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On the global well-posedness for the two-dimensional Boussinesq equations with horizontal dissipation

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In this paper, we study the global well-posedness for the two-dimensional nonlinear Boussinesq equations with horizontal dissipation. The method we adopted is the smoothing effect in horizontal direction and the low-high decomposition technique.

KEYWORDS

Boussinesq equations, fourier localization, global well-posedness, horizontal dissipation

1 | INTRODUCTION

In this paper, we consider the two-dimensional incompressible nonlinear Boussinesq equations with horizontal dissipation:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \partial_{x_1}^2 u + \nabla \Pi = F(\theta), \\ \partial_t \theta + u \cdot \nabla \theta - \kappa \partial_{x_1}^2 \theta = 0, \\ \mathrm{d}iv\, u = 0, \\ (u, \theta)|_{t=0} = (u_0, \theta_0). \end{cases} \tag{1.1}$$

Here, $u = (u_1, u_2)(t, x_1, x_2)$ denotes the velocity vector field, and $\theta = \theta(t, x_1, x_2)$ is the temperature. The pressure Π is governed by the following:

$$\Pi = \sum_{i,j=1}^{2} \frac{\partial_{x_i}\partial_{x_j}}{-\Delta}(u_i u_j) - \sum_{i=1}^{2} \frac{\partial_{x_i}}{-\Delta}(F_i(\theta)) =: \Pi_u + \Pi_\theta. \tag{1.2}$$

The function $F(\theta) = (F_1(\theta), F_2(\theta))$ is a vector-valued function such that $F \in C^2$ and $F(0) = 0$. And the positive parameters ν and κ denote the kinematic viscosity coefficient and the molecular diffusivity of the fluid, respectively. Without loss of generality, we assume $\nu = \kappa = 1$ and denote ∂_{x_i} by ∂_i with $i = 1, 2$.

In Pedlosky,[1] the Boussinesq system model is used to describe atmospheric or oceanographic turbulence whenever rotation and stratification play a key role. If we take $F(\theta) = \theta e_2, e_2 = (0, 1)$, and Laplacian replaced of horizontal dissipation, the syetem (1.1) is reduced to the classical one and has been intensively studied from a mathematical viewpoint due to its physical significance. Most of the papers are devoted to studying the global well-posedness, one can refer to previous works.[2-16]

If Laplacian is replaced by the horizontal dissipation or the vertical dissipation, this physical phenomenon is natural and one can refer to Chemin et al.[17] For the horizontal dissipation, Danchin and Paicu[18] firstly proved the global existence for the two-dimensional Boussinesq system in only one equation. Then Miao and Zheng[19,20] extended the above results to three dimensions under the assumption that the initial data are axisymmetric without swirl. Recently, Adhikari, Cao, and Wu also established some global results for the two-dimensional Boussinesq with vertical dissipation, see other works.[21-23] In the paper,[24] Wu and Zheng also discussed the vertical case for the two-dimensional nonlinear Boussinesq equations and obtained the global well-posedness result for the rough initial data. They established a growth estimate on vertical component of velocity and improved the above results.

In the present paper, adopted the method from Wu and Zheng,[24] we extend and improve the result of Danchin and Paicu[18] to general source term $F(\theta)$. Our main results are stated as follows:

Theorem 1.1. *Assume $F \in C^2$ satisfies $F(0) = 0$. Suppose that $u_0 \in H^1(\mathbb{R}^2)$ with $\partial_2\omega_0 \in L^2(\mathbb{R}^2)$, and $\theta_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $\partial_2\theta_0 \in L^2(\mathbb{R}^2)$, $\partial_2^2\theta_0 \in L^2(\mathbb{R}^2)$ and $F(\theta_0) \in L^2(\mathbb{R}^2)$. Then there is a unique global solution (u, θ) for system (1.1) such that*

$$u \in C(\mathbb{R}^+; H^1(\mathbb{R}^2)), \theta \in C(\mathbb{R}^+; L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)), \partial_2\theta \in C(\mathbb{R}^+; L^2(\mathbb{R}^2)),$$

$$(\partial_2\omega; \partial_2^2\theta) \in C(\mathbb{R}^+; L^2(\mathbb{R}^2)), (\partial_1 u; \partial_1\theta) \in L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)),$$

$$(\partial_1\partial_2\omega, \partial_1\partial_2^2\theta) \in L^2_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2))$$

and

$$(\nabla u, \partial_2\theta) \in L^4_{loc}(\mathbb{R}^+; L^\infty(\mathbb{R}^2)).$$

Remark 1.1. Compared with Danchin and Paicu,[18] we weaken the assumptions on initial data in horizontal direction and obtain the continuity of higher regularity norm of (u, θ).

Besides, we also have the following theorem.

Theorem 1.2. *Assume $F \in C^2$ satisfy $F(0) = 0$. Suppose that $u_0 \in H^1(\mathbb{R}^2)$, $\theta_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $\partial_2\theta_0 \in L^2(\mathbb{R}^2)$ and $F(\theta_0) \in L^2(\mathbb{R}^2)$. Then there is a unique global solution (u, θ) for system (1.1) such that*

$$\theta \in C_b(\mathbb{R}^+; L^p(\mathbb{R}^2)), \forall p \in [2, \infty), \quad \theta \in C_w(\mathbb{R}^+; L^\infty(\mathbb{R}^2)), \quad \partial_2\theta \in C_w(\mathbb{R}^+; L^2(\mathbb{R}^2)),$$

$$u \in C_w(\mathbb{R}^+; H^1(\mathbb{R}^2)), \quad \partial_1 u \in L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)), \quad \partial_1\partial_2\theta \in L^2_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2)).$$

Notation

Throughout the paper, $\mathbb{R}^+ = (0, \infty)$, C stands for a "generic" constant and C_0 depends only on the initial data. For $p, q \in [1, \infty]$, the usual Lebesgue space is denoted by $L^p(\mathbb{R}^2)$ and $\|\cdot\|_{L^q_t L^p}$ denotes the norm of $\left(\int_0^t \|\cdot\|_p^q d\tau\right)^{\frac{1}{q}}$.

2 | PRELIMINARIES

In this section, we are going to recall some basic facts on Littlewood-Paley theory, one may check Miao et al[25] for more details. And then we give two useful lemmas.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^n)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by the following:

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n.$$

Set $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and let $h_j = \mathcal{F}^{-1}\varphi_j$ and $\tilde{h} = \mathcal{F}^{-1}\chi$. Define the frequency localization operators:

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{nj} \int_{\mathbb{R}^n} h_j(y)u(x-y)dy, \quad \text{for} \quad j \in \mathbb{N},$$

and

$$\Delta_{-1}u = \chi(D)u.$$

We shall also use the following low-frequency cutoff:

$$S_j u := \chi(2^{-j}D)u \quad \text{for} \quad j \in \mathbb{N}.$$

The formal equality

$$u = \sum_{j\geq -1} \triangle_j u.$$

Formally, $\Delta_j = S_{j+1} - S_j$ is a frequency projection into the annulus $\{|\xi| \approx 2^j\}$, and S_j is a frequency projection into the ball $\{|\xi| \lesssim 2^j\}$. One easily verifies that with the above choice of φ

$$\Delta_{j'}\Delta_j u \equiv 0 \quad if \quad |j'-j| \geq 2 \quad \text{and} \quad \Delta'_j(S_{j-1}u\Delta_j u) \equiv 0 \quad \text{if} \quad |j'-j| \geq 5.$$

We now introduce the following definition of inhomogenous Besov spaces by means of Littlewood-Paley projection Δ_j and S_j:

Definition 2.1. Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, the inhomogenous Besov space $B^s_{p,q}$ is defined by the following:

$$B^s_{p,q} = \{f \in S'(\mathbb{R}^n); \quad \|f\|_{B^s_{p,q}} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}} = \begin{cases} \left(\sum_{j=-1}^{\infty} 2^{jsq}\|\Delta_j f\|^q_{L^p}\right)^{\frac{1}{q}}, & \text{for} \quad q < \infty, \\ \sup_{j\geq -1} 2^{js}\|\Delta_j f\|_{L^p}, & \text{for} \quad q = \infty. \end{cases}$$

Let us point out that $B^s_{2,2}$ is the usual Sobolev space H^s and that $B^s_{\infty,\infty}$ is the usual Hölder space C^s for $s \in \mathbb{R} \setminus \mathbb{Z}$.

We now recall the para-differential calculus which enables us to define a generalized product between distributions, which is continuous in many functional spaces where the usual product does not make sense (see Bony[26]). The paraproduct between u and v is defined by the following:

$$T_u v := \sum_j S_{j-1}u\Delta_j v.$$

Formally, we have the following Bony's decomposition:

$$uv = T_u v + T_v u + R(u, v),$$

with

$$R(u, v) = \sum_{|j'-j|\le 1} \Delta_j u \Delta_{j'} v,$$

and we also denote

$$T'_u v := T_u v + R(u, v).$$

Next, we introduce two important lemmas which will be useful throughout this paper.

Lemma 2.1. (Miao et al25)
Let $1 \le a \le b \le \infty$. Assume that $f \in L^a$, then there exists a constant C independent of f, j such that

$$\sup_{|\alpha|=k} ||\partial^\alpha S_q f||_{L^b} \le C^k 2^{q(k+n(\frac{1}{a}-\frac{1}{b}))} ||S_q f||_{L^a},$$
$$C^{-k} 2^{qk} ||\triangle_q f||_{L^a} \le \sup_{|\alpha=k|} ||\partial^\alpha \triangle_q f||_{L^a} \le C^k 2^{qk} ||\triangle_q f||_{L^a}. \tag{1.2}$$

Lemma 2.2. (Cao and Wu[23] and Wu and Zheng[24])
Let $p \in [2, \infty)$. There holds that

$$\int_{\mathbb{R}^2} fgh dx \le C_p ||f||_{L^2(\mathbb{R}^2)} ||g||^{1-\frac{1}{p}}_{L^2(\mathbb{R}^2)} ||\partial_2 g||^{\frac{1}{p}}_{L^2(\mathbb{R}^2)} ||h||^{1-\frac{1}{p}}_{L^{2(p-1)}(\mathbb{R}^2)} ||\partial_1 h||^{\frac{1}{p}}_{L^2(\mathbb{R}^2)}. \tag{2.1}$$

In particular,

$$\int_{\mathbb{R}^2} fgh dx \le C ||f||_{L^2(\mathbb{R}^2)} ||g||^{\frac{1}{2}}_{L^2(\mathbb{R}^2)} ||\partial_2 g||^{\frac{1}{2}}_{L^2(\mathbb{R}^2)} ||h||^{\frac{1}{2}}_{L^2(\mathbb{R}^2)} ||\partial_1 h||^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}. \tag{2.2}$$

3 | A PRIORI ESTIMATES

In this section, we give some estimates needed to prove Theorems 1.1 and 1.2. Let us begin with the following energy estimates.

Proposition 3.1. *Assume $u_0 \in L^2(\mathbb{R}^2)$ and $\theta_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with $F(\theta_0) \in L^2(\mathbb{R}^2)$. Suppose that (u, θ) is a smooth solution of (1.1). Then*

$$||\theta(t)||^2_{L^2} + 2\int_0^t ||\partial_1 \theta(\tau)||^2_{L^2} d\tau \le ||\theta_0||^2_{L^2}, \tag{3.1}$$

$$||F(\theta)(t)||^2_{L^2} + 2\int_0^t ||\partial_1 F(\theta)(\tau)||^2_{L^2} d\tau \le ||F(\theta_0)||^2_{L^2} + F''_M F_M ||\theta_0||^2_{L^2}, \tag{3.2}$$

and

$$||u(t)||^2_{L^2} + 2\int_0^t ||\partial_1 u(\tau)||^2_{L^2} d\tau \le 2||u_0||^2_{L^2} + 3t^2(||F(\theta_0)||^2_{L^2} + F''_M F_M ||\theta_0||^2_{L^2}), \tag{3.3}$$

where F_M,, and F'_M, and F''_M denote the maximum absolute value of $F(\cdot), F'(\cdot)$, and $F''(\cdot)$ on interval $[-||\theta_0||_{L^\infty}, ||\theta_0||_{L^\infty}]$.

Proof. Recall the temperature equation as follows:

$$\partial_t \theta + u \cdot \nabla\theta - \partial_1^2 \theta = 0.$$

Taking L^2-inner product of the above equation with θ yields (3.1).

Now, we calculate the temperature equation by a simple way as follows:

$$\partial_t F(\theta) + u \cdot \nabla F(\theta) - \partial_1^2 F(\theta) = F''(\theta)\partial_1\theta\partial_1\theta. \tag{3.4}$$

Multiplying (3.4) by $F(\theta)$ and integrating in space, we have the following:

$$\frac{1}{2}\frac{d}{dt}||F(\theta)(t)||_{L^2}^2 + ||\partial_1 F(\theta)(t)||_{L^2}^2 = -\int_{\mathbb{R}^2} F''(\theta)\partial_1\theta\partial_1\theta F(\theta)dx. \tag{3.5}$$

Using Hölder's inequality, the right hand side of (3.5) can be bounded by the following:

$$||F''(\theta)||_{L^\infty}||F(\theta)||_{L^\infty}||\partial_1\theta||_{L^2}^2 \le F_M''F_M||\partial_1\theta||_{L^2}^2.$$

Plugging the above inequality into (3.5) and integrating with respect to time variable, we get

$$||F(\theta)(t)||_{L^2}^2 + 2\int_0^t ||\partial_1 F(\theta)(\tau)||_{L^2}^2 d\tau \le ||F(\theta_0)||_{L^2}^2 + 2F_M''F_M\int_0^t ||\partial_1\theta(\tau)||_{L^2}^2 d\tau. \tag{3.6}$$

(3.6) together with (3.1) gives (3.2).

