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Elliptic Equations with Degenerate Coercivity: Gradient Regularity

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Abstract In this paper, we prove higher integrability results for the gradient of the solutions of some elliptic equations with degenerate coercivity whose prototype is

$$-\mathrm{div}\,(a(x,u)Du) = f \quad \text{in} \quad D'(\Omega), \quad f \in L^r(\Omega), \quad r > 1,$$

where, for example, $a(x,u) = (1+|u|)^{-\theta}$ with $\theta \in (0,1)$. We study the same problem for minima of functionals closely related to the previous equation.

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0 Introduction

The main aim of this paper is to study the integrability properties of the gradient of (global or local) solutions of problems like

$$\begin{cases} -\mathrm{div}(a(x,u)Du) = f & \text{in} \quad D'(\Omega), \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

We recall that in the classical case $a(x,s) \equiv 1$, by means of the Calderon-Zygmund regularity theorem [1], we have that if $f \in L^r(\Omega)$, then $a(x,u)Du = Du$ belongs to $L^m(\Omega)$, where $m = r^*$ if $1 < r < N$, otherwise m can be any number bigger than one.

The same happens (i.e. $a(x,u)Du \in L^{r^*}(\Omega)$), when $a(x,s) = a(s)$ is a function independent of the variable x. As a matter of fact, performing the following change of variable:

$$v = \int_0^u a(s)ds, \tag{2}$$

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we obtain that v is a solution of

$$\begin{cases} -\triangle v = f & \text{in} \quad D'(\Omega), \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Hence by the quoted Calderon-Zygmund theorem, the function v belongs to $W_0^{1,m}(\Omega)$, where m is as before, and thus $Dv = a(u)Du \in L^{r^*}(\Omega)$ if $r < N$. When the function a depends also on x, the previous change of variable cannot be applied and the problem to establish the integrability properties of the gradient of the solutions of (1) is, in general, still open. Here we will study the case when the coercivity can degenerate when u is too big. More precisely we assume that $f \in L^r(\Omega)$, $r > 1$, and $a(x, s)$ is a Carathéodory function satisfying

$$\frac{\alpha}{(1+|s|)^\theta} \le a(x,s) \le \beta, \qquad 0 \le \theta < 1. \tag{3}$$

Notice that our case includes the classical one.

We refer for the existence of solutions of (1) to [2].

If the problem is

$$\begin{cases} -\mathrm{div}(a(x,u)Du) = -\mathrm{div}(F) & \text{in} \quad D'(\Omega), \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

in [3] we prove the existence of solutions when $F \in L^\nu(\Omega)$, ν sufficiently close to 2. In this case the space for solutions is sharp and it depends on θ.

On the contrary, the existence result for (1), proved in [2], is not sharp. We will recall this in Section 2.

We will first prove here an higher integrability result for $a(x, u)Du$ with respect to [2], (u solution to (1)), and then also for Du when the datum f is sufficiently regular (see Theorems 1.1–1.7).

A counterexample of Meyers [4] shows that, just in the easier case $a(x, s) = a(x)$ with $\beta \ge a(x) \ge \alpha > 0$, if $r < N$, there are local solutions u with Du not belonging to $L^{r^*}_{\mathrm{loc}}(\Omega)$, but only to $L^{2+\epsilon}_{\mathrm{loc}}(\Omega)$, with ϵ depending on α and r. Thus in the general case (1) the goal is to prove this kind of regularity.

Let us point out that, if we assume a more restrictive condition

$$\frac{\alpha}{(1+|s|)^\theta} \le a(x,s) \le \frac{\beta}{(1+|s|)^\theta}, \qquad 0 < \theta < 1,$$

the problem will become easier, since the regularity of $a(x, u)Du$ is equivalent to that of $\frac{Du}{(1+|u|)^\theta}$, in which there is no explicit dependence on x (see Theorems 1.2–1.3).

Nevertheless, also in the general case (3), it is possible to achieve the result by giving a further assumption on $a(x, s)$, which is implied, for example, by

$$\frac{\partial a(x,s)}{\partial x} = \left(\frac{\partial a(x,s)}{\partial x_1}, \cdots, \frac{\partial a(x,s)}{\partial x_n} \right) \in (L^m(\Omega))^N, \quad m > N,$$

(see Theorems 1.5 and 1.6).

The higher integrability on $a(x, u)Du$ is interesting in itself, but can be used, for regular data, to deduce higher integrability on Du.

In the second part of the paper, we study an analogous problem for minima of functionals like

$$J(v) = \int_\Omega a(x,v)|Dv|^2 dx - \int_\Omega f v dx, \tag{5}$$

where $a(x,s)$ is a Carathéodory function such that

$$\frac{\beta_0}{(1+|s|)^{2\theta}} \le a(x,s) \le \beta_1, \qquad 0 \le \theta < \frac{1}{2}.$$

Functionals like (5) are differentiable just in some directions and, in any case, their Euler equations do not look like Equation (1).

1 Main Results

1.1 Elliptic Equations

Let Ω be a bounded open subset of $\mathbf{R}^N$. Consider the following nonlinear elliptic problem:

$$\begin{cases} -\mathrm{div}\,(a(x,u)Du) = f & \text{in} \quad D'(\Omega), \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6}$$

where $a(x,u) : \Omega \times \mathbf{R} \to \mathbf{R}$, is a measurable Carathéodory function satisfying

$$\frac{\alpha}{(1+|s|)^\theta} \le a(x,s) \le \beta, \tag{7}$$

where α and β are positive constants and θ is a real number such that

$$0 < \theta < 1. \tag{8}$$

We recall that in [2] an existence (and regularity) result for the solution of (6) is proved by using approximation techniques. In more details they prove the following:

Proposition 1.1 *Let f be a function in $L^r(\Omega)$, with $r > N/2$. Then there exists a function u in $H_0^1(\Omega) \cap L^\infty(\Omega)$ which is a solution of* (6)*. If, otherwise, r verifies*

$$\frac{2N}{N+2-\theta(N-2)} \le r < \frac{N}{2}, \tag{9}$$

then again there exists a function u that is a solution of (6) *and it belongs to $H_0^1(\Omega) \cap L^h(\Omega)$, where*

$$h = \frac{Nr(1-\theta)}{N-2r}. \tag{10}$$

Finally if $r > 1$ such that

$$\frac{N}{N+1-\theta(N-1)} < r < \frac{2N}{N+2-\theta(N-2)}, \tag{11}$$

then there exists a solution u in $W_0^{1,q}(\Omega)$, where

$$q = \frac{Nr(1-\theta)}{N-r(1+\theta)} < 2. \tag{12}$$

Moreover, if $r > \frac{N(2-\theta)}{N+2-N\theta}$, such a solution verifies the following regularity condition:

$$a(x,u)Du \in \left(L^2(\Omega)\right)^N. \tag{13}$$

Remark 1.1 Notice that when $1 < r < \frac{N}{N+1-\theta(N-1)}$, then $q < 1$. Anyway, it is still possible to give a definition of the solution (see [2]), but such a solution doesn't belong to any Sobolev space. This is the reason why we will not consider this case.

Remark 1.2 It is easy to verify that the L^∞ or L^h regularity on u holds true if u is any solution in $H_0^1(\Omega)$ (not necessarily got by approximation), and that we have

$$\|u\|_{L^\infty(\Omega)} \le C_0 = C_0(|\Omega|, N, r, \|f\|_{L^r(\Omega)}) \tag{14}$$

if $r > \frac{N}{2}$; while if r satisfies (9) then

$$\|u\|_{L^h(\Omega)} \le C_1(\alpha, N, \theta, \|f\|_{L^r(\Omega)}). \tag{15}$$

Moreover, these results hold true, with obvious modifications, for the following more general problem:

$$\begin{cases} -\mathrm{div}\left(a(x,u)|Du|^{p-2}Du\right) = f & \text{in} \quad D'(\Omega), \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{16}$$

with $p > 1$.

We are interested in the regularity of the gradient Du of a solution of (6).

The case of bounded solutions

Theorem 1.1 *Assume* (7) *and* (8) *hold and let $f \in L^r(\Omega)$, where $r > \frac{N}{2}$. If $u \in H_0^1(\Omega)$ is a solution of problem* (6) *then*

$$Du \in \left(L^{2+\epsilon}_{\mathrm{loc}}(\Omega)\right)^N, \tag{17}$$

where ϵ is a positive constant that depends only on α, β, r, N, $|\Omega|$ and $\|f\|_{L^r(\Omega)}$.

Remark 1.3 In general, as noticed in the introduction, under the assumptions of Theorem 1.1, if $r < N$, which is the interesting case, we cannot expect that Du belongs to $L^{r^*}_{\mathrm{loc}}(\Omega)$ as it happens for the Laplacian equation (i.e. in the particular case $a(x,s) = 1$). As a matter of fact, it is not true even if $a(x,s) = a(x)$, as it is showed by the counterexample of Meyers (see [4]).

The case of unbounded solutions

We start with a simpler case when, instead of (7) we assume that $a(x,u)$ is a measurable Carathéodory function satisfying the stronger condition

$$\frac{\alpha}{(1+|s|)^\theta} \le a(x,s) \le \frac{\beta}{(1+|s|)^\theta}, \qquad 0 < \theta < 1. \tag{18}$$

We have the following results:

Theorem 1.2 *Let* (18) *be satisfied and assume that $f \in L^r(\Omega)$, where r verifies* (9). *If $u \in H_0^1(\Omega)$ is a solution of problem* (6), *then there exists a positive constant ϵ that depends only on N, α and β (with $\epsilon \le r^* - 2$), such that*

$$a(x,u)Du \in \left(L^{2+\epsilon}_{\mathrm{loc}}(\Omega)\right)^N. \tag{19}$$

Finally, if $r > \frac{2N}{N+2-\theta(N-2)}$ then there exists a constant $s > 2$ such that

$$Du \in \left(L^s_{\mathrm{loc}}(\Omega)\right)^N. \tag{20}$$

Remark 1.4 We observe that the result of Theorem 1.2 (respectively 1.1) works in general if $u \in H_0^1(\Omega)$ is a solution of the following nonlinear problem:

$$\begin{cases} -\mathrm{div}(a(x,u,Du)) = f & \text{in} \quad D'(\Omega), \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{21}$$

where $a(x,u,Du) : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a measurable Carathéodory vector-valued function satisfying

$$|a(x,u,Du)| \le \beta \frac{|Du|}{(1+|u|)^\theta}, \quad \text{(respectively} \quad |a(x,u,Du)| \le c|Du|),$$
$$a(x,u,Du)Du \ge \frac{\alpha |Du|^2}{(1+|u|)^\theta}, \quad \theta \in (0,1). \tag{22}$$

Moreover, proceeding as in [5], if $\partial\Omega$ is sufficiently smooth, we can prove regularity up to the boundary.

