}(n, \lambda, A'_1, A'_2)$ such that

$$(41) \quad \begin{aligned} \|Du'\|_{L^{2,\lambda}(B^+(y_o, R_o))} &\leq c \left[(\min\{\bar{\rho}, R - R_o\})^{-\lambda/2} \|Du'\|_{L^2(B^+(y_o, R))} + \|g\|_{L^{2,\lambda}(B^+(y_o, R))} \right] \end{aligned}$$

for any $0 < R_o < R$.

Proof. The proof of the lemma follows the same procedure of the foregoing lemma. We will outline it briefly indicating different steps.

Firstly, let us fix $R_o \in]0, R[$, $y \in \Gamma(y_o, R_o)$ and $0 < \sigma < \rho < R - R_o$.

By Caccioppoli's inequality (see Lemma 5.III⁵ at p. 329 of [11]) we obtain

$$(42) \quad \int_{B^+(y, \sigma/2)} |Du'|^2 dx \leq c_1(A'_1, A'_2) \left[\frac{1}{\sigma^2} \int_{B^+(y, \sigma)} |u'|^2 dx + \int_{B^+(y, \sigma)} |g|^2 dx \right].$$

By Lemma 3.2 with $\bar{A} = (B_{ij})_{B^+(y, \rho)}$ we get a function $h \in W^{1,2}(B^+(y, \rho), \mathbb{R}^N)$ satisfying (18), (19) and (21).

Thus,

$$\begin{aligned} & \int_{B^+(y, \sigma/2)} |Du'|^2 dx \\ & \leq 2c_1 \frac{1}{\sigma^2} \left(\int_{B^+(y, \sigma)} |u' - h|^2 dx + \int_{B^+(y, \sigma)} |h|^2 dx \right) + c_1 \sigma^\lambda \|g\|_{L^{2,\lambda}(B^+(y_o, R))}^2. \end{aligned}$$

As $h = 0$ on $\Gamma(y, \rho)$, by Poincaré (see e.g. [30]) and Campanato inequalities⁶ (see e.g. also Proposition 2.7 at p. 206 of [28]) and (19) we deduce

$$\begin{aligned} I_1 & \equiv \int_{B^+(y, \sigma)} |h|^2 dx \leq c_2(n) \sigma^2 \int_{B^+(y, \sigma)} |Dh|^2 dx \\ & \leq c_3(n, A'_1, A'_2) \sigma^2 \left(\frac{\sigma}{\rho} \right)^n \int_{B^+(y, \rho)} |Dh|^2 dx \leq c_3 \sigma^2 \left(\frac{\sigma}{\rho} \right)^n \int_{B^+(y, \rho)} |Du'|^2 dx. \end{aligned}$$

The estimate of $I_2 \equiv \int_{B^+(y, \sigma)} |u' - h|^2 dx$ can be obtained as in (28), (29) and (30) of Lemma 3.3, taking use of Lemma 3.2 instead of Lemma 3.1.

⁵ The proof of this lemma remains unchanged in the case of several equations ($N \geq 2$).

⁶ Campanato's inequality

$$\int_{B^+(y, \sigma)} |Dh|^2 dx \leq c(A'_1, A'_2) \left(\frac{\sigma}{\rho} \right)^n \int_{B^+(y, \rho)} |Dh|^2 dx$$

follows immediately gathering together Corollary 11.I and Lemma 11.II at p. 352 of [11]. The proof of Corollary 11.I requires only the extension of Lemma 11.I at p. 350 of [11] to the case of several equations. To this goal it is enough to proceed as in the proof of the aforementioned Lemma 11.I just observing that the normal derivative $D_n D_n h$ satisfies the system of linear equations

$$(B_{nn})_{B^+(y, \rho)} D_n D_n h = - \left\{ \sum_{j=1}^{n-1} (B_{nj})_{B^+(y, \rho)} D_j D_n h + \sum_{i=1}^{n-1} \sum_{j=1}^n (B_{ij})_{B^+(y, \rho)} D_i D_j h \right\}$$

where the matrix $(B_{nn})_{B^+(y, \rho)}$ is non degenerate and such that $\|(B_{nn})_{B^+(y, \rho)}^{-1}\| \leq \frac{1}{A'_1}$.

So we obtain the following inequality analog to (33):

$$\begin{aligned}
 (43) \quad & \int_{B^+(y, \tau_o \rho)} |Du'|^2 dx \\
 & \leq \tau_o^\alpha \left[1/2 + c_4 C(\varepsilon_o) A'_2 V'(y, \rho) \tau_o^{-2-\alpha} \right] \int_{B^+(y, \rho)} |Du'|^2 dx \\
 & \quad + C_1(\varepsilon_o) \tau_o^{-2} \rho^\lambda \|g\|_{L^{2,\lambda}(B^+(y_o, R))}^2, \\
 & \quad \forall \rho \in]0, R - R_o[, \quad \forall y \in \Gamma(y_o, R_o)
 \end{aligned}$$

where the positive constants $\alpha, c_4, \varepsilon_o$ and $\tau_o \in]0, 1[$ now depend on n, λ, A'_1, A'_2 .

On the other hand there exists $\bar{\rho} = \bar{\rho}_{V'}(n, \lambda, A'_1, A'_2) > 0$ such that $c_4 C(\varepsilon_o) A'_2 V'(y, \rho) \cdot \tau_o^{-2-\alpha} \leq 1/2$ for any $\rho \in]0, \min\{\bar{\rho}, 2R\}]$ and $y \in \Gamma(y_o, R_o)$.

