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Leray–Lions operators with logarithmic growth

Lucio Boccardo, Luigi Orsina *

Dipartimento di Matematica, "Sapienza" Università di Roma, P.le A. Moro 5, 00185 Roma, Italy

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ABSTRACT

In this paper we prove existence, uniqueness, and regularity results for solutions of nonlinear elliptic equations in which the differential operator has logarithmic growth with respect to the gradient. The solution will belong to the Sobolev space $W_0^{1,1}(\Omega)$, to the Orlicz–Sobolev space generated by the function $t\log(1+|t|)$, but not to any Sobolev space $W_0^{1,p}(\Omega)$, with $p>1$.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 2$, and let $M : \Omega \to \mathbb{R}^{N^2}$ be a matrix-valued function such that there exist $0 < \alpha \leq \beta$ such that

$$M(x)\xi \cdot \xi \geq \alpha|\xi|^2, \qquad \big|M(x)\big| \leq \beta, \tag{1.1}$$

for almost every x in Ω, and for every ξ in $\mathbb{R}^N$. Moreover we assume that

$$f \text{ belongs to } L^N(\Omega). \tag{1.2}$$

In this paper we study the existence of distributional solutions belonging to $W_0^{1,1}(\Omega)$ (but not to $W_0^{1,p}(\Omega)$ for every $p > 1$) of the boundary value problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\log(1+|\nabla u|)}{|\nabla u|}M(x)\nabla u\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

* Corresponding author.
E-mail addresses: boccardo@mat.uniroma1.it (L. Boccardo), orsina@mat.uniroma1.it (L. Orsina).

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An existence result for solutions of (1.3) is a consequence of the results of [16,17] (see also [22]), concerning nonlinear elliptic problems in non-reflexive Banach spaces, but we will give here a direct proof of the existence of a solution u (see Theorem 3.1 in Section 3). We point out that one of the main difficulties lies in the non-reflexivity of the Orlicz–Sobolev space where we find the solution u. However, we overcome this difficulty using *a priori* estimates and the Dunford–Pettis theorem.

Furthermore, and thanks to the assumption (1.2), we will prove the existence of a solution belonging to $L^\infty(\Omega)$; to this aim, note that the assumption for the existence of a bounded solution for a p-laplacian ($p > 1$) type elliptic equation is that f belongs to $L^m(\Omega)$, with $m > \frac{N}{p}$; in our case, the growth of the operator is in between the case $p = 1$ and the case $p > 1$, and we obtain bounded solutions for data in $L^N(\Omega)$, i.e., under a stronger assumption than f in $L^m(\Omega)$, $m > \frac{N}{p}$, for some $p > 1$, but weaker than the assumption f in $L^m(\Omega)$, $m > N$ which is the one which corresponds to $p = 1$.

Note that if M is the identity matrix, our problem is variational, since it can be seen as the Euler–Lagrange equation for the functional

$$J(v) = \int_\Omega \big[\big(1 + |\nabla v|\big)\log\big(1 + |\nabla v|\big) - |\nabla v|\big] - \int_\Omega fv. \tag{1.4}$$

We recall that variational integrals of nearly linear growth (but different from the one in (1.4)) were introduced in [13] for the study of non-Newtonian fluids of Prandtl–Eyring type, and studied in [18,2,12,21,24,6,14,15]. In particular, in [18] regularity results for the gradients of minima are proved (with respect to the regularity of the data).

The plan of the paper is as follows: in the next section we will recall some results on Orlicz, and Orlicz–Sobolev spaces, while the main result will be proved in Section 3. In Section 4 we will prove an existence and uniqueness results for solutions of (1.3) if f belongs to $L^1(\Omega)$.

2. Some results on Orlicz spaces

Let $A : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex function, and let $L^A(\Omega)$ be the space of measurable functions u on Ω such that there exists $L > 0$ such that

$$\int_\Omega A\left(\frac{|u|}{L}\right) < +\infty.$$

In $L^A(\Omega)$ we can define the Luxemburg norm given by

$$\|u\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|u|}{\lambda}\right) \le 1\right\},$$

which makes $L^A(\Omega)$ a Banach space. Given a convex function A, we denote by $\tilde{A}$ its convex conjugate function, and we recall that

$$st \le A(s) + \tilde{A}(t), \quad \forall s, t \in \mathbb{R}^+, \tag{2.1}$$

and that, if

$$A(s) = \int_0^s a(t)\,dt,$$

then

$$\tilde{A}(s) = \int_0^s a^{-1}(t)\,dt.$$

We also recall that $L^{\tilde{A}}(\Omega)$ is the dual space of $L^A(\Omega)$ and that inequality (2.1) and the definition of norm in $L^A(\Omega)$ and $L^{\tilde{A}}(\Omega)$ imply

$$\begin{aligned}\int_\Omega \frac{u(x)}{\|u\|_{L^A(\Omega)}} \frac{v(x)}{\|v\|_{L^{\tilde{A}}(\Omega)}} &\le \int_\Omega A\left(\frac{u(x)}{\|u\|_{L^A(\Omega)}}\right) + \int_\Omega \tilde{A}\left(\frac{v(x)}{\|v\|_{L^{\tilde{A}}(\Omega)}}\right) \\ &\le 1 + 1,\end{aligned}$$

that is

$$\left|\int_\Omega uv\right| \le 2\|u\|_{L^A(\Omega)}\|v\|_{L^{\tilde{A}}(\Omega)}, \quad \forall u \in L^A(\Omega),\ \forall v \in L^{\tilde{A}}(\Omega). \tag{2.2}$$

