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Regularity results for the gradient of solutions of a class of linear elliptic systems with $L^{1,\lambda}$ data

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Abstract

We prove that if a vector-function f belongs to the Morrey space $L^{1,\lambda}(\Omega, \mathbb{R}^N)$, with $\Omega \subset \mathbb{R}^n$, $n \geq 3$, $N \geq 2$, $\lambda \in]0, n-2]$, and u is the solution of the system

$$\begin{cases} -D_i(A_{ij}(x)D_j u) = f & \text{in } \Omega \\ u \in W_0^{1,1}(\Omega, \mathbb{R}^N) \end{cases}$$

then Du belongs to the space $L^{q,n-q(n-\lambda-1)}(\Omega, \mathbb{R}^{nN})$, for any $q \in [1, \frac{n}{n-1}[$, provided the matrix of bounded measurable coefficients (A_{ij}) has sufficiently small dispersion of the eigenvalues.

1. Introduction

In this paper we study the regularity of the solution to the Dirichlet problem associated with the system[1]

$$\begin{cases} A(u) \equiv -D_i(A_{ij}(x)D_j\, u) = f \\ u \in W_0^{1,1}(\Omega, \mathbb{R}^N) \end{cases} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary, $A_{ij}(x)$ is an elliptic symmetric matrix with L^∞-coefficients and f belongs to the Morrey space $L^{1,\lambda}(\Omega, \mathbb{R}^N)$, $\lambda \in]0, n-2]$.

The study of linear elliptic equations $(N = 1)$ with L^1-data was started by G. Stampacchia (see [25,27,28]) who introduced, by means of a duality method, the notion of the *very weak solution*. The very weak solution is unique, belongs to the space $W_0^{1,q}(\Omega)$ for any $q \in [1, \frac{n}{n-1}[$ and satisfies (1) in the distributional sense.

Since the sixties these results have been generalized in many directions. Thus the existence and uniqueness of solutions to nonlinear elliptic equations with right-hand side measure have been studied in [3–6] and also in the framework of nonlinear potential theory (see [26] and for an overview also [20]).

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[1] Einstein's convention will be used throughout the paper.

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For semilinear elliptic equations and systems with smooth coefficients in the main part and right-hand sides or boundary data measures, analogous questions were answered in [1].

The same class of linear systems with L^∞-coefficients was investigated in [23], where through a duality and a $C^{0,\alpha}$ arguments the existence and uniqueness of a very weak solution to (1) were obtained for convex (generally nonsmooth) domains.

Results similar to those obtained in [3–6] for nonlinear elliptic systems with principal part behaving like a p-Laplace operator can be found in [14–17].

Here we will study higher regularity of the solution for smoother right-hand side under the same conditions as in [23]. Namely, we will prove that if $f \in L^{1,\lambda}(\Omega, \mathbb{R}^N)$, with $\lambda \in]0, n-2]$, and u is the vector-solution to (1), then Du belongs to Morrey's space $L^{q,n-q(n-\lambda-1)}(\Omega, \mathbb{R}^{nN})$ for any $q \in [1, \frac{n}{n-1}[$.

We point out that for $\lambda = 0$ the result has already been obtained in [23] while for $\lambda > n-2$ it holds that $L^{1,\lambda}(\Omega) \subset W^{-1,2}(\Omega)$ and this case need not be treated in the L^1-framework (see e.g. [19]).

Further comments about the space in which the right-hand side lies, the technique we are adopting and a wider bibliography on equations with measure-valued data can be found in the paper [12].

2. Main notation, function spaces, and statement of the results

In $\mathbb{R}^n$ ($n \geq 3$), with generic point $x = (x_1, x_2, \ldots, x_n)$, we shall denote by Ω a bounded open nonempty set with diameter d_Ω and C^1-boundary $\partial\Omega$.

For $\rho > 0$ and $x_o \in \mathbb{R}^n$ we define

$$B(x_o, \rho) = \{x \in \mathbb{R}^n : |x - x_o| < \rho\},$$
$$\Omega(x_o, \rho) = \Omega \cap B(x_o, \rho),$$
$$d(x_o, \partial\Omega) = \operatorname{dist}(x_o, \partial\Omega).$$

If $y_o = (y_{o1}, \ldots, y_{on-1}, 0)$ we define

$$B^+(y_o, \rho) = \{x \in B(y_o, \rho) : x_n > 0\},$$
$$\Gamma(y_o, \rho) = \{x \in B(y_o, \rho) : x_n = 0\}.$$

Moreover, if $u \in L^1(B, \mathbb{R}^N)$ we define

$$u_B = \frac{1}{|B|} \int_B u(x)\,\mathrm{d}x$$

where $|B|$ is the n-dimensional Lebesgue measure of B.

Definition 2.1 (*Morrey's Space*). Let $q \geq 1$ and $0 \leq \lambda < n$. By $L^{q,\lambda}(\Omega, \mathbb{R}^N)$ we denote the linear space formed by all vector-functions $u \in L^q(\Omega, \mathbb{R}^N)$ for which

$$\|u\|_{L^{q,\lambda}(\Omega)} = \sup_{x_o \in \Omega, 0 < \rho \leq d_\Omega} \left\{ \rho^{-\lambda} \int_{\Omega(x_o,\rho)} |u(x)|^q \mathrm{d}x \right\}^{1/q} < +\infty.$$

$L^{q,\lambda}(\Omega, \mathbb{R}^N)$ equipped with the above norm is a Banach space.

Definition 2.2 (*Campanato Space*). Let $q \geq 1$ and $0 \leq \lambda < n + q$. By $\mathcal{L}^{q,\lambda}(\Omega)$ we denote the space of all functions $u \in L^q(\Omega)$ such that

$$[u]_{\mathcal{L}^{q,\lambda}(\Omega)} = \sup_{x_o \in \Omega, 0 < \rho \leq d_\Omega} \left\{ \rho^{-\lambda} \int_{\Omega(x_o,\rho)} |u(x) - u_{\Omega(x_o,\rho)}|^q \mathrm{d}x \right\}^{1/q} < +\infty.$$

If $u : \Omega \to \mathbb{R}^N$, we set

$$D_i \equiv \frac{\partial}{\partial x_i}, \qquad Du = (D_i u^r)_{\substack{i=1,\ldots,n \\ r=1,\ldots,N}}.$$

Let $A_{ij}(x)$, $i, j = 1, 2, \ldots, n$, be matrix-functions for which the following conditions are satisfied:

$$A_{ij}(x) = \left(A_{ij}^{rs}(x)\right)_{r,s=1,\ldots,N} \in L^\infty(\Omega, \mathbb{R}^{N^2}),$$
$$A_{ij}^{rs}(x) = A_{ji}^{sr}(x) \quad \text{for a.e. } x \in \Omega, \tag{2}$$

and there exist two positive constants Λ_1 and Λ_2 such that

$$\Lambda_2|\xi|^2 \geq A_{ij}(x)\xi_i\xi_j \geq \Lambda_1 |\xi|^2 \quad \text{for a.e. } x \in \Omega,\ \forall \xi = (\xi_i^r) \in \mathbb{R}^{nN}. \tag{3}$$

The paper will be organized as follows. First we prove the following

Theorem 2.1. *Let Ω be a bounded domain with C^1-boundary and let the structural conditions (2) and (3) be satisfied. Let $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ be the weak solution of the system*

$$D_i(A_{ij}(x)D_j u) = D_i g_i \quad \text{in } \Omega$$

with $g_i \in L^{2,\kappa}(\Omega, \mathbb{R}^N)$, $0 \leq \kappa < \gamma$, where

$$\gamma = (n-1)\left[1 - \left(\frac{\Lambda_2 - \Lambda_1}{\Lambda_2 + \Lambda_1}\right)^2\right]. \tag{4}$$