Thus, as in (35), we deduce

$$\begin{aligned}
 (44) \quad & \int_{B^+(y, \theta)} |Du'|^2 dx \\
 & \leq c_5(\lambda, n, A'_1, A'_2) \theta^\lambda \left(\frac{1}{\rho^\lambda} \int_{B^+(y, \rho)} |Du'|^2 dx + \|g\|_{L^{2,\lambda}(B^+(y_o, R))}^2 \right)
 \end{aligned}$$

for all $0 < \theta < \rho < \min\{\bar{\rho}, R - R_o\}$.

Setting now $d_o = \frac{1}{2} \min\{\bar{\rho}, R - R_o\}$, for $\theta \in]0, d_o[$ the above formula yields⁷

$$\begin{aligned}
 (45) \quad & \int_{B^+(y_o, R_o) \cap B^+(y, \theta)} |Du'|^2 dx \\
 & \leq c_5 \theta^\lambda \left(\frac{1}{d_o^\lambda} \int_{B^+(y, d_o)} |Du'|^2 dx + \|g\|_{L^{2,\lambda}(B^+(y_o, R))}^2 \right) \\
 & \leq c_5 \theta^\lambda \left(\frac{1}{d_o^\lambda} \int_{B^+(y_o, R)} |Du'|^2 dx + \|g\|_{L^{2,\lambda}(B^+(y_o, R))}^2 \right).
 \end{aligned}$$

Let us now fix $y \in B^+(y_o, R_o)$, consider the point $y' = (y_1, \dots, y_{n-1}, 0)$ and choose $0 < \theta < \frac{1}{2} \min\{\bar{\rho}, R - R_o\}$ (the other case being obvious).

If $\theta \geq y_n$ then $B(y, \theta) \cap B^+(y_o, R_o) \subset \subset B^+(y', 2\theta)$ and we can use (45).

On the contrary, if $\theta < y_n$ then $B(y, \theta) \cap B^+(y_o, R_o) \subset \subset B^+(y_o, \frac{R+R_o}{2})$ and in this case we can apply Corollary 3.2.

Proof of Theorem 3.1

Since Ω is of class C^1 and bounded (see e.g. [31] p. 305), there exist a positive R and an open finite covering $\{\mathcal{B}(\bar{y}^j, R)\}_{j=1, \dots, \nu}$ of $\partial\Omega$ such that for all \bar{y}^j there exists a C^1 -function ζ^j , defined on a domain $D \subset \mathbb{R}^{n-1}$ such that with respect to a suitable system of coordinates $\{y_1, \dots, y_n\}$, with the origin at \bar{y}^j :

⁷ The case when $\theta \in [d_o, 2R_o]$ is clear.

- (a) the set $\partial\Omega \cap \mathcal{B}(\bar{y}^j, R)$ can be represented by an equation of the type $y_n = \zeta^j(y_1, \dots, y_{n-1})$,
 (b) each $y \in \Omega \cap \mathcal{B}(\bar{y}^j, R)$ satisfies $y_n > \zeta^j(y_1, \dots, y_{n-1})$.

Without loss of generality we can suppose that the system of coordinates is such that the hyperplane tangent to $\partial\Omega$ at \bar{y}^j has equation $y_n = 0$, that

$$(46) \quad \zeta^j(\bar{y}^j) = D\zeta^j(\bar{y}^j) = 0$$

and that R is such that $\max_{j=1, \dots, \nu} \max_{\overline{\mathcal{B}(\bar{y}^j, R) \cap \Omega}} |D\zeta^j| < 1/2$.

For such domains the portion of boundary within the ball $\mathcal{B}(\bar{y}^j, R)$ can be straightened by means of the smooth transformation⁸

$$(47) \quad \begin{cases} \psi_i(y) = y_i - (\bar{y}^j)_i & \text{for } i = 1, 2, \dots, n-1 \\ \psi_n(y) = y_n - \zeta^j(y_1, \dots, y_{n-1}). \end{cases}$$

It turns out that $\psi(y) = (\psi_1(y), \dots, \psi_n(y))$ is a $C^1(\mathcal{B}(\bar{y}^j, R))$ -diffeomorphism verifying the following properties (see e.g. [31] p. 305 or Theorem V at p. 375 of [11]):

- (i) $\psi(\mathcal{B}(\bar{y}^j, R) \cap \partial\Omega) = \{x \in \mathbb{R}^n : x_n = 0, |x_i| < R, \text{ for } i = 1, \dots, n-1\}$,
 (ii) $\frac{1}{2}|y - \bar{y}^j| \leq |\psi(y)| \leq \frac{3}{2}|y - \bar{y}^j|, \quad \forall y \in \mathcal{B}(\bar{y}^j, R) \cap \Omega$,
 (iii) $B^+(0, R/2) \subset \psi(\mathcal{B}(\bar{y}^j, R) \cap \Omega) \subset B^+(0, \frac{3}{2}R)$,
 $\mathcal{B}(\bar{y}^j, \frac{2}{3}R) \cap \Omega \subset \psi^{-1}(B^+(0, R)) \subset \mathcal{B}(\bar{y}^j, 2R) \cap \Omega$.

If $z \in B^+(0, R)$ we set

$$(48) \quad \begin{aligned} B_{ik}(z) &= A_{rs}(\psi^{-1}(z)) \frac{\partial \psi_i}{\partial y_r}(\psi^{-1}(z)) \frac{\partial \psi_k}{\partial y_s}(\psi^{-1}(z)), \\ g_i(z) &= f_r(\psi^{-1}(z)) \frac{\partial \psi_i}{\partial y_r}(\psi^{-1}(z)), \\ u'(z) &= u(\psi^{-1}(z)) \end{aligned}$$

where we have used the fact that the absolute value of the Jacobian determinant of $\psi^{-1}(z)$ is equal to 1.