Then, taking the L^2-inner product with the first equation of (1.1) by u and using Hölder's inequality, we have

$$\frac{1}{2}\frac{d}{dt}||u(t)||_{L^2}^2 + ||\partial_1 u(t)||_{L^2}^2 = \int_{R^2} F(\theta)u dx \le ||F(\theta)(t)||_{L^2}||u(t)||_{L^2}. \tag{3.7}$$

Consequently,

$$\frac{d}{dt}||u(t)||_{L^2} \le ||F(\theta)(t)||_{L^2}.$$

Inserting (3.2) into the above inequality and integrating with respect to time variable yield

$$||u(t)||_{L^2} \le ||u_0||_{L^2} + t\sqrt{||F(\theta_0)||_{L^2}{}^2 + F_M''F_M||\theta_0||_{L^2}{}^2},$$

which together with (3.2) and (3.7) gives,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||u(t)||_{L^2}{}^2 + ||\partial_1 u(t)||_{L^2}{}^2\\ &\le \sqrt{||F(\theta_0)||_{L^2}{}^2 + F_M''F_M||\theta_0||_{L^2}{}^2}(||u_0||_{L^2} + t\sqrt{||F(\theta_0)||_{L^2}{}^2 + F_M''F_M||\theta_0||_{L^2}{}^2}).\end{aligned}$$

Integrating the above inequality with respect to time variable, we have (3.3). □

Definition 3.1. (Taylor[27] and Yudovich[28])
A modulus of continuity is any nondecreasing nonzero continuous function Ω on $[0, \infty)$ such that $\Omega(0) = 0$, and satisfying that Ω does not vary too rapidly, ie, that $\Omega(2h) \le C\Omega(h), \forall h \in (0, \infty)$. The modulus of continuity Ω is a dual Osgood modulus of continuity if, in addition,

$$\int_1^\infty \frac{1}{\Omega(x)}dx = \infty. \tag{3.8}$$

Proposition 3.2. *Assume $u_0 \in H^1(\mathbb{R}^2)$, $\theta_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $\partial_2\theta_0 \in L^2(\mathbb{R}^2)$, and $F(\theta_0) \in L^2(\mathbb{R}^2)$. Suppose that (u, θ) is a smooth solution of (1.1). And we also suppose that*

$$\int_0^t \sup_{q\ge 2}\frac{||S_q u_1(\tau)||_{L^\infty}^2}{\Omega(q)}d\tau < \infty, \tag{3.9}$$

where $\Omega(\cdot)$ is a dual modulus of continuity. Then

$$||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2 + \int_0^t ||\partial_1\omega(\tau)||_{L^2}^2 d\tau + \int_0^t ||\partial_1\partial_2\theta(\tau)||_{L^2}^2 d\tau \le C_0(t). \tag{3.10}$$

Proof of Proposition. Taking the operator $\nabla\times$ for the first equation of (1.1), we obtain

$$\omega_t + u\cdot\nabla\omega - \partial_1^2\omega = \partial_1(F_2(\theta)) - \partial_2(F_1(\theta)).$$

Multiplying the above equation with ω and integrating with respect to space yields

$$\frac{1}{2}\frac{d}{dt}||\omega(t)||_{L^2}^2 + ||\partial_1\omega(t)||_{L^2}^2 = \int_{\mathbb{R}^2}\partial_1(F_2(\theta))\omega dx - \int_{\mathbb{R}^2}\partial_2(F_1(\theta))\omega dx. \tag{3.11}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
-\int_{\mathbb{R}^2}\partial_2(F_1(\theta))\omega dx &= -\int_{\mathbb{R}^2}F_1'(\theta)\partial_2\theta\omega dx \\
&\le F_M'||\partial_2\theta(t)||_{L^2}||\omega(t)||_{L^2} \\
&\le \frac{1}{2}(F_M')^2||\partial_2\theta||_{L^2}^2 + \frac{1}{2}||\omega(t)||_{L^2}^2.
\end{aligned}$$

Integrating by parts gives

$$\begin{aligned}
\int_{\mathbb{R}^2}\partial_1(F_2(\theta))\omega dx &= -\int_{\mathbb{R}^2}F_2(\theta)\partial_1\omega dx \\
&\le ||F_2(\theta)||_{L^2}||\partial_1\omega(t)||_{L^2} \\
&\le \frac{1}{2}||F_2(\theta)||_{L^2}^2 + \frac{1}{2}||\partial_1\omega(t)||_{L^2}^2.
\end{aligned}$$

It follows that

$$\frac{d}{dt}||\omega(t)||_{L^2}^2 + ||\partial_1\omega(t)||_{L^2}^2 \le (F_M'')^2||\partial_2\theta||_{L^2}^2 + ||\omega(t)||_{L^2}^2 + ||F_2(\theta)||_{L^2}^2. \tag{3.12}$$

Then, taking the differential operator ∂_2 for the temperature equation, we get

$$\partial_t\partial_2\theta + u\cdot\nabla\partial_2\theta - \partial_1^2\partial_2\theta = -\partial_2u_1\partial_1\theta - \partial_2u_2\partial_2\theta. \tag{3.13}$$

Multiplying the above equation with $\partial_2\theta$ and integrating in space, we obtain

$$\frac{1}{2}\frac{d}{dt}||\partial_2\theta(t)||_{L^2}^2 + ||\partial_2\partial_1\theta(t)||_{L^2}^2 = -\int_{\mathbb{R}^2}\partial_2u_1\partial_1\theta\partial_2\theta dx - \int_{\mathbb{R}^2}\partial_2u_2\partial_2\theta\partial_2\theta.$$

For the first term of the right hand of the above equation, integrating by parts and using Hölder's inequality, we know

$$\begin{aligned}
-\int_{\mathbb{R}^2}\partial_2u_1\partial_1\theta\partial_2\theta dx &= \int_{\mathbb{R}^2}\partial_1\partial_2u_1\theta\partial_2\theta dx + \int_{\mathbb{R}^2}\partial_2u_1\theta\partial_1\partial_2\theta dx \\
&\le ||\theta(t)||_{L^\infty}||\partial_1\partial_2u_1||_{L^2}||\partial_2\theta||_{L^2} + ||\theta(t)||_{L^\infty}||\partial_2u_1||_{L^2}||\partial_1\partial_2\theta||_{L^2} \\
&\le C||\theta_0||_{L^\infty}||\partial_1\omega||_{L^2}||\partial_2\theta||_{L^2} + ||\theta_0||_{L^\infty}||\omega||_{L^2}||\partial_1\partial_2\theta||_{L^2} \\
&\le C||\theta_0||_{L^\infty}^2(||\partial_2\theta||_{L^2}^2 + ||\omega||_{L^2}^2) + \frac{1}{8}||\partial_1\omega||_{L^2}^2 + \frac{1}{4}||\partial_1\partial_2\theta||_{L^2}^2,
\end{aligned}$$

where we use the following fact $\partial_1\omega = -\triangle u_2$ and

$$||\partial_1\partial_2u_1||_{L^2} = ||-\partial_2\partial_2u_2||_{L^2} \le C||\partial_1\omega||_{L^2}.$$

For the second term, from the incompressible condition, we obtain

$$-\int_{\mathbb{R}^2} \partial_2 u_2 \partial_2\theta\partial_2\theta dx = \int_{\mathbb{R}^2} \partial_1 u_1 \partial_2\theta\partial_2 dx = -2\int_{\mathbb{R}^2} u_1\partial_1\partial_2\theta\partial_2\theta dx.$$

By Littlewood-Paley theory, we decompose the integral term as follows:

$$\begin{aligned}
2\int_{\mathbb{R}^2} u_1\partial_1\partial_2\theta\partial_2\theta dx &= 2\int_{\mathbb{R}^2} S_{N+1}u_1\partial_2\theta\partial_1\partial_2\theta dx + 2\int_{\mathbb{R}^2}\sum_{j>N}\Delta_j u_1\cdot\partial_2\theta\partial_1\partial_2\theta dx \\
&\leq 16\Omega(N+1)\sup_{q\geq 2}\frac{||S_q u_1||_{L^\infty}^2}{\Omega(q)}||\partial_2\theta||_{L^2}^2 + \frac{1}{16}||\partial_1\partial_2\theta||_{L^2}^2 \\
&\quad + 2\sum_{j>N}||\Delta_j u_1||_{L^\infty}||\partial_2\theta||_{L^2}||\partial_1\partial_2\theta||_{L^2},
\end{aligned} \tag{3.14}$$

where the positive integer N will be fixed later.

Using the interpolation theorem, Minkowsik's inequality and Hölder's inequality, we get

$$\begin{aligned}
||\Delta_j u_1||_{L^\infty(\mathbb{R}^2)} &\leq \left\| ||\Delta_j u_1||_{L^\infty_{x_1}(\mathbb{R})}\right\|_{L^\infty_{x_2}(R)} \\
&\leq \left\| ||\Delta_j u_1||^{\frac{1}{2}}_{L^2_{x_1}(\mathbb{R})}||\partial_1\Delta_j u_1||^{\frac{1}{2}}_{L^2_{x_1}(\mathbb{R})}\right\|_{L^\infty_{x_2}(\mathbb{R})} \\
&\leq \left\| ||\Delta_j u_1||_{L^\infty_{x_2}(\mathbb{R})}\right\|^{\frac{1}{2}}_{L^2_{x_1}(\mathbb{R})}\left\| ||\partial_1\Delta_j u_1||_{L^\infty_{x_2}(\mathbb{R})}\right\|^{\frac{1}{2}}_{L^2_{x_1}(\mathbb{R})}.
\end{aligned}$$

Since

$$||\Delta_j u_1(x_1,\cdot)||_{L^\infty_{x_2}(\mathbb{R})} \leq ||\Delta_j u_1(x_1,\cdot)||^{\frac{1}{2}}_{L^2_{x_2}(\mathbb{R})}||\partial_2\Delta_j u_1(x_1,\cdot)||^{\frac{1}{2}}_{L^2_{x_2}(\mathbb{R})},$$

and

$$||\partial_1\Delta_j u_1(x_1,\cdot)||_{L^\infty_{x_2}(\mathbb{R})} \leq ||\partial_1\Delta_j u_1(x_1,\cdot)||^{\frac{1}{3}}_{L^2_{x_2}(\mathbb{R})}||\Lambda^{\frac{3}{4}}_{x_2}\partial_1\Delta_j u_1(x_1,\cdot)||^{\frac{2}{3}}_{L^2_{x_2}(\mathbb{R})},$$

where we use the fractional operators $\Lambda := \sqrt{-(\partial_1^2+\partial_2^2)}$, $\quad \Lambda_{x_i} := \sqrt{-\partial_i^2}$ with $i = 1, 2$. Then,

$$\begin{aligned}
||\Delta_j u_1||_{L^\infty} &\leq ||\Delta_j u_1||^{\frac{1}{4}}_{L^2}||\partial_2\Delta_j u_1||^{\frac{1}{4}}_{L^2}||\partial_1\Delta_j u_1||^{\frac{1}{6}}_{L^2}||\Lambda^{\frac{3}{4}}_{x_2}\partial_1\Delta_j u_1||^{\frac{1}{3}}_{L^2} \\
&\leq C||\Delta_j u_1||^{\frac{1}{4}}_{L^2}||\Delta_j\omega||^{\frac{5}{12}}_{L^2}||\Lambda^{\frac{3}{4}}_{x_2}\partial_1\Delta_j u_1||^{\frac{1}{3}}_{L^2}.
\end{aligned}$$

By the Bernstein lemma, we have

$$\begin{aligned}
2\sum_{j>N}||\Delta_j u_1||_{L^\infty} &\leq C||u_1||^{\frac{1}{4}}_{L^2}||\omega||^{\frac{5}{12}}_{L^2}\sum_{j>N}||\Lambda^{\frac{3}{4}}_{x_2}\partial_1\Delta_j u_1||^{\frac{1}{3}}_{L^2} \\
&\leq C||u_1||^{\frac{1}{4}}_{L^2}||\omega||^{\frac{5}{12}}_{L^2}\sum_{j>N}2^{-\frac{j}{12}}(2^{\frac{j}{4}}||\Lambda^{\frac{3}{4}}_{x_2}\partial_1\Delta_j u_1||_{L^2})^{\frac{1}{3}} \\
&\leq C2^{-\frac{N}{12}}||u_1||^{\frac{1}{4}}_{L^2}||\omega||^{\frac{5}{12}}_{L^2}||\partial_1\Lambda^{\frac{1}{4}}\Lambda^{\frac{3}{4}}_{x_2}u_1||^{\frac{1}{3}}_{L^2} \\
&\leq C2^{-\frac{N}{12}}||u_1||^{\frac{1}{4}}_{L^2}||\omega||^{\frac{5}{12}}_{L^2}||\partial_1\omega||^{\frac{1}{3}}_{L^2} \\
&\leq C||u_1||_{L^2} + \frac{1}{4}2^{-\frac{N}{9}}||\omega||_{L^2} + \frac{1}{8}2^{-\frac{N}{9}}||\partial_1\omega||_{L^2}.
\end{aligned}$$

Inserting the above inequality into (3.14) and using Cauchy-Schwarz's inequality, we infer that

$$
\begin{aligned}
2\int_{\mathbb{R}^2} u_1\partial_1\partial_2\theta\partial_2\theta dx \le{}& 16\Omega(N+1)\sup_{q\ge 2}\frac{||S_q u_1||_{L^\infty}^2}{\Omega(q)}||\partial_2\theta||_{L^2}^2 + C||u_1||_{L^2}||\partial_2\theta||_{L^2}||\partial_1\partial_2\theta||_{L^2}\\
&+\frac{1}{4}2^{-\frac{N}{9}}||\omega||_{L^2}||\partial_2\theta||_{L^2}||\partial_1\partial_2\theta||_{L^2} + \frac{1}{8}2^{-\frac{N}{9}}||\partial_1\omega||_{L^2}||\partial_2\theta||_{L^2}||\partial_1\partial_2\theta||_{L^2}\\
&+\frac{1}{16}||\partial_1\partial_2\theta||_{L^2}^2\\
\le{}& 16\Omega(N+1)\sup_{q\ge 2}\frac{||S_q u_1||_{L^\infty}^2}{\Omega(q)}||\partial_2\theta||_{L^2}^2 + C||u_1||_{L^2}^2||\partial_2\theta||_{L^2}^2\\
&+\frac{1}{4}2^{-\frac{2N}{9}}||\omega||_{L^2}^2||\partial_2\theta||_{L^2}^2 + \frac{1}{8}2^{-\frac{2N}{9}}||\partial_1\omega||_{L^2}^2||\partial_2\theta||_{L^2}^2 + \frac{1}{4}||\partial_1\partial_2\theta||_{L^2}^2.
\end{aligned}
$$

Now we set a fixed positive integer N such that $2^{-\frac{2N}{9}}(||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2) \le 1$, namely,

$$
N \ge \left[\frac{9}{2\log 2}\log^+(||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2)\right] + 1,
$$

where $\log^+(\cdot) = \log(e+\cdot)$ and $[\cdot]$ is the rounding function.