Then

$$u \in \mathcal{L}^{2,\kappa+2}(\Omega, \mathbb{R}^N), \qquad Du \in L^{2,\kappa}(\Omega, \mathbb{R}^{nN})$$

and

$$[u]_{\mathcal{L}^{2,\kappa+2}(\Omega)} + \|Du\|_{L^{2,\kappa}(\Omega)} \leq c(n, \Lambda_1, \Lambda_2, \kappa, \partial\Omega)\|g\|_{L^{2,\kappa}(\Omega)}. \tag{5}$$

In particular, if the condition

$$\frac{\Lambda_1}{\Lambda_2} > \frac{\sqrt{n-1}-1}{\sqrt{n-1}+1} \tag{6}$$

holds and

$$\kappa \in]n-2, \gamma[, \tag{7}$$

then $u \in C^{0,\mu}(\overline{\Omega}, \mathbb{R}^N)$, with $\mu = 1 - \frac{n-\kappa}{2}$, and the inequality

$$[u]_{C^{0,\mu}(\overline{\Omega})} \leq c(n, \Lambda_1, \Lambda_2, \kappa, \partial\Omega)\|g\|_{L^{2,\kappa}(\Omega)} \tag{8}$$

holds.

We recall now the notion of a very weak solution for such linear systems.

Definition 2.3. Let $f \in L^1(\Omega, \mathbb{R}^N)$. We say that a vector-function $u \in W_0^{1,1}(\Omega, \mathbb{R}^N)$ is a very weak solution (for short a Stampacchia solution) of the system (1) if it satisfies

$$\int_\Omega u A(\varphi)\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x, \quad \forall \varphi \in \Phi = \{\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N) \cap C^0(\bar{\Omega}, \mathbb{R}^N) : A(\varphi) \in C^0(\bar{\Omega}, \mathbb{R}^N)\}.$$

As a consequence of the previous theorem we deduce

Theorem 2.2. *Let Ω be a bounded domain with C^1-boundary and let the right-hand side $f \in L^1(\Omega, \mathbb{R}^N)$. Let conditions (2), (3) and (6) be satisfied.*

Then there exists a unique Stampacchia solution u of the system (1) such that $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$ for all $q < \frac{n}{n-1}$.[2]

[2] The existence of the Stampacchia solution can be proved also under the assumption of convexity of Ω (see [23]).

Moreover, u satisfies the estimate

$$\|u\|_{W_0^{1,q}(\Omega)} \leq c(n, q, \Lambda_1, \Lambda_2) d_\Omega^{1-n\left(1-\frac{1}{q}\right)} \|f\|_{L^1(\Omega)}. \tag{9}$$

Moreover, for the Stampacchia solution we have the following regularity properties:

Theorem 2.3. *Assume Ω to be bounded domain with C^1-boundary and hypotheses (2), (3) and (6) to be satisfied. Assume moreover that*

$$f \in L^{1,\lambda}(\Omega, \mathbb{R}^N), \quad \lambda \in]0, n-2]. \tag{10}$$

and let u be the Stampacchia solution of the problem (1).

Then

$$Du \in L^{q,\nu}(\Omega, \mathbb{R}^N) \quad \forall q \in \left[1, \frac{n}{n-1}\right[,$$

with $\nu = n - q(n - \lambda - 1)$, and there exists a positive constant c depending on $n, q, \lambda, \Lambda_1, \Lambda_2, \Omega$ such that

$$\|Du\|_{L^{q,\nu}(\Omega)} \leq c\|f\|_{L^{1,\lambda}(\Omega)}.$$

Corollary 2.1. *Under the same assumptions as for Theorem 2.3 we have*

$$u \in L^{\beta,\nu}(\Omega, \mathbb{R}^N)$$

for all $\beta \in \left[1, \frac{q(n-\lambda-1)}{n-\lambda-2}\right[$.

3. Auxiliary results

In this section we consider a weak solution v of the linear system

$$-D_i(A_{ij}(x)D_j v) = 0 \quad \text{in } \Omega, \tag{11}$$

that is, a function $v \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega A_{ij}(x) D_j v D_i \varphi \, \mathrm{d}x = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N).$$

Analogously, for fixed $y_o = (y_{o1}, y_{o2}, \ldots, y_{on-1}, 0)$ and $R > 0$, let us take into account the system

$$\begin{cases} -D_i(B_{ij}(x)D_j v) = 0 & \text{in } B^+(y_o, R), \\ v = 0 & \text{on } \Gamma(y_o, R) \end{cases} \tag{12}$$

under the structural assumptions

$$\begin{aligned} &B_{ij}(x) = (B_{ij}^{rs}(x))_{r,s=1,\ldots,N} \in L^\infty(B^+(y_o, R), \mathbb{R}^{N^2}), \\ &B_{ij}^{rs}(x) = B_{ji}^{sr}(x) \quad \text{for a.e. } x \in B^+(y_o, R), \end{aligned} \tag{13}$$

and there exist two positive constants Λ_1' and Λ_2' such that

$$\Lambda_2' |\xi|^2 \geq B_{ij}(x)\xi_i\xi_j \geq \Lambda_1' |\xi|^2 \quad \text{for a.e. } x \in B^+(y_o, R), \ \forall \xi = (\xi_i^r) \in \mathbb{R}^{nN}. \tag{14}$$

Definition 3.1. A vector-function $v \in W^{1,2}(B^+(y_o, R), \mathbb{R}^N)$ is a weak solution of the system (12) if

$$\begin{cases} \displaystyle\int_{B^+(y_o,R)} B_{ij}(x) D_j v D_i \varphi, \mathrm{d}x = 0, & \forall \varphi \in W_0^{1,2}(B^+(y_o, R), \mathbb{R}^N) \\ v = 0 & \text{on } \Gamma(y_o, R). \end{cases}$$

The key step in our paper will be proving the following

Theorem 3.1 (*Saint-Venant Principle on B*). *Let conditions (2), (3) and (6) be satisfied.*

Then, there exist two positive constants $\mu = \mu(n, \Lambda_1, \Lambda_2, \Omega) \in]0, 1[$ *and* $c = c(n, q, \Lambda_1, \Lambda_2)$ *such that, for any weak solution* $v \in W^{1,2}(\Omega, \mathbb{R}^N)$ *to the system (11), it holds that*

$$\|Dv\|^q_{L^q(B(x_o,\rho_1))} \le c \left(\frac{\rho_1}{\rho_2}\right)^{n-q+\mu q} \|Dv\|^q_{L^q(B(x_o,\rho_2))}$$

for all $x_o \in \Omega$, $0 < \rho_1 \le \rho_2 < d(x_o, \partial\Omega)$ *and for any* $q \in [1, 2]$.

Theorem 3.2 (*Saint-Venant Principle on* B^+). *Let conditions (13), (14) and*

$$\frac{\Lambda'_1}{\Lambda'_2} > \frac{\sqrt{n-1}-1}{\sqrt{n-1}+1}$$

be satisfied.

Then there exist two positive constants $\mu' = \mu'(n, \Lambda'_1, \Lambda'_2) \in]0, 1[$ *and* $c = c(n, q, \Lambda'_1, \Lambda'_2)$ *such that, for any weak solution* $v \in W^{1,2}(B^+(0, R), \mathbb{R}^N)$ *to the system (12), it holds that*

$$\|Dv\|^q_{L^q(B(y_o,\rho_1)\cap B^+(0,R))} \le c \left(\frac{\rho_1}{\rho_2}\right)^{n-q+\mu' q} \|Dv\|^q_{L^q(B(y_o,\rho_2)\cap B^+(0,R))} \tag{15}$$

for all $y_o \in \overline{B^+(0, R/2)}$, $0 < \rho_1 \le \rho_2 < R/2$ *and for any* $q \in [1, 2]$.