For example, if $A(s) = \frac{s^p}{p}$, then $\tilde{A}(s) = \frac{s^{p'}}{p'}$, with $p' = \frac{p}{p-1}$, and $L^A(\Omega)$ and $L^{\tilde{A}}(\Omega)$ are the "standard" Lebesgue spaces $L^p(\Omega)$ and its dual $L^{p'}(\Omega)$. If, instead, $A(s) = \mathrm{e}^s - s - 1$, then, since

$$A(s) = \int_0^s a(t)\,dt,$$

with $a(s) = \mathrm{e}^s - 1$, we have $a^{-1}(t) = \log(1 + t)$, so that

$$\tilde{A}(s) = \int_0^s \log(1 + t)\,dt = (1 + s)\log(1 + s) - s.$$

Thus, by (2.1), we have

$$st \le \mathrm{e}^s - s - 1 + (1 + t)\log(1 + t) - t, \quad \forall s, t \in \mathbb{R}^+,$$

which then implies, dropping negative terms,

$$st \le \mathrm{e}^s - 1 + t\log(1 + t), \quad \forall s, t \in \mathbb{R}^+. \tag{2.3}$$

If u belongs to $L^A(\Omega)$, one cannot in general control the norm of u with the integral of $A(u)$. One has however the estimate

$$\|u\|_{L^A(\Omega)} \le \max\left\{1, \int_\Omega A(u)\right\} = \max\left\{1, \left\|A(u)\right\|_{L^1(\Omega)}\right\}. \tag{2.4}$$

If A satisfies the so-called Δ_2 condition, i.e., if there exist $s_0 \ge 0$ and $C > 0$ such that

$$A(2s) \le CA(s), \quad \forall s \ge s_0,$$

then $L^\infty(\Omega)$ is dense in $L^A(\Omega)$. Once one has defined the Orlicz space $L^A(\Omega)$, one can define the Orlicz–Sobolev space $W^{1,A}(\Omega)$ as the space of $L^A(\Omega)$ functions whose distributional derivatives belong to $L^A(\Omega)$, and $W_0^{1,A}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,A}$-norm. As in the "standard" Lebesgue–Sobolev case, $W_0^{1,A}(\Omega)$ is continuously embedded in $L^B(\Omega)$ (see [10]), with B given as follows: first we define

$$H(s) = \left[\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{1}{N-1}} dt\right]^{\frac{N-1}{N}},$$

and then define $B(s) = A(H^{-1}(s))$. The embedding between norms becomes, once written as integrals,

$$\int_\Omega B\left(\frac{|u|}{[M\int_\Omega A(|\nabla u|)]^{\frac{1}{N}}}\right) \le \int_\Omega A(|\nabla u|), \quad \forall u \in W_0^{1,A}(\Omega), \tag{2.5}$$

for some $M > 0$ independent of u. If $A(s) = s\log(1+s)$, then it can be proved (see [10]) that one can choose $B(s) = [s\log(1+s)]^{\frac{N}{N-1}}$ (see also [11]).

In the following, we will use the continuous function $\mathcal{L} : \mathbb{R}^N \to \mathbb{R}^N$, defined by

$$\mathcal{L}(\xi) = \frac{\log(1+|\xi|)}{|\xi|}\xi, \quad \forall \xi \in \mathbb{R}^N,$$

and the convex, Δ_2 function $A : \mathbb{R}^N \to \mathbb{R}$ defined by

$$A(\xi) = \mathcal{L}(\xi)\cdot\xi = |\xi|\log\bigl(1+|\xi|\bigr), \quad \forall \xi \in \mathbb{R}^N.$$

Note that, since (2.3) holds for A, then, choosing $\tilde{A}(t) = \mathrm{e}^t - 1$, it follows that (2.2) holds for A and $\tilde{A}$.

3. Existence

In this section we prove the main result of this paper.

Theorem 3.1. *Assume (1.1) and (1.2). Then there exists a unique solution of (1.3), that is, a function $u \in W_0^{1,A} \cap L^\infty(\Omega)$ such that*

$$\int_\Omega \frac{\log(1+|\nabla u|)}{|\nabla u|} M(x)\nabla u \cdot \nabla v = \int_\Omega f v, \quad \forall v \in W_0^{1,A}(\Omega). \tag{3.1}$$

Proof. Let u_ε be a weak solution of the Dirichlet problem

$$\begin{cases} \varepsilon L(u_\varepsilon) - \operatorname{div}\left(\dfrac{\log(1+|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|} M(x)\nabla u_\varepsilon\right) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $L = -\Delta$. The existence of u_ε in $W_0^{1,2}(\Omega)$ is a consequence of the Leray–Lions theory (see [20,9,7,19]); furthermore, since f belongs to $L^N(\Omega)$, and $N > \frac{N}{2}$, every u_ε belongs to $L^\infty(\Omega)$ by the results of Stampacchia (see [23]), although not uniformly in ε.

Step 1: Existence and uniqueness in $W_0^{1,A}(\Omega)$.