Let us observe that Lemma 2.1 guarantees that the coefficients $B_{ik}(z)$ still satisfy hypothesis (39).

Moreover, from the definition (47) and the fact that $\max_{\overline{\mathcal{B}(\bar{y}^j, R) \cap \Omega}} |D\zeta^j| < 1/2$, it follows that

$$(49) \quad (1/2)^2 A_1 |\eta|^2 \leq B_{ik} \eta_i \eta_k \leq (3/2)^2 A_2 |\eta|^2 \quad \forall \eta = (\eta_i) \in \mathbb{R}^{nN}.$$

⁸ For the sake of simplicity we will drop the index j relative to the diffeomorphism ψ^j .

Thus, a change of variables in the system (7) yields

$$(50) \quad \begin{cases} u' \in W^{1,2}(B^+(0, R)) \\ u' = 0 \text{ on } \Gamma(0, R) \\ \int_{B^+(0, R)} B_{ik} D_k u' D_i \phi dz = \int_{B^+(0, R)} D_{ig} D_i \phi dz, \quad \forall \phi \in W_0^{1,2}(B^+(0, R)). \end{cases}$$

To the problem (50) we apply Lemma 3.4 and so we conclude that Du' lies in $L^{2,\lambda}(B^+(0, R_o))$, $R_o \in]0, R[$, with norm estimate (41).

As a consequence, the matrix-function $Du'(\psi(y))$, $y \in \mathcal{B}(\overline{\mathcal{Y}}^j, r) \cap \Omega$, $r \in]0, \frac{2}{3}R_o[$, belongs to $L^{2,\lambda}(\mathcal{B}(\overline{\mathcal{Y}}^j, r) \cap \Omega)$ that is, by the chain rule, $Du \in L^{2,\lambda}(\mathcal{B}(\overline{\mathcal{Y}}^j, r) \cap \Omega)$.

Thus, by changing back coordinates in (41) we deduce

$$(51) \quad \|Du\|_{L^{2,\lambda}(\mathcal{B}(\overline{\mathcal{Y}}^j, r) \cap \Omega)} \leq c \left[\|Du\|_{L^2(\Omega)} + \|f\|_{L^{2,\lambda}(\Omega)} \right],$$

where $c = c_V(n, \lambda, A_1, A_2, R, R - R_o)$.

Since R_o is arbitrary it can be chosen sufficiently close to R so that the family $\{\mathcal{B}(\overline{\mathcal{Y}}^j, r)\}_{j=1, \dots, \nu}$ still covers $\partial\Omega$.

On the other hand, set

$$\delta := \min_{\partial\Omega} d(x, \mathbb{R}^n \setminus \bigcup_{j=1}^{\nu} \mathcal{B}(\overline{\mathcal{Y}}^j, r)) > 0,$$

the open set

$$H = \{x \in \Omega : d(x, \partial\Omega) > \delta/2\} \subset\subset \Omega$$

is such that $H, \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_\nu$ cover $\overline{\Omega}$.⁹

The aforementioned remarks, the use of Corollary 3.2 and Lax–Milgram theorem prove the theorem (see e.g. [11] p. 365–366 or [1] p. 252–255).

4 Existence and uniqueness of the Stampacchia solution

G. Stampacchia proved in [39, 33], by duality method, the existence and uniqueness of the weak solution to the Dirichlet boundary problem for elliptic equations with non smooth coefficients and right-hand side measure.

For a right-hand side $f \in L^1(\Omega, \mathbb{R}^N)$ and an operator A satisfying conditions (4) and (5) his procedure can be modified as follows.

Definition 4.1. Let $f \in L^1(\Omega, \mathbb{R}^N)$. We say that a vector-function $u \in W_0^{1,1}(\Omega, \mathbb{R}^N)$ is a *very weak solution* (briefly Stampacchia solution) of the system (1) if it satisfies

$$(52) \quad \begin{aligned} \int_{\Omega} u A(\varphi) dx &= \int_{\Omega} f \varphi dx, \\ \forall \varphi \in \Phi &= \left\{ \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N) \cap C^0(\bar{\Omega}, \mathbb{R}^N) : A(\varphi) \in C^0(\bar{\Omega}, \mathbb{R}^N) \right\}. \end{aligned}$$

⁹ It is worth noticing that $d(\overline{H}, \partial\Omega) \geq \delta/2$ and that δ depends only on the covering of $\partial\Omega$.

The proof of the existence and uniqueness of the Stampacchia solution to (1) follows the same steps as in papers [32, 17] and we give only an outline of it.

Theorem 4.1. *Let Ω be a bounded domain with C^1 -boundary and $f \in L^1(\Omega, \mathbb{R}^N)$. Let conditions (4) and (5) be satisfied.*

Then there exists a unique Stampacchia solution u of the problem (1) such that $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$ for any $q < \frac{n}{n-1}$.