Before proving the previous theorems let us premise some useful lemmata which are interesting in themselves.

Let us set

$$\gamma' = (n-1)\left[1 - \left(\frac{\Lambda'_2 - \Lambda'_1}{\Lambda'_2 + \Lambda'_1}\right)^2\right], \tag{16}$$

$$F_{x_o}(\rho) := \rho^{-\gamma} \int_{B(x_o,\rho)} |Dv|^2 \,\mathrm{d}x, \quad \rho \in]0, d(x_o, \partial\Omega)[\,,\ x_o \in \Omega,$$

$$G_{y_o}(\sigma) := \sigma^{-\gamma'} \int_{B^+(y_o,\sigma)} |Dv|^2 \,\mathrm{d}x, \quad \sigma \in]0, R/2[\,,\ y_o \in \Gamma(0, R/2).$$

Remark 1. By Theorems 3.1 and 3.2 from [22] it follows respectively that $F_{x_o}(\rho)$ and $G_{y_o}(\sigma)$ are monotone nondecreasing functions in their domains.

The following lemmata are analogous of De Giorgi's theorem [13] for the class of systems considered.

Lemma 2. *Let the hypotheses of Theorem 3.1 be satisfied.*

Then there exists a constant $c = c(n, \Lambda_1, \Lambda_2) > 0$ *such that, for any weak solution* $v \in W^{1,2}(\Omega, \mathbb{R}^N)$ *to the system (11), it holds that*

$$[v]_{C^{0,1-\frac{n-\gamma}{2}}(\overline{B(x_o,\rho)})} \le c(n, \Lambda_1, \Lambda_2)\, \rho^{-1-\gamma/2} \|v\|_{L^2(B(x_o,6\rho))} \tag{17}$$

for any $x_o \in \Omega$ *and for all* $0 < \rho < \frac{1}{6} d(x_o, \partial\Omega)$.

Proof. Let us fix x_o, ρ as in the statement and let $\tau \in]0, 2\rho]$.

By the monotonicity of the function $F_{y_o}(\tau)$, $y_o \in B(x_o, \rho)$, we have

$$F_{y_o}(\tau) \le F_{y_o}(2\rho) \le c\, \rho^{-\gamma}\, \|Dv\|^2_{L^2(B(x_o,3\rho))}. \tag{18}$$

On the other hand, by Morrey's theorem (see e.g. [18] p. 43) it follows that

$$\begin{aligned}[v]_{C^{0,1-\frac{n-\gamma}{2}}(\overline{B(x_o,\rho)})} &\le c(n, \Lambda_1, \Lambda_2)\, \|Dv\|_{L^{2,\gamma}(B(x_o,\rho))} \\ &= c(n, \Lambda_1, \Lambda_2) \sup_{y_o \in B(x_o,\rho), \tau \le 2\rho} (F_{y_o}(\tau))^{1/2}. \end{aligned} \tag{19}$$

Inequalities (19), (18) and Caccioppoli's one (see e.g. [9] p. 46) then prove the theorem. □

Lemma 3. *Let the hypotheses of Theorem 3.2 be satisfied.*

Then there exists a constant $c = c(n, \Lambda_1', \Lambda_2') > 0$ such that, for any weak solution $v \in W^{1,2}(B^+(0,R),\mathbb{R}^N)$ to the system (12), it holds that

$$[v]_{C^{0,1-\frac{n-\gamma'}{2}}\left(\overline{B^+(z_o,\rho)}\right)} \le c(n,\Lambda_1',\Lambda_2')\,\rho^{-1-\gamma'/2}\|v\|_{L^2(B^+(z_o,10\rho))} \tag{20}$$

for any $z_o \in \Gamma(0,R/2)$ and for all $0<\rho\le \frac{1}{10}R$.

Proof. Let us fix ρ as in the statement and let $y_o \in \overline{B^+(z_o,\rho)}$.

If $y_{on}=0$ then, $\forall \sigma \in]0,2\rho]$, we have

$$\begin{aligned}\sigma^{-\gamma'}\|Dv\|^2_{L^2(B(y_o,\sigma)\cap B^+(z_o,\rho))} &\le \sigma^{-\gamma'}\|Dv\|^2_{L^2(B^+(y_o,\sigma))} \le G_{y_o}(2\rho)\\ &\le c\,\rho^{-\gamma'}\|Dv\|^2_{L^2(B^+(z_o,3\rho))}.\end{aligned} \tag{21}$$

If $y_{on}>0$, then consider the point $y_o'=(y_{o1},\dots,y_{on-1},0)$ and fix $\sigma \in]0,2\rho]$.

If $\sigma < y_{on}$ and thus $B(y_o,\sigma)\cap B^+(z_o,\rho) \subset\subset B^+(0,R)$, by Lemma 3.1 from [22] we have

$$\begin{aligned}\sigma^{-\gamma'}\|Dv\|^2_{L^2(B(y_o,\sigma)\cap B^+(z_o,\rho))} &\le \sigma^{-\gamma'}\|Dv\|^2_{L^2(B(y_o,\sigma))}\\ &\le y_{on}^{-\gamma'}\|Dv\|^2_{L^2(B(y_o,y_{on}))}\\ &\le 2^{\gamma'}(2y_{on})^{-\gamma'}\|Dv\|^2_{L^2(B(y_o',2y_{on}))}\\ &\le 2^{\gamma'}G_{y_o'}(2\rho)\\ &\le c\,\rho^{-\gamma'}\|Dv\|^2_{L^2(B^+(z_o,3\rho))}.\end{aligned} \tag{22}$$

Finally, if $\sigma \ge y_{on}$, again by monotonicity of $G_{y_o}(\rho)$ we have

$$\begin{aligned}\sigma^{-\gamma'}\|Dv\|^2_{L^2(B(y_o,\sigma)\cap B^+(z_o,\rho))} &\le \sigma^{-\gamma'}\|Dv\|^2_{L^2(B^+(y_o',2\sigma))}\\ &\le G_{y_o'}(4\rho)\\ &\le c\,\rho^{-\gamma'}\|Dv\|^2_{L^2(B^+(z_o,5\rho))}.\end{aligned} \tag{23}$$

Gathering together (21)–(23) we conclude that $Dv \in L^{2,\gamma'}(B^+(0,\rho))$.

The assertion now follows as in the proof of previous theorem via Morrey's theorem and Caccioppoli's inequality. $\square$

Lemma 4. *Let the hypotheses of Theorem 3.1 be satisfied.*

Then there exists a constant $c = c(n,q,\Lambda_1,\Lambda_2,\Omega) > 0$ such that

$$\max_{B(x_o,\rho)}|v| \le c\,\rho^{-n/q}\|v\|_{L^q(B(x_o,6\rho))} \tag{24}$$

for any $x_o\in\Omega$, for all $0<\rho<\frac{1}{6^2}d(x_o,\partial\Omega)$ and for any $q\in[1,2]$.

Proof. Let us fix x_o, ρ and q as in the statement.