First of all, using u_ε as a test function in (3.2), we have, dropping nonnegative terms, and using (1.1),

$$\alpha\int_\Omega |\nabla u_\varepsilon|\log\bigl(1+|\nabla u_\varepsilon|\bigr) \le \|f\|_{L^N(\Omega)}\|u_\varepsilon\|_{L^{1^*}(\Omega)} \le \mathcal{S}\|f\|_{L^N(\Omega)}\int_\Omega |\nabla u_\varepsilon|,$$

where $\mathcal{S}$ is the Sobolev constant for the embedding of $W_0^{1,1}(\Omega)$ in $L^{1^*}(\Omega)$. Therefore, if $S > 0$ we have

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon| &\le S|\Omega| + \frac{1}{\log(1+S)} \int_{\{|\nabla u_\varepsilon| \ge S\}} |\nabla u_\varepsilon| \log\bigl(1+|\nabla u_\varepsilon|\bigr) \\ &\le S|\Omega| + \frac{\mathcal{S}\|f\|_{L^N(\Omega)}}{\alpha \log(1+S)} \int_\Omega |\nabla u_\varepsilon|,\end{aligned}$$

which implies, choosing S large enough, that there exists $R > 0$ such that

$$\int_\Omega |\nabla u_\varepsilon| \le R, \qquad \int_\Omega |\nabla u_\varepsilon| \log\bigl(1+|\nabla u_\varepsilon|\bigr) \le R. \tag{3.3}$$

By the compactness of the Sobolev embedding, there exists a subsequence (not relabelled) such that

$$u_\varepsilon \text{ converges in } L^1(\Omega) \text{ and a.e. to a function } u. \tag{3.4}$$

Let now $E \subset \Omega$ be measurable, and $S > 0$; we have

$$\begin{aligned}\int_E \left|\frac{\partial u_\varepsilon}{\partial x_i}\right| &\le \int_E |\nabla u_\varepsilon| \\ &\le S|E| + \frac{1}{\log(1+S)} \int_{\{|\nabla u_\varepsilon| \ge S\}} |\nabla u_\varepsilon| \log\bigl(1+|\nabla u_\varepsilon|\bigr) \\ &\le S|E| + \frac{R}{\log(1+S)}.\end{aligned} \tag{3.5}$$

Choosing first S in such a way that the second term of the right hand side is small, and then $|E|$ so that the first term of the right hand side is small as well, we have that $\{\frac{\partial u_\varepsilon}{\partial x_i}\}$ is equiintegrable. Thus, by the Dunford–Pettis theorem, and up to subsequences (not relabelled), there exists Y_i in $L^1(\Omega)$ such that $\frac{\partial u_\varepsilon}{\partial x_i}$ weakly converges to Y_i in $L^1(\Omega)$.

Now we want to prove that

$$u_\varepsilon \text{ weakly converges to } u \text{ in } W_0^{1,1}(\Omega), \tag{3.6}$$

and we follow [3,5]. Since $\frac{\partial u_\varepsilon}{\partial x_i}$ is the distributional partial derivative of u_ε, we have

$$\int_\Omega \frac{\partial u_\varepsilon}{\partial x_i}\varphi = -\int_\Omega u_\varepsilon \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Passing to the limit, using that $\frac{\partial u_\varepsilon}{\partial x_i}$ weakly converges to Y_i in $L^1(\Omega)$, and that u_ε strongly converges to u in $L^1(\Omega)$, we obtain

$$\int_\Omega Y_i\varphi = -\int_\Omega u \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega),$$

which implies that $Y_i = \frac{\partial u}{\partial x_i}$, so that u belongs to $W_0^{1,1}(\Omega)$.

The convergence (3.6) and the second of (3.3) imply (see [8])

$$\int_\Omega |\nabla u| \log\bigl(1+|\nabla u|\bigr) \le R, \tag{3.7}$$

which then implies that u belongs to $W_0^{1,A}(\Omega)$, as desired.

Since by (2.3) we have

$$|\nabla u| \log\bigl(1+|\nabla u_\varepsilon|\bigr) \le |\nabla u| \log\bigl(1+|\nabla u|\bigr) + |\nabla u_\varepsilon|,$$

integrating on Ω, and using (3.3) and (3.7), we get

$$\int_\Omega |\nabla u| \log\bigl(1+|\nabla u_\varepsilon|\bigr) \le 2R. \tag{3.8}$$

Note that it is not possible to use the function u as a test function in (3.2) since it does not belong to $W_0^{1,2}(\Omega)$. However, thanks to the density of $C_0^\infty(\Omega)$ in both $W_0^{1,1}(\Omega)$ and $W_0^{1,A}(\Omega)$, for every $k>0$ there exists a $C_0^\infty(\Omega)$ function U_k such that

$$\|u-U_k\|_{W_0^{1,1}(\Omega)} \le \frac{1}{k}, \qquad \bigl\|\nabla(u-U_k)\bigr\|_{L^A(\Omega)} \le \frac{1}{k}.$$

Using $u_\varepsilon - U_k$ as a test function in (3.2), we have

$$\bigl\langle \varepsilon L(u_\varepsilon), u_\varepsilon - U_k\bigr\rangle + \int_\Omega M(x)\mathcal{L}(\nabla u_\varepsilon)\cdot\nabla(u_\varepsilon - U_k) = \int_\Omega f(u_\varepsilon - U_k).$$

Dropping the positive term $\varepsilon\langle L(u_\varepsilon), u_\varepsilon\rangle$, we obtain

$$\bigl\langle \varepsilon L(u_\varepsilon), -U_k\bigr\rangle + \int_\Omega M(x)\mathcal{L}(\nabla u_\varepsilon)\cdot\nabla(u_\varepsilon - U_k) \le \int_\Omega f(u_\varepsilon - U_k).$$