Moreover, there exists a positive constant $c = c_V(n, q, \Lambda_1, \Lambda_2, \Omega)$ such that

$$(53) \quad \|u\|_{W_0^{1,q}(\Omega)} \leq c \|f\|_{L^1(\Omega)}.$$

Proof. If A satisfies (4) then by Lax–Milgram theorem, there exists a linear continuous operator $G : W^{-1,2}(\Omega, \mathbb{R}^N) \rightarrow W_0^{1,2}(\Omega, \mathbb{R}^N)$ such that $\tilde{u} = G(T)$ is the unique weak solution of the equation

$$A(\tilde{u}) = T.$$

For $p > n$ consider $T = D_i g_i$ with $g_i \in L^p$. Then, by Hölder inequality, $g_i \in L^{2,\kappa}(\Omega, \mathbb{R}^N)$ with $0 \leq \kappa \leq n(1 - 2/p)$ and

$$(54) \quad \|g\|_{L^{2,\kappa}(\Omega)} \leq c(n, p) d_\Omega^{n(1/2-1/p)-\kappa/2} \|g\|_{L^p(\Omega)}.$$

Hence, for $p > n$ and A satisfying (4) and (5) we can take $\kappa > n - 2$ and by Theorem 3.1 and (54) we have

$$(55) \quad \max_{\bar{\Omega}} |u| \leq c_V(n, p, \kappa, \Lambda_1, \Lambda_2, \Omega) d_\Omega^{1-\frac{n-\kappa}{2}+n(1/2-1/p)-\kappa/2} \|g\|_{L^p(\Omega)}.$$

As the inequality (55) holds for any representation $T = D_i g_i$ we have

$$(56) \quad \max_{\bar{\Omega}} |u| \leq c_V(n, p, \kappa, \Lambda_1, \Lambda_2, \Omega) d_\Omega^{1-\frac{n-\kappa}{2}+n(1/2-1/p)-\kappa/2} \|T\|_{W^{-1,p}(\Omega)}.$$

Thus G maps continuously $W^{-1,p}(\Omega, \mathbb{R}^N)$ into $C^0(\bar{\Omega}, \mathbb{R}^N)$.

On the other hand, any continuous function T can be represented as $T = D_i g_i$ with $g_i \in L^p(\Omega, \mathbb{R}^N)$ so that

$$\|g\|_{L^p(\Omega)} \leq c(|\Omega|) \max_{\bar{\Omega}} |T|$$

and so (52) holds if and only if

$$(57) \quad \int_{\Omega} u \psi \, dx = \int_{\Omega} f G(\psi) \, dx, \quad \forall \psi \in C^0(\bar{\Omega}, \mathbb{R}^N)$$

i.e. if and only if $u = G^*(f)$ for G^* adjoint of G .

Since G maps continuously $W^{-1,p}(\Omega, \mathbb{R}^N)$ into $C^0(\bar{\Omega}, \mathbb{R}^N)$, then G^* is a continuous linear operator from $L^1(\Omega, \mathbb{R}^N)$ into $W_0^{1,p'}(\Omega, \mathbb{R}^N)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\|G^*\| \leq \|G\|$ which implies the thesis.

Corollary 4.1. *Let $\Omega = B(x_o, R)$ with $0 < R \leq 1$ and assume the hypotheses of the theorem.*

Then there exists a unique Stampacchia solution u of the problem (1) such that $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$ for any $q < \frac{n}{n-1}$.

Moreover, there exists a positive constant $c = c_V(n, q, \Lambda_1, \Lambda_2)$, which is independent of R , such that

$$(58) \quad \|u\|_{W_0^{1,q}(\Omega)} \leq c R^{1-n(1-1/q)} \|f\|_{L^1(\Omega)}.$$

Proof. The proof follows readily from (56) taking into account (11) from Corollary 3.1.

5 Saint-Venant's principle

In this section we consider weak solutions of the homogeneous systems

$$(59) \quad -D_i(A_{ij}(x)D_j v) = 0 \quad \text{in } \Omega.$$

As the right-hand side $g \equiv 0$, Lemmas 3.3 and 3.4 hold with any $\lambda \geq 0$. For $\lambda > n - 2$ any weak solution $v \in W^{1,2}$ to problems (59) is locally Hölder continuous on Ω .

We start obtaining better estimates of Hölder seminorm.

Lemma 5.1. *Let v be a solution to problem (59), let conditions (4) and (5) be satisfied and assume $\lambda \in]n - 2, n[$, $\gamma = 1 - \frac{n-\lambda}{2}$, $q \in [1, 2[$.*

Then there exists a positive constant $c = c(n, \lambda, \Lambda_1, \Lambda_2)$ such that it holds

$$(60) \quad [v]_{C^{0,\gamma}(\overline{B(x_o, \rho)})} \leq c \rho^{n(1/2-1/q)-\lambda/2} \|Dv\|_{L^q(B(x_o, 16\rho))}$$

where $x_o \in \Omega$ and $0 < \rho < \frac{1}{16} \min\{\rho_o, d(x_o, \partial\Omega)\}$ with ρ_o as in Lemma 3.3.

Proof. Let us fix $x_o \in \Omega$ and recall that, according to (24) and embedding of Morrey space, $\forall 0 < R < \min\{\rho_o, d(x_o, \partial\Omega)\}$ we have

$$(61) \quad [v]_{C^{0,\gamma}(\overline{B(x_o, R/4)})}^2 \leq c_1(n, \lambda, \Lambda_1, \Lambda_2) R^{-\lambda} \|Dv\|_{L^2(B(x_o, R))}^2.$$

The above formula, together with (2.5) at p. 14 of [12], Caccioppoli's inequality and the fact that $v - z$ is still a solution of problem (59) for any $z \in \mathbb{R}^N$, gives

$$\begin{aligned}
 (62) \quad & \sup_{\overline{B(x_o, R/8)}} |v - z| \\
 & \leq c_2(n) \left(R^\gamma [v]_{C^{0,\gamma}(\overline{B(x_o, R/8)})} + R^{-n/2} \|v - z\|_{L^2(B(x_o, R/8))} \right) \\
 & \leq c_3(n, \lambda, \Lambda_1, \Lambda_2) \left(R^{\gamma-\lambda/2} \|Dv\|_{L^2(B(x_o, R/2))} + R^{-n/2} \|v - z\|_{L^2(B(x_o, R/8))} \right) \\
 & \leq c_3 \left(R^{\gamma-\lambda/2-1} \|v - z\|_{L^2(B(x_o, R))} + R^{-n/2} \|v - z\|_{L^2(B(x_o, R))} \right) \\
 & = c_3 R^{-n/2} \|v - z\|_{L^2(B(x_o, R))}.
 \end{aligned}$$