From (2.5) p. 14 of [9] and (17) we deduce

$$\begin{aligned}\max_{B(x_o,\rho)}|v| &\le [v]_{C^{0,1-\frac{n-\gamma}{2}}\left(\overline{B(x_o,\rho)}\right)}\rho^{1-\frac{n-\gamma}{2}} + c(n)\rho^{-n/2}\|v\|_{L^2(B(x_o,\rho))}\\ &\le c\,\rho^{-n/2}\|v\|_{L^2(B(x_o,6\rho))}.\end{aligned}$$

We now exploit the above inequality and Young's inequality proceeding as in [18], p. 81:

$$\max_{B(x_o,\rho)}|v| \le c\,\rho^{-n/2}\|v\|_{L^2(B(x_o,6\rho))}$$

$$\leq c\,\rho^{-n/2}\left(\max_{B(x_o,6\rho)}|v|\right)^{\frac{2-q}{2}}\|v\|^{q/2}_{L^q(B(x_o,6\rho))}$$
$$\leq \varepsilon\max_{B(x_o,6\rho)}|v| + c\,c_1(\varepsilon)\,\rho^{-n/q}\|v\|_{L^q(B(x_o,6\rho))},\quad \forall\varepsilon>0$$

whence the theorem follows by Lemma 5.1, p. 81, from [18]. □

Analogously we have the following

Lemma 5. *Let the hypotheses of Theorem 3.2 be satisfied.*
Then there exists a constant $c = c(n, q, \Lambda_1', \Lambda_2', \Omega) > 0$ such that

$$\max_{B^+(z_o,\rho)}|v| \leq c\rho^{-n/q}\|v\|_{L^q(B^+(z_o,10\rho))} \tag{25}$$

for any $z_o \in \Gamma(0, R/2)$, for all $0 < \rho \leq \frac{1}{10^2}R$ and for all $q \in [1, 2]$.

Proof. It is sufficient to repeat the steps of previous lemma using inequality (20) instead of (17). □

Proof of Theorem 3.1. Let us fix x_o, ρ_1, ρ_2 and q as in the statement.
It will be enough to prove our inequality for $\rho_1 \in]0, \frac{1}{2\cdot 6^2}\rho_2[$.
By means of the Hölder inequality, the monotonicity of the function $F_{x_o}(\rho)$ and Caccioppoli's inequality we deduce

$$\begin{aligned}\|Dv\|_{L^q(B(x_o,\rho_1))} &\leq \rho_1^{n(1/q-1/2)}\|D\,v\|_{L^2(B(x_o,\rho_1))}\\ &\leq c\rho_1^{n(1/q-1/2)}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma/2}\|D\,v\|_{L^2(B(x_o,\rho_2/72))}\\ &\leq c\rho_1^{n(1/q-1/2)}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma/2}\rho_2^{-1}\|v-v_{B(x_o,\rho_2)}\|_{L^2(B(x_o,\rho_2/6^2))}.\end{aligned} \tag{26}$$

On the other hand, as for all $h \in \mathbb{R}^N$ the vector-function $v - h$ is still a solution of problem (11), by formula (24) we obtain

$$\max_{B(x_o,\rho_2/6^2)}|v - v_{B(x_o,\rho_2)}| \leq c\rho_2^{-n/q}\|v - v_{B(x_o,\rho_2)}\|_{L^q(B(x_o,\rho_2))}.$$

The above inequality and Poincaré's one yield

$$\begin{aligned}\|v - v_{B(x_o,\rho_2)}\|_{L^2(B(x_o,\rho_2/6))} &\leq c\,\rho_2^{-n/q+n/2}\|v - v_{B(x_o,\rho_2)}\|_{L^q(B(x_o,\rho_2))}\\ &\leq c\,\rho_2^{-n/q+n/2+1}\|D\,v\|_{L^q(B(x_o,\rho_2))}.\end{aligned} \tag{27}$$

Joining together formulas (26) and (27) we deduce

$$\|Dv\|_{L^q(B(x_o,\rho_1))} \leq c\,\rho_1^{n(1/q-1/2)}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma/2}\rho_2^{-1}\rho_2^{-n/q+n/2+1}\|D\,v\|_{L^q(B(x_o,\rho_2))}$$

which gives the expected result setting $\mu = 1 - \frac{n-\gamma}{2}$. □

Proof of Theorem 3.2. Let us fix y_o, ρ_1, ρ_2 and q as in the statement.
It will be enough to prove (15) for $\rho_1 \in]0, \frac{1}{(2\cdot 10)^2}\rho_2[$.
We will proceed as in the proof of Lemma 3.
If $y_{on} = 0$ then we have

$$\begin{aligned}\rho_1^{-\gamma'}\|Dv\|^2_{L^2(B(y_o,\rho_1)\cap B^+(0,R))} &= \rho_1^{-\gamma'}\|Dv\|^2_{L^2(B^+(y_o,\rho_1))}\\ &\leq G_{y_o}\left(\frac{1}{2\cdot 10^2}\rho_2\right).\end{aligned} \tag{28}$$

Hence, by means of Hölder's inequality, (28) and Caccioppoli's inequality we deduce

$$\begin{aligned}\|Dv\|_{L^q(B(y_o,\rho_1)\cap B^+(0,R))} &\le \rho_1^{n(1/q-1/2)}\|D v\|_{L^2(B(y_o,\rho_1)\cap B^+(0,R))}\\ &\le c\,\rho_1^{n(1/q-1/2)}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma'/2}\|D v\|_{L^2(B^+(y_o,\frac{1}{2\cdot 10^2}\rho_2))}\\ &\le c\,\rho_1^{n(1/q-1/2)}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma'/2}\rho_2^{-1}\|v\|_{L^2(B^+(y_o,\frac{1}{10^2}\rho_2))}.\end{aligned} \tag{29}$$

On the other hand, by formula (25) we obtain

$$\max_{B^+(y_o,\frac{1}{10^2}\rho_2)}|v| \le c\rho_2^{-n/q}\|v\|_{L^q(B^+(y_o,\frac{1}{10}\rho_2))}.$$

The above inequality and Poincaré's one yield

$$\begin{aligned}\|v\|_{L^2(B^+(y_o,\frac{1}{10^2}\rho_2))} &\le c\rho_2^{-n/q+n/2}\|v\|_{L^q(B^+(y_o,\frac{1}{10}\rho_2))}\\ &\le c\,\rho_2^{-n/q+n/2+1}\|D v\|_{L^q(B^+(y_o,\frac{1}{10}\rho_2))}.\end{aligned} \tag{30}$$

Joining together formulas (29) and (30) we deduce

$$\|Dv\|_{L^q(B(y_o,\rho_1)\cap B^+(0,R))} \le c\left(\frac{\rho_1}{\rho_2}\right)^{n(1/q-1/2)+\gamma'/2}\|D v\|_{L^q(B(y_o,\rho_2)\cap B^+(0,R))}$$

which gives the expected result on setting $\mu = 1 - \frac{n-\gamma'}{2}$.

Assume now $y_{on} > 0$ and consider the point $y'_o = (y_{o1}, \ldots, y_{on-1}, 0)$.

If $\rho_1 > y_{on}$ then we have

$$\begin{aligned}\rho_1^{-\gamma'}\|Dv\|^2_{L^2(B(y_o,\rho_1)\cap B^+(0,R))} &\le \rho_1^{-\gamma'}\|Dv\|^2_{L^2(B^+(y'_o,2\rho_1))}\\ &\le 2^{\gamma'}G_{y'_o}(2\rho_2).\end{aligned}$$

If $\rho_1 \le y_{on} \le \rho_2$ then we have

$$\begin{aligned}\rho_1^{-\gamma'}\|Dv\|^2_{L^2(B(y_o,\rho_1)\cap B^+(0,R))} &= \rho_1^{-\gamma'}\|Dv\|^2_{L^2(B(y_o,\rho_1))}\\ &\le y_{on}^{-\gamma'}\|Dv\|^2_{L^2(B(y_o,y_{on}))}\\ &\le 2^{\gamma'}(2y_{on})^{-\gamma'}\|Dv\|^2_{L^2(B^+(y'_o,2y_{on}))}\\ &\le 2^{\gamma'}G_{y'_o}(2\rho_2).\end{aligned}$$

Hence, for the latter two cases the theorem follows as for the first one.