Since $M(x)\mathcal{L}(\nabla u)\cdot\nabla(u_\varepsilon - u)$ belongs to (and is bounded in) $L^1(\Omega)$ thanks to (3.8), we have

$$\begin{aligned}&\bigl\langle L(u_\varepsilon), -\varepsilon U_k\bigr\rangle + \int_\Omega M(x)\bigl[\mathcal{L}(\nabla u_\varepsilon) - \mathcal{L}(\nabla u)\bigr]\cdot\nabla(u_\varepsilon - u) + \int_\Omega M(x)\mathcal{L}(\nabla u_\varepsilon)\cdot\nabla(u - U_k)\\ &\quad\le \int_\Omega f(u_\varepsilon - u) + \int_\Omega f(u - U_k) - \int_\Omega M(x)\mathcal{L}(\nabla u)\cdot\nabla(u_\varepsilon - u).\end{aligned} \tag{3.9}$$

We have

$$\bigl\langle L(u_\varepsilon), -\varepsilon U_k\bigr\rangle = -\varepsilon\int_\Omega \nabla u_\varepsilon\cdot\nabla U_k,$$

and so, since the sequence $\{u_\varepsilon\}$ is bounded in $W_0^{1,1}(\Omega)$, and U_k belongs to $C_0^\infty(\Omega)$, we have

$$\lim_{\varepsilon\to 0}\bigl\langle L(u_\varepsilon), -\varepsilon U_k\bigr\rangle = 0. \tag{3.10}$$

Furthermore,

$$\lim_{\varepsilon\to 0}\int_{\Omega} f(u_\varepsilon - u) = 0, \tag{3.11}$$

and

$$\left|\int_{\Omega} f(u - U_k)\right| \le \frac{C_1}{k}\|f\|_{L^N(\Omega)}. \tag{3.12}$$

Furthermore, the use of (2.2) with $A(s) = s\log(1+s)$ and $\tilde{A}(s) = \mathrm{e}^s - 1$ (note that (2.2) holds true since $st \le A(s) + \tilde{A}(t)$ by (2.3)) yields

$$\begin{aligned}\left|\int_{\Omega} M(x)\mathcal{L}(\nabla u_\varepsilon)\cdot\nabla(u - U_k)\right| &\le 2\beta\big\|\nabla(u - U_k)\big\|_{L^A(\Omega)}\big\|\log\big(1 + |\nabla u_\varepsilon|\big)\big\|_{L^{\tilde{A}}(\Omega)} \\ &\le 2\beta\frac{1}{k}\max\big(1, \|u_\varepsilon\|_{W_0^{1,1}(\Omega)}\big) \le \frac{C_2}{k}. \end{aligned}\tag{3.13}$$

We also have

$$\begin{aligned}\int_{\Omega} M(x)\mathcal{L}(\nabla u)\cdot\nabla(u_\varepsilon - u) &= \int_{\Omega} M(x)\big[\mathcal{L}(\nabla u) - \mathcal{L}(\nabla U_k)\big]\cdot\nabla(u_\varepsilon - u) \\ &\quad + \int_{\Omega} M(x)\mathcal{L}(\nabla U_k)\cdot\nabla(u_\varepsilon - u) = I_1 + I_2\end{aligned}$$

and, thanks to Lemma A.1 in Appendix A, and to (2.2),

$$\begin{aligned}|I_1| &\le 2\beta\int_{\Omega}\log\big(1 + \big|\nabla(u - U_k)\big|\big)\big|\nabla(u_\varepsilon - u)\big| \\ &\le 4\beta\big\|\log\big(1 + \big|\nabla(u - U_k)\big|\big)\big\|_{L^{\tilde{A}}(\Omega)}\big\|\nabla(u_\varepsilon - u)\big\|_{L^A(\Omega)} \le \frac{C_3}{k},\end{aligned}$$

while

$$\lim_{\varepsilon\to 0} I_2 = \lim_{\varepsilon\to 0}\int_{\Omega} M(x)\mathcal{L}(\nabla U_k)\cdot\nabla(u_\varepsilon - u) = 0,$$

since $M(x)\mathcal{L}(\nabla U_k)$ is fixed in $L^\infty(\Omega)$, and $\nabla(u_\varepsilon - u)$ weakly converges to zero in $(L^1(\Omega))^N$. Thus,

$$\limsup_{\varepsilon\to 0}\int_{\Omega} M(x)\mathcal{L}(\nabla u)\cdot\nabla(u_\varepsilon - u) \le \frac{C_3}{k}. \tag{3.14}$$

Using (3.10)–(3.14), we have proved that

$$\limsup_{\varepsilon\to 0}\int_{\Omega} M(x)\big[\mathcal{L}(\nabla u_\varepsilon) - \mathcal{L}(\nabla u)\big]\cdot\nabla(u_\varepsilon - u) \le \frac{C_4}{k},$$

which, together with the monotonicity of $\mathcal{L}(\xi)$, implies

$$\lim_{\varepsilon\to 0}\int_\Omega M(x)\big[\mathcal{L}(\nabla u_\varepsilon)-\mathcal{L}(\nabla u)\big]\cdot\nabla(u_\varepsilon-u)=0. \tag{3.15}$$

Adapting the proof of a result by Leray and Lions (see [20]), from (3.15) it follows that

$$\nabla u_\varepsilon(x)\to\nabla u(x) \quad \text{a.e. in } \Omega. \tag{3.16}$$