Thus,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\partial_2\theta(t)||_{L^2}^2 + \frac{1}{2}||\partial_2\partial_1\theta(t)||_{L^2}^2\\
&\le 16\Omega\left(\left[\frac{9}{2\log 2}\log^+(||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2)\right]+1\right)\sup_{q\ge 2}\frac{||S_q u_1(t)||_{L^\infty}^2}{\Omega(q)}||\partial_2\theta(t)||_{L^2}^2\\
&+C||\theta_0||_{L^\infty}^2(||\omega||_{L^2}^2 + ||\partial_2\theta||_{L^2}^2) + C||u_1(t)||_{L^2}^2||\partial_2\theta(t)||_{L^2}^2 + \frac{1}{4}||\partial_1\omega(t)||_{L^2}^2 + \frac{1}{4}||\omega(t)||_{L^2}^2.
\end{aligned}
$$

This together with (3.12) yields

$$
\begin{aligned}
&\frac{d}{dt}(||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2) + \frac{1}{2}(||\partial_1\omega(t)||_{L^2}^2 + ||\partial_2\partial_1\theta(t)||_{L^2}^2)\\
&\le (F_M')^2||\partial_2\theta(t)||_{L^2}^2 + ||F_2(\theta)(t)||_{L^2}^2 + C||\theta_0||_{L^\infty}^2(||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2)\\
&\quad + 32\Omega\left(\left[\frac{9}{2\log 2}\log^+\left(||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2\right)\right]+1\right)\sup_{q\ge 2}\frac{||S_q u_1||_{L^\infty}^2}{\Omega(q)}||\partial_2\theta(t)||_{L^2}^2\\
&\quad + C||u_1(t)||_{L^2}^2||\partial_2\theta(t)||_{L^2}^2 + 2||\omega(t)||_{L^2}^2.
\end{aligned}
$$

According to the properties of the dual Osgood modulus of continuity, we obtain

$$
\Omega\left(\left[\frac{9}{2\log 2}\log^+\left(||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2\right)\right]+1\right) \le C\Omega(\log^+(||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2)).
$$

Moreover,

$$
\begin{aligned}
&\frac{d}{dt}(e + ||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2)\\
&\le C\left(1 + (F_M')^2 + ||\theta_0||_{L^\infty}^2 + ||u(t)||_{L^2}^2 + ||F_2(\theta)(t)||_{L^2}^2 + \sup_{q\ge 2}\frac{||S_q u_1(t)||_{L^\infty}^2}{\Omega(q)}\right)\\
&\quad\times\Omega(\log^+(||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2))(e + ||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2)).
\end{aligned}
$$

We denote

$$f(t) := \log^+(||\omega(t)||_{L^2}^2 + ||\partial_2\theta(t)||_{L^2}^2),$$

$$g(t) := 1 + (F'_M)^2 + ||\theta_0||_{L^\infty}^2 + ||u(t)||_{L^2}^2 + ||F_2(\theta)(t)||_{L^2}^2,$$

$$h(t) := \sup_{q\geq 2} \frac{||S_q u_1||_{L^\infty}^2}{\Omega(q)},$$

then

$$\frac{d}{dt} f(t) \leq C(h(t) + g(t))\Omega(f(t)).$$

If $M(x) := \int_1^x \frac{1}{\Omega(\tau)} d\tau$, we have

$$\frac{d}{dt} M(f(t)) \leq C(h(t) + g(t)).$$

Consequently,

$$M(f(t)) \leq M(f(0)) + C\int_0^t (h(\tau) + g(\tau))d\tau =: l(t).$$

So we can conclude that M is bijective mapping from $[1, +\infty]$ to $[0, +\infty]$ by using the property of the dual Osgood modulus of continuity that $\int_1^\infty \frac{1}{\Omega(x)} dx = \infty$. Hence, there exists a unique inverse function $M^{-1}(\cdot)$ of $M(\cdot)$ such that

$$f(t) \leq M^{-1}(l(t)).$$

This completes the proof of the desired result. □

Proposition 3.3. *Assume $u_0 \in L^2(\mathbb{R}^2) \cap L^4(\mathbb{R}^2)$ and $\theta_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with $F(\theta_0) \in L^2(\mathbb{R}^2)$. Suppose that (u, θ) is a smooth solution of (1.1). Then, we have*

$$||u(t)||_{L^4}{}^4 + \int_0^t \int_{\mathbb{R}^2} |u(\tau, x)|^2 |\partial_1 u(\tau, x)|^2 dx d\tau \leq C_0 e^{(1+t)^4}, \tag{3.15}$$

$$||\Pi_u(t)||_{L^2}{}^2 + \int_0^t ||\nabla\Pi_u(\tau)||_{L^2}{}^2 d\tau \leq C_0 e^{(1+t)^4}. \tag{3.16}$$

Proof of Proposition. By the Equation 1.2, we have

$$\begin{aligned}\Pi_u &= \frac{\partial_1\partial_1}{-\Delta}(u_1u_1) + 2\frac{\partial_1\partial_2}{-\Delta}(u_1u_2) + \frac{\partial_2\partial_2}{-\Delta}(u_2u_2) \\ &= \frac{\partial_1\partial_1}{-\Delta}(u_1u_1) + 2\frac{\partial_1\partial_2}{-\Delta}(u_1u_2) - 2\frac{\partial_2}{-\Delta}(u_2\partial_1 u_1).\end{aligned} \tag{3.17}$$

It follows that

$$||\partial_2\Pi_u||_{L^2} \leq C\left(\int_{\mathbb{R}^2} |u|^2|\partial_1 u|^2 dx\right)^{\frac{1}{2}}. \tag{3.18}$$

Multiplying the first equation of (1.1) by $|u|^2 u$ and using (1.2), we obtain that

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}||u(t)||_{L^4}{}^4 + \int_{\mathbb{R}^2} |u|^2|\partial_1 u|^2 dx \leq &- \int_{\mathbb{R}^2} \nabla\Pi_u \cdot |u|^2 u dx - \int_{\mathbb{R}^2} \nabla\Pi_\theta \cdot |u|^2 u dx \\ &+ \int_{\mathbb{R}^2} F(\theta)|u|^2 u dx,\end{aligned} \tag{3.19}$$

where we have used the fact

$$-\int_{\mathbb{R}^2} |u|^2 u \cdot \partial_1^2 u dx \geq \int_{\mathbb{R}^2} |u|^2 |\partial_1 u|^2 dx.$$

For the first term of right hand of (3.19), we have

$$-\int_{\mathbb{R}^2} \nabla\Pi_u \cdot |u|^2 u dx = \int_{\mathbb{R}^2} \Pi_u |u|^2 \partial_1 u_1 dx + 2\int_{\mathbb{R}^2} \Pi_u u_1 u \cdot \partial_1 u dx - \int_{\mathbb{R}^2} \partial_2 \Pi_u |u|^2 u_2 dx. \tag{3.20}$$

By (3.18), (2.2) of Lemma 2.2 and Young's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^2} \partial_2\Pi_u |u|^2 u_2 dx &\le C||\partial_2\Pi_u||_{L^2}||u_2||_{L^2}^{\frac{1}{2}}||\partial_2 u_2||_{L^2}^{\frac{1}{2}}|||u|^2||_{L^2}^{\frac{1}{2}}||\partial_1|u|^2||_{L^2}^{\frac{1}{2}}\\ &\le C|||u|\partial_1 u||_{L^2}||u_2||_{L^2}^{\frac{1}{2}}||\partial_1 u_1||_{L^2}^{\frac{1}{2}}||u||_{L^4}|||u||\partial_1 u|||_{L^2}^{\frac{1}{2}}\\ &\le C||u_2||_{L^2}^2||\partial_1 u_1||_{L^2}^2||u||_{L^4}^4 + \frac{1}{4}|||u||\partial_1 u|||_{L^2}^2.\end{aligned}$$

In a similar way, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \Pi_u |u|^2 \partial_1 u_1 dx &\le C||\partial_1 u_1||_{L^2}|||u|^2||_{L^2}{}^{\frac{1}{2}}||\partial_1|u|^2||_{L^2}{}^{\frac{1}{2}}||\Pi_u||_{L^2}{}^{\frac{1}{2}}||\partial_2\Pi_u||_{L^2}{}^{\frac{1}{2}}\\ &\le C||\partial_1 u_1||_{L^2}||u||_{L^4}|||u||\partial_1 u|||_{L^2}^{\frac{1}{2}}||u||_{L^4}|||u||\partial_1 u|||_{L^2}{}^{\frac{1}{2}}\\ &\le C||\partial_1 u_1||_{L^2}{}^2||u||_{L^4}{}^4 + \frac{1}{4}|||u||\partial_1 u|||_{L^2}{}^2,\end{aligned}$$

and

$$\int_{\mathbb{R}^2} \Pi_u u_1 u \cdot \partial_1 u dx \le C||\partial_1 u||_{L^2}{}^2||u||_{L^4}{}^4 + \frac{1}{4}|||u||\partial_1 u|||_{L^2}{}^2.$$

For the last two integral terms of (3.19), using Hölder's inequality and Young's inequality, we conclude

$$-\int_{\mathbb{R}^2} \nabla\Pi_\theta |u|^2 u dx + \int_{\mathbb{R}^2} F(\theta)|u|^2 u dx \le CF_M^{\frac{1}{2}}(||F(\theta_0)||_{L^2}^2 + F_M'' F_M||\theta_0||_{L^2}^2)^{\frac{1}{4}}||u||_{L^4}^3.$$

Summing up above estimates, we deduce that

$$\frac{d}{dt}||u(t)||_{L^4} \le C(||u||_{L^2}^2|\partial_1 u||_{L^2}^2 + ||\partial_1 u||_{L^2}^2)||u||_{L^4} + CF_M^{\frac{1}{2}}(||F(\theta_0)||_{L^2}^2 + F_M'' F_M||\theta_0||_{L^2}^2)^{\frac{1}{4}}.$$

The Gronwall inequality gives

$$||u(t)||_{L^4} \le e^{C(||u||_{L_t^\infty L^2}^2+1)\int_0^t ||\partial_1 u(\tau)||_{L^2}^2 d\tau}(||u_0||_{L^4} + CF_M^{\frac{1}{2}}(||F(\theta_0)||_{L^2}^2 + F_M'' F_M||\theta_0||_{L^2}^2)^{\frac{1}{4}} t).$$

Plugging this into (3.19) together with Proposition 3.1 yields (3.15).

Finally, by (1.2), (3.15) and the property of Calderón-Zygmund operator, we have (3.16). □

Proposition 3.4. *Assume $u_0 \in \mathbb{L}(\mathbb{R}^2)$ and $\theta_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with $F(\theta_0) \in L^2(\mathbb{R}^2)$, where $||u_0||_{\mathbb{L}(\mathbb{R}^2)} := sup_{2\le p<\infty}\frac{||u_0||_{L^p(\mathbb{R}^2)}}{p}$. Suppose that (u, θ) is a smooth solution of (1.1). Then, we have*

$$||u_i(t)||_{\mathbb{L}(\mathbb{R}^2)} \le ||u_i(0)||_{\mathbb{L}(\mathbb{R}^2)} e^{C_0(1+t)^4} \quad \textit{for } i = 1, 2. \tag{3.21}$$

Proof of Proposition. We only prove the result for u_1. The estimates for u_2 can be obtained in a similar way.