If $\rho_2 < y_{on}$, since $B(y_o,\rho_2) \subset\subset B^+(0,R)$, the assertion follows immediately from Theorem 3.1. □

4. Interior and boundary estimates with right-hand side in $L^{2,\kappa}$

We now give the interior and boundary estimates for a vector-function u which is a weak solution of some auxiliary problems.

Theorem 4.1. *Let the assumptions of Theorem 3.1 be satisfied and let $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ be a weak solution of the system*

$$D_i(A_{ij}(x)D_j u) = D_i g_i \quad \text{in } \Omega$$

where $g_i = (g_i^r) \in L^{2,\kappa}(\Omega,\mathbb{R}^N)$ with $0 \le \kappa \le \gamma$.

Then, for every open set $\Omega' \subset\subset \Omega$, we have

$$Du \in L^{2,\kappa}(\Omega', \mathbb{R}^{nN})$$

and the inequality

$$\|Du\|_{L^{2,\kappa}(\Omega')} \le c\left[\|Du\|_{L^2(\Omega)} + \|g\|_{L^{2,\kappa}(\Omega)}\right] \tag{31}$$

holds, where the constant $c > 0$ depends also on $\mathrm{dist}(\overline{\Omega'}, \partial\Omega)$.

Proof. The proof can be attained via the standard Campanato technique (see e.g. [8] or [7], p. 59, or [22]) using Theorem 3.1 with $q = 2$. □

Theorem 4.2. *Let the assumptions of Theorem 3.2 be satisfied and let $u \in W^{1,2}(B^+(0, R_1), \mathbb{R}^N)$ be a weak solution of the system*

$$D_i(B_{ij}(x)D_j u) = D_i g_i \quad \text{in } B^+(0, R_1),$$
$$u = 0 \quad \text{on } \Gamma(0, R_1)$$

where $g_i = (g_i^r) \in L^{2,\kappa}(B^+(0, R_1), \mathbb{R}^N)$ with $0 \le \kappa \le \gamma'$.

Then, for every $R < \frac{R_1}{2}$, we have

$$Du \in L^{2,\kappa}(B^+(0, R), \mathbb{R}^{nN})$$

and the inequality

$$\|Du\|_{L^{2,\kappa}(B^+(0,R))} \le c\left[\|Du\|_{L^2(B^+(0,R_1))} + \|g\|_{L^{2,\kappa}(B^+(0,R_1))}\right]$$

holds, where the constant $c > 0$ depends also on R_1.

Proof. The proof can be acquired as well via the standard Campanato technique (see e.g. [8] or [7], p. 59, or [10], p. 312, or [22]) exploiting Theorem 3.2 with $q = 2$. □

5. Global regularity with right-hand side in $L^{2,\kappa}$: Proof of Theorem 2.1

In this section we will prove a global regularity result for the system

$$\begin{cases} D_i(A_{ij}(x)D_j u) = D_i g_i & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{32}$$

where $g_i \in L^{2,\kappa}(\Omega, \mathbb{R}^N)$ with $0 \le \kappa < \gamma$.

The claimed result can be proved as in papers [10,22]. We will reproduce the basic steps here for the reader's convenience.

Let Ω be of class C^1 and $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be the solution of the Dirichlet problem (32).

Since Ω is of class C^1, for each $y_o \in \partial\Omega$ there is a ball $\mathcal{B}(y_o, R_o)$ and a C^1-function ζ defined on a domain $D \subset \mathbb{R}^{n-1}$ such that with respect to a suitable system of coordinates $\{y_1, \dots, y_n\}$, with the origin at y_o:

(a) the set $\partial\Omega \cap \mathcal{B}(y_o, R_o)$ can be represented by an equation of the type

$$y_n = \zeta(y_1, \dots, y_{n-1}),$$

(b) each $y \in \Omega \cap \mathcal{B}(y_o, R_o)$ satisfies

$$y_n < \zeta(y_1, \dots, y_{n-1}).$$

Without loss of generality we can suppose that the system of coordinates is such that the hyperplane tangent to $\partial\Omega$ at y_o has equation $y_n = 0$ and

$$\zeta(y_{o1}, \dots, y_{on-1}) = 0, \qquad D\zeta(y_{o1}, \dots, y_{on-1}) = 0. \tag{33}$$

For such domains the boundary can be locally straightened by means of the C^1 transformation:

$$\begin{cases} \psi_i(y) = y_i - y_{o\,i} & \text{for } i = 1, 2, \dots, n-1 \\ \psi_n(y) = y_n - \zeta(y_1, \dots, y_{n-1}). \end{cases} \tag{34}$$

It turns out that $\psi(y) = (\psi_1(y), \dots, \psi_n(y))$ is a C^1-diffeomorphism verifying the following properties:

(i) $\psi(y_o) = 0$ (see $(33)_1$),
(ii) $\psi(\mathcal{B}(y_o, R_o) \cap \partial\Omega) = \{x \in \mathbb{R}^n : x_n = 0,\ |x_i| < R_o, \text{for } i = 1, \dots, n-1\}$,
(iii) there exist two positive constants α_1 and α_2, with $\alpha_1 \le \alpha_2$, such that

$$\begin{aligned} &\alpha_1\, |y - y_o| \le |\psi(y)| \le \alpha_2\, |y - y_o|, \quad \forall y \in \mathcal{B}(y_o, R_o) \cap \Omega, \\ &B^+(0, \alpha_1 R_o) \subset \psi(\mathcal{B}(y_o, R_o) \cap \Omega) \subset B^+(0, \alpha_2 R_o), \\ &\mathcal{B}(y_o, \alpha_1/\alpha_2 R_o) \cap \Omega \subset \psi^{-1}(B^+(0, \alpha_1 R_o)) \subset \mathcal{B}(y_o, R_o) \cap \Omega. \end{aligned} \tag{35}$$

Remark 6. The fact that $\zeta \in C^1$ and the condition $(33)_2$ allow us to choose R_o so that $|D\zeta|$ is sufficiently small in $\mathcal{B}(y_o, R_o) \cap \bar{\Omega}$.

Put $R_1 = \alpha_1 R_o$; if $z \in B^+(0, R_1)$ we set

$$\begin{aligned} &\tilde{A}_{ij}(z) = A_{ij}(\psi^{-1}(z)), \\ &B_{ij}(z) = \tilde{A}_{hk}(z) \frac{\partial \psi_i}{\partial y_h}(\psi^{-1}(z)) \frac{\partial \psi_j}{\partial y_k}(\psi^{-1}(z)), \\ &U(z) = u(\psi^{-1}(z)), \\ &G_i(z) = g_h(\psi^{-1}(z)) \frac{\partial \psi_i}{\partial y_h}(\psi^{-1}(z)). \end{aligned} \tag{36}$$

Let us observe that $B_{ij}(z)$ still satisfies hypotheses (13).
Moreover, by the definitions (34) and (36) it follows that

$$\frac{\partial \psi_i}{\partial y_h} = \begin{cases} \delta_{ih} & \text{if } i = 1, \dots, n-1, h = 1, \dots, n \\ \delta_{ih} - \dfrac{\partial \zeta}{\partial y_h} & \text{if } i = n, h = 1, \dots, n \end{cases} \tag{37}$$

and that

$$\Lambda_1 \sum_{h=1}^{n} \left(\frac{\partial \psi_i}{\partial y_h} \eta_i \right)^2 \le B_{ij} \eta_i \eta_j \le \Lambda_2 \sum_{h=1}^{n} \left(\frac{\partial \psi_i}{\partial y_h} \eta_i \right)^2, \quad \forall \eta = (\eta_i) \in \mathbb{R}^{nN}. \tag{38}$$