This almost everywhere convergence and (3.5) allow to use the Vitali theorem to prove that

$$\nabla u_\varepsilon\to\nabla u \quad \text{strongly in } \big(L^1(\Omega)\big)^N, \tag{3.17}$$

so that, thanks to the growth of $\mathcal{L}(\xi)$, we have

$$\mathcal{L}(\nabla u_\varepsilon)\to\mathcal{L}(\nabla u) \quad \text{strongly in } \big(L^r(\Omega)\big)^N, \text{ for every } r\geq 1. \tag{3.18}$$

Thus, choosing φ in $W_0^{1,\infty}(\Omega)$ as a test function in (3.2), we obtain

$$\varepsilon\big\langle L(u_\varepsilon),\varphi\big\rangle+\int_\Omega M(x)\mathcal{L}(\nabla u_\varepsilon)\cdot\nabla\varphi=\int_\Omega f\varphi.$$

Passing to the limit as ε tends to zero, which is possible by the above results, yields

$$\int_\Omega M(x)\mathcal{L}(\nabla u)\cdot\nabla\varphi=\int_\Omega f\varphi, \tag{3.19}$$

for every φ in $W_0^{1,\infty}(\Omega)$. Since u belongs to $W_0^{1,1}(\Omega)$, if $\tilde{A}=\mathrm{e}^t-1$, we have

$$\int_\Omega \tilde{A}\big(|\mathcal{L}(\nabla u)|\big)=\int_\Omega|\nabla u|<+\infty,$$

so that $|\mathcal{L}(\nabla u)|$ belongs to $L^{\tilde{A}}(\Omega)$. Therefore, since $W_0^{1,\infty}(\Omega)$ is dense in $W_0^{1,A}(\Omega)$, and since $W_0^{1,A}(\Omega)$ functions also belong to $L^{\frac{N}{N-1}}(\Omega)$, from (3.19) it follows that (3.1) holds true.

Once (3.1) holds true, if u and v are two solutions in $W_0^{1,A}(\Omega)$ of (1.3), uniqueness follows from the monotonicity of $\mathcal{L}(\xi)$.

Step 2: $L^\infty(\Omega)$ estimates.

To prove that u belongs to $L^\infty(\Omega)$, let $G_k(s)=(|s|-k)_+\operatorname{sgn}(s)$ for $k>0$ and s in $\mathbb{R}$. Since $G_k(u_\varepsilon)$ belongs to $W_0^{1,2}(\Omega)$, we can choose it as a test function in (3.2) to obtain

$$\alpha\int_\Omega A\big(|\nabla G_k(u_\varepsilon)|\big)\leq\int_\Omega fG_k(u_\varepsilon)\leq\int_\Omega|f|\big|G_k(u_\varepsilon)\big|. \tag{3.20}$$

Recalling the Orlicz–Sobolev embedding (2.5) for $W_0^{1,A}(\Omega)$, we have

$$\int_\Omega B\bigg(\frac{|G_k(u_\varepsilon)|}{[M\int_\Omega A(|\nabla G_k(u_\varepsilon)|)]^{\frac{1}{N}}}\bigg)\leq\int_\Omega A\big(|\nabla G_k(u_\varepsilon)|\big), \tag{3.21}$$

where $B(s) = [s \log(1+s)]^{1^*}$. If we define

$$\lambda^{-N} = \frac{M}{\alpha} \|f\|_{L^N(\Omega)} \|G_k(u_\varepsilon)\|_{L^{1^*}(\Omega)},$$

by the Hölder inequality and (3.20) we have,

$$M \int_\Omega A(|\nabla G_k(u_\varepsilon)|) \le \frac{M}{\alpha} \int_\Omega |f||G_k(u_\varepsilon)| \le \lambda^{-N},$$

so that (3.21) becomes, thanks to the fact that B is increasing,

$$\int_\Omega B(\lambda |G_k(u_\varepsilon)|) \le \int_\Omega A(|\nabla G_k(u_\varepsilon)|) \le \frac{1}{\alpha} \int_\Omega |f||G_k(u_\varepsilon)| \le \frac{\lambda^{-N}}{M}, \tag{3.22}$$

that is,

$$\int_\Omega \left[\lambda |G_k(u_\varepsilon)| \log(1 + \lambda |G_k(u_\varepsilon)|)\right]^{1^*} \le \frac{\lambda^{-N}}{M},$$

which implies

$$\int_\Omega \left[|G_k(u_\varepsilon)| \log(1 + \lambda |G_k(u_\varepsilon)|)\right]^{1^*} \le \frac{1}{M} \left[\lambda^{-N}\right]^{1^*}.$$

Recalling the definition of λ, we obtain

$$\int_\Omega \left[|G_k(u_\varepsilon)| \log(1 + \lambda |G_k(u_\varepsilon)|)\right]^{1^*} \le \frac{M^{\frac{1}{N-1}}}{\alpha^{1^*}} \|f\|_{L^N(\Omega)}^{1^*} \int_\Omega |G_k(u_\varepsilon)|^{1^*}.$$

Let now $\delta > 0$, and let $s_0 > 0$ be such that $\delta \log(1 + \lambda s_0) = 1$; i.e., $s_0 = (\mathrm{e}^{1/\delta} - 1)\lambda^{-1}$. Then

$$s = \delta s \log(1 + \lambda s_0) \le \delta s \log(1 + \lambda s_0) + s_0 \le \delta s \log(1 + \lambda s) + s_0,$$

which can be rewritten, recalling the expression for s_0, as

$$s \le \delta s \log(1 + \lambda s) + (\mathrm{e}^{1/\delta} - 1)\lambda^{-1}.$$