Following now the idea of [27] p. 80–81, by (62), we deduce for any $\tau \in]0, R[$, $q \in [1, 2[$ and $z \in \mathbb{R}^N$

$$(63) \quad \sup_{\overline{B(x_o, \tau)}} |v - z| \leq c_3 (R - \tau)^{-n/q} \|v - z\|_{L^q(B(x_o, R))}.$$

Then from (61), (63), Caccioppoli and Poincaré inequalities it follows, $\forall 0 < \rho < \frac{1}{16} \min\{\rho_o, d(x_o, \partial\Omega)\}$,

$$\begin{aligned}
 (64) \quad & [v]_{C^{0,\gamma}(\overline{B(x_o, \rho)})}^2 \leq c_3 \rho^{-\lambda} \|Dv\|_{L^2(B(x_o, 4\rho))}^2 \\
 & \leq c_3 \rho^{-2-\lambda} \|v - v_{B(x_o, 8\rho)}\|_{L^2(B(x_o, 8\rho))}^2 \\
 & \leq c_3 \rho^{-2-\lambda} \sup_{\overline{B(x_o, 8\rho)}} |v - v_{B(x_o, 8\rho)}|^{2-q} \|v - v_{B(x_o, 8\rho)}\|_{L^q(B(x_o, 8\rho))}^q \\
 & \leq c_3 \rho^{n(1-2/q)-\lambda} \|Dv\|_{L^q(B(x_o, 16\rho))}^2.
 \end{aligned}$$

As a consequence of previous lemma we obtain the following

Theorem 5.1 (Saint-Venant Principle). *Let conditions (4) and (5) be satisfied.*

Then, there exists a positive constant $c = c(n, \lambda, \Lambda_1, \Lambda_2)$ such that, for any weak solution v to the system (59), it holds

$$(65) \quad \|Dv\|_{L^q(B(x_o, \rho_1))}^q \leq c \left(\frac{\rho_1}{\rho_2} \right)^{n-q+\gamma q} \|Dv\|_{L^q(B(x_o, \rho_2))}^q$$

$\forall 0 \leq \rho_1 \leq \rho_2 \leq \min\{\rho_o, d(x_o, \partial\Omega)\}$ with ρ_o as in Lemma 3.3. and $\forall q \in [1, 2[$, where $\gamma = 1 - \frac{n-\lambda}{2}$.

Proof. The assertion is obvious for $\rho_1 \geq \frac{1}{32}\rho_2$. Hence assume that $0 < \rho_1 < \frac{1}{32}\rho_2$.

Then by Hölder's inequality, (35) and Caccioppoli's inequality we have

$$\begin{aligned}
 (66) \quad & \|Dv\|_{L^q(B(x_o, \rho_1))}^q \\
 & \leq c_1(n) \rho_1^{n(1-q/2)} \|Dv\|_{L^2(B(x_o, \rho_1))}^q \\
 & \leq c_2(n, \lambda, A_1, A_2) \left(\frac{\rho_1}{\rho_2}\right)^{\lambda q/2} \rho_1^{n(1-q/2)} \|Dv\|_{L^2(B(x_o, \rho_2/32))}^q \\
 & \leq c_2 \left(\frac{\rho_1}{\rho_2}\right)^{\lambda q/2} \rho_2^{-q} \rho_1^{n(1-q/2)} \|v - v_{B(x_o, \rho_2/16)}\|_{L^2(B(x_o, \rho_2/16))}^q.
 \end{aligned}$$

As in the proof of Lemma 5.1 we estimate

$$\begin{aligned}
 & \|v - v_{B(x_o, \rho_2/16)}\|_{L^2(B(x_o, \rho_2/16))}^2 \\
 & \leq c_3(n) \left(\rho_2^\gamma [v]_{C^{0,\gamma}(\overline{B(x_o, \rho_2/16)})}\right)^{2-q} \|v - v_{B(x_o, \rho_2/16)}\|_{L^q(B(x_o, \rho_2/16))}^q.
 \end{aligned}$$

Using Poincaré's inequality and (60) we get

$$\begin{aligned}
 & \|v - v_{B(x_o, \rho_2/16)}\|_{L^2(B(x_o, \rho_2/16))}^2 \\
 & \leq c_4(n, \lambda, A_1, A_2) \rho_2^{\gamma(2-q)+q+(2-q)[n(1/2-1/q)-\lambda/2]} \|Dv\|_{L^q(B(x_o, \rho_2))}^2.
 \end{aligned}$$

Inserting the above estimate into (66) we obtain

$$(67) \quad \|Dv\|_{L^q(B(x_o, \rho_1))}^q \leq c_5(n, \lambda, A_1, A_2) \left(\frac{\rho_1}{\rho_2}\right)^{n+\frac{q}{2}(\lambda-n)} \|Dv\|_{L^q(B(x_o, \rho_2))}^q$$

which proves the theorem.

6 Local regularity of the Stampacchia solution

In this section we will gather the techniques developed in [9] with the nowadays classical method of S. Campanato as we have done in paper [15]. We reproduce this procedure here for reader's convenience.