Since

$$\partial_t u_1 + u \cdot \nabla u_1 - \partial_1^2 u_1 + \partial_1 \Pi = F_1(\theta).$$

For any $p \in (4, \infty)$, multiplying the above equation by $|u_1|^{p-2}u_1$ and integrating with respect to the space variable, we obtain

$$\frac{1}{p}||u_1||_{L^p}^p + (p-1)\int_{\mathbb{R}^2} |u_1|^{p-2}|\partial_1 u_1|^2 dx = -\int_{\mathbb{R}^2} \partial_1 \Pi |u_1|^{p-2}u_1 dx + \int_{\mathbb{R}^2} F_1(\theta)|u_1|^{p-2}u_1 dx. \tag{3.22}$$

By Hölder's inequality, we have

$$\int_{\mathbb{R}^2} F_1(\theta)|u_1|^{p-2}u_1 dx \le ||F(\theta)||_{L^p}||u_1||_{L^p}^{p-1} \le ||F(\theta)||_{L^2\cap L^\infty}||u_1||_{L^p}^{p-1}.$$

Integrating by parts gives

$$-\int_{\mathbb{R}^2} \partial_1 \Pi |u_1|^{p-2}u_1 dx = (p-1)\int_{\mathbb{R}^2} \Pi_u |u_1|^{p-2}\partial_1 u_1 dx - \int_{\mathbb{R}^2} \partial_1 \Pi_\theta |u_1|^{p-2}u_1 dx.$$

Note that

$$\partial_1 \Pi_\theta = -\sum_i \frac{\partial_i}{-\triangle}\partial_1(F_i(\theta)).$$

Using the following fact

$$\sup_{p\ge 2}\frac{||u||_{L^p(\mathbb{R}^2)}}{\sqrt{p}} \le C||u||_{H^1(\mathbb{R}^2)} \tag{3.23}$$

and Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^2} \partial_1 \Pi_\theta |u_1|^{p-2}u_1 dx &\le ||\partial_1\Pi_\theta||_{L^p}||u_1||_{L^p}^{p-1}\\ &\le C\sqrt{p}(||\nabla\partial_1\Pi_\theta||_{L^2} + ||\partial_1\Pi_\theta||_{L^2})||u_1||_{L^p}^{p-1}\\ &\le C\sqrt{p}(||\partial_1 F(\theta)||_{L^2} + ||F(\theta)||_{L^2})||u_1||_{L^p}^{p-1}.\end{aligned}$$

Then, we deal with $\int_{\mathbb{R}^2}\Pi_u|u_1|^{p-2}\partial_1 u_1 dx$. According to (3.23), we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \Pi_u |u_1|^{p-2}\partial_1 u_1 dx &\le ||\Pi_u||_{L^p}||u_1||_{L^p}^{\frac{p-2}{2}}|||u_1|^{\frac{p-2}{2}}\partial_1 u_1||_{L^2}\\ &\le Cp(||\nabla\Pi_u||_{L^2}^2 + ||\Pi_u||_{L^2}^2)||u_1||_{L^p}^{p-2} + \frac{1}{4}|||u_1|^{\frac{p-2}{2}}\partial_1 u_1||_{L^2}^2.\end{aligned}$$

Inserting these estimates into (3.22), we have

$$\begin{aligned}\frac{d}{dt}\frac{||u_1(t)||_{L^p}^2}{p^2} &\le C||F(\theta_0)||_{L^2\cap L^\infty}\frac{||u_1(t)||_{L^p}}{p} + C(||\partial_1 F(\theta)(t)||_{L^2} + ||F(\theta)||_{L^2})\frac{||u_1(t)||_{L^p}}{p}\\ &\quad + C(||\nabla\Pi_u(t)||_{L^2}^2 + ||\Pi_u(t)||_{L^2}^2)\\ &\le C\frac{||u_1(t)||_{L^p}^2}{p^2} + C(||\partial_1 F(\theta)(t)||_{L^2}^2 + ||F(\theta)||_{L^2}^2 + ||F(\theta_0)||_{L^2\cap L^\infty}^2\\ &\quad + ||\nabla\Pi_u(t)||_{L^2}^2 + ||\Pi_u(t)||_{L^2}^2).\end{aligned}$$

Using the Gronwall inequality yields

$$\begin{aligned}\frac{||u_1(t)||_{L^p}^2}{p^2} \le e^{C_0 t}\Bigg(&\frac{||u_1(0)||_{L^p}^2}{p^2} + C\int_0^t (||\partial_1 F(\theta)(\tau)||_{L^2}^2 + ||F(\theta)||_{L^2}^2 + ||F(\theta_0)||_{L^2\cap L^\infty}^2\\ &+||\nabla\Pi_u(\tau)||_{L^2}^2 + ||\Pi_u(\tau)||_{L^2}^2)d\tau\Big).\end{aligned}$$

Taking the supermum over p and using Proposition 3.1 and Proposition 3.3 gives the desired result. □

Proposition 3.5. *Assume $u_0 \in L^2(\mathbb{R}^2) \cap L^r(\mathbb{R}^2)$ for some $r \in (4, \infty)$, $u_1(0) \in \sqrt{L\log L}(\mathbb{R}^2)$, $\theta_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with $F(\theta_0) \in L^2(\mathbb{R}^2)$, where*

$$||u_1(0)||_{\sqrt{L\log L}(\mathbb{R}^2)} := \sup_{p\geq 2} \frac{||u_1(0)||_{L^p(\mathbb{R}^2)}}{\sqrt{p\log p}}.$$

Suppose that (u, θ) is a smooth solution of (1.1). Then, we have

$$||u_1(t)||_{\sqrt{L\log L}} := \sup_{p\geq 2} \frac{||u_1(t)||_{L^p(\mathbb{R}^2)}}{\sqrt{p\log p}} \leq ||u_1(0)||_{\sqrt{L\log L}} e^{C_0(1+t)^4}. \tag{3.24}$$

Proof of Proposition. Using Bony's paraproduct decomposition, we know

$$\begin{aligned}\int_{\mathbb{R}^2} \Pi_u |u_1|^{p-2}\partial_1 u_1 dx = &\int_{\mathbb{R}^2} \Pi_u \sum_{k\geq 1} S_{k-1}(|u_1|^{\frac{p-2}{2}})\triangle_k(|u_1|^{\frac{p-2}{2}}\partial_1 u_1)dx \\ &+ \int_{\mathbb{R}^2} \Pi_u \sum_{k\geq 1} S_{k-1}(|u_1|^{\frac{p-2}{2}}\partial_1 u_1)\triangle_k(|u_1|^{\frac{p-2}{2}})dx \\ &+ \int_{\mathbb{R}^2} \Pi_u \sum_{k\geq -1} \triangle_k(|u_1|^{\frac{p-2}{2}})\tilde{\triangle}_k(|u_1|^{\frac{p-2}{2}}\partial_1 u_1)dx.\end{aligned}$$

By Hölder's inequality, Young's inequality and $H^1(\mathbb{R}^2) \hookrightarrow B^0_{\infty,2}(\mathbb{R}^2)$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} \Pi_u \sum_{k\geq 1} S_{k-1}(|u_1|^{\frac{p-2}{2}})\triangle_k(|u_1|^{\frac{p-2}{2}}\partial_1 u_1)dx \\ &= \sum_j \int_{\mathbb{R}^2} \triangle_j \Pi_u \sum_{k\sim j} S_{k-1}(|u_1|^{\frac{p-2}{2}})\triangle_k(|u_1|^{\frac{p-2}{2}}\partial_1 u_1)dx \\ &\leq C\sum_j ||\triangle_j \Pi_u||_{L^\infty} \sum_{k\sim j} ||S_{k-1}(|u_1|^{\frac{p-2}{2}})||_{L^2}||\triangle_k(|u_1|^{\frac{p-2}{2}}\partial_1 u_1)||_{L^2} \\ &\leq C||\Pi_u||_{B^0_{\infty,2}} ||u_1||_{L^2}^{\frac{2}{p-2}} ||u_1||_{L^p}^{\frac{p(p-4)}{2(p-2)}} |||u_1|^{\frac{p-2}{2}}\partial_1 u_1||_{L^2} \\ &\leq C||\Pi_u||^2_{H^1} ||u_1||_{L^2}^{\frac{4}{p-2}} ||u_1||_{L^p}^{p-2-\frac{4}{p-2}} + \frac{1}{4}|||u_1|^{\frac{p-2}{2}}\partial_1 u_1||^2_{L^2},\end{aligned}$$

where we have used the following interpolation inequality

$$|||u_1|||_{L^{p-2}} \leq ||u_1||_{L^2}^{\frac{4}{(p-2)^2}} ||u_1||_{L^p}^{\frac{p(p-4)}{(p-2)^2}} \quad \text{for} \quad 4 < p < \infty. \tag{3.25}$$

By the same way, the second integral term can be estimated as follows:

$$\int_{\mathbb{R}^2} \Pi_u \sum_{k\geq 1} S_{k-1}(|u_1|^{\frac{p-2}{2}}\partial_1 u_1)\triangle_k(|u_1|^{\frac{p-2}{2}})dx \leq C||\Pi_u||^2_{H^1} ||u_1||_{L^2}^{\frac{4}{p-2}} ||u_1||_{L^p}^{p-2-\frac{4}{p-2}} + \frac{1}{4}|||u_1|^{\frac{p-2}{2}}\partial_1 u_1||^2_{L^2}.$$

For the last integral term, we get

$$\begin{aligned}&\int_{\mathbb{R}^2} \Pi_u \sum_{k\geq -1} \triangle_k(|u_1|^{\frac{p-2}{2}})\tilde{\triangle}_k(|u_1|^{\frac{p-2}{2}}\partial_1 u_1)dx \\ &= \int_{\mathbb{R}^2} \sum_{j\geq -1} \triangle_j \Pi_u \sum_{k\geq j-3} \triangle_k(|u_1|^{\frac{p-2}{2}})\tilde{\triangle}_k(|u_1|^{\frac{p-2}{2}}\partial_1 u_1)dx \\ &= \int_{\mathbb{R}^2} \sum_{j<\log R} \triangle_j \Pi_u \sum_{k\geq j-3} \triangle_k(|u_1|^{\frac{p-2}{2}})\tilde{\triangle}_k(|u_1|^{\frac{p-2}{2}}\partial_1 u_1)dx\end{aligned}$$

$$+\int_{\mathbb{R}^2}\sum_{j\geq\log R}\triangle_j\Pi_u\sum_{k\geq j-3}\triangle_k(|u_1|^{\frac{p-2}{2}})\tilde{\triangle}_k(|u_1|^{\frac{p-2}{2}}\partial_1u_1)dx=:III^\alpha+III^\beta,$$

where the integer R is to be fixed later.

Using (3.25), Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
III^\alpha&\leq\sum_{-1\leq k<\log R}||\triangle_k(|u_1|^{\frac{p-2}{2}})||_{L^2}||\tilde{\triangle}_k(|u_1|^{\frac{p-2}{2}}\partial_1u_1)||_{L^2}\sum_{j<\log R}||\triangle_j\Pi_u||_{L^\infty}\\
&\leq\sum_{k\geq-1}||\triangle_k(|u_1|^{\frac{p-2}{2}})||_{L^2}||\tilde{\triangle}_k(|u_1|^{\frac{p-2}{2}}\partial_1u_1)||_{L^2}(\sum_{-1\leq k<\log R})^{\frac12}||\Pi_u||_{H^1}\\
&\leq C\sqrt{\log R}||\Pi_u||_{H^1}||u_1||_{L^2}^{\frac{2}{p-2}}||u_1||_{L^p}^{\frac{p(p-4)}{2(p-2)}}|||u_1|^{\frac{p-2}{2}}\partial_1u_1||_{L^2}\\
&\leq C\log R||\Pi_u||_{H^1}^2||u_1||_{L^2}^{\frac{4}{p-2}}||u_1||_{L^p}^{p-2-\frac{4}{p-2}}+\frac14|||u_1|^{\frac{p-2}{2}}\partial_1u_1||_{L^2}^2.
\end{aligned}$$

By Lemma 2.2, for fixed $q\in(2,\infty)$, we obtain

$$\begin{aligned}
III^\beta&\leq C_q\sum_{k\geq-1}||\tilde{\triangle}_k(|u_1|^{\frac{p-2}{2}}\partial_1u_1)||_{L^2}||\triangle_k(|u_1|^{\frac{p-2}{2}})||_{L^2}^{1-\frac1q}||\triangle_k\partial_1(|u_1|^{\frac{p-2}{2}})||_{L^2}^{\frac1q}\\
&\quad\times\sum_{j\geq\log R}||\triangle_j\Pi_u||_{L^{2(q-1)}}^{1-\frac1q}||\partial_2\triangle_j\Pi_u||_{L^2}^{\frac1q}\\
&\leq C_q(p-2)^{\frac1q}|||u_1|^{\frac{p-2}{2}}\partial_1u_1||_{L^2}||u_1||_{L^{p-2}}^{\frac{(p-2)(q-1)}{2q}}|||u_1|^{\frac{p-4}{2}}\partial_1u_1||_{L^2}^{\frac1q}\\
&\quad\times\sum_{j\geq\log R}||\triangle_j\Pi_u||_{L^{2(q-1)}}^{1-\frac1q}||\partial_2\triangle_j\Pi_u||_{L^2}^{\frac1q}.
\end{aligned}\tag{3.26}$$

Since

$$|||u_1|^{\frac{p-4}{2}}\partial_1u_1||_{L^2}\leq|||u_1|^{\frac{p-2}{2}}\partial_1u_1||_{L^2}^{\frac{p-4}{p-2}}||\partial_1u_1||_{L^2}^{\frac{2}{p-2}},$$

and for $q\in(2,\infty)$,

$$\begin{aligned}
&\sum_{j\geq\log R}||\triangle_j\Pi_u||_{L^{2(q-1)}}^{1-\frac1q}||\partial_2\triangle_j\Pi_u||_{L^2}^{\frac1q}\\
&\leq C\sum_{j\geq\log R}||\triangle_j\Pi_u||_{L^{2(q-1)}}^{\frac1q}(2^{-\frac{1}{q-1}j}||\nabla\triangle_j\Pi_u||_{L^2})^{1-\frac2q}||\partial_2\triangle_j\Pi_u||_{L^2}^{\frac1q}\\
&\leq C||\Pi_u||_{L^{2(q-1)}}^{\frac1q}\sum_{j\geq\log R}2^{-\frac{(q-2)j}{q(q-1)}}||\nabla\triangle_j\Pi_u||_{L^2}^{1-\frac1q}\\
&\leq C_q2^{-\frac{(q-2)\log R}{q(q-1)}}||\Pi_u||_{L^{2(q-1)}}^{\frac1q}||\nabla\Pi_u||_{L^2}^{1-\frac1q}\\
&\leq C_qR^{-\frac{(q-2)}{q(q-1)}}||\Pi_u||_{L^{2(q-1)}}^{\frac1q}||\nabla\Pi_u||_{L^2}^{1-\frac1q}.
\end{aligned}$$