Theorem 1.3 *Let u be a solution of problem* (6) *that belongs to $W_0^{1,q}(\Omega)$, where q is as in* (12)*. Assume that $a(x,u)$ satisfies* (18)*. If*

$$a(x,u)Du \in \left(L^2_{\mathrm{loc}}(\Omega)\right)^N, \tag{23}$$

and if f belongs to $L^r(\Omega)$, where $r > \frac{2N}{N+2}$ verifies (11)*, then there exists a positive constant ϵ that depends only on N, α and β (with $\epsilon \le r^* - 2$), such that*

$$a(x,u)Du \in \left(L^{2+\epsilon}_{\mathrm{loc}}(\Omega)\right)^N. \tag{24}$$

Remark 1.5 We notice that the solutions obtained in [2] satisfy (23) if r satisfies the further condition $r > \frac{N(2-\theta)}{N+2-N\theta}$ (see Proposition 1.1). Moreover, the condition $r > \frac{2N}{N+2}$ is not restrictive since if $r < \frac{2N}{N+2}$ we will prove in the following Theorem 1.7 that $\frac{Du}{(1+|u|)^\theta} \in (L^{r^*}(\Omega))^N$ just in the general case (7).

Remark 1.6 We point out that under the assumptions of Theorem 1.3 one has $q^* = h$ and it is not possible to improve the summability of Du (see Remark 3.3 at the end of the proof of Theorem 1.3).

Moreover the results of Theorems 1.1, 1.2 and 1.3 have a local version, as shown by the following result:

Theorem 1.4 *Assume $u \in H^1_{\mathrm{loc}}(\Omega)$ or $W^{1,q}_{\mathrm{loc}}(\Omega)$ is a local solution of problem* (6)*, where $a(x,u)$ satisfies* (18)*. If $f \in L^r_{\mathrm{loc}}(\Omega)$, with $r > \frac{2N}{N+2-\theta(N-2)}$, there exists a positive constant ϵ that depends only on N, α and β (with $\epsilon \le r^* - 2$), such that*

$$a(x,u)Du \in \left(L^{2+\epsilon}_{\mathrm{loc}}(\Omega)\right)^N. \tag{25}$$

Moreover if $r > N/2$, then u belongs to $L^\infty_{\mathrm{loc}}(\Omega)$, while if r verifies (9)*, then u is in $L^h_{\mathrm{loc}}(\Omega)$, where h is as in* (10).

Remark 1.7 We point out that the proof of the $L^h_{\mathrm{loc}}(\Omega)$ regularity of the local solutions of problem (6) requires the assumption (18) while when we have a global solution, the proof of the $L^h(\Omega)$ regularity needs just the weaker condition (7).

We return now to the general case when the weaker condition (7) (with respect to (18)) is assumed and we study what happens when u, in general, is an unbounded solution. We have:

Theorem 1.5 *Let* (7), (8) *and* (9) *hold. Assume that* $u \in H_0^1(\Omega)$ *is a solution of problem* (6) *that satisfies*

$$\int_0^u \frac{\partial a(x,s)}{\partial x} ds \in L^{2+\sigma}_{\rm loc}(\Omega), \qquad \sigma > 0. \tag{26}$$

Then there exists a positive constant ϵ *that depends only on* N *and* σ *(with* $\epsilon \le r^* - 2$*) such that*

$$a(x,u)Du \in \left(L^{2+\epsilon}_{\rm loc}(\Omega)\right)^N. \tag{27}$$

Moreover if $r > \frac{2N}{N+2-\theta(N-2)}$ *then there exists* $s > 2$ *such that*

$$Du \in \left(L^{s}_{\rm loc}(\Omega)\right)^N. \tag{28}$$

Remark 1.8 We observe that assumption (26) is verified if, for example, we have

$$\left|\frac{\partial a(x,s)}{\partial x}\right| \le h(x) \quad \text{a.e.,} \quad x \in \Omega, \quad \forall s \in \mathbf{R}, \tag{29}$$

where $h(x)$ belongs to $L^m(\Omega)$ with $m > N$.

Theorem 1.6 *Let* (7), (8) *hold and* $f \in L^r(\Omega)$ *with* r *as in* (11). *Assume* u *is a solution of problem* (6) *belonging to* $W_0^{1,q}(\Omega)$ *and verifying*

$$g(x) = \int_0^u \frac{\partial a(x,s)}{\partial x} ds \in \left(L^{q+\sigma}_{\rm loc}(\Omega)\right)^N, \qquad \sigma > 0, \tag{30}$$

where q *is as in* (12). *Then there exist two positive constants* r_0 *and* ϵ, *where* $r_0 > \frac{N}{N+1-\theta(N-1)}$ *depends only on* N *and* θ, *while* ϵ *depends only on* N, θ, σ *and* r, *such that if we assume*

$$r_0 < r < \frac{2N}{N+2-\theta(N-2)}, \tag{31}$$

then

$$a(x,u)Du \in \left(L^{q+\epsilon}_{\rm loc}(\Omega)\right)^N. \tag{32}$$

Moreover if $g(x)$ *belongs to* $L^{2+\sigma}(\Omega)$, *then there exists a positive constant* γ *such that*

$$a(x,u)Du \in \left(L^{2+\gamma}_{\rm loc}(\Omega)\right)^N. \tag{33}$$

Remark 1.9 The assumption $r > r_0$ is taken to guarantee that q is "near" 2 so that it is possible to use the techniques related to the Hodge decomposition (see Lemma 2.2 below).

Remark 1.10 Let us now point out that, in some particular case, the solution can reach a better regularity.

For example if $a(x,u) = a(u)$ then under the weak conditions (7) and (8) it is easy to prove every solution $u \in H_0^1(\Omega)$ belongs to $W_0^{1,r^*}(\Omega)$ if $N/2 < r < N$ and to $W_0^{1,q}(\Omega)$ if $\frac{2N}{N+2-\theta(N-2)} < r < \frac{N}{2}$, where q is, as before, given by the following formula:

$$q = q(r) = \frac{Nr(1-\theta)}{N - r(1+\theta)}.$$

We notice that when $r > \frac{2N}{N+2-\theta(N-2)}$ we have $q > 2$. Moreover, we have $q \to r^*$ as $r \to N/2$.

In all the previous results we have assumed that the summability r of the data f verifies the following condition:

$$r > \frac{2N}{N+2}. \tag{34}$$

If condition (34) is violated, it is possible to prove by using the techniques of [6], the following regularity result:

Theorem 1.7 *Let* (7) *and* (8) *hold and* $f \in L^r(\Omega)$, *where* r *verifies*

$$\frac{N}{N+1-\theta(N-1)} < r < \frac{2N}{N+2}. \tag{35}$$

Then every solution $u \in W_0^{1,q}(\Omega)$ (*where* q *is as in* (12)) *of problem* (6) *satisfies*

$$\frac{Du}{(1+|u|)^{\theta}} \in \left(L^{r^*}(\Omega)\right)^N. \tag{36}$$

1.2 Minima of Functionals

Let v be in $W_0^{1,p}(\Omega)$ and consider the following functional:

$$J(v) = \int_\Omega a(x,v) j(Dv)dx - \int_\Omega f v dx. \tag{37}$$

Here $a : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that

$$\frac{\beta_0}{(1+|s|)^{\theta p}} \le a(x,s) \le \beta_1, \qquad 0 < \theta < \frac{1}{p}, \tag{38}$$

where $0 < \beta_0 \le \beta_1$, and j is a convex function, where $j : \mathbf{R}^N \to \mathbf{R}$, $j(0) = 0$, satisfying

$$\beta_2 |\xi|^p \le j(\xi) \le \beta_3 \left(|\xi|^p + 1\right), \qquad \beta_3 \ge \beta_2 > 0. \tag{39}$$

Moreover, f is a function in $L^r(\Omega)$, where r verifies

$$r \ge [p^*(1-\theta)]'. \tag{40}$$

We notice that the functional in (37) is differentiable just in the directions of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see [7]), and in this case its Euler equation is not (6).

By the assumptions (38)–(40), J turns out to be defined on the whole space $W_0^{1,p}(\Omega)$. We extend J to a larger space, $W_0^{1,q}(\Omega)$ with $q = \frac{Np(1-\theta)}{N-\theta p}$ by

$$I(v) = \begin{cases} J(v) & \text{if} \quad J(v) \quad \text{is finite,} \\ +\infty & \text{otherwise.} \end{cases} \tag{41}$$

We recall the following result proved in [8]:

Proposition 1.2 *Let* (39)–(40) *hold. Then there exists a minimum* u *of* $I(v)$ *in* $W_0^{1,q}(\Omega)$.

Moreover if $r > \frac{N}{p}$ *then any minimum* u *of* I *on* $W_0^{1,q}(\Omega)$ *belongs to* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; *thus* J *attains its minimum in* $W_0^{1,p}(\Omega)$.

If $\left(\frac{p^*}{1+\theta p}\right)' \le r < \frac{N}{p}$, *then* u *belongs to* $W_0^{1,p}(\Omega) \cap L^h(\Omega)$, *with*

$$h = \frac{Nr[p(1-\theta)-1]}{N-rp}, \tag{42}$$

and thus, again J *attains its minimum on* $W_0^{1,p}(\Omega)$.