First of all let us introduce the truncation operator. For a given constant $k > 0$ we define the cut off function $T_k : \mathbb{R} \rightarrow \mathbb{R}$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k \operatorname{sign}(s) & \text{if } |s| > k. \end{cases}$$

For a vector-function $f = (f^r(x))_{r=1,\dots,N}$, $x \in \Omega$, we define the truncated vector-function $f_k = (T_k(f^r))_{r=1,\dots,N}$ pointwise: for every $x \in \Omega$ the value of f_k at x is just $(T_k(f^r(x)))_{r=1,\dots,N}$.

Throughout this section we shall assume the right-hand side of (1) as

$$f \in L^{1,\lambda}(\Omega, \mathbb{R}^N), \quad \lambda \in [0, n-2].$$

For such a vector-function let us consider a sequence of functions $\{f_k\}_{k \in \mathbb{N}}$ such that

- (i) $f_k \in W^{-1,2}(\Omega, \mathbb{R}^N) \cap L^{1,\lambda}(\Omega, \mathbb{R}^N), \quad \forall k \in \mathbb{N},$
- (ii) $f_k \rightarrow f$ in $L^1(\Omega, \mathbb{R}^N)$ as $k \rightarrow +\infty,$
- (iii) $\|f_k\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \quad \forall k \in \mathbb{N},$
- (iv) $\|f_k\|_{L^{1,\lambda}(\Omega)} \leq \|f\|_{L^{1,\lambda}(\Omega)}, \quad \forall k \in \mathbb{N}.$

An example of a sequence satisfying the above requirements is the sequence $\{T_k(f)\}_{k \in \mathbb{N}}$.

For fixed $k \in \mathbb{N}$, let u_k be the weak solution of the system

$$(68) \quad -D_i(A_{ij}(x)D_j u_k) = f_k \quad \text{in } \Omega$$

that is,

$$\begin{cases} u_k \in W_0^{1,2}(\Omega, \mathbb{R}^N) \\ \int_{\Omega} A_{ij}(x)D_j u_k D_i \phi \, dx = \int_{\Omega} f_k \phi \, dx, \quad \forall \phi \in W_0^{1,2}(\Omega, \mathbb{R}^N). \end{cases}$$

We will prove, at first, the following

Theorem 6.1. *Assume that hypotheses (4), (5) hold and let u_k be the weak solution of problem (68).*

Then

$$Du_k \in L_{\text{loc}}^{q,\nu}(\Omega, \mathbb{R}^{nN}), \quad \forall q \in \left[1, \frac{n}{n-1}\right], \quad \forall k \in \mathbb{N},$$

with $\nu = n - q(n - \lambda - 1)$, and for all $H \subset\subset \Omega$ there exists a positive constant $c = c_V(n, \lambda, q, A_1, A_2, d(\overline{H}, \partial\Omega))$ such that

$$(69) \quad \|Du_k\|_{L^{q,\nu}(H)} \leq c \left[\|Du_k\|_{L^q(\Omega)} + \|f\|_{L^{1,\lambda}(\Omega)} \right], \quad \forall k \in \mathbb{N}.$$

Proof. We will follow the idea of the proof of Theorem 4.1 of [15].

Fix $k \in \mathbb{N}$, $x_o \in \Omega$ and $\rho \in]0, \min\{1, \rho_o, d(x_o, \partial\Omega)\}[$ with ρ_o as in Lemma 3.3.

In $B(x_o, \rho)$ we can write $u_k = v_k + w_k$ where $v_k \in W^{1,2}(B(x_o, \rho), \mathbb{R}^N)$ is a weak solution of the problem

$$-D_i(A_{ij}(x)D_j v_k) = 0 \quad \text{in } B(x_o, \rho)$$

and $w_k \in W_0^{1,2}(B(x_o, \rho), \mathbb{R}^N)$ is a weak solution of the Dirichlet problem

$$(70) \quad \begin{cases} -D_i(A_{ij}(x)D_j w_k) = f_k & \text{in } B(x_o, \rho) \\ w_k = 0 & \text{on } \partial B(x_o, \rho). \end{cases}$$

Since any weak solution of the problem (70) is also a very weak solution of the same problem then, by (58) and by item (iv) it follows that, for any $q \in [1, \frac{n}{n-1}]$,

$$(71) \quad \begin{aligned} \|w_k\|_{W^{1,q}(B(x_o, \rho))}^q &\leq c_1(n, q, A_1, A_2) \rho^{n+q-nq} \|f_k\|_{L^1(B(x_o, \rho))}^q \\ &\leq c_1 \rho^{n+q+\lambda q-nq} \|f\|_{L^{1,\lambda}(\Omega)}^q. \end{aligned}$$

Gathering together (65) and (71) we deduce, for any $\sigma < \rho$,

$$\begin{aligned} &\|Du_k\|_{L^q(B(x_o, \sigma))}^q \\ &\leq c_1 \left[\left(\frac{\sigma}{\rho} \right)^{n-q+\gamma q} \|Dv_k\|_{L^q(B(x_o, \rho))}^q + \rho^{n+q+\lambda q-nq} \|f\|_{L^{1,\lambda}(\Omega)}^q \right] \\ &\leq c_1 \left[\left(\frac{\sigma}{\rho} \right)^{n-q+\gamma q} \|Du_k\|_{L^q(B(x_o, \rho))}^q + \rho^{n+q+\lambda q-nq} \|f\|_{L^{1,\lambda}(\Omega)}^q \right]. \end{aligned}$$

An application of Lemma 1.1 p. 7 of [12]¹⁰ to the above inequality gives

$$(72) \quad \begin{aligned} &\|Du_k\|_{L^q(B(x_o, \sigma))}^q \\ &\leq c_1 \left[\left(\frac{\sigma}{\rho} \right)^{n+q+\lambda q-nq} \|Du_k\|_{L^q(B(x_o, \rho))}^q + \sigma^{n+q+\lambda q-nq} \|f\|_{L^{1,\lambda}(\Omega)}^q \right]. \end{aligned}$$

The proof can now be completed as in Corollary 3.2 making use of (72).