Raising to the power 1^*, we then have

$$s^{1^*} \le C\delta \left[s \log(1 + \lambda s)\right]^{1^*} + C_\delta \lambda^{-1^*}.$$

Therefore, we have

$$\int_\Omega |G_k(u_\varepsilon)|^{1^*} \le C\delta \int_\Omega \left[|G_k(u_\varepsilon)| \log(1 + \lambda |G_k(u_\varepsilon)|)\right]^{1^*} + C_\delta |A_k| \lambda^{-1^*},$$

since the integral is on the set $A_k = \{|u_\varepsilon| \ge k\}$. Thus, choosing $\delta < 1$ such that $C\delta M^{\frac{1}{N-1}} \|f\|_{L^N(\Omega)}^{1^*} \le \frac{\alpha^{1^*}}{2}$, we have

$$\int_\Omega \left[|G_k(u_\varepsilon)| \log(1 + \lambda |G_k(u_\varepsilon)|)\right]^{1^*} \le C_1 |A_k| \lambda^{-1^*}.$$

We now have

$$\int_\Omega \big[\big|G_k(u_\varepsilon)\big|\log\big(1+\lambda\big|G_k(u_\varepsilon)\big|\big)\big]^{1^*} \geq \big[\log(2)\big]^{1^*} \int_{\{\lambda|G_k(u_\varepsilon)|\geq 1\}} \big|G_k(u_\varepsilon)\big|^{1^*},$$

and so

$$\int_{\{\lambda|G_k(u_\varepsilon)|\geq 1\}} \big|G_k(u_\varepsilon)\big|^{1^*} \leq C_2|A_k|\lambda^{-1^*}. \tag{3.23}$$

On the other hand,

$$\int_{\{\lambda|G_k(u_\varepsilon)|\leq 1\}} \big|G_k(u_\varepsilon)\big|^{1^*} \leq \int_{\{\lambda|G_k(u_\varepsilon)|\leq 1\}\cap A_k} \lambda^{-1^*} \leq |A_k|\lambda^{-1^*},$$

so that, summing with (3.23), and recalling the definition of λ,

$$\int_\Omega \big|G_k(u_\varepsilon)\big|^{1^*} \leq C_3|A_k|\lambda^{-1^*} = C_4|A_k|\bigg[\int_\Omega \big|G_k(u_\varepsilon)\big|^{1^*}\bigg]^{\frac{1}{N}}.$$

Therefore,

$$\int_\Omega \big|G_k(u_\varepsilon)\big|^{1^*} \leq C_5|A_k|^{1^*}.$$

Choosing $h > k$, we arrive at

$$|A_h| \leq \frac{C_5}{(h-k)^{1^*}}|A_k|^{1^*},$$

which then implies, since $1^* > 1$, that there exists $k_0 = k_0(\Omega, N, f)$, such that $|A_{k_0}| = 0$ (see [23]). Hence, u_ε belongs to $L^\infty(\Omega)$, uniformly with respect to ε, and so u is in $L^\infty(\Omega)$ as well. □

4. $L^1(\Omega)$ data

In this section we prove existence and uniqueness of entropy solutions for (1.3) if f only belongs to $L^1(\Omega)$. First of all, following [1], we give the definition of entropy solution of (1.3).

Definition 4.1. A measurable function u such that $T_k(u)$ belongs to $W_0^{1,A}(\Omega)$ for every $k > 0$ is an entropy solution of (1.3) if

$$\int_\Omega \frac{\log(1+|\nabla u|)}{|\nabla u|} M(x)\nabla u \cdot \nabla T_k(u-\varphi) \leq \int_\Omega f T_k(u-\varphi), \tag{4.1}$$

for every φ in $W_0^{1,A}(\Omega) \cap L^\infty(\Omega)$.

Theorem 4.2. *Let f be a function in $L^1(\Omega)$. Then there exists a unique entropy solution u of (1.3). Furthermore,* $\log(1+|\nabla u|)$ *belongs to $L^q(\Omega)$, for every $q < \frac{N}{N-1}$.*

Proof. Let $n \in \mathbb{N}$, and consider the solution u_n (given by Theorem 3.1) of the following problem:

$$\begin{cases} -\operatorname{div}\left(\dfrac{\log(1+|\nabla u_n|)}{|\nabla u_n|} M(x)\nabla u_n\right) = T_n(f) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Choosing $v = T_k(u_n)$ as a test function in the formulation (3.1) of Eq. (4.2), we obtain

$$\int_\Omega A\big(\big|\nabla T_k(u_n)\big|\big) \le \int_\Omega |f|\big|T_k(u_n)\big| \le k\|f\|_{L^1(\Omega)}. \tag{4.3}$$

By the Orlicz–Sobolev inequality (3.21) with $B(s) = [s\log(1+s)]^{1^*}$, we have

$$\int_\Omega \left[\frac{|T_k(u_n)|}{[Mk\|f\|_{L^1(\Omega)}]^{\frac{1}{N}}} \log\left(1 + \frac{|T_k(u_n)|}{[Mk\|f\|_{L^1(\Omega)}]^{\frac{1}{N}}}\right)\right]^{1^*} \le k\|f\|_{L^1(\Omega)},$$

which implies

$$\int_\Omega \left[\big|T_k(u_n)\big| \log\left(1 + \frac{|T_k(u_n)|}{[Mk\|f\|_{L^1(\Omega)}]^{\frac{1}{N}}}\right)\right]^{1^*} \le Mk^{1^*}\|f\|_{L^1(\Omega)}^{1^*}.$$