Finally, if $[p^*(1-\theta)]' \le r < \left(\frac{p^*}{1+\theta p}\right)'$*, then* u *belongs to* $W_0^{1,\rho}(\Omega)$*, with*

$$\rho = \frac{Nr[p(1-\theta)-1]}{N-r(1+\theta p)}. \tag{43}$$

Remark 1.11 We observe that $\rho^* = h$, so there is continuity with respect to the regularity of u in the two cases above. Moreover, we have $\rho = p$ for $r = \left(\frac{p^*}{1+\theta p}\right)'$, and if r tends to $\frac{N}{p}$ then h tends to $+\infty$.

As before, we are interested in the regularity of the gradient of solutions depending on the regularity of the datum f. As in previous sections we have to take into account the upper growth condition on $a(x,s)$. More precisely, let us first suppose that, instead of (38), $a(x,s)$ satisfies the stronger condition

$$\beta_0 a_1(x) a_2(u) \le a(x,s) \le \beta_1 a_1(x) a_2(u), \tag{44}$$

where

$$0 < \alpha_1 \le a_1(x) \le \alpha_2, \tag{45}$$

$$\frac{\gamma_1}{(1+|s|)^{p\theta}} \le a_2(s) \le \gamma_2. \tag{46}$$

A simple example in which these hypotheses are satisfied is when

$$\frac{\beta_0}{(1+|s|)^{\theta p}} \le a(x,s) \le \frac{\beta_1}{(1+|s|)^{\theta p}}, \quad 0 < \beta_0 \le \beta_1, \quad 0 < \theta < \frac{1}{p}. \tag{47}$$

Assume that r verifies

$$r > [p^*(1-\theta)]'. \tag{48}$$

Theorem 1.8 *Let* (48)*,* (39) *and* (44)*–*(46) *hold. Then every minimum* $u \in W_0^{1,q}(\Omega)$ *satisfies*

$$[a(x,u)]^{\frac{1}{p}} |Du| \in L^{p+\epsilon}_{\mathrm{loc}}(\Omega), \tag{49}$$

where ϵ *is a positive constant depending only on the data. Besides, if* r *verifies*

$$r > \left(\frac{p^*}{1+\theta p}\right)', \tag{50}$$

then there exists a positive constant ν *such that*

$$|Du| \in L^{p+\nu}_{\mathrm{loc}}(\Omega). \tag{51}$$

Remark 1.12 We do not know if a higher regularity for Du holds true, when $r \le \left(\frac{p^*}{1+\theta p}\right)'$.

Remark 1.13 We observe that, in general, all the previous results hold for quasi-minima of (37).

Remark 1.14 We recall that if $a(x,s) = a(s)$ then (44) is equivalent to (38) and that, in general, this equivalence doesn't hold if a depends also on the x variable, as shown by the following example:

$$a(x,s) = \frac{1}{(1+\alpha(x)|s|)^{p\theta}}, \tag{52}$$

where, for example, $\alpha(x)$ is bounded with $\inf_\Omega \alpha(x) = 0$.

Let us now return to the general assumption (38). Define

$$H(x,\tau) = \int_0^\tau (a(x,s))^{1/p} ds. \tag{53}$$

We notice that $H(x,\cdot) : \mathbf{R} \to \mathbf{R}$ is strictly increasing and $H(\mathbf{R}) = \mathbf{R}$. Hence the inverse function is well defined, with respect to the variable τ, $C(x,t)$. We have the following result:

Theorem 1.9 *Let (48), (38) and (39) hold true. Assume that there exists $\frac{\partial C}{\partial x}(x,t)$ a.e. in Ω, and that*

$$\frac{\partial C}{\partial x}(x,t)/_{t=\overline{z}(x)} \in L^q(\Omega), \qquad \forall \overline{z} \in W_0^{1,p}(\Omega), \tag{54}$$

and that the distributional derivative $\frac{\partial a^{1/p}(x,s)}{\partial x}$ is a measurable function verifying

$$\begin{cases} \left| \int_0^\tau \dfrac{\partial a^{1/p}(x,s)}{\partial x} ds \right| \leq \overline{c_1} |\tau|^\gamma + \overline{c_2} \quad a.e. \ in \quad \Omega, \\ 0 \leq \gamma < \dfrac{q^*}{p} = \dfrac{N(1-\theta)}{N-p}. \end{cases} \tag{55}$$

Then there exists a positive constant ϵ, which depends only on α, β, N, $|\Omega|$ and $\|f\|_{L^r(\Omega)}$, such that every minimum $u \in W_0^{1,q}(\Omega)$ with $J(u)$ finite satisfies

$$a^{1/p}(x,u)|Du| \in L^{p+\epsilon}_{\mathrm{loc}}(\Omega). \tag{56}$$

Moreover, if r verifies (50) there exists a positive constant ν such that

$$|Du| \in L^{p+\nu}_{\mathrm{loc}}(\Omega). \tag{57}$$

Remark 1.15 Notice that we can weaken the assumption (55) so that it becomes

$$\left| \int_0^\tau \frac{\partial a^{1/p}(x,s)}{\partial x} ds \right| \leq \overline{c_1}(x) |\tau|^\gamma + \overline{c_2}, \quad \text{a.e. in} \quad \Omega, \tag{58}$$

where $\overline{c_1}(x) \in L^\lambda(\Omega)$ with $\frac{\gamma}{p^*(1-\theta)} + \frac{1}{\lambda} < \frac{1}{p}$.

Remark 1.16 An example of a functional verifying the assumptions (38), (54) and (55) is the model case (52) with the function $\alpha(x)$ positive, bounded in Ω and verifying, for example, the following further regularity condition:

$$\frac{|D\alpha|}{\alpha^2} \in L^m(\Omega), \quad m > N. \tag{59}$$

We point out that the model case (52) with $\alpha(x) = \chi_E(x)$ ($\chi_E(x)$ characteristic function of $E \subset \Omega$ measurable) does not satisfy condition (55). In this case the problem of further regularity of Du is still open.

2 Notations and Preliminaries

In this section, we give some preliminary results that will play an essential role in what follows. We start with the following theorem (see [9]) whose proof is the same as the one by Gehring ([10]):

Lemma 2.1 *Let Q be an N-cube and set $Q_R = \{x \in R^N : |x_i - x_{0i}| < R, i = 1, ..., N\}$.*

Assume that $w \in L^1(Q)$ *and that we have*

$$\fint_{Q_R} w(x)dx \le \gamma_0 \left\{ \left(\fint_{Q_{2R}} w^m dx \right)^{\frac{1}{m}} + \fint_{Q_{2R}} g dx \right\}, \tag{60}$$

for each $x_0 \in Q$ *and each* $R < \frac{1}{2}\min\{\mathrm{dist}(x_0, \partial Q), R_0\}$, *where* R_0 *and* γ_0 *are positive constants,* $0 < m < 1$ *and we have set* $\fint_Q v = \frac{1}{|Q|}\int_Q v$. *Assume that the function* g *belongs to* $L^s_{\mathrm{loc}}(Q)$ *for some* $s > 1$. *Then there exists a constant* $d > 1$ *such that* $w \in L^d_{\mathrm{loc}}(Q)$, *and we have*

$$\fint_{Q_{R/2}} w(x)^d dx \le \gamma_1 \left\{ \left(\fint_{Q_R} w dx \right)^d + \fint_{Q_R} g^d dx \right\} \tag{61}$$

for $Q_{2R} \subset Q$, $R < R_0$, *where* γ_1 *is a constant that depends only on the data.*

We will need the following decomposition theorem (see [11] or [12]):

Lemma 2.2 *Let* $Q_\sigma = Q(x_0, \sigma)$ *be an open* N*-cube centered in* x_0 *of side* 2σ, $v \in W_0^{1,r}(Q_\sigma)$, $r > 1$, *and let* $-1 < \gamma < r - 1$. *Then there exist* $\phi : Q_\sigma \to \mathbf{R}$ *and* $H : Q_\sigma \to \mathbf{R}^N$ *such that* $H \in \left(L^{\frac{r}{1+\gamma}}(Q_\sigma) \right)^N$, $\mathrm{div} H = 0$, $\phi \in W_0^{1,\frac{r}{1+\gamma}}(Q_\sigma)$ *and*

$$|Dv|^\gamma Dv = D\phi + H, \tag{62}$$

$$\|H\|_{L^{\frac{r}{1+\gamma}}(Q_\sigma)} \le c(r,N)|\gamma| \cdot \|Dv\|^{1+\gamma}_{L^r(Q_\sigma)}, \tag{63}$$

$$\|D\phi\|_{L^{\frac{r}{1+\gamma}}(Q_\sigma)} \le (1 + c(r,N)|\gamma|)\|Dv\|^{1+\gamma}_{L^r(Q_\sigma)}. \tag{64}$$

Remark 2.1 We notice explicitly that the constant $c = c(r, N)$ which appears in the formulas (63) and (64) doesn't depend on x_0, neither on σ, nor on r if r belongs to a compact set.