Now we are in the position to prove the following

Theorem 6.2. Assume Ω be a bounded domain with C^1 -boundary and hypotheses (4), (5) be satisfied. Let moreover u be the Stampacchia solution of the problem (1).

Then

$$Du \in L_{\text{loc}}^{q,\nu}(\Omega, \mathbb{R}^{nN}), \quad \forall q \in \left[1, \frac{n}{n-1} \right],$$

¹⁰ Set $\varphi(\rho) = \|Du_k\|_{L^q(B(x_o, \rho))}^q$, $A = c_1$, $\alpha = n-q+\gamma q$, $\beta = n+q+\lambda q-nq$, $\Phi(\rho) = c_1 \|f\|_{L^{1,\lambda}(\Omega)}^q$, $\varepsilon = \gamma q + q(n-\lambda-2)$.

with $\nu = n - q(n - \lambda - 1)$, and for all $H \subset\subset \Omega$ there exists a positive constant $c = c_V(n, \lambda, q, \Lambda_1, \Lambda_2, d(\overline{H}, \partial\Omega))$ such that

$$(73) \quad \|Du\|_{L^{q,\nu}(H)} \leq c \left[\|Du\|_{L^q(\Omega)} + \|f\|_{L^{1,\lambda}(\Omega)} \right].$$

Proof. We have already remarked (see Theorem 2.1 formula (53)) that

$$\|Du_k\|_{L^q(\Omega)} \leq c_V(n, q, \Lambda_1, \Lambda_2, \Omega) \|f\|_{L^1(\Omega)}, \quad \forall k \in \mathbb{N}, \quad \forall q \in \left[1, \frac{n}{n-1}\right].$$

This information allows us to deduce that there exists a subsequence $\{u_{n_k}\} \subset \{u_k\}$ such that

- (a) $u_{n_k} \rightharpoonup v$ in $W^{1,q}(\Omega, \mathbb{R}^N)$ as $k \rightarrow +\infty$, $\forall q \in [1, \frac{n}{n-1}[$,
- (b) $u_{n_k} \rightarrow v$ in $L^q(\Omega, \mathbb{R}^N)$ and a.e. in Ω as $k \rightarrow +\infty$, $\forall q \in [1, \frac{n}{n-1}[$,
- (c) the function v is a Stampacchia solution of the Dirichlet problem (1).

By the uniqueness of the Stampacchia solution we can conclude that $v = u$.

To achieve the thesis we need only to show that $Du \in L_{\text{loc}}^{q,\nu}(\Omega)$.

To this purpose let us fix $H \subset\subset \Omega$, $x_o \in H$ and $\rho \in]0, d_H]$.

Hence, by (a), we have

$$Du_{n_k} \rightharpoonup Du \quad \text{in } L^q(H(x_o, \rho), \mathbb{R}^N).$$

By virtue of Proposition 3.5 in [5] p. 53 (see also [42] Ch. V, Theorem 1) and (69) we obtain

$$\begin{aligned} & \|Du\|_{L^q(H(x_o, \rho))}^q \\ & \leq \liminf_{k \rightarrow +\infty} \|Du_{n_k}\|_{L^q(H(x_o, \rho))}^q \\ & \leq \rho^\nu \liminf_{k \rightarrow +\infty} \|Du_{n_k}\|_{L^{q,\nu}(H)}^q \\ & \leq c_V(n, \lambda, q, \Lambda_1, \Lambda_2, d(\overline{H}, \partial\Omega)) [\|Du\|_{L^q(\Omega)} + \|f\|_{L^{1,\lambda}(\Omega)}] \rho^\nu. \end{aligned}$$

The above inequality proves the theorem.

Corollary 6.1. Assume the same hypotheses of Theorem 6.2 and that $\lambda \in]0, n-2[$. Then

$$u \in L_{\text{loc}}^{\beta,\nu}(\Omega, \mathbb{R}^N)$$

$$\text{for all } \beta \in \left[1, \frac{q(n-\lambda-1)}{n-\lambda-2}\right].$$

The proof of Corollary 6.1 is an easy consequence of the following useful lemma applied to each component of u .

Lemma 6.1. Let $v \in W_0^{1,p}(\Omega, \mathbb{R})$ such that $Dv \in L_{\text{loc}}^{p,\kappa}(\Omega, \mathbb{R}^n)$, with $\kappa \in]0, n-p[$. Then

$$v \in L_{\text{loc}}^{p,\kappa}(\Omega, \mathbb{R})$$

$$\text{where } \frac{1}{p_\kappa} = \frac{1}{p} - \frac{1}{n-\kappa}.$$

Moreover, for all $H \subset\subset \Omega$ there exist $H' \subset\subset \Omega$ such that $H \subset\subset H'$ and a positive constant $c = c(n, p, \kappa, H, H')$ such that

$$(74) \quad \|v\|_{L^{p, \kappa}(H)} \leq c [\|Dv\|_{L^p(\Omega)} + \|Dv\|_{L^{p, \kappa}(H')}].$$

Proof. Observe that by Lemma 4.22 of [3], for any fixed $H \subset\subset \Omega$, there exists $H' \subset\subset \Omega$ with the cone property such that $H \subset\subset H'$.