Since $|T_k(u_n)| = k$ on the set $A_k = \{|u_n| \ge k\}$, we have

$$\big[k\log\big(1 + \big(M\|f\|_{L^1(\Omega)}\big)^{\frac{1}{N}} k^{\frac{N-1}{N}}\big)\big]^{1^*} |A_k| \le Mk^{1^*}\|f\|_{L^1(\Omega)}^{1^*}.$$

Hence,

$$|A_k| \le \frac{M\|f\|_{L^1(\Omega)}^{1^*}}{[\log(1 + (M\|f\|_{L^1(\Omega)})^{\frac{1}{N}} k^{\frac{N-1}{N}})]^{1^*}}.$$

Moreover, from (4.3) it follows that, for every $\rho > 0$,

$$\int_{\{|\nabla u_n| \ge \rho\}} A\big(\big|\nabla T_k(u_n)\big|\big) \le k\|f\|_{L^1(\Omega)},$$

we deduce that

$$\big|\{|\nabla u_n| \ge \rho\}\big| \le \frac{M\|f\|_{L^1(\Omega)}^{1^*}}{[\log(1 + (M\|f\|_{L^1(\Omega)})^{\frac{1}{N}} k^{\frac{N-1}{N}})]^{1^*}} + \frac{k\|f\|_{L^1(\Omega)}}{\rho\log(1+\rho)}.$$

With the choice

$$k = \frac{\rho}{[\log(1+\rho)]^{\frac{1}{N-1}}},$$

we obtain (after some simple calculations)

$$\big|\{|\nabla u_n| \ge \rho\}\big| \le \frac{C}{[\log(1+\rho)]^{\frac{N}{N-1}}},$$

which implies that $\log(1 + |\nabla u_n|)$ is bounded in $L^q(\Omega)$, for every $q < \frac{N}{N-1}$.

Using a result proved in [4] for general monotone operators, we have that there exists a measurable function u, with $T_k(u)$ belonging to $W_0^{1,A}(\Omega)$ for every $k > 0$, such that both u_n almost everywhere converges to u, and ∇u_n almost everywhere converges to ∇u (for the definition of ∇u, see [1]). The almost everywhere convergence of ∇u_n and the boundedness of $\log(1 + |\nabla u_n|)$ in $L^q(\Omega)$ imply that $\log(1 + |\nabla u|)$ belongs to $L^q(\Omega)$, for every $q < \frac{N}{N-1}$, as desired.

If we define

$$\mathcal{Q}(x,\xi) = \frac{\log(1+|\xi|)}{|\xi|} M(x)\xi,$$

we have, if ψ belongs to $W_0^{1,\infty}(\Omega)$, and $k > 0$,

$$\int_\Omega \big[\mathcal{Q}(x,\nabla u_n) - \mathcal{Q}(x,\nabla \psi)\big] \cdot \nabla T_k(u_n - \psi) + \int_\Omega \mathcal{Q}(x,\nabla \psi) \cdot \nabla T_k(u_n - \psi) = \int_\Omega T_n(f) T_k(u_n - \varphi).$$

Using the monotonicity of $\mathcal{Q}$, the almost everywhere convergence of ∇u_n, and the Fatou lemma, we obtain

$$\int_\Omega \big[\mathcal{Q}(x,\nabla u) - \mathcal{Q}(x,\nabla \psi)\big] \cdot \nabla T_k(u - \psi) \le \liminf_{n\to+\infty} \int_\Omega \big[\mathcal{Q}(x,\nabla u_n) - \mathcal{Q}(x,\nabla \psi)\big] \cdot \nabla T_k(u_n - \psi),$$

while the Lebesgue theorem (and the almost everywhere convergence of u_n) implies

$$\lim_{n\to+\infty} \int_\Omega T_n(f) T_k(u_n - \psi) = \int_\Omega f T_k(u - \psi).$$

We observe now that $|\nabla T_k(u_n - \psi)|$ is bounded in $L^A(\Omega)$, and is almost everywhere convergent. Thus, by the Vitali theorem, $\nabla T_k(u_n - \psi)$ strongly converges in $(L^1(\Omega))^N$ to $\nabla T_k(u - \psi)$. Therefore, recalling that ψ is Lipschitz continuous, so that $|\mathcal{Q}(x,\nabla \psi)|$ is in $L^\infty(\Omega)$,

$$\lim_{n\to+\infty} \int_\Omega \mathcal{Q}(x,\nabla \psi) \cdot \nabla T_k(u_n - \psi) = \int_\Omega \mathcal{Q}(x,\nabla \psi) \cdot \nabla T_k(u - \psi).$$

Thus, putting together all the results, we obtain

$$\int_\Omega \mathcal{Q}(x,\nabla u) \cdot \nabla T_k(u - \psi) \le \int_\Omega f T_k(u - \psi), \tag{4.4}$$

for every ψ in $W_0^{1,\infty}(\Omega)$. Recalling that $W_0^{1,\infty}(\Omega)$ is dense in $W_0^{1,A}(\Omega) \cap L^\infty(\Omega)$, for every fixed φ in $W_0^{1,A}(\Omega) \cap L^\infty(\Omega)$, there exists a sequence ψ_n in $W_0^{1,\infty}(\Omega)$ such that

$$\|\psi_n - \varphi\|_{W_0^{1,A}(\Omega)} \le \frac{1}{n}, \qquad \|\psi_n\|_{L^\infty(\Omega)} \le \|\varphi\|_{L^\infty(\Omega)} + 1.$$