On the other hand, exploiting (37), we obtain

$$\sum_{h=1}^{n} \left(\frac{\partial \psi_i}{\partial y_h} \eta_i \right)^2 = |\eta|^2 + \eta_n^2 |D\zeta|^2 - 2 \sum_{h=1}^{n-1} \frac{\partial \zeta}{\partial y_h} \eta_h \eta_n := I$$

whence, since $\max_{\mathcal{B}(y_o, R_o) \cap \bar{\Omega}} |D\,\zeta| < 1$,

$$(1 - |D\zeta|)^2\, |\eta|^2 \le I \le (1 + |D\,\zeta|)^2\, |\eta|^2.$$

Gathering together the last inequality and (38) we deduce

$$\Lambda_1 (1 - |D\zeta|)^2\, |\eta|^2 \le B_{ij} \eta_i \eta_j \le \Lambda_2 (1 + |D\zeta|)^2\, |\eta|^2, \quad \forall \eta = (\eta_i) \in \mathbb{R}^{nN}.$$

The above inequality and formula (16) yield

$$\begin{aligned} \gamma' &= \gamma'(R_o) \\ &= (n-1) \left[1 - \left(\frac{\Lambda_2 (1 + \max_{\mathcal{B}(y_o, R_o) \cap \bar{\Omega}} |D\zeta|)^2 - \Lambda_1 (1 - \min_{\mathcal{B}(y_o, R_o) \cap \bar{\Omega}} |D\zeta|)^2}{\Lambda_2 (1 + \max_{\mathcal{B}(y_o, R_o) \cap \bar{\Omega}} |D\zeta|)^2 + \Lambda_1 (1 - \min_{\mathcal{B}(y_o, R_o) \cap \bar{\Omega}} |D\zeta|)^2} \right)^2 \right]. \end{aligned}$$

With the change of coordinates $z = \psi(y)$, since u is the solution of the system (32) in $\Omega \cap \mathcal{B}(y_o, R_o)$, then U becomes a solution of the problem[3]

$$\begin{cases} U \in W^{1,2}(B^+(0, R_1), \mathbb{R}^N) \\ D_i(B_{ij}(z) D_j U) = D_i G_i(z) \quad \text{in } B^+(0, R_1) \\ U = 0 \quad \text{on } \Gamma(0, R_1). \end{cases} \tag{39}$$

Remark 7. Since ψ is of class C^1 and $g_i \in L^{2,\kappa}(\Omega \cap \mathcal{B}(y_o, R_o), \mathbb{R}^N)$ then, by virtue of Theorem V of [8], $G_i \in L^{2,\kappa}(B^+(0, R_1), \mathbb{R}^N)$ and

$$\|G\|_{L^{2,\kappa}(B^+(0,R_1))} \le c(\psi) \|g\|_{L^{2,\kappa}(\Omega \cap \mathcal{B}(y_o,R_o))}. \tag{40}$$

Fix now $\kappa \in [0, \gamma[$[4] and let $g_i \in L^{2,\kappa}(\Omega, \mathbb{R}^n)$.

Since $\lim_{R_o \to 0^+} \gamma'(R_o) = \gamma$, Remark 6 implies that we can choose a positive $R_o = R_o(\partial\Omega, 1, \gamma, \kappa)$ so that $\gamma' > \kappa$.

If now $U \in W^{1,2}(B^+(0, R_1), \mathbb{R}^N)$ is a solution of the problem (39), by Theorem 4.2 we have, for every $R \in]0, R_1/2[$,

$$DU \in L^{2,\kappa}(B^+(0, R), \mathbb{R}^{nN})$$

and the inequality

$$\|DU\|_{L^{2,\kappa}(B^+(0,R))} \le c \left[\|DU\|_{L^2(B^+(0,R_1))} + \|G\|_{L^{2,\kappa}(B^+(0,R_1))} \right] \tag{41}$$

holds.

From (41) and the Poincaré inequality we achieve for all $R \in]0, R_1[$,

$$[U]_{\mathcal{L}^{2,\kappa+2}(B^+(0,R))} \le c \left[\|DU\|_{L^2(B^+(0,R_1))} + \|G\|_{L^{2,\kappa}(B^+(0,R_1))} \right].$$

The above inequality, changing back to the old coordinates (see Theorem V from [8]), gives (see (35))

$$[u]_{\mathcal{L}^{2,\kappa+2}(\Omega \cap \mathcal{B}(y_o,R_o))} \le c \left[\|Du\|_{L^2(\Omega)} + \|g\|_{L^{2,\kappa}(\Omega)} \right]. \tag{42}$$

Inequalities (41) and (42) yield

$$[u]_{\mathcal{L}^{2,\kappa+2}(\Omega \cap \mathcal{B}(y_o,R_o))} + \|Du\|_{L^{2,\kappa}(\Omega \cap \mathcal{B}(y_o,R_o))} \le c \left[\|Du\|_{L^2(\Omega)} + \|g\|_{L^{2,\kappa}(\Omega)} \right]. \tag{43}$$

Proof of Theorem 2.1. Since Ω is of class C^1, around every $y_o \in \partial\Omega$ there exists a ball $\mathcal{B}(y_o, R_o)$ and a corresponding diffeomorphism $\psi : \mathcal{B}(y_o, R_o) \to \mathbb{R}^n$ such that (33) and (34), (i), (ii), (iii) are satisfied.

Because $\partial\Omega$ is compact, only a finite number of such balls is needed to cover it, say $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\nu$. For each $\mathcal{B}_\iota$ we suppose that its radius is small enough (see Remark 6).

Then there exists an open set $\Omega' \subset\subset \Omega$ such that $\Omega', \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\nu$ cover $\overline{\Omega}$.

Exploiting inequalities (31) and (43) we obtain

$$[u]_{\mathcal{L}^{2,\kappa+2}(\Omega')} + \|Du\|_{L^{2,\kappa}(\Omega')} \le c \left[\|Du\|_{L^2(\Omega)} + \|g\|_{L^{2,\kappa}(\Omega)} \right] \tag{44}$$

and, for $\iota = 1, 2, \ldots, \nu$,

$$[u]_{\mathcal{L}^{2,\kappa+2}(\Omega \cap \mathcal{B}_\iota)} + \|Du\|_{L^{2,\kappa}(\Omega \cap \mathcal{B}_\iota)} \le c \left[\|Du\|_{L^2(\Omega)} + \|g\|_{L^{2,\kappa}(\Omega)} \right]. \tag{45}$$

On the other hand, Theorem 1.III from [9] p. 42 yields

$$\|Du\|_{L^2(\Omega)} \le c \|g\|_{L^2(\Omega)} \tag{46}$$

and thus inequality (5) is achieved putting together inequalities (44)–(46).

In particular, (5)–(7) yield the required Hölder continuity of u and inequality (8). □

[3] By D_i we denote $\frac{\partial}{\partial z_i}$.

[4] The same γ as is defined in (4).