Another tool is the following lemma of real analysis, whose proof is very easy and can be found in [10]:

Lemma 2.3 *Let* $f(\tau)$ *be a non-negative bounded function defined for* $0 \le R_0 \le t \le R_1$. *Suppose that for* $R_0 \le \tau < t \le R_1$ *we have*

$$f(\tau) \le A(t-\tau)^{-\alpha} + B + \gamma f(t), \tag{65}$$

where A, B, α, γ *are non-negative constants, and* $\gamma < 1$. *Then there exists a constant* c, *depending only on* α *and* γ *such that for every* $\rho, R, R_0 \le \rho < R \le R_1$ *we have*

$$f(\rho) \le c\left[A(R-\rho)^{-\alpha} + B\right]. \tag{66}$$

We now state the following regularity theorem proved in [13] (Theorem 2.1):

Theorem 2.1 *Let* $v \in W^{1,\overline{p}}_{\mathrm{loc}}(\Omega)$, $\phi_0 \in L^m_{\mathrm{loc}}(\Omega)$, *where* $1 < \overline{p} < N$ *and* m *satisfies*

$$1 < m < \frac{N}{\overline{p}}. \tag{67}$$

Assume that for all $B_{R_1} \subset\subset \Omega$ *the following integral estimate holds:*

$$\int_{A_{k,\rho}} |Dv|^{\overline{p}} \le c_1 \left[\int_{A_{k,R}} \phi_0 dx + (R-\rho)^{-\lambda} \int_{A_{k,R}} |v|^{\overline{p}} \right], \tag{68}$$

for every $k \in \mathbf{N}$ *and* $R_0 \le \rho < R \le R_1$, *where* $A_{k,\rho} = B_\rho \cap \{|v| > k\}$. *Here* $c_1 = c_1$ $(N, \overline{p}, m, R_0, R_1, |\Omega|)$, *and* λ *is a real positive constant. Then we have* $v \in L^s_{\mathrm{loc}}(\Omega)$, *where*

$s = (\overline{p}m)^*$.

Finally, we conclude this section with a technical lemma that can be easily proved by induction.

Lemma 2.4 *Let Y_n be a sequence of non-negative numbers satisfying*

$$Y_{n+1} \le cb^n Y_n^{1+\delta}, \quad \forall n = 0, 1, \dots, \tag{69}$$

where c, b and δ are positive constants and $b \ge 1$. If

$$Y_0 \le c^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}}, \tag{70}$$

then $Y_n \to 0$ when $n \to +\infty$.

3 Proofs of Theorems 1.1–1.7 (Elliptic Equations)

In this section, we prove the results stated in Section 2 concerning the regularity of Du, when u is a solution of the Dirichlet problem (6).

Proof of Theorem 1.1 We recall, as noticed in Remark 1.2, that every solution $u \in H_0^1(\Omega)$ verifies the following estimate:

$$\|u\|_{L^\infty(\Omega)} \le C_0 = C_0(\Omega, N, r, \|f\|_{L^r(\Omega)}). \tag{71}$$

Let Q_R be an N-cube contained in Ω. Choose $\Psi = (u - u_R)\eta^2$ as a test function in (6), where $\eta \in C_0^\infty(Q_R), 0 \le \eta \le 1, \eta \equiv 1$ on $Q_{R/2}, |D\eta| \le c/R$, and $u_R = \frac{1}{|Q_R|}\int_{Q_R} u$. Using (7) and the fact that $u \in L^\infty(\Omega)$ we obtain

$$\begin{aligned} \frac{\alpha}{(1+\|u\|_{L^\infty(\Omega)})^\theta}\int_\Omega |Du|^2\eta^2 &\le \alpha \int_\Omega \frac{|Du|^2\eta^2}{(1+|u|)^\theta} \\ &\le 2\beta \int_\Omega |Du|\eta|D\eta||u-u_R| + \int_\Omega f(u-u_R)\eta^2. \end{aligned} \tag{72}$$

Thus we can conclude that

$$\begin{aligned} \int_\Omega |Du|^2\eta^2 \quad &\le \frac{1+\|u\|_{L^\infty(\Omega)}}{\alpha}\left(2\beta\int_\Omega |Du|\eta|D\eta||u-u_R| + \int_\Omega f(u-u_R)\eta^2\right) \\ &\le c_0\left(\int_\Omega |Du|\eta|D\eta||u-u_R| + \int_\Omega f(u-u_R)\eta^2\right), \end{aligned} \tag{73}$$

where $c_0 = \frac{1+C_0}{\alpha}\max\{2\beta, 1\}$, with C_0 as in (71). From now on, the proof is the same as that of the uniformly elliptic case (see [10]) and we have

$$⨍_{Q_{R/2}} |Du|^2 \le c_1\left[\frac{1}{R^N}\left(\int_{Q_R}|f|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}} + \frac{1}{R^{N+2}}\left(\int_{Q_R}|Du|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}}\right],$$

where $c_1 = c(N, |\Omega|, r, \|f\|_{L^r(\Omega)})$. Applying Lemma 2.1 with $w(x) = |Du(x)|^2$, $m = \frac{N}{N+2}$ and $g = (\int_\Omega |f|^{\frac{2N}{N+2}})^{\frac{2}{N}}|f|^{\frac{2N}{N+2}}$ we get the result.

Remark 3.1 It is easy to see that Theorem 1.1 holds true if f is supposed to belong to $W^{-1,\rho}(\Omega)$ where $\rho > N$.

Proof of Theorem 1.2 We point out that under the assumption (18) the regularity result (19) is equivalent to the following:

$$\frac{|Du|}{(1+|u|)^{\theta}} \in L^{2+\epsilon}_{\rm loc}(\Omega). \tag{74}$$

Define

$$v(x) = \frac{1}{1-\theta}\,\mathrm{sign}(u)\left((1+|u|)^{1-\theta}-1\right). \tag{75}$$

Let $Q_R \subset \Omega$ and η be a cut-off function as in the proof of Theorem 1.1. Choose as a test function $w = (v-(v)_R)\eta^2$, where v is as in (75). Using assumption (18) and (75) we obtain

$$\begin{aligned}\alpha\int_{Q_R}|Dv|^2\eta^2 &= \alpha\int_{Q_R}\frac{\eta^2|Du|^2}{(1+|u|)^{2\theta}}\\ &\le \beta\int_{Q_R}2\eta|D\eta||Dv||v-(v)_R| + \int_{Q_R} f\eta^2\left(v-(v)_R\right),\end{aligned} \tag{76}$$

that is, (73) with u replaced by v. Thus proceeding as in the proof of Theorem 1.1 we obtain that there exists a positive constant ϵ such that $Dv \in L^{2+\epsilon}_{\rm loc}(\Omega)$, i.e. (74) holds true.

Assume now that $r > \frac{2N}{N+2-\theta(N-2)}$. It remains to prove the regularity stated in (20). To do this, we will use the regularity stated in (74). We notice that proceeding as in the proof of Lemma 2.3 in [2] we can prove that there exists a constant c_0, depending only on the data, such that

$$\int_{\Omega}|Du|^2|u|^{2(\gamma-1)} \le c_0, \quad \gamma = \frac{(N-2)(1-\theta)r}{2(N-2r)}. \tag{77}$$

Let $p = 2+\epsilon$ and $s = 2\lambda + (1-\lambda)p > 2$, where ϵ is as in (74) and $\lambda \in (0,1)$ to be determined later. Using Young's inequality we obtain

$$\begin{aligned}\int_{Q_R}|Du|^s &= \int_{Q_R}\frac{|Du|^{(1-\lambda)p}}{(1+|u|)^{2\lambda(\gamma-1)}}|Du|^{2\lambda}(1+|u|)^{2\lambda(\gamma-1)}\\ &\le \int_{Q_R}|Du|^2(1+|u|)^{2(\gamma-1)} + \int_{Q_R}\frac{|Du|^p}{(1+|u|)^{\frac{2\lambda(\gamma-1)}{1-\lambda}}} \le c_1,\end{aligned} \tag{78}$$

where c_1 is a constant that depends only on the data, and where we have set $\frac{2\lambda(\gamma-1)}{1-\lambda} = \theta p$, that is, $\lambda = \frac{\theta p}{\theta p + 2(\gamma-1)}$. Notice that we have $\lambda < 1$ as it is equivalent to the assumption $r > \frac{2N}{N+2-\theta(N-2)}$.

Remark 3.2 We observe that it is possible to show that the thesis (19) of Theorem 1.2 is also true if we replace the assumption $f \in L^r(\Omega)$ (r verifying (9)) with a weaker hypothesis that f belongs to $W^{-1,\rho}(\Omega)$, where $\rho > 2$.

Proof of Theorem 1.3 Let $v(x)$ be as in (75). The function $w = (v-v_R)\eta^2 \in H^1_0(Q_R)$, by the assumption (23) and the fact that $2(1-\theta) < q^*$. Thus it is possible to proceed exactly as in the proof of Theorem 1.2 and to conclude that (24) holds.

Remark 3.3 If $f \in L^r(\Omega)$ with r verifying (11) then, as just noticed in Section 2, we cannot expect a better regularity for Du as shown by the following model case:

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{(1+|u|)^{\theta}}\right) = f & \text{in} \quad D'(\Omega),\\ u = 0 & \text{on} \quad \partial\Omega.\end{cases} \tag{79}$$

We have $|Du| = |Dv|(1+|u|)^\theta = |Dv|[|v|(1-\theta)+1]^{\frac{\theta}{1-\theta}}$, where $v = \int_0^u \frac{ds}{(1+|s|)^\theta}$. By the Calderon-Zygmund theorem, the sharp regularity for v is $W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ which gives $Du \in L^q(\Omega)$.

Proof of Theorem 1.4 The proof of (25) is exactly the same as that given in Theorems 1.2 and 1.3. Hence it remains to show that if $u \in H^1_{\mathrm{loc}}(\Omega)$ and $r > N/2$ then $u \in L^\infty_{\mathrm{loc}}(\Omega)$, while if f verifies (9) then u belongs to $L^h_{\mathrm{loc}}(\Omega)$, where h is as in (10).

