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Orlicz-Sobolev Spaces and Imbedding Theorems

THOMAS K. DONALDSON

Department of Mathematics, Australian National University, Canberra, A.C.T., Australia

AND

NEIL S. TRUDINGER

Department of Mathematics, University of Queensland St. Lucia, Queensland, Australia

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1. INTRODUCTION

In the following we shall denote by α the multi-index of integers $[\alpha_1, ..., \alpha_n]$ and by $|\alpha|$ the sum $\sum_{i=1}^{n} \alpha_i$ according to the standard notation used, e.g., by Lions in [7]. Let Ω be a domain in Euclidean n space, E^n and consider the classes $W^mL_B(\Omega)$ or $W^mE_B(\Omega)$ consisting of all functions u in the Orlicz spaces $L_B(\Omega)$ or $E_B(\Omega)$ such that the distributional derivatives $D^\alpha u$ are contained in $L_B(\Omega)$ or $E_B(\Omega)$, respectively, for all α with $|\alpha| \leqslant m$. The classes of such functions may be given a norm

$$\| u \|_{W^mL_B(\ )} = \max_{\alpha}\{\| D^\alpha u \|_B\}, \tag{1.1}$$

where $\| \cdot \|_B$ is a suitable norm in L_B such as the Luxemburg norm defined below. These classes will be Banach spaces under this norm. We shall refer to spaces of the forms $W^mL_B(\Omega)$ or $W^mE_B(\Omega)$ as *Orlicz-Sobolev spaces*. They form a generalization of Sobolev spaces in much the same way as Orlicz spaces form a generalization of L_p spaces.

In the following parts of this paper we shall deal with three of the questions which arise in the application of these spaces to differential equations: the separability of these spaces, the imbedding of Orlicz-Sobolev spaces in Orlicz spaces or spaces of continuous functions and, the relationship of complementarity for linear functionals on these spaces. Our results extend analogous results for Sobolev spaces. In particular in Section 3, we obtain a series of imbedding results for the spaces $W^1L_B(\Omega)$ for arbitrary N functions B, which generalize directly the various imbedding results for Sobolev spaces, including imbeddings into spaces defined on lower-dimensional hyperplanes (see [1] or [10]). The following three sections of the paper, concerning separability, imbeddings and complementarity, respectively, are independent and each may be studied separately. Section 2 is a straightforward extension of the standard density results for Sobolev spaces. However, the methods of Sections 3 and 4 are considerably different from the Sobolev space case and are, we feel, of intrinsic interest. Some of the results of this paper have been used by the first author in his work on the existence of solutions of nonlinear elliptic boundary value problems [4] and by the second author for regularity considerations [15].

We devote the rest of this section to listing briefly some basic definitions and properties of Orlicz spaces to serve as a reference for the following sections. The reader desirous of more information should consult a work such as Krasnoselski–Rutickii [6].

Let $A(t)$ be a real-valued continuous, convex, even function of the real variable t, satisfying

$$\lim_{t\to 0} \frac{A(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{A(t)}{t} = \infty. \tag{1.2}$$

Then the Orlicz class $L_A'(\Omega)$ consists of all functions u such that

$$\int_\Omega A(u(x))\,dx < \infty.$$

The *Orlicz space* $L_A(\Omega)$ may be defined as the linear hull of $L_A'(\Omega)$ together with the Luxemburg norm

$$\| u \|_{L_A(\Omega)} = \inf \left\{ k; \int_\Omega A\left(\frac{u}{k}\right) dx \leqslant 1 \right\}. \tag{1.3}$$

$L_A(\Omega)$ is a Banach space under Eq. (1.3). To simplify notation, we we will generally write $\| u \|_{L_A(\Omega)} = \| u \|_A$, as in Eq. (1.1). A defining function for an Orlicz space, which has the above properties, is called an *N function.* Associated with any N function A, we have the *complementary function* $\bar{A}$ given by

$$\bar{A}(x) = \int_0^{|x|} \bar{a}(t)\,dt, \tag{1.4}$$

where

$$\bar{a}(t) = \sup\{s;\, a(s) \leqslant t\}, \qquad a(s) = A'(s).$$

Clearly, $\bar{\bar{A}} = A$ and A, $\bar{A}$ are said to be *complementary* to each other. Furthermore, the following inequalities hold

$$xy \leqslant A(x) + \bar{A}(y) \qquad \text{(Young's inequality)} \tag{1.5}$$

$$y \leqslant A^{-1}(y)\, \bar{A}^{-1}(y) \leqslant 2y \tag{1.6}$$

$$\int uv\, dx \leqslant 2 \| u \|_A \| v \|_{\bar{A}}\,, \qquad u \in A,\, v \in \bar{A} \qquad \text{(Hölder's inequality)} \tag{1.7}$$

provided meas $\Omega < \infty$.

For meas $\Omega < \infty$, we will require the following two comparison relations for N functions.