6. Existence and uniqueness of the Stampacchia solution: Proof of Theorem 2.2

As in paper [23] we will use the duality method introduced by Stampacchia in papers [27,28].
To this end we observe the following

Remark 8. Let Ω be bounded. If $g \in L^p(\Omega, \mathbb{R}^N)$ with $p > n \geq 2$ then $g \in L^{2,\kappa}(\Omega, \mathbb{R}^N)$ with $0 \leq \kappa \leq n(1 - 2/p)$ and

$$\|g\|_{L^{2,\kappa}(\Omega)} \leq c(n)\, d_\Omega^{n(1/2-1/p)-\kappa/2} \|g\|_{L^p(\Omega)}. \tag{47}$$

Proof of Theorem 2.2. By the Lax–Milgram theorem, there exists a linear continuous operator

$$G : W^{-1,2}(\Omega, \mathbb{R}^N) \to W_0^{1,2}(\Omega, \mathbb{R}^N)$$

such that $\tilde{u} = G(T)$ is the unique weak solution of the equation

$$A(\tilde{u}) = T.$$

The function G is the Green operator of A.
If $p > n$ and A satisfies conditions (6) and (7), then by virtue of (8) and (47)[5], G continuously maps $W^{-1,p}(\Omega, \mathbb{R}^N)$ into $C^0(\bar{\Omega}, \mathbb{R}^N)$ (see Lemma 4.I, p. 30, from [9]) and

$$\max_{\bar{\Omega}} |G(\psi)| \leq c(n, \Lambda_1, \Lambda_2)\, d_\Omega^{1-\frac{n-\kappa}{2}+n(1/2-1/p)-\kappa/2} \|\psi\|_{W^{-1,p}(\Omega)}, \quad \forall \psi \in C^0(\bar{\Omega}, \mathbb{R}^N), \tag{48}$$

where $\psi = D_i g_i$.
Thus, u is the Stampacchia solution of the system (1) if and only if

$$\int_\Omega u\, \psi \,\mathrm{d}x = \int_\Omega f\, G(\psi)\,\mathrm{d}x, \quad \forall \psi \in C^0(\bar{\Omega}, \mathbb{R}^N). \tag{49}$$

From (48) and (49) we have for all $\psi \in C^0(\bar{\Omega}, \mathbb{R}^N)$,

$$\left| \int_\Omega u\, \psi \,\mathrm{d}x \right| \leq c(n, \Lambda_1, \Lambda_2)\, d_\Omega^{1-n/p} \|\psi\|_{H^{-1,p}(\Omega)} \|f\|_{L^1(\Omega)}.$$

Since $C^0(\bar{\Omega}, \mathbb{R}^N)$ is dense in $W^{-1,p}(\Omega, \mathbb{R}^N)$ we get

$$\|u\|_{W_0^{1,q}(\Omega)} \leq c(n, \Lambda_1, \Lambda_2)\, d_\Omega^{1-n(1-1/q)} \|f\|_{L^1(\Omega)}$$

with $\frac{1}{q} = 1 - \frac{1}{p}$, $p > n$.

The mapping $f \mapsto u$ is the adjoint $G^\star$ of G, that is $u = G^\star(f)$.
Since G continuously maps $W^{-1,p}(\Omega, \mathbb{R}^N)$ into $C^0(\bar{\Omega}, \mathbb{R}^N)$, then $G^\star$ is a continuous linear operator from $L^1(\Omega, \mathbb{R}^N)$ into $W_0^{1,q}(\Omega, \mathbb{R}^N)$, for all $q < \frac{n}{n-1}$.

7. Global regularity of the Stampacchia solution: Proof of Theorem 2.3

In this section we will gather together the technique developed in [6] with the nowadays classical method of S. Campanato, as we did in paper [12]. We reproduce this procedure here for the reader's convenience. Let us introduce a truncation operator. For a given constant $k > 0$ we define the cut-off function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k \,\mathrm{sign}(s) & \text{if } |s| > k. \end{cases}$$

[5] Since $p > n$ then $n - 2 < n(1 - 2/p)$ and we can choose e.g.

$$\kappa = \frac{1}{2}\{n - 2 + \min\{n(1 - 2/p), \gamma\}\}.$$

For a vector-function $f = (f^r(x))_{r=1,\dots,N}$, $x \in \Omega$, we define the truncated vector-function $f_k = (T_k(f^r))_{r=1,\dots,N}$ pointwise.

For a given vector-function $f \in L^{1,\lambda}(\Omega, \mathbb{R}^N)$ let us consider a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ such that

(i) $f_k \in W^{-1,2}(\Omega, \mathbb{R}^N) \cap L^{1,\lambda}(\Omega, \mathbb{R}^N)$, $\forall k \in \mathbb{N}$,
(ii) $f_k \to f$ in $L^1(\Omega, \mathbb{R}^N)$ as $k \to +\infty$,
(iii) $\|f_k\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}$, $\forall k \in \mathbb{N}$,
(iv) $\|f_k\|_{L^{1,\lambda}(\Omega)} \le \|f\|_{L^{1,\lambda}(\Omega)}$, $\forall k \in \mathbb{N}$.

An example of a sequence satisfying the above requirements is the sequence $\{T_k(f)\}_{k\in\mathbb{N}}$.

For fixed $k \in \mathbb{N}$, let $u_k \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ be the weak solution of the system

$$-D_i(A_{ij}(x)D_j u_k) = f_k \quad \text{in } \Omega, \tag{50}$$

that is,

$$\begin{cases} u_k \in W_0^{1,2}(\Omega, \mathbb{R}^N) \\ \displaystyle\int_\Omega A_{ij}(x) D_j u_k \, D_i\varphi \, \mathrm{d}x = \int_\Omega f_k \varphi \mathrm{d}x, \quad \forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N). \end{cases}$$

We will prove, first, the following

Theorem 7.1. *Assume that hypotheses (2), (3), (6) and (10) hold and let u_k be the weak solution of problem (50). Then*

$$D u_k \in L^{q,\nu}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN}), \quad \forall q \in \left[1, \frac{n}{n-1}\right[, \forall k \in \mathbb{N},$$

with $\nu = n - q(n - \lambda - 1)$, moreover for all $\Omega' \subset\subset \Omega$ there exists a positive constant c depending on n, q, λ, Λ_1, Λ_2, d_Ω, $\operatorname{dist}(\overline{\Omega'}, \partial\Omega)$ such that

$$\|Du_k\|_{L^{q,\nu}(\Omega')} \le c \left[\|Du_k\|_{L^q(\Omega)} + \|f\|_{L^{1,\lambda}(\Omega)}\right], \quad \forall k \in \mathbb{N}. \tag{51}$$

Proof. We can proceed as in the proof of Theorem 4.1 from [12] exploiting Theorem 3.1 for $q < \frac{n}{n-1}$. $\square$

Analogously, fixed $k \in \mathbb{N}$, we take into consideration a weak solution $u_k \in W^{1,2}(B^+(0, R_1), \mathbb{R}^N)$ of the following system:

$$\begin{cases} D_i(B_{ij}(x)D_j u_k) = f_k & \text{in } B^+(0, R_1), \\ u_k = 0 & \text{on } \Gamma(0, R_1). \end{cases} \tag{52}$$

Theorem 7.2. *Let assumptions of Theorem 3.2 and (10) be satisfied and let a function $u_k \in W^{1,2}(B^+(0, R_1), \mathbb{R}^N)$ be a weak solution of the system (52).*

Then, for every $R < R_1/2$, we have

$$D u_k \in L^{q,\nu}_{\mathrm{loc}}(B^+(0, R), \mathbb{R}^{nN}), \quad \forall q \in \left[1, \frac{n}{n-1}\right[, \quad \forall k \in \mathbb{N},$$

with $\nu = n - q(n - \lambda - 1)$, and the inequality

$$\|D u_k\|_{L^{q,\nu}(B^+(0,R))} \le c \left[\|D u_k\|_{L^q(B^+(0,R_1))} + \|f\|_{L^{1,\lambda}(B^+(0,R_1))}\right]$$

holds, where the constant $c > 0$ depends also on R_1.