L^∞_{loc}**-regularity** We notice that our aim is equivalent to prove that $v \in L^\infty_{\mathrm{loc}}(\Omega)$, where v is as in (75). Denote by B_r the ball centered at x_0 and of radius r. Let $B_{2\rho} \subset\subset \Omega$, $\rho_n = \rho + \frac{\rho}{2^n}$ and define $B_n = B_{\rho_n}$. We have $B_n \supset B_{n+1}$. Moreover, let k be a number bigger than one that will be chosen later and denote $k_n = k - \frac{k}{2^n}$. We introduce the following sequence of cut-off function μ_n in $C_0^\infty(\Omega)$ verifying:

$$0 \le \mu_n \le 1, \quad |D\mu_n| \le \frac{2^{n+2}}{\rho}, \quad \mu_n = 0 \ \text{ outside } B_n, \quad \mu_n \equiv 1 \ \text{ in } B_{n+1}.$$

Choose $\mu_n^2(v-k_{n+1})_+$ as a test function in (6). We obtain

$$\begin{aligned}\alpha \int_{B_n\cap\{v>k_{n+1}\}} |Dv|^2\mu_n^2 &\le \int_{B_n\cap\{v>k_{n+1}\}} a(x,u)DuDv\mu_n^2 \\ &= -2\int_{B_n\cap\{v>k_{n+1}\}} \mu_n a(x,u)DuD\mu_n(v-k_{n+1})_+ + \int_{B_n} f\mu_n^2(v-k_{n+1})_+. \end{aligned} \tag{80}$$

Moreover, using assumption (18) and Young's inequality we have

$$\begin{aligned}&\left|2\int_{B_n\cap\{v>k_{n+1}\}} \mu_n a(x,u)DuD\mu_n(v-k_{n+1})_+\right| \\ &\quad\le 2\beta \int_{B_n\cap\{v>k_{n+1}\}} |Dv|\mu_n|D\mu_n|(v-k_{n+1})_+ \\ &\quad\le \frac{\alpha}{2}\int_{B_n\cap\{v>k_{n+1}\}} |Dv|^2\mu_n^2 + \frac{2\beta^2 2^{2(n+2)}}{\alpha\rho^2}\int_{B_n\cap\{v>k_{n+1}\}} (v-k_{n+1})_+^2. \end{aligned} \tag{81}$$

Putting together all the previous estimates we obtain

$$\frac{\alpha}{2}\int_{B_n\cap\{v>k_{n+1}\}} |Dv|^2\mu_n^2 \le \frac{2\beta^2 2^{2(n+2)}}{\alpha\rho^2}\int_{B_n\cap\{v>k_{n+1}\}} (v-k_{n+1})_+^2 + \int_{B_n} f\mu_n^2(v-k_{n+1})_+. \tag{82}$$

Now we show that from the integral estimate (82) it follows that $v \le k$ in B_ρ for a suitable choice of k. Using the Sobolev inequality and (82) we have

$$\begin{aligned}&\int_{B_{n+1}} (v-k_{n+1})_+^{2^*} \le \int_{B_n} [\mu_n(v-k_{n+1})_+]^{2^*} \\ &\quad\le c_s\left(\int_{B_n} |D[\mu_n(v-k_{n+1})_+]|^2\right)^{\frac{2^*}{2}} \\ &\quad\le c_s\left(\int_{B_n\cap\{v>k_{n+1}\}} |Dv|^2\mu_n^2 + \int_{B_n} |D\mu_n|^2(v-k_{n+1})_+^2\right)^{\frac{N}{N-2}} \\ &\quad\le c_0\left(\frac{2^{2(n+2)}}{\rho^2}\int_{B_n\cap\{v>k_{n+1}\}} (v-k_{n+1})_+^2 + \int_{B_n} |f|\mu_n^2(v-k_{n+1})_+\right)^{\frac{N}{N-2}}, \end{aligned} \tag{83}$$

where $c_0 = c_s\left[\frac{2}{\alpha}\left(\frac{2\beta^2}{\alpha}+1\right)\right]^{\frac{N}{N-2}}$ and $c_s = c_s(N)$ is the Sobolev constant. We now distinguish the two cases: $r' \geq 2$ and $r' < 2$, where $1/r + 1/r' = 1$. If $r' \geq 2$, we have

$$\int_{B_n} |f|\mu_n^2 (v - k_{n+1})_+ \leq \|f\|_{L^r(\Omega)} Y_n^{1-\frac{1}{r}}, \quad Y_n \equiv \int_{B_n} (v - k_n)_+^{r'}. \tag{84}$$

We observe that when we assume $r > N/2$ we have $r' < \frac{N}{N-2} < 2^*$, and thus, using (83) and (84) we derive that

$$\begin{aligned} Y_{n+1} &\leq \left(\int_{B_{n+1}} (v - k_{n+1})_+^{2^*} \right)^{\frac{r'}{2^*}} |A_{n+1}|^{1-\frac{r'}{2^*}} \\ &\leq c_1 \left[\frac{2^{2(n+2)}}{\rho^2} \int_{B_n} (v - k_{n+1})_+^2 + \|f\|_{L^r(\Omega)} Y_n^{1-\frac{1}{r}} \right]^{\frac{Nr'}{(N-2)2^*}} |A_{n+1}|^{1-\frac{r'}{2^*}}, \end{aligned} \tag{85}$$

where $c_1 = c_0^{\frac{r'}{2^*}}$ and $A_{n+1} = B_{n+1} \cap \{v > k_{n+1}\}$. Then

$$Y_n \geq \int_{B_n \cap \{v > k_{n+1}\}} (k_{n+1} - k_n)^{r'} = \frac{k^{r'}}{2^{(n+1)r'}} |B_n \cap [v > k_{n+1}]|,$$

from which it follows that

$$|A_{n+1}| \leq |B_n \cap \{v > k_{n+1}\}| \leq \frac{2^{(n+1)r'}}{k^{r'}} Y_n. \tag{86}$$

Notice that $r' \geq 2$ is equivalent to $r \leq 2$. Then using (86) we have

$$\int_{B_n} (v - k_{n+1})_+^2 \leq Y_n^{\frac{2}{r'}} |B_n \cap \{v > K_{n+1}\}|^{1-\frac{2}{r'}} \leq \frac{2^{(n+1)r'(1-\frac{2}{r'})} c_2}{k^{r'(1-\frac{2}{r'})}} Y_n^{1-\frac{1}{r}},$$

where $c_2 = (\int_{B_{2\rho}} |v|^{r'})^{1/r}$. From (85) and (86) we deduce (being $k \geq 1$) that (69) holds true with $c = \frac{c_1 \max\{\frac{c_2}{\rho^2}, \|f\|_{L^r(\Omega)}\}^{r'/2} 2^{4(r'+1)}}{k^{r'(1-\frac{r'}{2^*})}}$, $b = 2^{2(r'+1)}$ and $\delta = (1-\frac{1}{r})\frac{r'}{2} - \frac{r'}{2^*}$. We notice that $\delta > 0$ is equivalent to the assumption $r > N/2$. If we choose k verifying

$$\int_{B_{2\rho}} |v|^{r'} \leq \frac{k^{r'(1-\frac{r'}{2^*})\frac{1}{\delta}}}{(c_1 \max\{\frac{c_2}{\rho^2}, \|f\|_{L^r(\Omega)}\}^{r'/2} 2^{4(r'+1)})^{\frac{1}{\delta}}} b^{\frac{-1}{\delta^2}},$$

(70) is also satisfied. Hence we can apply Lemma 2.4 and conclude that $Y_n \to 0$ as $n \to +\infty$, that is $v \leq k$ in B_ρ. To prove also $v \geq k_0$, $(k_0 > 0)$, we notice that if $\overline{u} = -u$ then $\overline{v} \equiv \int_0^{\overline{u}} \frac{ds}{(1+|s|)^\theta} = -v$. Hence it is sufficient to apply the previous result to the function $\overline{v} = -v$. As a matter of fact, $\overline{u}$ verifies a problem of the same kind of u. When $r' < 2$ we set $Y_n = \int_{B_n} (v - k_n)_+^2$ and the proof is similar to that of the previous case $r' \geq 2$, and so we omit it.

L^h_{loc}-regularity Let us now assume that r verifies (9). Let $B_{R_1} \subset\subset \Omega$ and $0 \leq R_0 \leq \tau < t \leq R_1 \leq 1$ be arbitrarily fixed. Choose $\eta(v - T_k(v))$ as a test function in (6), where v is as defined before in (75) and η is a cut-off function in $C_0^\infty(B_t)$ such that $0 \leq \eta \leq 1, \eta \equiv 1$ on $B_\tau, |D\eta| \leq C(t-\tau)^{-1}$. Notice that when $r < N$ we have that $f = -\mathrm{div}(f_1)$ with $f_1 \in L^{r^*}(\Omega)$. Hence we obtain

$$\int_{B_t} a(x,u) DuD[\eta(v - T_k(v))] = \int_{B_t} f_1 D[\eta(v - T_k(v))]. \tag{87}$$

We estimate now the integrals in (87). Applying (18) we deduce that

$$\int_{B_t} a(x,u) DuD[\eta(v - T_k(v))] \geq \alpha \int_{A_{k,\tau}} |Dv|^2 - \frac{\beta C}{t-\tau} \int_{A_{k,t} - A_{k,\tau}} |Dv||v - T_k(v)|$$

$$\geq \alpha \int_{A_{k,\tau}} |Dv|^2 - \delta \int_{A_{k,t}-A_{k,\tau}} |Dv|^2 - \frac{\beta^2 C^2}{\delta(t-\tau)^2} \int_{A_{k,t}-A_{k,\tau}} v^2, \tag{88}$$

where $A_{k,t}$ is as in Theorem 2.1 and δ is a positive constant to be determined. Besides, using Young's inequality we can increase the right-hand side of (87) as follows:

$$\int_{B_t} f_1 D[\eta(v - T_k(v))]$$
$$\leq \int_{A_{k,t}-A_{k,\tau}} |f_1|^2 + C^2 \int_{A_{k,t}-A_{k,\tau}} \frac{v^2}{(t-\tau)^2} + \frac{1}{\delta} \int_{A_{k,t}} |f_1|^2 + \delta \int_{A_{k,t}} |Dv|^2, \tag{89}$$

where δ is as before. Using the previous estimates in (87) we have

$$\int_{A_{k,\tau}} |Dv|^2 \leq \frac{2\delta}{\alpha} \int_{A_{k,t}} |Dv|^2 + c_3 \int_{A_{k,t}} \frac{v^2}{(t-\tau)^2} + c_4 \int_{A_{k,t}} \phi_0,$$

where $c_3 = \frac{C^2}{\alpha}(1 + \frac{\beta^2}{\delta})$, $c_4 = \frac{1}{\alpha}\left(1 + \frac{1}{\delta}\right)$ and $\phi_0 = f_1^2$. Notice that ϕ_0 belongs to $L^m_{\mathrm{loc}}(\Omega)$ with $m = r^*/2$. Now we want to eliminate the first term on the right-hand side including Dv. To do this, we proceed exactly as in the proof of Theorem 5.1 of [13]. Hence choose δ such that $\gamma = \frac{2\delta}{\alpha} < 1$ and let ρ, R be arbitrarily fixed with $R_0 \leq \rho < R \leq R_1$. Thus we can deduce that for every t and τ such that $\rho \leq \tau < t \leq R$, we have

$$\int_{A_{k,\tau}} |Dv|^2 \leq \gamma \int_{A_{k,t}} |Dv|^2 + \frac{c_3}{(t-\tau)^2} \int_{A_{k,R}} v^2 + c_4 \int_{A_{k,R}} \phi_0. \tag{90}$$