For $p>4$ and $q>2$, plugging these estimates into (3.26) and using Young's inequality yields

$$\begin{aligned}
III^\beta&\leq C_q(p-2)^{\frac1q}R^{-\frac{(q-2)}{q(q-1)}}||\Pi_u||_{L^{2(q-1)}}^{\frac1q}||\nabla\Pi_u||_{L^2}^{1-\frac1q}\\
&\quad\times||u_1||_{L^2}^{\frac{2(q-1)}{q(p-2)}}||u_1||_{L^p}^{\frac{p(p-4)(q-1)}{2q(p-2)}}||\partial_1u_1||_{L^2}^{\frac{2}{q(p-2)}}|||u_1|^{\frac{p-2}{2}}\partial_1u_1||_{L^2}^{1+\frac{(p-4)}{q(p-2)}}\\
&\leq(\frac{2(q(p-2)+p-4)}{q(p-2)})^{1+\frac{2(p-4)}{q(p-2)-p+4}}(\frac{q(p-2)-p+4}{2q(p-2)})
\end{aligned}$$

$$
\begin{aligned}
&\times C_q^{\frac{2q(p-2)}{q(p-2)-p+4}}(p-2)^{\frac{2(p-2)}{q(p-2)-p+4}}(R^{-\frac{(q-2)}{q(q-1)}})^{\frac{2q(p-2)}{q(p-2)-p+4}}||\Pi_u||_{L^{2(q-1)}}^{\frac{2(p-2)}{q(p-2)-p+4}}||\nabla\Pi_u||_{L^2}^{\frac{2(q-1)(p-2)}{q(p-2)-p+4}}\\
&\times||u_1||_{L^2}^{\frac{4(q-1)}{q(p-2)-p+4}}||u_1||_{L^p}^{\frac{p(p-4)(q-1)}{q(p-2)-p+4}}||\partial_1 u_1||_{L^2}^{\frac{4}{q(p-2)-p+4}}+\frac{1}{4}|||u_1|^{\frac{p-2}{2}}\partial_1 u_1||_{L^2}^2\\
\le\;&(2(1+\frac{1}{q}))^{1+\frac{2}{q-1}}C_q^{2+\frac{2}{q-1}}(p-2)^{\frac{2(p-2)}{q(p-2)-p+4}}(R^{-\frac{q-2}{q-1}})^{\frac{2(p-2)}{q(p-2)-p+4}}||\Pi_u||_{L^{2(q-1)}}^{\frac{2(p-2)}{q(p-2)-p+4}}\\
&\times||\nabla\Pi_u||_{L^2}^{\frac{2(q-1)(p-2)}{q(p-2)-p+4}}||u_1||_{L^2}^{\frac{4(q-1)}{q(p-2)-p+4}}||u_1||_{L^p}^{\frac{p(p-4)(q-1)}{q(p-2)-p+4}}||\partial_1 u_1||_{L^2}^{\frac{4}{q(p-2)-p+4}}+\frac{1}{4}|||u_1|^{\frac{p-2}{2}}\partial_1 u_1||_{L^2}^2,
\end{aligned}
$$

where we have used the facts

$$
\frac{2q(p-2)}{q(p-2)-p+4}\le 2+\frac{2}{q-1}
$$

and

$$
\left(\frac{2(q(p-2)+p-4)}{q(p-2)}\right)^{1+\frac{2(p-4)}{q(p-2)-p+4}}\frac{q(p-2)-p+4}{2q(p-2)}\le\left(2+\frac{2}{q}\right)^{1+\frac{2}{q-1}}.
$$

Setting $R=(p-2)^{\frac{q-1}{q-2}}$, we have

$$
\begin{aligned}
III^\beta\le\;&\tilde{C}_q|\Pi_u||_{L^{2(q-1)}}^{\frac{2(p-2)}{q(p-2)-p+4}}||\nabla\Pi_u||_{L^2}^{\frac{2(q-1)(p-2)}{q(p-2)-p+4}}||u_1||_{L^2}^{\frac{4(q-1)}{q(p-2)-p+4}}||u_1||_{L^p}^{p-2+\frac{2(p+2q-4)}{q(p-2)-p+4}}||\partial_1 u_1||_{L^2}^{\frac{4}{q(p-2)-p+4}}\\
&+\frac{1}{4}|||u_1|^{\frac{p-2}{2}}\partial_1 u_1||_{L^2}^2,
\end{aligned}
$$

and

$$
\log R=\frac{q-1}{q-2}\log(p-2)\le C_q\log p.
$$

Hence,

$$
\begin{aligned}
III^\alpha+III^\beta\le\;&\tilde{C}_q|\Pi_u||_{L^{2(q-1)}}^{\frac{2(p-2)}{q(p-2)-p+4}}||\nabla\Pi_u||_{L^2}^{\frac{2(q-1)(p-2)}{q(p-2)-p+4}}||u_1||_{L^2}^{\frac{4(q-1)}{q(p-2)-p+4}}||\partial_1 u_1||_{L^2}^{\frac{4}{q(p-2-p+4)}}||u_1||_{L^p}^{p-2+\frac{2(p+2q-4)}{q(p-2)-p+4}}\\
&+C_q\log p||\Pi_u||_{H^1}^2||u_1||_{L^2}^{\frac{4}{p-2}}||u_1||_{L^p}^{p-2-\frac{4}{p-2}}+\frac{1}{2}|||u_1|^{\frac{p-2}{2}}\partial_1 u_1||_{L^2}^2.
\end{aligned}
$$

Next, we suppose that $||u_1||_{L^p}\ge 1$. Collecting these estimates with (3.22), we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}||u_1(t)||_{L^p}^2\le\;&C||F(\theta)(t)||_{L^2\cap L^\infty}||u_1(t)||_{L^p}+C\sqrt{p}(||\partial_1 F(\theta)(t)||_{L^2}+||F(\theta)(t)||_{L^2})||u_1(t)||_{L^p}\\
&+Cp||\Pi_u(t)||_{H^1}^2||u_1(t)||_{L^2}^{\frac{4}{p-2}}+C_q p\log p||\Pi_u(t)||_{H^1}^2||u_1(t)||_{L^2}^{\frac{4}{p-2}}\\
&+\tilde{C}_q p||\Pi_u(t)||_{L^{2(q-1)}}^{\frac{2(p-2)}{q(p-2)-p+4}}||\nabla\Pi_u(t)||_{L^2}^{\frac{2(q-1)(p-2)}{q(p-2)-p+4}}||u_1(t)||_{L^2}^{\frac{4(q-1)}{q(p-2)-p+4}}||\partial_1 u_1(t)||_{L^2}^{\frac{4}{q(p-2-p+4)}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{d}{dt}\frac{||u_1(t)||_{L^p}^2}{p\log p}\le\;&C\frac{1}{\log p}(||F(\theta)||_{L^2\cap L^\infty}+||\partial_1 F(\theta)(t)||_{L^2}^2)+\frac{||u_1(t)||_{L^p}^2}{p\log p}\\
&+C\frac{1}{\log p}||\Pi_u(t)||_{H^1}^2||u_1(t)||_{L^2}^{\frac{4}{p-2}}+C_q||\Pi_u(t)||_{H^1}^2||u_1(t)||_{L^2}^{\frac{4}{p-2}}\\
&+\frac{1}{\log p}\tilde{C}_q||\Pi_u(t)||_{L^{2(q-1)}}^{\frac{2(p-2)}{q(p-2)-p+4}}||u_1(t)||_{L^2}^{\frac{4(q-1)}{q(p-2)-p+4}}\\
&\times||\nabla\Pi_u(t)||_{L^2}^{\frac{2(q-1)(p-2)}{q(p-2)-p+4}}||\partial_1 u_1(t)||_{L^2}^{\frac{4}{q(p-2)-p+4}}.
\end{aligned}
$$

Using Gronwall's inequality and taking supremum over p, we have

$$
||u_1(t)||_{\sqrt{L\log L}}\le(||u_1(0)||_{\sqrt{L\log L}}+\int_0^t H(\tau)+G(\tau)d\tau)e^t, \tag{3.27}
$$

where

$$H(\tau) : = C(||F(\theta)||_{L^2\cap L^\infty} + ||\partial_1 F(\theta)(t)||_{L^2}^2 + ||\Pi_u(t)||_{H^1}^2 ||u_1(t)||_{L^2}^{\frac{4}{p-2}}) + C_q||\Pi_u(t)||_{H^1}^2 ||u_1(t)||_{L^2}^{\frac{4}{p-2}},$$

and

$$G(\tau) := \tilde{C}_q||\Pi_u(t)||_{L^{2(q-1)}}^{\frac{2(p-2)}{q(p-2)-p+4}} ||u_1(t)||_{L^2}^{\frac{4(q-1)}{q(p-2)-p+4}} ||\nabla\Pi_u(t)||_{L^2}^{\frac{2(q-1)(p-2)}{q(p-2)-p+4}} ||\partial_1 u_1(t)||_{L^2}^{\frac{4}{q(p-2)-p+4}}.$$

Note that

$$\frac{2(q-1)(p-2)}{q(p-2)-p+4} + \frac{4}{q(p-2)-p+4} = 2.$$

By Hölder's inequality and taking $q = \frac{r}{4} + 1$, we obtain

$$\begin{aligned}
&\int_0^t ||\Pi_u(\tau)||_{L^{2(q-1)}}^{\frac{2(p-2)}{q(p-2)-p+4}} ||u_1(\tau)||_{L^2}^{\frac{4(q-1)}{q(p-2)-p+4}} ||\nabla\Pi_u(\tau)||_{L^2}^{\frac{2(q-1)(p-2)}{q(p-2)-p+4}} ||\partial_1 u_1(\tau)||_{L^2}^{\frac{4}{q(p-2)-p+4}} d\tau \\
&\leq ||\Pi_u||_{L_t^\infty L^{2(q-1)}}^{\frac{2(p-2)}{q(p-2)-p+4}} ||u_1||_{L_t^\infty L^2}^{\frac{4(q-1)}{q(p-2)-p+4}} ||\nabla\Pi_u||_{L_t^2 L^2}^{\frac{2(q-1)(p-2)}{q(p-2)-p+4}} ||\partial_1 u_1||_{L_t^2 L^2}^{\frac{4}{q(p-2)-p+4}} \\
&\leq ||u||_{L_t^\infty L^{4(q-1)}}^{\frac{4(p-2)}{q(p-2)-p+4}} ||u_1||_{L_t^\infty L^2}^{\frac{4(q-1)}{q(p-2)-p+4}} ||\nabla\Pi_u||_{L_t^2 L^2}^{\frac{2(q-1)(p-2)}{q(p-2)-p+4}} ||\partial_1 u_1||_{L_t^2 L^2}^{\frac{4}{q(p-2)-p+4}}.
\end{aligned} \tag{3.28}$$

Finally, we can infer that $H(t)$ and $G(t)$ are locally integrable on R^+ by Propositions 3.3 and 3.4. This completes the proof. □

Proposition 3.6. *Assume $u_0 \in H^1(\mathbb{R}^2), \partial_2\omega_0 \in L^2(\mathbb{R}^2), \theta_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), \partial_2\theta_0 \in L^2(\mathbb{R}^2), \partial_2^2\theta_0 \in L^2(\mathbb{R}^2)$, and $F(\theta_0) \in L^2(\mathbb{R}^2)$. Suppose that (u, θ) is a smooth solution of (1.1). Then we have*

$$||\partial_2\omega(t)||_{L^2}^2 + ||\partial_2^2\theta(t)||_{L^2}^2 + \int_0^t ||\partial_1\partial_2\omega(\tau)||_{L^2}^2 d\tau + \int_0^t ||\partial_1\partial_2^2\theta(\tau)||_{L^2}^2 d\tau \leq C_0(t), \tag{3.29}$$

$$\int_0^t ||\nabla u(\tau)||_{L^\infty}^4 d\tau + \int_0^t ||\partial_2\theta(\tau)||_{L^\infty}^4 d\tau \leq C_0(t). \tag{3.30}$$

Proof of Proposition. Taking the operator ∂_2 to the vorticity equality, we get

$$\partial_t\partial_2\omega + u\cdot\nabla\partial_2\omega - \partial_1^2\partial_2\omega = \partial_1\partial_2(F_2(\theta)) - \partial_2^2(F_1(\theta)) - \partial_2 u_1\partial_1\omega - \partial_2 u_2\partial_2\omega.$$

Multiplying the above equality with $\partial_2\omega$ and integrating in space yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\partial_2\omega(t)||_{L^2}^2 + ||\partial_1\partial_2\omega(\tau)||_{L^2}^2 = &\int_{\mathbb{R}^2} \partial_1\partial_2(F_2(\theta))\partial_2\omega dx - \int_{\mathbb{R}^2} \partial_2^2(F_1(\theta))\partial_2\omega dx \\
&- \int_{\mathbb{R}^2} \partial_2 u_1\partial_1\omega\partial_2\omega dx - \int_{\mathbb{R}^2} \partial_2 u_2\partial_2\omega\partial_2\omega dx.
\end{aligned} \tag{3.31}$$

As for the first integral, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \partial_1\partial_2(F_2(\theta))\partial_2\omega dx &= -\int_{\mathbb{R}^2} F_2'(\theta)\partial_2\theta\partial_1\partial_2\omega dx \\
&\leq 2(F_M')^2||\partial_2\theta||_{L^2}^2 + \frac{1}{8}||\partial_1\partial_2\omega||_{L^2}^2.
\end{aligned}$$

For the second integral term, using Cauchy-Schwarz's inequality together with Lemma 2.2 yields

$$
\begin{aligned}
-\int_{\mathbb{R}^2}\partial_2^2F_1(\theta)\partial_2\omega dx &= -\int_{\mathbb{R}^2}(F_1'(\theta))\partial_2^2\theta\partial_2\omega dx-\int_{\mathbb{R}^2}F_1''(\theta)\partial_2\theta\partial_2\theta\partial_2\omega dx\\
&\leq \frac{1}{2}F_M'(||\partial_2\omega||_{L^2}^2+||\partial_2^2\theta||_{L^2}^2)+F_M''\int_{\mathbb{R}^2}|\partial_2\theta||\partial_2\theta||\partial_2\omega|dx\\
&\leq \frac{1}{2}F_M'(||\partial_2\omega||_{L^2}^2+||\partial_2^2\theta||_{L^2}^2)+CF_M''||\partial_2\theta||_{L^2}^{\frac{3}{2}}||\partial_2|\partial_2\theta|||_{L^2}^{\frac{1}{2}}||\partial_2\omega||_{L^2}^{\frac{1}{2}}||\partial_1|\partial_2\omega|||_{L^2}^{\frac{1}{2}}\\
&\leq \frac{1}{2}F_M'(||\partial_2\omega||_{L^2}^2+||\partial_2^2\theta||_{L^2}^2)+CF_M''||\partial_2\theta||_{L^2}^{\frac{3}{2}}||\partial_2^2\theta||_{L^2}^{\frac{1}{2}}||\partial_2\omega||_{L^2}^{\frac{1}{2}}||\partial_1\partial_2\omega||_{L^2}^{\frac{1}{2}}\\
&\leq C(1+F_M')(||\partial_2\omega||_{L^2}^2+||\partial_2^2\theta||_{L^2}^2)+C(F_M'')^4||\partial_2\theta||_{L^2}^6+\frac{1}{8}||\partial_1\partial_2\omega||_{L^2}^2,
\end{aligned}
$$

where we have used the following fact that

$$
|\partial_2|\partial_2\theta||\leq|\partial_2^2\theta|,\qquad |\partial_1|\partial_2\omega||\leq\partial_1|\partial_2\omega|\quad for\quad ae\quad x\in\mathbb{R}^2.
$$