Proof. We can proceed as in the proof of Theorem 4.1 from [12] exploiting Theorem 3.2 for $q < \frac{n}{n-1}$. $\square$

Theorem 7.3. *Let Ω be a bounded domain with C^1-boundary. Assume that hypotheses (2), (3), (6) and (10) hold and let u_k be the weak solution of problem (50). Then*

$$D u_k \in L^{q,\nu}(\Omega, \mathbb{R}^{nN}), \quad \forall q \in \left[1, \frac{n}{n-1}\right[, \forall k \in \mathbb{N},$$

with $\nu = n - q(n-\lambda-1)$, and there exists a positive constant c depending on n, q, λ, Λ_1, Λ_2, d_Ω such that

$$\|Du_k\|_{L^{q,\nu}(\Omega)} \le c\left[\|Du_k\|_{L^q(\Omega)} + \|f\|_{L^{1,\lambda}(\Omega)}\right], \quad \forall k \in \mathbb{N}.$$

Proof. The proof follows from the previous two theorems, exploiting the idea of Section 5. □

Proof of Theorem 2.3. We have already remarked (see Theorem 2.1 formula (9)) that

$$\|Du_k\|_{L^q(\Omega)} \le c(n,q,\Lambda_1,d_\Omega)\|f\|_{L^1(\Omega)}, \quad \forall k \in \mathbb{N}, \forall q \in \left[1, \frac{n}{n-1}\right[.$$

This information allows us to deduce that there exists a subsequence $\{u_{n_k}\} \subset \{u_k\}$ such that

(a) $u_{n_k} \rightharpoonup v$ in $W^{1,q}(\Omega,\mathbb{R}^N)$ as $k \to +\infty$, $\forall q \in [1, \frac{n}{n-1}[$,
(b) $u_{n_k} \to v$ in $L^q(\Omega,\mathbb{R}^N)$ and a.e. in Ω as $k \to +\infty$, $\forall q \in [1, \frac{n}{n-1}[$,
(c) the function v is the Stampacchia solution of the Dirichlet problem (1).

By the uniqueness of the Stampacchia solution we can conclude that $v = u$.
To prove the theorem we need only to show that $Du \in L^{q,\nu}(\Omega)$.
To this end let us fix $x_o \in \Omega$ and $\rho \in]0, d_\Omega]$.
Since, by (a), we have

$$Du_{n_k} \rightharpoonup Du \quad \text{in } L^q(\Omega(x_o,\rho),\mathbb{R}^N),$$

by virtue of Proposition 3.5 in [2], p. 53 (see also [29], Ch. V, Theorem 1), and (51) we obtain

$$\begin{aligned}\|Du\|^q_{L^q(\Omega(x_o,\rho))} &\le \liminf_{k\to+\infty}\|Du_{n_k}\|^q_{L^q(\Omega(x_o,\rho))}\\ &\le \rho^\nu \liminf_{k\to+\infty}\|Du_{n_k}\|^q_{L^{q,\nu}(\Omega)}\\ &\le c(n,q,\lambda,\Lambda_1,\Lambda_2,d_\Omega,\partial\Omega)\|f\|_{L^{1,\lambda}(\Omega)}\,\rho^\nu.\end{aligned}$$

The above inequality and (c) prove the theorem. □

Proof of the Corollary 2.1. The proof of the Corollary 2.1 is an easy consequence of the following useful.

Lemma 9. *Let Ω have C^1-boundary and let $v \in W^{1,p}_0(\Omega)$ be such that $Dv \in L^{p,\lambda}(\Omega)$, with $\lambda \in]0, n-p[$, $p < n$. Then*

$$v \in L^{p_\lambda,\lambda}(\Omega)$$

where $\frac{1}{p_\lambda} = \frac{1}{p} - \frac{1}{n-\lambda}$ and there exists a positive constant $c = c(n,p,\lambda,\Omega)$ such that

$$\|v\|_{L^{p_\lambda,\lambda}(\Omega)} \le c\|Dv\|_{L^{p,\lambda}(\Omega)}.$$

Proof. By a well known representation formula, we have

$$v(x) \le c(n)\int_{\mathbb{R}^n}\frac{V(y)}{|x-y|^{n-1}}\mathrm{d}y, \quad \text{for a.a. } x \in \Omega \tag{53}$$

where

$$V(y) = \begin{cases}|Dv(y)| & y \in \Omega\\ 0 & y \in \mathbb{R}^n \setminus \Omega.\end{cases}$$

The theorem can be proved by applying to formula (53) a slight modification of the proof of Theorem 2 from [11]. □

8. Application to quasilinear problems

The previous results can be extended to a class of quasilinear elliptic systems. Let $A_{ij} : \Omega \times \mathbb{R}^N \to \mathbb{R}^{N^2}$ be matrix-valued functions satisfying Carathéodory conditions and such that

$$A_{ij}(x,u) = \left(A^{rs}_{ij}(x,u)\right)_{r,s=1,\dots,N} \in L^\infty(\Omega\times\mathbb{R}^N,\mathbb{R}^{N^2}),$$

$$A^{rs}_{ij}(x,u) = A^{sr}_{ji}(x,u) \quad \text{for a.e. } x \in \Omega, \forall u \in \mathbb{R}^N,$$

and there exist two positive constants Λ_1 and Λ_2 such that

$$\Lambda_2 |\xi|^2 \geq A_{ij}(x,u)\xi_i\xi_j \geq \Lambda_1 |\xi|^2 \quad \text{for a.e. } x \in \Omega, \forall u \in \mathbb{R}^N, \forall \xi = (\xi_i^r) \in \mathbb{R}^{nN}. \tag{54}$$

Then the following theorem holds:

Theorem 8.1. *Let Ω be open domain with C^1-boundary, $f \in L^{1,\lambda}(\Omega,\mathbb{R}^N)$, $0 < \lambda \leq n-2$, and let condition (6) be satisfied.*

Then there exists a distributional solution u to the Dirichlet problem

$$\begin{cases} -D_i(A_{ij}(x,u)D_j u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{55}$$

such that

$$Du \in L^{q,\nu}(\Omega,\mathbb{R}^{nN}), \quad \forall q \in \left[1, \frac{n}{n-1}\right[$$

with $\nu = n - q(n-\lambda-1)$, moreover there exists a positive constant c depending on n, q, λ, Λ_1, Λ_2, d_Ω such that

$$\|Du\|_{L^{q,\nu}(\Omega)} \leq c\,\|f\|_{L^{1,\lambda}(\Omega)}.$$

Proof. We will only briefly sketch the proof.

Let us denote by u_k a weak solution of (55) corresponding to the right-hand side f_k. The existence of u_k can be proved by Galerkin approximation following Ch. IV, par. 8, or Ch. VIII, par. 5, from the book [21] or using the Leray–Lions Theorem (see [24]).

Reasoning as in the proof of Theorem 7.3 we can obtain a uniform estimate of $\|Du_k\|_{L^{q,\lambda}(\Omega)}$ for any $q \in [1, \frac{n}{n-1}[$. Note that, even if $A_{ij}(x,u_k(x))$ depend on k, the norm estimates of Du_k are uniform with respect to k by assumption (54).

As a consequence of the uniform norm estimate of Du_k in $L^{q,\lambda}(\Omega,\mathbb{R}^{nN})$ we deduce that $\{u_k\}$ is bounded in $W^{1,q}(\Omega,\mathbb{R}^N)$ for any $q \in [1, \frac{n}{n-1}[$.

Thus there exists a subsequence $\{u_{n_k}\} \subset \{u_k\}$ such that

(a') $u_{n_k} \rightharpoonup u$ in $W^{1,q}(\Omega,\mathbb{R}^N)$ as $k \to +\infty$, $\forall q \in [1, \frac{n}{n-1}[$,
(b') $u_{n_k} \to u$ in $L^q(\Omega,\mathbb{R}^N)$ and a.e. in Ω as $k \to +\infty$, $\forall q \in [1, \frac{n}{n-1}[$.

Then, for any $\varphi \in C_0^\infty(\Omega,\mathbb{R}^{nN})$, $A_{ij}(x,u_{n_k}(x))D\varphi \to A_{ij}(x,u(x))D\varphi$ in $L^{q'}(\Omega,\mathbb{R}^{nN})$ and so u is a distributional solution to the problem (55).

The proof can be now finished as in Theorem 2.3. □

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