Applying Lemma 2.3 in (90) we conclude that

$$\int_{A_{k,\rho}} |Dv|^2 \leq \frac{cc_3}{(R-\rho)^2} \int_{A_{k,R}} v^2 + cc_4 \int_{A_{k,R}} \phi_0, \tag{91}$$

where c is the constant given by Lemma 2.3, that is, a constant depending only on γ. Thus v verifies the assumptions of Theorem 2.1 of Section 3 with $\overline{p} = \lambda = 2$. The condition (67) is equivalent to $r < N/2$, which is true by assumption. Hence $v \in L^s_{\mathrm{loc}}(\Omega)$, where $s = (r^*)^* = \frac{Nr}{N-2r}$, which implies $u \in L^h_{\mathrm{loc}}(\Omega)$ with h as in (10).

Proof of Theorem 1.5 Let us define

$$v(x) = \int_0^u a(x,s)ds. \tag{92}$$

Then

$$Dv(x) = a(x,u)Du + \int_0^u \frac{\partial a(x,s)}{\partial x} ds. \tag{93}$$

Let $Q_R \subset \Omega$ and η be a cut-off function as in the proof of Theorem 1.1. Choose as a test function $\psi = (v - v_R)\eta^2$. We obtain

$$\int_{Q_R} a(x,u)DuDv\eta^2 dx \leq 2 \int_{Q_R} a(x,u)|Du||v - v_R|\eta|D\eta| + \int_{Q_R} f(v - v_R)\eta^2.$$

We now estimate the terms in the previous inequality. Then we have

$$\int_{Q_R} a(x,u)DuDv\eta^2 = \int_{Q_R} (a(x,u))^2|Du|^2\eta^2 + \int_{Q_R} a(x,u)Du\eta^2 \int_0^u \frac{\partial a(x,s)}{\partial x} ds, \tag{94}$$

and for the last integral in (94), using the assumption (26), we have

$$\left| \int_{Q_R} a(x,u)Du\eta^2 \int_0^u \frac{\partial a(x,s)}{\partial x} ds \right| \leq \frac{1}{2} \int_{Q_R} (a(x,u))^2|Du|^2\eta^2 + \frac{1}{2} \int_{Q_R} \eta^2 g(x)^2, \tag{95}$$

where we have set

$$g(x)=\int_0^u \frac{\partial a(x,s)}{\partial x}ds. \tag{96}$$

Besides, we have

$$\begin{aligned}&2\int_{Q_R} a(x,u)|Du||v-v_R|\eta|D\eta|\\&\qquad\le 2\left[\frac{1}{2}\delta\int_{Q_R}(a(x,u))^2|Du|^2\eta^2+\frac{1}{2\delta}\int_{Q_R}|D\eta|^2|v-v_R|^2\right],\end{aligned} \tag{97}$$

where δ is a positive constant to be chosen. Putting together the previous estimates we deduce that

$$\begin{aligned}&\left(\frac{1}{2}-\delta\right)\int_{Q_R}(a(x,u))^2|Du|^2\eta^2\\&\qquad\le \frac{1}{2}\int_{Q_R}\eta^2 g(x)^2+\frac{1}{\delta}\int_{Q_R}|D\eta|^2|v-v_R|^2+\int_{Q_R}f(v-v_R)\eta^2.\end{aligned} \tag{98}$$

Let us choose $\delta=\frac{1}{4}$. From (98) and (93) we obtain

$$\begin{aligned}\int_{Q_{R/2}}|Dv|^2&\le 2\int_{Q_R}(a(x,u))^2|Du|^2\eta^2+2\int_{Q_R}g(x)^2\eta^2\\&\le 6\int_{Q_R}g(x)^2\eta^2+\frac{32c^2}{R^2}\int_{Q_R}|v-v_R|^2+8\int_{Q_R}f(v-v_R)\eta^2.\end{aligned} \tag{99}$$

Now observing that, by the assumption (26), $|g(x)|^2$ belongs to $L^\gamma(\Omega)$, where $\gamma=\frac{2+\sigma}{2}$ is bigger than one; we can proceed as in [10] and conclude that there exists a positive constant ϵ_0 such that $Dv\in L^{2+\epsilon_0}_{\mathrm{loc}}(\Omega)$ which implies (27) with $\epsilon=\min\{\sigma,\epsilon_0\}$. Thus proceeding exactly as in the proof of Theorem 1.2 we can conclude that (28) also holds.

Proof of Theorem 1.6 As a first step we prove that $a(x,u)Du\in L^{q+\epsilon}(\Omega)$. Let v be as in (92). Let $Q_R\subset\Omega$ as before and $R/2\le\rho<\sigma\le R$ and $\eta\in C_0^\infty(Q_\sigma)$, $0\le\eta\le 1$, $\eta\equiv 1$ on Q_ρ, with $|D\eta|\le C/(\sigma-\rho)$. Applying Lemma 2.2 to the function $D[\eta^2(v-v_R)]$ we obtain that there exists a function $\varphi\in W_0^{1,q'}(Q_\sigma)$, where q is as in the statement of the theorem and

$$\begin{cases} D\varphi=|D[\eta^2(v-v_R)]|^{q-2}D[\eta^2(v-v_R)]+H,\\ \mathrm{div}(H)=0,\end{cases} \tag{100}$$

$$\|H\|_{L^{q'}(Q_\sigma)}\le c(q,N)|q-2|\|D[\eta^2(v-v_R)]\|^{q-1}_{L^q(Q_\sigma)}, \tag{101}$$

$$\|D\varphi\|_{L^{q'}(Q_\sigma)}\le (1+c(q,N)|q-2|)\,\|D[\eta^2(v-v_R)]\|^{q-1}_{L^q(Q_\sigma)}. \tag{102}$$

Choose φ as a test function in (6). We obtain

$$\int_{Q_\sigma}a(x,u)DuD\varphi=\int_{Q_\sigma}f\varphi, \tag{103}$$

from which, using (93), we deduce that

$$\int_{Q_\sigma}DvD\varphi\le\int_{Q_\sigma}|g(x)||D\varphi|+\int_{Q_\sigma}f\varphi, \tag{104}$$

where g is as in (30). From (104), using Young's inequality, we obtain

$$\int_{Q_\sigma}D[\eta^2(v-v_R)]D\varphi\le\int_{Q_\sigma}\left|D\left[(\eta^2-1)(v-v_R)\right]\right||D\varphi|+\int_{Q_\sigma}|g(x)||D\varphi|+\int_{Q_\sigma}|f\varphi|$$

$$\leq \int_{Q_\sigma \setminus Q_\rho} \left[2\eta|D\eta||v-v_R| + (1-\eta^2)|Dv|\right] |D\varphi| + \int_{Q_\sigma} |g(x)||D\varphi| + \int_{Q_\sigma} |f\varphi|. \quad (105)$$

Moreover, using inequality (102), we have

$$\begin{aligned}
&\int_{Q_\sigma \setminus Q_\rho} \left[2\eta|D\eta||v-v_R| + (1-\eta^2)|Dv|\right] |D\varphi| + \int_{Q_\sigma} |g(x)||D\varphi| \\
&\quad \leq \|2\eta|D\eta||v-v_R| + (1-\eta^2)|Dv|\|_{L^q(Q_\sigma\setminus Q_\rho)} \|D\varphi\|_{L^{q'}(Q_\sigma)} + \|g(x)\|_{L^q(Q_\sigma)} \|D\varphi\|_{L^{q'}(Q_\sigma)} \\
&\quad \leq c_0 \left[\frac{2C}{\sigma-\rho}\|v-v_R\|_{L^q(Q_\sigma\setminus Q_\rho)} + \|Dv\|_{L^q(Q_\sigma\setminus Q_\rho)} + \|g(x)\|_{L^q(Q_\sigma)}\right] \\
&\qquad \times \|D[\eta^2(v-v_R)]\|^{q-1}_{L^q(Q_\sigma)} \\
&\quad \leq c_1 \left[\frac{C^q}{(\sigma-\rho)^q}\int_{Q_\sigma\setminus Q_\rho} |v-v_R|^q + \int_{Q_\sigma\setminus Q_\rho} |Dv|^q + \int_{Q_\sigma} |g(x)|^q\right] \\
&\qquad + \delta \int_{Q_\sigma} |D[\eta^2(v-v_R)]|^q, \qquad (106)
\end{aligned}$$

where $c_0 = 1 + \max_{q\in[1,3]} c(q,N)$, $c_1 = \frac{c_0^q 2^q 3^q}{\delta^{q-1}}$ and δ is a positive constant that will be chosen later. Using (100), (105) and (106) we get