Similarly, we obtain

$$
\begin{aligned}
-\int_{\mathbb{R}^2}\partial_2u_1\partial_1\omega\partial_2\omega dx &\leq C||\partial_2\omega||_{L^2}||\partial_2u_1||_{L^2}^{\frac{1}{2}}||\partial_1\partial_2u_1||_{L^2}^{\frac{1}{2}}||\partial_1\omega||_{L^2}^{\frac{1}{2}}||\partial_2\partial_1\omega||_{L^2}^{\frac{1}{2}}\\
&\leq C||\partial_2\omega||_{L^2}||\omega||_{L^2}^{\frac{1}{2}}||\partial_1\omega||_{L^2}||\partial_2\partial_1\omega||_{L^2}^{\frac{1}{2}}\\
&\leq C||\omega||_{L^2}^2+C||\partial_2\omega||_{L^2}^2||\partial_1\omega||_{L^2}^2+\frac{1}{16}||\partial_2\partial_1\omega||_{L^2}^2.
\end{aligned}
$$

The last term of the right hand side with (3.31) can be estimated as follows:

$$
\begin{aligned}
-\int_{\mathbb{R}^2}\partial_2u_2\partial_2\omega\partial_2\omega dx &= \int_{\mathbb{R}^2}\partial_1u_1\partial_2\omega\partial_2\omega dx\\
&\leq C||\partial_2\omega||_{L^2}^{\frac{3}{2}}||\partial_1u_1||_{L^2}^{\frac{1}{2}}||\partial_2\partial_1u_1||_{L^2}^{\frac{1}{2}}||\partial_1\partial_2\omega||_{L^2}^{\frac{1}{2}}\\
&\leq C||\partial_1u_1||_{L^2}^{\frac{2}{3}}||\partial_2\partial_1u_1||_{L^2}^{\frac{2}{3}}||\partial_2\omega||_{L^2}^2+\frac{1}{8}||\partial_1\partial_2\omega||_{L^2}^2\\
&\leq C||\omega||_{L^2}^{\frac{2}{3}}||\partial_1\omega||_{L^2}^{\frac{2}{3}}||\partial_2\omega||_{L^2}^2+\frac{1}{8}||\partial_1\partial_2\omega||_{L^2}^2.
\end{aligned}
$$

Then, applying the operator ∂_2 to (3.13) gives

$$
\partial_t\partial_2^2\theta+u\cdot\nabla\partial_2^2\theta-\partial_1^2\partial_2^2\theta=-2\partial_2u_2\partial_2^2\theta-2\partial_2u_1\partial_1\partial_2\theta-\partial_2^2u_1\partial_1\theta-\partial_2^2u_2\partial_2\theta.
$$

In the same way, we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\partial_2^2\theta(t)||_{L^2}^2+||\partial_1\partial_2^2\theta(t)||_{L^2}^2 &= -2\int_{\mathbb{R}^2}\partial_2u_2\partial_2^2\theta\partial_2^2\theta dx-2\int_{\mathbb{R}^2}\partial_2u_1\partial_1\partial_2\theta\partial_2^2\theta dx\\
&\quad-\int_{\mathbb{R}^2}\partial_2^2u_1\partial_1\theta\partial_2^2\theta dx-\int_{\mathbb{R}^2}\partial_2^2u_2\partial_2\theta\partial_2^2\theta dx.
\end{aligned}\tag{3.32}
$$

Using Lemma 2.2 and Young's inequality, we have

$$
\begin{aligned}
-2\int_{\mathbb{R}^2}\partial_2u_2\partial_2^2\theta\partial_2^2\theta dx &= 2\int_{\mathbb{R}^2}\partial_1u_1\partial_2^2\theta\partial_2^2\theta dx\\
&\leq C||\partial_2^2\theta||_{L^2}^{\frac{3}{2}}||\partial_1u_1||_{L^2}^{\frac{1}{2}}||\partial_1\partial_2u_1||_{L^2}^{\frac{1}{2}}||\partial_1\partial_2^2\theta||_{L^2}^{\frac{1}{2}}\\
&\leq C||\partial_1u_1||_{L^2}^{\frac{2}{3}}||\partial_1\partial_2u_1||_{L^2}^{\frac{2}{3}}||\partial_2^2\theta||_{L^2}^2+\frac{1}{8}||\partial_1\partial_2^2\theta||_{L^2}^2\\
&\leq C||\omega||_{L^2}^{\frac{2}{3}}||\partial_1\omega||_{L^2}^{\frac{2}{3}}||\partial_2^2\theta||_{L^2}^2+\frac{1}{8}||\partial_1\partial_2^2\theta||_{L^2}^2.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
-2\int_{\mathbb{R}^2}\partial_2u_1\partial_1\partial_2\theta\partial_2^2\theta dx &\le C||\partial_2^2\theta||_{L^2}||\partial_2u_1||_{L^2}^{\frac12}||\partial_1\partial_2u_1||_{L^2}^{\frac12}||\partial_1\partial_2\theta||_{L^2}^{\frac12}||\partial_1\partial_2^2\theta||_{L^2}^{\frac12}\\
&\le C||\partial_2^2\theta||_{L^2}||\omega||_{L^2}^{\frac12}||\partial_1\omega||_{L^2}^{\frac12}||\partial_1\partial_2\theta||_{L^2}^{\frac12}||\partial_1\partial_2^2\theta||_{L^2}^{\frac12}\\
&\le C||\omega||_{L^2}||\partial_1\omega||_{L^2}||\partial_2^2\theta||_{L^2}^2+||\partial_1\partial_2\theta||_{L^2}^2+\frac18||\partial_1\partial_2^2\theta||_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_{\mathbb{R}^2}\partial_2^2u_1\partial_1\theta\partial_2^2\theta dx &= \int_{\mathbb{R}^2}\partial_1\partial_2^2u_1\theta\partial_2^2\theta dx+\int_{\mathbb{R}^2}\partial_2^2u_1\theta\partial_1\partial_2^2\theta dx\\
&\le ||\theta||_{L^\infty}||\partial_1\partial_2^2u_1||_{L^2}||\partial_2^2\theta||_{L^2}+||\theta||_{L^\infty}||\partial_2^2u_1||_{L^2}||\partial_1\partial_2^2\theta||_{L^2}\\
&\le ||\theta_0||_{L^\infty}||\partial_1\partial_2\omega||_{L^2}||\partial_2^2\theta||_{L^2}+||\theta_0||_{L^\infty}||\partial_2\omega||_{L^2}||\partial_1\partial_2^2\theta||_{L^2}\\
&\le C||\theta_0||_{L^\infty}^2(||\partial_2\omega||_{L^2}^2+||\partial_2^2\theta||_{L^2}^2)+\frac1{16}||\partial_1\partial_2\omega||_{L^2}^2+\frac18||\partial_1\partial_2^2\theta||_{L^2}.
\end{aligned}
$$

As for the last integral term of (3.32), we have

$$
\begin{aligned}
-\int_{\mathbb{R}^2}\partial_2^2u_2\partial_2\theta\partial_2^2\theta dx &= \int_{\mathbb{R}^2}\partial_2\partial_1u_1\partial_2\theta\partial_2^2\theta dx\\
&\le C||\partial_2\partial_1u_1||_{L^2}||\partial_2\theta||_{L^2}^{\frac12}||\partial_2^2\theta||_{L^2}||\partial_1\partial_2^2\theta||_{L^2}^{\frac12}\\
&\le C||\partial_1\omega||_{L^2}||\partial_2\theta||_{L^2}^{\frac12}||\partial_2^2\theta||_{L^2}||\partial_1\partial_2^2\theta||_{L^2}^{\frac12}\\
&\le C||\partial_1\omega||_{L^2}^2||\partial_2^2\theta||_{L^2}^2+||\partial_2\theta||_{L^2}^2+\frac18||\partial_1\partial_2^2\theta||_{L^2}^2.
\end{aligned}
$$

Summing up the above estimates, we get

$$
\begin{aligned}
&\frac{d}{dt}(||\partial_2\omega(t)||_{L^2}^2+||\partial_2^2\theta(t)||_{L^2}^2)+||\partial_1\partial_2\omega(t)||_{L^2}^2+||\partial_1\partial_2^2\theta(t)||_{L^2}^2\\
&\le C(1+F_M'+||\theta_0||_{L^\infty}^2+||\omega(t)||_{L^2}^2+||\partial_1\omega(t)||_{L^2}^2)(||\partial_2\omega(t)||_{L^2}^2+||\partial_2^2\theta(t)||_{L^2}^2)\\
&+C(F_M'')^4||\partial_2\theta(t)||_{L^2}^6+C(1+(F_M')^2)||\partial_2\theta(t)||_{L^2}^2+C||\omega(t)||_{L^2}^2+||\partial_1\partial_2\theta(t)||_{L^2}^2.
\end{aligned}
$$

By Gronwall's inequality, we obtain

$$
||\partial_2\omega(t)||_{L^2}^2+||\partial_2^2\theta(t)||_{L^2}^2\le e^{C\int_0^tA(\tau)d\tau}(||\partial_2\omega(0)||_{L^2}^2+||\partial_2^2\theta(0)||_{L^2}^2+C\int_0^tB(\tau)d\tau),
$$

where

$$
A(t):=1+F_M'+||\theta_0||_{L^\infty}^2+||\omega(t)||_{L^2}^2+||\partial_1\omega(t)||_{L^2}^2,
$$

and

$$
B(t):=(F_M'')^4||\partial_2\theta(t)||_{L^2}^6+(1+(F_M')^2)||\partial_2\theta(t)||_{L^2}^2+||\omega(t)||_{L^2}^2+||\partial_1\partial_2\theta(t)||_{L^2}^2.
$$

Since $A(t)$ and $B(t)$ are both locally integrable on $\mathbb{R}^+$, we have the desired result (3.29).

It remains to prove (3.30). By virtue of the interpolation theorem, we get

$$
||\nabla u(t)||_{L^\infty(\mathbb{R}^2)}\le C||\omega(t)||_{L^2(\mathbb{R}^2)}^{\frac14}||\partial_2\omega(t)||_{L^2(\mathbb{R}^2)}^{\frac14}||\partial_1\omega(t)||_{L^2(\mathbb{R}^2)}^{\frac14}||\partial_1\partial_2\omega(t)||_{L^2(\mathbb{R}^2)}^{\frac14}.
$$

Taking L^4-norm on interval $[0,t]$ in time, we obtain

$$
\begin{aligned}
\int_0^t||\nabla u(\tau)||_{L^\infty(\mathbb{R}^2)}^4d\tau &\le C\int_0^tC||\omega(\tau)||_{L^2(\mathbb{R}^2)}||\partial_2\omega(\tau)||_{L^2(\mathbb{R}^2)}||\partial_1\omega(\tau)||_{L^2(\mathbb{R}^2)}||\partial_1\partial_2\omega(\tau)||_{L^2(\mathbb{R}^2)}d\tau\\
&\le C||\omega||_{L_t^\infty L^2}||\partial_2\omega||_{L_t^\infty L^2}||\partial_1\omega||_{L_t^2L^2}||\partial_1\partial_2\omega||_{L_t^2L^2}.
\end{aligned}
$$

The same process yields

$$\int_0^t ||\partial_2\theta(\tau)||^4_{L^\infty(\mathbb{R}^2)}d\tau \le C||\partial_2\theta||_{L^\infty_t L^2}||\partial_2^2\theta||_{L^\infty_t L^2}||\partial_2\partial_1\theta||_{L^2_t L^2}||\partial_1\partial_2^2\theta||_{L^2_t L^2}.$$

This completes the proof of Proposition 3.6. □

4 | PROOF OF THEOREM 1.1

Before proving Theorem 1.1, we give a useful Proposition as follows:

Proposition 4.1. *Assume $(u_0, \theta_0) \in H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ and $F \in C^2(\mathbb{R}^2)$ satisfies $F(0) = 0$. Then there is a unique global solution (u, θ) for system (1.1) such that*

$$u(t,x), \theta(t,x) \in C(\mathbb{R}^+, H^2(\mathbb{R}^2)) \quad \textit{and} \quad \partial_1 u(t,x), \partial_1\theta(t,x) \in L^2_{loc}((\mathbb{R}^+, H^2(\mathbb{R}^2)).$$

Proof of Proposition. We construct an approximate system to (1.1) as follows:

$$\begin{cases} \partial_t u_{n,k} + S_n \mathrm{div}(S_n u_{n,k} \otimes S_n u_{n,k}) - \partial_1^2 S_n u_{n,k} = -\nabla\Pi_{n,k} + S_n(F(S_k\theta_{n,k})), \\ \partial_t\theta_{n,k} + S_n\mathrm{div}(S_n u_{n,k} S_n\theta_{n,k}) - \partial_1^2 S_n\theta_{n,k} = 0, \\ \mathrm{div} u_{n,k} = 0, \\ (\theta_{n,k}, u_{n,k})|_{t=0} = S_k(\theta_0, u_0), \end{cases} \tag{4.1}$$

where S_n is a low-frequency cutoff operator defined in Section 2.