Therefore, choosing $\psi = \psi_n$ in (4.4), we have

$$\int_\Omega \mathcal{Q}(x,\nabla u) \cdot \nabla T_k(u - \psi_n) \le \int_\Omega f T_k(u - \psi_n).$$

Using the convergence of ψ_n to φ, it is easy to pass to the limit in the right hand side. We remark now that the integral in the left hand side is on the set $\{|u - \psi_n| \le k\}$, which is contained in the set $\{|u| \le k + \|\psi_n\|_{L^\infty(\Omega)}\}$;

thanks to the assumptions on ψ_n, and setting $M = k + \|\psi\|_{L^\infty(\Omega)} + 1$, this set is contained in the set $\{|u| \le M\}$. Thus, the left hand side is equal to

$$\int_{\{|u-\psi_n|\le k\}} \mathcal{Q}\big(x, \nabla T_M(u)\big) \cdot \big(\nabla T_M(u) - \nabla\psi_n\big).$$

Using the Fatou lemma, we have

$$\int_{\{|u-\psi|\le k\}} \mathcal{Q}\big(x, \nabla T_M(u)\big) \cdot \nabla T_M(u) \le \liminf_{n\to+\infty} \int_{\{|u-\psi_n|\le k\}} \mathcal{Q}\big(x, \nabla T_M(u)\big) \cdot \nabla T_M(u).$$

As for the second term, we have

$$\left| \int_{\{|u-\psi_n|\le k\}} \mathcal{Q}\big(x, \nabla T_M(u)\big) \cdot (\nabla\psi_n - \nabla\varphi) \right| \le 2\beta \big\| \mathcal{Q}\big(x, \nabla T_M(u)\big) \big\|_{L^{\tilde{A}}(\Omega)} \|\psi_n - \varphi\|_{W_0^{1,A}(\Omega)}.$$

Since

$$\big\| \mathcal{Q}\big(x, \nabla T_M(u)\big) \big\|_{L^{\tilde{A}}(\Omega)} \le \max\left(1, \int_\Omega \big|\nabla T_M(u)\big|\right) \le C,$$

we have

$$\lim_{n\to+\infty} \int_{\{|u-\psi_n|\le k\}} \mathcal{Q}\big(x, \nabla T_M(u)\big) \cdot \nabla\psi_n = \int_{\{|u-\psi_n|\le k\}} \mathcal{Q}\big(x, \nabla T_M(u)\big) \cdot \nabla\varphi.$$

Summing up, and recalling the definition of $\mathcal{Q}$, we have

$$\int_\Omega \frac{\log(1 + |\nabla u|)}{|\nabla u|} M(x)\nabla u \cdot \nabla T_k(u - \varphi) \le \int_\Omega f T_k(u - \varphi),$$

for every φ in $W_0^{1,A}(\Omega) \cap L^\infty(\Omega)$, i.e., u is an entropy solution of (1.3).

As for uniqueness, the choice $\varphi = T_h(u)$ in (4.1) gives

$$\int_{\{h\le|u|<h+k\}} \frac{\log(1 + |\nabla u|)}{|\nabla u|} M(x)\nabla u \cdot \nabla u \le \int_\Omega f T_k\big[u - T_h(u)\big],$$

which implies

$$\lim_{h\to\infty} \int_{\{h\le|u|<h+k\}} \frac{\log(1 + |\nabla u|)}{|\nabla u|} M(x)\nabla u \cdot \nabla u = 0, \quad \forall k > 0. \tag{4.5}$$

Thanks to the monotonicity of the differential operator and (4.5), it is possible to repeat the proof of [1] to prove the uniqueness of the entropy solution u. $\square$

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Appendix A

Lemma A.1. *For every s, t in $\mathbb{R}$ we have*

$$\left| s\frac{\log(1+|s|)}{|s|} - t\frac{\log(1+|t|)}{|t|} \right| \leq 2\log\bigl(1+|s-t|\bigr) \tag{A.1}$$

Proof. By symmetry, it is enough to prove the result for $s \geq t$. If $s \geq t \geq 0$, we have to prove that

$$\log(1+s) - \log(1+t) \leq 2\log(1+s-t).$$

If we define

$$g(s) = 2\log(1+s-t) - \log(1+s) + \log(1+t),$$

we have

$$g'(s) = \frac{2}{1+s-t} - \frac{1}{1+s} \geq 0,$$

so that $g(s) \geq g(t) = 0$, and the result is proved. If $t \leq 0 \leq s$, since $|s-t| = s-t = s+|t|$, we have to prove that

$$\log(1+s) + \log\bigl(1+|t|\bigr) \leq 2\log\bigl(1+s+|t|\bigr),$$

which is equivalent to

$$(1+s)\bigl(1+|t|\bigr) \leq \bigl(1+s+|t|\bigr)^2.$$

Expanding the expressions, we have to prove that

$$1+s+|t|+s|t| \leq 1+s^2+t^2+2s+2|t|+2s|t|,$$

which is clearly true. Finally, if $t \leq s \leq 0$, since $|s-t| = s-t = |t|-|s|$, we have to prove that

$$\log\bigl(1+|t|\bigr) - \log\bigl(1+|s|\bigr) \leq 2\log\bigl(1+|t|-|s|\bigr),$$

which is true by the first step since $|t| \geq |s| \geq 0$. $\square$

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