$$\begin{aligned}
\int_{Q_\sigma} \left|D[\eta^2(v-v_R)]\right|^q &= \int_{Q_\sigma} D[\eta^2(v-v_R)]D\varphi - \int_{Q_\sigma} D[\eta^2(v-v_R)]H \\
&\leq c_1 \left\{\frac{C^q}{(\sigma-\rho)^q}\int_{Q_\sigma\setminus Q_\rho} |v-v_R|^q + \int_{Q_\sigma\setminus Q_\rho} |Dv|^q + \int_{Q_\sigma} |g(x)|^q\right\} \\
&\quad + \int_{Q_\sigma} |f\varphi| + 2\delta \int_{Q_\sigma} \left|D[\eta^2(v-v_R)]\right|^q + \gamma_0(\delta)\|H\|^{q'}_{L^{q'}(Q_\sigma)}, \qquad (107)
\end{aligned}$$

where c_1 is as before, i.e. it is a positive constant that depends only on δ, N and q in a continuous way, and $\gamma_0(\delta) = \delta^{\frac{1}{1-q}}$. Let us choose $s = \frac{Nr(1-\theta)}{N-2r\theta} = ((q')*)'$. Notice that $r > s$ since $r < N/2$. Thus using the Hölder and the Young's inequalities and (100) again, we have

$$\begin{aligned}
\int_{Q_\sigma} |f\varphi| &\leq \left(\int_{Q_\sigma} |f|^s\right)^{\frac{1}{s}} \left(\int_{Q_\sigma} |\varphi|^{s'}\right)^{\frac{1}{s'}} \leq c_s \left(\int_{Q_\sigma} |f|^s\right)^{\frac{1}{s}} \left(\int_{Q_\sigma} |D\varphi|^{q'}\right)^{\frac{1}{q'}} \\
&\leq 2c_s \left(\int_{Q_\sigma} |f|^s\right)^{\frac{1}{s}} \left(\int_{Q_\sigma} \left|D[\eta^2(v-v_R)]\right|^q + |H|^{q'}\right)^{\frac{1}{q'}} \\
&\leq (2c_s)^q \gamma_1(\delta) \left(\int_{Q_\sigma} |f|^s\right)^{\frac{q}{s}} + \delta \int_{Q_\sigma} \left|D[\eta^2(v-v_R)]\right|^q + \delta\|H\|^{q'}_{L^{q'}(Q_\sigma)}, \qquad (108)
\end{aligned}$$

where δ is as in (107), c_s is the Sobolev constant and $\gamma_1(\delta) = \delta^{1-q}$. Using in (107) the estimates (108), (101) and (102) we obtain

$$\begin{aligned}
\int_{Q_\sigma} \left|D[\eta^2(v-v_R)]\right|^q &\leq c_2 \left\{\frac{C^q}{(\sigma-\rho)^q}\int_{Q_\sigma\setminus Q_\rho} |v-v_R|^q \right. \\
&\quad \left. + \int_{Q_\sigma\setminus Q_\rho} |Dv|^q + \int_{Q_\sigma} |g(x)|^q + \left(\int_{Q_\sigma} |f|^s\right)^{\frac{q}{s}}\right\} \\
&\quad + \left[3\delta + (\gamma_0(\delta)+\delta)c(q,N)^{q'}|q-2|^{q'}\right] \int_{Q_\sigma} \left|D[\eta^2(v-v_R)]\right|^q, \qquad (109)
\end{aligned}$$

where $c_2 = \max\{c_1, 2^q\gamma_1(\delta)c_s^q\}$. Choose $\delta = 1/6$ and observe that if $r \to \frac{2N}{N+2-\theta(N-2)}$ then

$q = q(r) \to 2$. Then it is possible to choose r_0 such that if r satisfies (31) then the corresponding $q = q(r)$ verifies

$$\left[\gamma_0\left(\frac{1}{6}\right) + \frac{1}{6}\right] c(q,N)^{q'}|q-2|^{q'} \le \frac{1}{4}, \tag{110}$$

that is, if we choose, for example, r_0 satisfying $q(r_0) \ge 3/2$ such that

$$\left[\gamma_0\left(\frac{1}{6}\right) + \frac{1}{6}\right] \max\left\{\left(\max_{\frac{3}{2}\le q\le 3} c(q,N)\right)^{\left(\frac{3}{2}\right)'}, 1\right\}|q(r_0)-2| \le \frac{1}{4}, \tag{111}$$

then we have

$$\begin{aligned}\frac{1}{4}\int_{Q_\rho}|Dv|^q \le c_2\Bigg\{&\frac{C^q}{(\sigma-\rho)^q}\int_{Q_\sigma\setminus Q_\rho}|v-v_R|^q \\ &+\int_{Q_\sigma\setminus Q_\rho}|Dv|^q + \int_{Q_\sigma}|g(x)|^q + \left(\int_{Q_\sigma}|f|^s\right)^{\frac{q}{s}}\Bigg\}.\end{aligned} \tag{112}$$

Adding to both sides the quantity $c_2\int_{Q_\rho}|Dv|^q$, and applying Lemma 2.3, we have

$$\int_{Q_{R/2}}|Dv|^q \le c_3(q)\left\{\frac{C^q}{R^q}\int_{Q_R}|v-v_R|^q + \int_{Q_R}|g(x)|^q + \left(\int_{Q_R}|f|^s\right)^{\frac{q}{s}}\right\}, \tag{113}$$

where c_3 is a positive constant that depends only (in a continuous way) on q and N. Notice that when $q = (\frac{qN}{N+q})^*$, by the Sobolev inequality we get

$$\int_{Q_R}|v-v_R|^q \le c_s\left(\int_{Q_R}|Dv|^{\frac{qN}{N+q}}\right)^{\frac{N+q}{N}}, \tag{114}$$

where c_s is again the Sobolev constant. We point out that, since $q \le 2$,

$$c_s \le \left(\frac{(N-1)p}{N-p}\right)^q, \qquad p = \frac{qN}{N+q} \le 2,$$

(see for example [14]), we deduce that $c_s \le c_4 = c_4(N) = (\frac{(N-1)2}{N-2})^2$. Using (114) in (113) and dividing by R^N, we obtain

$$\begin{aligned}\fint_{Q_{R/2}}|Dv|^q \quad &\le 2^N c_3(q)\left\{C^q c_4\left(\fint_{Q_R}|Dv|^{\frac{qN}{N+q}}\right)^{\frac{N+q}{N}} + \fint_{Q_R}|g(x)|^q + \fint_{Q_R}|F|^{\frac{q}{s}}\right\} \\ &\le c_5(q)\left\{\left(\fint_{Q_R}|Dv|^{\frac{qN}{N+q}}\right)^{\frac{N+q}{N}} + \fint_{Q_R}|g(x)|^q + \fint_{Q_R}|F|^{\frac{q}{s}}\right\},\end{aligned} \tag{115}$$

where c_5 is a constant that depends only on N and in a continuous way on q, $F = \left(\int_\Omega |f|^s\right)^{1-s/q} |f|^{s^2/q}$ and $|F|^{q/s}$ belongs to $L^{r/s}(\Omega)$, $r/s > 1$. By Lemma 2.1 and (93) we deduce that

$$Dv \in L^{q+\epsilon}_{\mathrm{loc}}(\Omega), \quad a(x,u)Du \in L^{\nu}_{\mathrm{loc}}(\Omega), \tag{116}$$

where $\epsilon = \epsilon(q,N,\sigma,r)$ and $\nu = q + \min\{\epsilon,\sigma\}$. It remains to show that if $g(x)$ is more regular, that is, if it belongs to $L^{2+\sigma}(\Omega)$, $\sigma > 0$, then (33) also holds true. As a first step we will prove that Dv is in $L^2_{\mathrm{loc}}(\Omega)$ and hence by (93) $a(x,u)Du$ also belongs to $L^2_{\mathrm{loc}}(\Omega)$. To do this, it is sufficient to show the following equality:

$$A = \left\{m \in [q(r_0),2] : v \in W^{1,m}_{\mathrm{loc}}(\Omega)\right\} = [q(r_0),2].$$

This last result can easily be proved by following [5], thus showing that A is an open and closed subset of $[q(r_0), 2]$. Finally to prove (33), as now we know that $Dv \in L^2_{\text{loc}}(\Omega)$, it is possible to proceed as in the proof of the previous theorem.

Remark 3.4 We notice that if $r > \frac{N(2-\theta)}{N+2-N\theta}$, it is proved in [2] that any distributional solution found by approximation can be used as a test function in (6) and, in particular, $a(x,u)Du \in L^2(\Omega)$. So if we suppose a better regularity on $\int_0^u \frac{\partial a(x,s)}{\partial x} ds$, that is, that this term belongs to $L^{2+\sigma}(\Omega)$, $\sigma > 0$, then we can avoid the use of the Hodge decomposition technique and prove, by following the outline of the proof of Theorem 1.5, that the results of Theorem 1.6 hold for every solution obtained by approximation as in [2].

Proof of Theorem 1.7 Take as a test function in (6) $w = \varphi(v)$, where v is as in (75) and $\varphi(s) = T_1(s - T_n(s))$. This implies the following integral estimate:

$$\alpha \int_{n<|v|<n+1} |Dv|^2 \leq \int_{|v|>n} |f|.$$

Now in order to prove the quoted regularity results it is sufficient to apply the following result that can be easily proved by using the techniques in [6]:

Proposition 3.1 *Let w be a measurable function such that $T_k(w) \in W_0^{1,p}(\Omega)$, $1 < p < N$, for every positive k. If $F \in L^m(\Omega)$ and we have*

$$\int_{n<|w|\leq n+1} |\nabla w|^p \leq c_0 \int_{|w|>n} |F|,$$

where $1 < m < (p^*)' = \frac{Np}{Np-N+p}$, then $|Dw|^{p-1} \in L^{m^*}(\Omega)$.

4 Proofs of Theorems 1.8–1.9 (Minima of Functionals)

In this section we prove the results stated in Section 2 concerning the regularity of Du, where u is a minimum of the functional (37).