Following Leray, we eliminate the pressure $\Pi_{n,k}$ by projecting the first equation of the above system onto Banach space $V^2(\mathbb{R}^2)$ with

$$V^2(\mathbb{R}^2) := \left\{u \in H^2(\mathbb{R}^2) | \mathrm{div} u = 0\right\}.$$

If k is fixed, then problem (4.1) reduce to an ODE in the Banach space $V^2(\mathbb{R}^2)$:

$$\begin{cases} \dfrac{d}{dt}\tilde{u}_n = F_n(\tilde{u}_n), \\ \tilde{u}_n|_{t=0} = S_n\tilde{u}_0, \end{cases} \tag{4.2}$$

where

$$\tilde{u}_n = \begin{pmatrix} u_{n,k} \\ \theta_{n,k} \end{pmatrix} \quad \text{and} \quad F_n(\tilde{u}_n) = \begin{pmatrix} \mathcal{P}S_n(F(S_k\theta_{n,k})) - \mathcal{P}S_n\mathrm{div}(S_n u_{n,k}\otimes S_n u_{n,k}) + \partial_1^2 S_n u_{n,k} \\ -S_n\mathrm{div}(S_n u_{n,k} S_n\theta_{n,k}) + \partial_1^2 S_n\theta_{n,k} \end{pmatrix}.$$

Step 1:
If an initial $(u_0, \theta_0) \in V^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$, then the system (4.2) has a unique solution $(u_{n,k}, \theta_{n,k}) \in C^1(\mathbb{R}^+; V^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2))$ for all time.
By the standard process,[29] we obtain that F_n is locally Lipschitz continuous on open set:

$$O^M := \{\tilde{u} \in \tilde{V}^2 \quad | \quad ||\tilde{u}||_{H^2} < M\} \quad \text{and} \quad \tilde{V}^2 := \begin{pmatrix} V^2 \\ H^2 \end{pmatrix}.$$

Hence, the Picard theorem of ODEs on Banach spaces ensures that for every $(u_0, \theta_0) \in H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ and $F \in W^{2,\infty}(\mathbb{R}^2)$ satisfying $F(0) = 0$, there exists a unique solution $(u_{n,k}, \theta_{n,k}) \in C^1([0, T_n); O^M)$ for some $T_n > 0$ being the maximal existence time.

Now, we are left to show that $T_n = \infty$. Using $F(0) = 0$, the Bernstein lemma and commutator estimates, we have

$$||\theta_{n,k}(t)||_{L^2}^2 + 2\int_0^t ||\partial_1 S_n\theta_{n,k}(\tau)||_{L^2}^2 d\tau \le ||\theta_0||_{L^2}^2,$$

$$\begin{aligned}||u_{n,k}(t)||_{L^2}^2 + 2\int_0^t ||\partial_1 S_n u_{n,k}(\tau)||_{L^2}^2 d\tau &\le 2||u_0||_{L^2}^2 + ||F(S_k\theta_{n,k})||_{L^\infty}^2 t^2 \\ &\le 2||u_0||_{L^2}^2 + |F'(||S_k\theta_{n,k}||_{L^\infty})|^2 t^2 \\ &\le 2||u_0||_{L^2}^2 + |F'(C2^k||\theta_0||_{L^2})|^2 t^2,\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}||(u_{n,k},\theta_{n,k})(t)||_{H^2} \le\ & C(||(u_0,\theta_0)||_{L^2}, |F'(C2^k||\theta_0||_{L^2})|, |F\prime'(C2^k||\theta_0||_{L^2})|, n, k, t) \\ &\times ||(u_{n,k},\theta_{n,k})||_{H^2},\end{aligned}$$

which implies that $||(u_{n,k},\theta_{n,k})(t)||_{H^2} \le e^{Ct}$. Thus, solutions can be continued for all time by invoking the continuation property of ODEs on a Banach space.[29]

Step 2:

Local existence.

$$\begin{cases}\partial_t u_k + \mathrm{div}(u_k\otimes u_k) - \partial_1^2 u_k = -\nabla\Pi_k + F(S_k\theta_k), \\ \partial_t\theta_k + \mathrm{div}(u_k\theta_k) - \partial_1^2\theta_k = 0, \\ \mathrm{div} u_k = 0, \\ (\theta_k, u_k)|_t = S_k(u_0,\theta_0).\end{cases} \tag{4.3}$$

Using Lemma 2.2, we get

$$\begin{aligned}\frac{d}{dt}||(u_{n,k},\theta_{n,k})(t)||_{H^2} &\le C(|F'(C2^k||\theta_0||_{L^2})|, |F\prime'(C2^k||\theta_0||_{L^2})|)(||u_{n,k}||_{H^2} + |\theta_{n,k}||_{H^2})||(u_{n,k},\theta_{n,k})(t)||_{H^2} \\ &\le C||(u_{n,k},\theta_{n,k})(t)||_{H^2}^2.\end{aligned}$$

Therefore, for all $n \in \mathbb{N}^+$, we have

$$\sup_{t\in[0,T]} ||(u_{n,k},\theta_{n,k})(t)||_{H^2} \le \frac{||(u_0,\theta_0)||_{H^2}}{1 - CT||(u_0,\theta_0)||_{H^2}}.$$

Moreover, by system (4.1), we can conclude that $(\partial_t u_{n,k}, \partial_t\theta_{n,k})$ is uniformly bounded in L^2.

Next, we prove that the solutions $\tilde{u}_n$ form a Cauchy sequence in $C([0,T]; L^2)$. Specifically, given any $n, m \in \mathbb{N}^+$ and a constant C depending only on $||(u_0,\theta_0)||_{H^2}$ and T, we get

$$\sup_{0<t<T} ||\tilde{u}_{n,k} - \tilde{u}_{m,k}||_{L^2} \le C\max(\frac{1}{2^n}, \frac{1}{2^m}).$$

Then, $(u_{n,k},\theta_{n,k})$ converges strongly to a pair (u_k,θ_k) in $C([0,T]; L^2)$. From the Fatou lemma, we infer $(u_{n,k},\theta_{n,k})$ is uniformly bounded in $L^\infty([0,T], H^2)$. Using the following interpolation inequality

$$||\tilde{u}||_{H^s} \le C_s||\tilde{u}||_{L^2}^{1-\frac{s}{2}}||\tilde{u}||_{H^2}^{\frac{s}{2}}, \quad \text{for} \quad s \in (0,2),$$

we know that the sequence $(u_{n,k},\theta_{n,k})$ converges strongly to a pair (u_k,θ_k) in $C([0,T]; H^s)$ for any $s \in (0,2)$, when n tends to ∞. And (u_k,θ_k) is a solution of (4.3).

Step 3:

Global existence to (4.3).

As the same process of Propositions 3.1, 3.2, 3.5, and 3.6, we readily get that

$$\sup_{t\in[0,t]} (||u_k(t)||_{H^2} + ||\theta_k(t)||_{H^2}) \le C(t),$$

and (u_k,θ_k) is global-in-time solution to (4.3).

Step 4:
Global existence to (1.1).
By (4.3), we can conclude that

$$||\theta_k(t)||_{L^2}^2 + 2\int_0^t ||\partial_1\theta_k(\tau)||_{L^2}^2 d\tau \leq ||\theta_0||_{L^2}^2, ||\theta_k(t)||_{L^p} \quad \leq ||\theta_0||_{L^p}, \quad \text{for} \quad p \in [1,\infty],$$

and

$$\begin{aligned} ||u_k(t)||_{L^2}^2 + 2\int_0^t ||\partial_1 u_k(\tau)||_{L^2}^2 d\tau &\leq 2||u_0||_{L^2}^2 + ||F(S_k\theta_k)||_{L^\infty}^2 t^2, \\ &\leq 2||u_0||_{L^2}^2 + |F'(C||\theta_0||_{L^\infty})|^2 t^2. \end{aligned}$$

Similarly, we know that (u_k, θ_k) converges strongly to a pair (u, θ) in $C(\mathbb{R}^+; L^2)$ and (u_k, θ_k) is uniformly bounded in $L^\infty(\mathbb{R}^+; H^2)$. Furthermore, we know that the sequence (u_k, θ_k) converges strongly to a pair (u, θ) in $C(\mathbb{R}^+; H^s)$ for any $s \in (0, 2)$, when k tends to ∞. And (u, θ) satisfies (1.1).

Finally, we can get continuity $|| \cdot ||_{H^2}$ norm and uniqueness of the solution using the argument in Majda and Bertozzi.[29] □

Now, we prove Theorem 1.1. We first create inductively approximation solution (u_n, θ_n) for solving (1.1) as follows:

$$\begin{cases} \partial_t u_n + \operatorname{div}(u_n \otimes u_n) - \partial_1^2 u_n = -\nabla\Pi_n + F(\theta_n), \\ \partial_t\theta_n + \operatorname{div}(u_n\theta_n) - \partial_1^2\theta_n = 0, \\ \operatorname{div} u_n = 0, \\ (\theta_n, u_n)|_t = S_n(u_0, \theta_0). \end{cases} \tag{4.4}$$

By virtue of Proposition 4.1, we infer that there is a unique solution $(u, \theta) \in C(\mathbb{R}^+, H^2)$ for system (4.4). Therefore, by Propositions 3.1, 3.2, 3.5, and 3.6, we can get that (u_n, θ_n) is uniformly bounded in following work spaces,

$$\begin{gathered} \theta_n \in L^\infty(\mathbb{R}^+; L^2 \cap L^\infty) \\ (u_n, \theta_n) \in L^\infty_{loc}(\mathbb{R}^+, H^1) \quad \text{and} \quad (\partial_1 u_n, \partial_1\theta_n) \in L^2_{loc}(\mathbb{R}^+, H^1); \\ (\partial_2^2 u_n, \partial_2^2\theta_n) \in L^\infty_{loc}(\mathbb{R}^+, L^2) \quad \text{and} \quad (\partial_1\partial_2^2 u_n, \partial_1\partial_2^2\theta_n) \in L^2_{loc}(\mathbb{R}^+, L^2); \\ \nabla u_n \in L^4_{loc}(\mathbb{R}^+, L^\infty) \quad \text{and} \quad \partial_2\theta_n \in L^4_{loc}(\mathbb{R}^+, L^\infty). \end{gathered} \tag{4.5}$$

It follows that $\partial_t u_n \in L^2_{loc}(\mathbb{R}^+, L^2)$ and $\partial_t\theta_n \in L^2_{loc}(\mathbb{R}^+, H^{-1})$. Since the embeddings $H^1 \hookrightarrow L^2$ and $L^2 \hookrightarrow H^{-1}$ are locally compact, the classical Aubin-Lions argument and Cantor's diagonal process enables us to conclude that, up to extration, sequence $(\theta_n, u_n)_{n\in\mathbb{N}}$ has a limit (θ, u) satisfying the system (1.1) and that

$$\begin{gathered} \theta \in L^\infty(\mathbb{R}^+; L^2 \cap L^\infty); \\ (u, \theta) \in L^\infty_{loc}(\mathbb{R}^+; H^1) \quad \text{and} \quad (\partial_1 u, \partial_1\theta) \in L^2_{loc}(\mathbb{R}^+; H^1); \\ (\partial_2^2 u_n, \partial_2^2\theta_n) \in L^\infty_{loc}(\mathbb{R}^+, L^2) \quad \text{and} \quad (\partial_1\partial_2^2 u, \partial_1\partial_2^2\theta) \in L^2_{loc}(\mathbb{R}^+; L^2); \\ \nabla u \in L^4_{loc}(\mathbb{R}^+; L^\infty) \quad \text{and} \quad \partial_2\theta \in L^4_{loc}(\mathbb{R}^+; L^\infty). \end{gathered} \tag{4.6}$$

For the time continuity of (θ, u), the proof is standard and one can refer to Wu and Zheng.[24]

Next, we prove the uniqueness of solution (θ, u, Π). Assume (θ, u, Π) and $(\tilde\theta, \tilde u, \tilde\Pi)$ satisfy (1.1) with the same initial data. Let the difference $\delta u = \tilde u - u, \delta\theta = \tilde\theta - \theta$ and $\delta\Pi = \tilde\Pi - \Pi$ solve the following equations:

$$\begin{cases} (\delta u)_t + \tilde u \nabla\delta u - \partial_1^2\delta u + \nabla\delta\Pi_n = -\delta u \cdot \nabla u + F(\tilde\theta) - F(\theta), \\ (\delta\theta)_t + \tilde u\nabla\delta\theta - \partial_1^2\delta\theta = -\delta u \cdot \nabla\theta, \\ (\delta u, \delta\theta)|_t = (0, 0). \end{cases} \tag{4.7}$$

Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\delta u(t)||_{L^2}^2+||\partial_1\delta u(t)||_{L^2}^2&=-\int_{\mathbb{R}^2}\delta u\cdot\nabla u\delta u dx+\int_{\mathbb{R}^2}(F(\tilde{\theta})-F(\theta))\delta u dx\\&\le||\nabla u(t)||_{L^\infty}||\delta u(t)||_{L^2}^2+||F(\tilde{\theta})(t)-F(\theta)(t)||_{L^2}||\delta u(t)||_{L^2}\\&\le||\nabla u(t)||_{L^\infty}||\delta u(t)||_{L^2}^2+F'_M||\delta\theta(t)||_{L^2}||\delta u(t)||_{L^2},\end{aligned}\tag{4.8}$$

and

$$\frac{1}{2}\frac{d}{dt}||\delta\theta(t)||_{L^2}^2+||\partial_1\delta\theta(t)||_{L^2}^2=-\int_{\mathbb{R}^2}\delta u_1\partial_1\theta\delta\theta dx-\int_{\mathbb{R}^2}\delta u_2\partial_2\theta\delta\theta dx.$$

Using Hölder's inequality, we get

$$\begin{aligned}-\int_{\mathbb{R}^2}\delta u_1\partial_1\theta\delta\theta dx&=\int_{\mathbb{R}^2}\partial_1\delta u_1\theta\delta\theta dx+\int_{\mathbb{R}^2}\delta u_1\theta\partial_1\delta\theta dx\\&\le||\theta||_{L^\infty}||\delta\theta||_{L^2}||\partial_1\delta u_1||_{L^2}+||\theta||_{L^\infty}||\delta u_1||_{L^2}||\partial_1\delta\theta||_{L^2}\\&\le\frac{1}{2}||\theta_0||_{L^\infty}^2(||\delta\theta||_{L^2}^2+||\delta u||_{L^2}^2)+\frac{1}{2}||\partial_1\delta u||_{L^2}^2+\frac{1}{2}||\partial_1\delta\theta||_{L^2}^2.\end{aligned}$$

Similarly, we obtain

$$-\int_{\mathbb{R}^2}\delta u_2\partial_2\theta\delta\theta dx\le||\partial_2\theta||_{L^\infty}||\delta u||_{L^2}||\delta\theta||_{L^2}\le\frac{1}{2}||\partial_2\theta||_{L^\infty}(||\delta u||_{L^2}^2+||\delta\theta||_{L^2}^2).$$

Therefore,

$$\frac{d}{dt}||\delta\theta(t)||_{L^2}^2+||\partial_1\delta\theta(t)||_{L^2}^2\le(||\partial_2\theta||_{L^\infty}+||\theta_0||_{L^\infty}^2)(||\delta\theta(t)||_{L^2}^2+||\delta u(t)||_{L^2}^2)+||\partial_1\delta u||_{L^2}^2.\tag{4.9}$$

Collecting (4.8) and (4.9) yields

$$\frac{d}{dt}(||\delta\theta(t)||_{L^2}^2+||\delta u(t)||_{L^2}^2)\le(2||\nabla u||_{L^\infty}+F'_M+||\partial_2\theta(t)||_{L^\infty}+||\theta_0||_{L^\infty}^2)(||\delta\theta||_{L^2}^2+||\delta u||_{L^2}^2).$$

The Gronwall inequality gives that $(\delta u,\delta\theta)\equiv 0$.