Moreover, since $v \in W_0^{1,p}(\Omega)$, following the proof of Corollary 2.1 in [15], we will prove that $v \in L^{p, \kappa}(H')$ with norm estimate

$$(75) \quad \|v\|_{L^{p, \kappa}(H')} \leq c_1(n, p, \kappa, H') [\|Dv\|_{L^p(\Omega)} + \|Dv\|_{L^{p, \kappa}(H')}].$$

Indeed, by Sobolev embedding theorem and the standard properties of Morrey spaces it turns out that

$$v \in L^{p^*, 0}(H') \subset L^{p, \mu_o}(H') \quad \forall \mu_o \in]0, p],$$

with norm estimate

$$(76) \quad \|v\|_{L^{p, \mu_o}(H')} \leq c_1 \|Dv\|_{L^p(\Omega)}.$$

- (i) If $\kappa \in]0, p]$ then $v \in L^{p, \kappa}(H')$ and (76) holds with κ instead of μ_o . Thus (75) is achieved.
- (ii) If $\kappa > p$ then $v, Dv \in L^{p, \mu_o}(H')$ and, by virtue of Theorem 1.2 at p. 72 of [10], we have

$$v \in L^{p, \mu}(H'), \quad \forall \mu < p + \mu_o,$$

with norm estimate

$$(77) \quad \begin{aligned} \|v\|_{L^{p, \mu}(H')} &\leq c_1 [\|v\|_{L^{p, \mu_o}(H')} + \|Dv\|_{L^{p, \mu_o}(H')}] \\ &\leq c_1 [\|Dv\|_{L^{p, \mu_o}(H')} + \|Dv\|_{L^p(\Omega)}] \end{aligned}$$

by virtue of (76).

If now $\kappa < p + \mu_o$ then $v, Dv \in L^{p, \kappa}(H')$ and we have the estimate (77) with κ instead of μ_o and μ . Thus (75) is achieved again.

If instead $\kappa \geq p + \mu_o$ then $v, Dv \in L^{p, \mu_1}(H')$, with $\mu_1 = \frac{p}{2} + \mu_o$, and a new application of the quoted theorem yields

$$v \in L^{p, \mu}(H'), \quad \forall \mu < p + \mu_1,$$

with norm estimate

$$(78) \quad \begin{aligned} \|v\|_{L^{p, \mu}(H')} &\leq c_1 [\|v\|_{L^{p, \mu_1}(H')} + \|Dv\|_{L^{p, \mu_1}(H')}] \\ &\leq c_1 [\|Dv\|_{L^{p, \mu_1}(H')} + \|Dv\|_{L^p(\Omega)}] \end{aligned}$$

by virtue of (77) and the embedding properties of Morrey spaces.

Iterating the above procedure we can prove that

$$v, Dv \in L^{p, \mu_m}(H'), \quad \forall m \in \mathbb{N}$$

with

$$\mu_m = \begin{cases} \kappa & \text{if } \kappa < p + \mu_{m-1} \\ \frac{p}{2} + \mu_{m-1} & \text{if } \kappa \geq p + \mu_{m-1} \end{cases}$$

and norm estimate

$$\|v\|_{L^{p, \mu_m}(H')} \leq c_1 [\|Dv\|_{L^{p, \mu_m}(H')} + \|Dv\|_{L^p(\Omega)}], \quad \forall m \in \mathbb{N}.$$

Since, for all $m \in \mathbb{N}$,

$$\mu_m = \begin{cases} \kappa & \text{if } \kappa < p + \mu_{m-1} \\ m\frac{p}{2} + \mu_o & \text{if } \kappa \geq p + \mu_{m-1}, \end{cases}$$

after a finite number of steps we obtain (75).

Let $\eta \in C_0^\infty(\Omega)$ such that $0 \leq \eta \leq 1$ in Ω , $\eta \equiv 1$ in H and $\text{supp}(\eta) \subset\subset H'$. Set $w = v\eta$ on Ω and $w = 0$ otherwise. Then $w \in W_0^{1,p}(\mathbb{R}^n)$, $w, Dw \in L^{p, \kappa}(\mathbb{R}^n)$, $\text{supp}(w) \subset H'$ and $w = v$ on H .¹¹

Thus, an application of Theorem 2 from [14] (see also Theorem 3.1 of [2]) to w completes the proof.

Corollary 6.2. *Assume the hypotheses of the Theorem 6.2 be satisfied and suppose that $\lambda = n - 2$. Then the solution u of the problem (1) belongs to $BMO(Q)$.*

Proof. Let $Q \subset\subset \Omega$ be a cube, $x_o \in Q$ and $\rho \in]0, d_Q]$. Then, Poincaré inequality implies

$$\begin{aligned} \int_{Q(x_o, \rho)} |u - u_{Q(x_o, \rho)}|^q dx \\ \leq c(n) \rho^q \int_{Q(x_o, \rho)} |Du|^q dx \leq c(n) \rho^{q+(n-q)} \|Du\|_{L^{q, n-q}(\Omega)}^q. \end{aligned}$$

Thus, $u \in \mathcal{L}^{q, n}(Q) \cong BMO(Q)$.

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¹¹ Observe that $\|w\|_{L^{p, \kappa}(\mathbb{R}^n)} + \|Dw\|_{L^{p, \kappa}(\mathbb{R}^n)} \leq c(H, H', n, p, \kappa) (\|v\|_{L^{p, \kappa}(H')} + \|Dv\|_{L^{p, \kappa}(H')})$.

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