Proof of Theorem 1.8 By assumption (44) and the minimality of u we have

$$\beta_0\alpha_1 \int_\Omega \left(|Du| a_2^{\frac{1}{p}}(u)\right)^p - \int_\Omega fu \leq \alpha_2\beta_1 \int_\Omega \left(|Dw| a_2^{\frac{1}{p}}(w)\right)^p - \int_\Omega fw, \tag{117}$$

for every $w \in W_0^{1,p}(\Omega)$ with $J(w)$ finite. Let us define

$$\overline{H}(\tau) = \frac{1}{\gamma_1^{1/p}} \int_0^\tau a_2^{\frac{1}{p}}(s) ds, \tag{118}$$

where γ_1 is the positive constant that appears in the assumption (46). Since $\overline{H}(\tau)$ is strictly increasing and $\overline{H}(\mathbf{R}) = \mathbf{R}$ by (46), the inverse function $\overline{H}^{-1}(t)$ is well defined on $\mathbf{R}$ and the following estimate holds true:

$$\left|\overline{H}^{-1}(t)\right| \leq \left|G^{-1}(t)\right|, \quad \text{where} \quad G(\tau) = \int_0^\tau \frac{1}{(1+|s|)^\theta} ds. \tag{119}$$

The estimate (119) follows from the fact that $\left|\overline{H}(\tau)\right| \geq |G(\tau)|$. Notice that by (119) we deduce that

$$\left|\overline{H}^{-1}(t)\right| \leq [|t|(1-\theta)+1]^{\frac{1}{1-\theta}} - 1. \tag{120}$$

Moreover, using assumption (46) we have

$$\left|\left(\overline{H}^{-1}\right)'(t)\right| \le \left(1 + \left|\overline{H}^{-1}(t)\right|\right)^{\theta}. \tag{121}$$

Denote $\overline{H}(w) = \overline{w}$. Let us point out that $\overline{H}$ is bijective between

$$A = \left\{w \in W_0^{1,q}(\Omega) : J(w) < +\infty\right\} \tag{122}$$

and $W_0^{1,p}(\Omega)$ as we can easily see by differentiating $w = \overline{H}^{-1}(\overline{w})$ that

$$|Dw| = \left|\left(\overline{H}^{-1}\right)'(\overline{w})\right| |D\overline{w}|, \tag{123}$$

from which, using (121) and (120), we have

$$|Dw| \le \left(1 + [|\overline{w}|(1-\theta) + 1]^{\frac{1}{1-\theta}}\right)^{\theta} |D\overline{w}|. \tag{124}$$

Indeed, from (124) we see that $Dw \in L^q(\Omega)$ if $D\overline{w} \in L^p(\Omega)$, and then $w \in W_0^{1,q}(\Omega)$ since $\overline{H}^{-1}(0) = 0$. Moreover, $J(w) < +\infty$ since $D\overline{w} = \frac{1}{\gamma_1^{1/p}} a_2^{1/p}(w) Dw \in L^p(\Omega)$. We can rewrite (117) as follows:

$$\beta_0\alpha_1\gamma_1 \int_\Omega |Dv|^p - \int_\Omega f\overline{H}^{-1}(v) \le \max\{1, \beta_1\alpha_2\gamma_1\} \left[\int_\Omega |D\overline{w}|^p - \int_\Omega f\overline{H}^{-1}(\overline{w})\right], \tag{125}$$

where $v = \overline{H}(u)$. Hence v is a quasi-minimum in $W_0^{1,p}(\Omega)$ of

$$F(\overline{w}) = \int_\Omega |D\overline{w}|^p - \int_\Omega f\overline{H}^{-1}(\overline{w}), \tag{126}$$

and the regularity (49) follows from Theorem 6.7 of [15]. Now we show, using (49), that if r satisfies (50) then (51) also holds true. To do this we notice that, proceeding as in the proof of Theorem 1.4 in [8], it follows, by using (50), that there exists c_0 ($c_0 > 0$), depending only on the data, such that

$$\int_\Omega |Du|^p(1+|u|)^{(\lambda-\theta)p} dx \le c_0,$$

where $\lambda > \theta$ is defined by the formula $\lambda p = \frac{Nr[p(1-\theta)-1](r-1)}{(N-rp)r} - 1$. Let $\gamma = tp + (1-t)(p+\epsilon)$ where $t \in (0,1)$ is to be determined later. Using Young's inequality we obtain

$$\begin{aligned}\int_{B_R} |Du|^\gamma &= \int_{B_R} |Du|^{tp}(1+|u|)^{(\lambda-\theta)pt} \frac{|Du|^{(1-t)(p+\epsilon)}}{(1+|u|)^{(\lambda-\theta)pt}} \\ &\le \int_{B_R} |Du|^p(1+|u|)^{(\lambda-\theta)p} + \int_{B_R} \frac{|Du|^{p+\epsilon}}{(1+|u|)^{\frac{(\lambda-\theta)pt}{1-t}}} \le c_1,\end{aligned} \tag{127}$$

where $c_1 > c_0$ is a constant that depends only on the data and where we have set $\frac{(\lambda-\theta)pt}{1-t} = \theta(p+\epsilon)$, that is, $t = \frac{\theta(p+\epsilon)}{(\lambda-\theta)p+\theta(p+\epsilon)} \in (0,1)$. Hence (51) holds true with $\nu = \gamma - p$.

Proof of Theorem 1.9 Our first goal is to prove that the map $\underline{H} : A \to W_0^{1,p}(\Omega)$ defined by $\underline{w} = \underline{H}(w) = H(x, w(x))$, where H is as in (53) and A as in (122), is well defined and bijective. Indeed, using (55) we have

$$D\underline{w} = a^{1/p}(x,w)Dw + \int_0^w \frac{\partial\left(a^{1/p}(x,s)\right)}{\partial x} ds \in L^p(\Omega), \tag{128}$$

for every $w \in A$. Moreover, for every $\underline{z} \in W_0^{1,p}(\Omega)$ there exists $z \in A$, such that $\underline{z} = \underline{H}(z)$. As a matter of fact, if we define $z(x) = C(x, \underline{z}(x))$, then

$$Dz = \frac{\partial C}{\partial x}(x, \underline{z}) + \frac{\partial C}{\partial t}(x, \underline{z}) D\underline{z}, \quad \text{a.e. in} \quad \Omega.$$

Let us observe that using assumption (38) we have

$$\left| \frac{\partial C}{\partial t}(x, \underline{z}) D\underline{z} \right| = \frac{|D\underline{z}|}{\left| \frac{\partial H}{\partial \tau}(x, C(x, \underline{z})) \right|} = \frac{|D\underline{z}|}{a^{1/p}(x, C(x, \underline{z}))} \leq \beta_0^{1/p} \left(1 + |C(x, \underline{z})|\right)^{\theta} |D\underline{z}|.$$

when $|H(x, \tau)| \geq |G(\tau)|$, where G is as in (119), we have

$$|C(x, t)| \leq |G^{-1}(t)|, \tag{129}$$

and thus

$$\left| \frac{\partial C}{\partial t}(x, \underline{z}) D\underline{z} \right| \leq \beta_0^{1/p} \left(1 + [1 + |\underline{z}|(1 - \theta)]^{\frac{\theta}{1-\theta}} \right) |D\underline{z}|,$$

where the right-hand side belongs to $L^q(\Omega)$. We can now conclude, using (54), that Dz belongs to $L^q(\Omega)$.

Using the definition of $\underline{H}$ and the minimality of u we have

$$\int_\Omega F(x, \underline{u}, D\underline{u}) \leq \int_\Omega F(x, \underline{v}, D\underline{v}), \quad \forall \underline{v} \in W_0^{1,p}(\Omega), \tag{130}$$

where we have set

$$F(x, y, \xi) = \left| \xi - \int_0^{C(x,y)} \frac{\partial \left(a^{1/p}(x, s) \right)}{\partial x} ds \right|^p - fC(x, y),$$

and $\underline{u} = \underline{H}(x, u(x))$. In order to apply the quoted result in [15] it is sufficient to verify the growth conditions of F. We have, using (55), that

$$\begin{aligned} c_0|\xi|^p + c_1|C(x, y)|^{\gamma p} + \frac{|f|^\nu}{\nu} + \frac{|C(x, y)|^{\nu'}}{\nu'} + c_2 &\geq F(x, y, \xi) \\ \geq c_3|\xi|^p - c_4|C(x, y)|^{\gamma p} - \frac{|f|^\nu}{\nu} - \frac{|C(x, y)|^{\nu'}}{\nu'} - c_5, & \end{aligned}$$

where $\nu < r$ is to be chosen. Notice that, by (129), we have

$$|C(x, y)|^{\gamma p} \leq \left([|y|(1 - \theta) + 1]^{\frac{1}{1-\theta}} + 1 \right)^{\gamma p} \leq c_6|y|^{\frac{\gamma p}{1-\theta}} + c_7,$$

where $\frac{\gamma p}{1-\theta} < p^*$, by the assumption on γ. Besides, we have

$$|C(x, y)|^{\nu'} \leq \left([|y|(1 - \theta) + 1]^{\frac{1}{1-\theta}} + 1 \right)^{\nu'} \leq c_8|y|^{\frac{\nu'}{1-\theta}} + c_9.$$

We notice that $\frac{r'}{1-\theta} < p^*$, by (48), and thus we can choose $\nu < r$ such that $\frac{\nu'}{1-\theta} < p^*$. Hence it follows that

$$|C(x, y)|^{\gamma p} + |C(x, y)|^{\nu'} \leq c_{10}|y|^{\overline{\gamma}} + c_{11},$$

where $\overline{\gamma} = \max \left\{ \frac{\nu'}{1-\theta}, \frac{\gamma p}{1-\theta} \right\} < p^*$. Thus the hypothesis of the quoted theorem is satisfied and then the stated results follow.

It remains to prove that under the assumption (50) the regularity result (57) holds true but we omit such a proof as it is similar to the proof of (51) in Theorem 1.8.

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