5 | PROOF OF THEOREM 1.2

A similar process of proving Theorem 1.1, we know that (θ_n, u_n) is uniformly bounded in the following work spaces,

$$\theta_n\in L^\infty(\mathbb{R}^+;L^2\cap L^\infty),\quad \partial_2\theta_n\in L^\infty_{loc}(\mathbb{R}^+;L^2)\quad\text{and}\quad\partial_1\partial_2\theta_n\in L^2_{loc}(\mathbb{R}^+;L^2);$$

$$u_n\in L^\infty_{loc}(\mathbb{R}^+;H^1)\quad\text{and}\quad\partial_1 u_n\in L^2_{loc}(\mathbb{R}^+;H^1).$$

Then, using Aubin-Lions compactness and Cantor's diagonal process, up to extraction, sequence $(\theta_n, u_n)_{n\in\mathbb{N}}$ convergence strongly to a limit (θ, u) satisfying the system (1.1) and that

$$\theta\in L^\infty(\mathbb{R}^+;L^2\cap L^\infty),\quad \partial_2\theta\in L^\infty_{loc}(\mathbb{R}^+;L^2)\quad\text{and}\quad\partial_1\partial_2\theta\in L^2_{loc}(\mathbb{R}^+;L^2);$$

$$u\in L^\infty_{loc}(\mathbb{R}^+;H^1)\quad\text{and}\quad\partial_1 u\in L^2_{loc}(\mathbb{R}^+;H^1).\tag{5.1}$$

In addition, the time continuity of (θ, u) can be proved in a similar way as in Wu and Zheng.[24]

As for the uniqueness, by (4.7), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\delta u(t)||_{L^2}^2 + ||\delta\theta(t)||_{L^2}^2) + ||\partial_1\delta u(t)||_{L^2}^2 + ||\partial_1\delta\theta(t)||_{L^2}^2\\
=& -\int_{\mathbb{R}^2}\delta u_1\partial_1 u\delta u dx - \int_{\mathbb{R}^2}\delta u_2\partial_2 u\delta u dx + \int_{\mathbb{R}^2}(F(\tilde{\theta}) - F(\theta))\delta u dx\\
&- \int_{\mathbb{R}^2}\delta u_1\partial_1\theta\delta\theta dx - \int_{\mathbb{R}^2}\delta u_2\partial_2\theta\delta\theta dx.
\end{aligned}$$

By Lemma 2.2 and Young's inequality, we obtain

$$\begin{aligned}
-\int_{\mathbb{R}^2}\delta u_1\partial_1 u\delta u dx &\le C||\delta u||_{L^2}||\delta u_1||_{L^2}^{\frac{1}{2}}||\partial_1\delta u_1||_{L^2}^{\frac{1}{2}}||\partial_1 u||_{L^2}^{\frac{1}{2}}|\partial_2\partial_1 u||_{L^2}^{\frac{1}{2}}\\
&\le C||\delta u||_{L^2}^{\frac{3}{2}}||\partial_1\delta u||_{L^2}^{\frac{1}{2}}||\omega||_{L^2}^{\frac{1}{2}}||\partial_1\omega||_{L^2}^{\frac{1}{2}}\\
&\le ||\omega||_{L^2}^{\frac{2}{3}}||\partial_1\omega||_{L^2}^{\frac{2}{3}}||\delta u||_{L^2}^2 + \frac{1}{4}||\partial_1\delta u||_{L^2}^2.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
-\int_{\mathbb{R}^2}\delta u_2\partial_2 u\delta u dx &\le ||\delta u||_{L^2}||\delta u_2||_{L^2}^{\frac{1}{2}}||\partial_2\delta u_2||_{L^2}^{\frac{1}{2}}||\partial_2 u||_{L^2}^{\frac{1}{2}}||\partial_1\partial_2 u||_{L^2}^{\frac{1}{2}}\\
&\le C||\delta u||_{L^2}^{\frac{3}{2}}||\partial_1\delta u||_{L^2}^{\frac{1}{2}}||\omega||_{L^2}^{\frac{1}{2}}||\partial_1\omega||_{L^2}^{\frac{1}{2}}\\
&\le C||\omega||_{L^2}^{\frac{2}{3}}||\partial_1\omega||_{L^2}^{\frac{2}{3}}||\delta u||_{L^2}^2 + \frac{1}{4}||\partial_1\delta u||_{L^2}^2.
\end{aligned}$$

By virtue of Taylor's formula and Cauchy-Schwarz's inequality, we conclude

$$\int_{\mathbb{R}^2}(F(\tilde{\theta}) - F(\theta))\delta u dx \le F'_M||\delta\theta||_{L^2}||\delta u||_{L^2}.$$

Using Lemma 2.2 yields

$$\begin{aligned}
-\int_{\mathbb{R}^2}\delta u_2\partial_2\theta\delta\theta dx &\le ||\delta\theta||_{L^2}||\delta u_2||_{L^2}^{\frac{1}{2}}||\partial_2\delta u_2||_{L^2}^{\frac{1}{2}}||\partial_2\theta||_{L^2}^{\frac{1}{2}}||\partial_1\partial_2\theta||_{L^2}^{\frac{1}{2}}\\
&\le C||\delta\theta||_{L^2}||\delta u||_{L^2}^{\frac{1}{2}}||\partial_1\delta u_1||_{L^2}^{\frac{1}{2}}||\partial_2\theta||_{L^2}^{\frac{1}{2}}||\partial_1\partial_2\theta||_{L^2}^{\frac{1}{2}}\\
&\le C||\partial_2\theta||_{L^2}||\partial_1\partial_2\theta||_{L^2}||\delta\theta||_{L^2}^2 + ||\delta u||_{L^2}^2 + \frac{1}{4}||\partial_1\delta u||_{L^2}^2.
\end{aligned}$$

By Hölder's inequality, we infer that

$$\begin{aligned}
-\int_{\mathbb{R}^2}\delta u_1\partial_1\theta\delta\theta dx &= \int_{\mathbb{R}^2}\partial_1\delta u_1\theta\delta\theta dx + \int_{\mathbb{R}^2}\delta u_1\theta\partial_1\delta\theta dx\\
&\le ||\theta||_{L^\infty}||\partial_1\delta u_1||_{L^2}||\delta\theta||_{L^2} + ||\theta||_{L^\infty}||\delta u_1||_{L^2}||\partial_1\delta\theta||_{L^2}\\
&\le ||\theta_0||_{L^\infty}^2(||\delta u||_{L^2}^2 + ||\delta\theta||_{L^2}^2) + \frac{1}{4}||\partial_1\delta u||_{L^2}^2 + \frac{1}{4}||\partial_1\delta\theta||_{L^2}^2.
\end{aligned}$$

Summing up these estimates, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\delta u(t)||_{L^2}^2 + ||\delta\theta(t)||_{L^2}^2) + ||\partial_1\delta u(t)||_{L^2}^2 + ||\partial_1\delta\theta(t)||_{L^2}^2\\
\le& C(1 + (F'_M)^2 + ||\omega(t)||_{L^2}^{\frac{2}{3}}||\partial_1\omega(t)||_{L^2}^{\frac{2}{3}} + ||\partial_2\theta(t)||_{L^2}||\partial_1\partial_2\theta(t)||_{L^2} + ||\theta_0||_{L^\infty}^2)\\
&\times(||\delta u(t)||_{L^2}^2 + ||\delta\theta(t)||_{L^2}^2).
\end{aligned}$$

The Gronwall inequality gives that $(\delta u, \delta\theta) \equiv 0$.

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REFERENCES

1. Pedlosky J. *Geophysical Fluids Dynamics*. NewYork: Springer-Verlag; 1987.
2. Chae D. Global regularity for the 2-D Boussinesq equations with partial viscous terms. *Adv Math*. 2006;203(2):497-513.
3. Danchin R, Paicu M. Global well-posedness issues for the inviscid Boussinesq system with Yudovich type data. *Comm Math Phys*. 2009;290:1-14.
4. Hmidi T, Keraani S. On the global well-posedness of the Boussinesq system with zero viscosity. *Adv Differ Eq*. 2007;12(4):461-480.
5. Hmidi T, Keraani S. On the global well-posedness of the Boussinesq system with zero viscosity. *Indiana Univ Math J*. 2009;58(4):1591-1618.
6. Hmidi T, Keraani S, Rousset F. Global well-posedness for Euler-Boussinesq system with critical dissipation. *Comm Partial Differ Eq*. 2011;36:420-445.
7. Hmidi T, Rousset F. Global well-posedness for the Navier-Stokes-Boussinesq system with axisymmetric data. *Ann Inst H Poincaŕe Anal Non Liné,aire*. 2010;27:1227-1246.
8. Hmidi T, Rousset F. Global well-posedness for the Euler-Boussinesq system with axisymmetric data. *J Funct Anal*. 2011;260:745-796.
9. Hmidi T, Zerguine M. On the global well-posedness of the Euler-Boussinesq system with fractional dissipation, in: Evolution Equations in Pure and Applied Sciences. *Phys D*. 2010;239:1387-1401.
10. Hou T, Li C. Global well-posedness of the viscous Boussinesq equations. *Discrete Contin Dyn Syst*. 2005;12:1-12.
11. Miao C, Xue L. On the global well-posedness of a class of Boussinesq-Navier-Stokes systems. *NoDEA Nonlinear Differ Equ Appl*. 2011;18:707-735.
12. Wu J, Xu X, Ye Z. The 2D Boussinesq equations with fractional horizontal dissipation and thermal diffusion. *J Math Pures Appl*. 2018;115(9):187-217.
13. Adhikari D, Cao C, Shang H, Wu J, Xu X, Ye Z. Global regularity results for the 2D Boussinesq equations with partial dissipation. *J Differ Equ*. 2016;260(2):1893-1917.
14. Wu J, Xu X, Xue L, Ye Z. Regularity results for the 2D Boussinesq equations with critical or supercritical dissipation. *Commun Math Sci*. 2016;14(7):1963-1997.
15. Jiu Q, Miao C, Wu J, Zhang Z. The 2D incompressible Boussinesq equations with general critical dissipation. *SIAM J Math Anal*. 2014;46:3426-3454.
16. Ye Z. Global smooth solution to the 2D Boussinesq equations with fractional dissipation. *Math Methods Appl Sci*. 2017;40(12):4595-4612.
17. Chemin JY, Desjardins B, Gallagher I, Grenier E. *Mathematical Geophysics: An Introduction to Rotating Fluids and to the Navier-Stokes Equations*. Oxford: Oxford University Press; 2006.
18. Danchin R, Paicu M. Global existence results for the anisotropic Boussinesq system in dimension two. *Math Models Methods Appl Sci*. 2011;21:421-457.
19. Miao C, Zheng X. On the global well-posedness for the Boussinesq system with horizontal dissipation. *Comm Math Phys*. 2013;321(1):33-67.
20. Miao C, Zheng X. Global well-posedness for axisymmetric Boussinesq system with horizontal viscosity. *J Math Pures Appl*. 2014;101:842-872.
21. Adhikari A, Cao C, Wu J. The 2-D Boussinesq equations with vertical viscosity and vertical diffusivity. *J Differ Equ*. 2010;249:1078-1088.
22. Adhikari A, Cao C, Wu J. Global regularity results for the 2-D Boussinesq equations with vertical dissipation. *J Differ Equ*. 2011;251:1637-1655.
23. Cao C, Wu J. Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation. *Arch Ration Mech Anal*. 2013;208(3):985-1004.
24. Wu G, Zheng X. Global well-posedness for the two-dimensional nonlinear Boussinesq equations with vertical dissipation. *J Differ Equ*. 2013;255:2891-2926.
25. Miao C, Wu J, Zhang Z. *Littlewood-Paley Theory and Applications to Fluid Dynamics Equations*, Monogr. Modern Pure Math., vol. 142. Beijing: Science Press; 2012.
26. Bony JM. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann Sci École Norm Sup*. 1981;14:209-246.

27. Taylor M. *Tools for PDE, Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials*. Rhode Island, USA: American Mathematical Society; 2000.
28. Yudovich VI. Nonsrtationary flow of an ideal incompressible liquid. *Zh Vychisl Mat*. 1963;3:1032-1066.
29. Majda A, Bertozzi AL. *Vorticity and Incompressible Flow*, Cambridge Texts Appl. Math., vol. 27. Cambridge: Cambridge University Press; 2002.

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