If for any two N functions A_1, A_2, there exist positive numbers x_0 and k such that for all $x \geqslant x_0$,

$$A_1(x) \leqslant A_2(kx), \tag{1.8}$$

then we write $A_1 \prec A_2$. Note that this means that $L_{A_2} \subset L_{A_1}$. If $A_1 \prec A_2$, $A_2 \succ A_1$, we write $A_1 \sim A_2$ and hence $L_{A_1} = L_{A_2}$. Finally, if we have a stronger condition than Eq. (1.8), viz.,

$$\lim_{x\to\infty} \frac{A_2\lambda(x)}{A_1(x)} = \infty \tag{1.9}$$

for every $\lambda > 0$, then we write $A_1 \ll A_2$. This latter condition implies that $L_{A_2} \subsetneqq L_{A_1}$. In Section 3, we will make use of the following convergence criterion [6; pp. 99, 115]. If a sequence $u_n \in L_A(\Omega)$ converges in measure and is bounded in $L_A(\Omega)$, then u_n converges in $L_B(\Omega)$ for any $B \ll A$ (assuming, of course, meas $\Omega < \infty$).

The space $E_A(\Omega)$ is defined to be the closure of the bounded functions in the $L_A(\Omega)$ norm. It follows that $E_A(\Omega) \subset L_A{}'(\Omega)$ and $W^m E_B(\Omega) \subset W^m L_B(\Omega)$. The spaces E_A are separable.

An important subclass of N functions is the class satisfying the Δ_2 condition. An N function, A, satisfies the Δ_2 *condition* if there exist positive constants x_0, k such that

$$A(2x) \leqslant kA(x) \qquad \text{for} \quad x \geqslant x_0\,. \tag{1.10}$$

The spaces L_A and E_A coincide if and only if A satisfies the Δ_2 condition and the space L_A is reflexive if and only if A and $\bar{A}$ both satisfy the Δ_2 condition. In general L_A is the dual of $E_{\bar{A}}$. The Orlicz–Sobolev spaces will, therefore, not in general be reflexive.

Some further properties of Orlicz spaces, not proved in [6], will be established in the particular sections of this paper where they are employed.

2. Separability Properties

We establish below some theorems concerning the approximation of elements in $W^m E_B(\Omega)$ by smooth functions. By considering the natural imbedding of $W^m E_B(\Omega)$ into the product space $\prod^N E_B(\Omega)$ where N is the number of multi-indices α, $|\alpha| \leqslant m$, we see that $W^m E_B(\Omega)$ is separable since $E_B(\Omega)$ is separable. The spaces $W^m L_B(\Omega)$ will clearly not be separable in general. In order to discuss separability properties of the spaces $W^m E_B(\Omega)$, a lemma on approximation is first necessary (see also Dankert [2]).

LEMMA 2.1. *Let ρ be a $C_0^\infty(E^n)$ function satisfying $\rho \geqslant 0$ and $\int \rho(t)\,dt = 1$. Define a sequence of $C_0^\infty(E^n)$ functions ρ_k, $k = 1, 2, \ldots$ by $\rho_k(t) = k\rho(kt)$. Let B be an N function, $f \in E_B$. Then the convolutions $\rho_k * f$ are in E_B and $\| \rho_k * f - f \|_B \to 0$ as $k \to \infty$.*

Proof. Let $\bar{B}$ be the complementary N-function to B and let $g \in L_{\bar{B}}$ with $\| g \|_{\bar{B}} = 1$. Then

$$\int |\rho_k * f(x) - f(x)|\,|g(x)|\,dx \leqslant \int \left\{ \int |f(x-t) - f(x)|\,|g(x)|\,dx \right\} \rho_k(t)\,dt$$
$$\leqslant 2 \int \| f_t - f \|_B\, \rho_k(t)\,dt,$$

where $f_t(x) = f(x - t)$. Hence

$$\| \rho_k * f - f \|_B \leqslant 2 \int \| f_t - f \|_B\, \rho_k(t)\,dt$$
$$= 2 \int \| f_{t/k} - f \|_B\, \rho(t)\,dt.$$

Since $f \in E_B$ and ρ has compact support, for every $\epsilon > 0$ there exists k sufficiently large such that

$$\int_{E^n} \| f_{t/k} - f \|_B\, \rho(t)\,dt \leqslant \epsilon \int \rho(t)\,dt = \epsilon$$

and the lemma is proved. Q.E.D.

If $\Omega \nsubseteq E^n$, as in the L^p–Sobolev spaces, the C_0^∞ functions are not necessarily dense in the space. However, it is true that if $\Omega = E^n$, then the $C_0^\infty(E^n)$ functions are dense in the space $W^mE_B(E^n)$ (although not, of course, in $W^mL_B(E^n)$). To show this, one has the first density theorem:

THEOREM 2.1. *$C_0^\infty(E^n)$ is dense in $W^mE_B(E^n)$.*

Proof. We may first assume that u has compact support. For we may consider a sequence of C_0^∞ functions $M_R(x)$ such that $M_R(x) = 1$ for $x \leqslant R$ and $M_R(x) = 0$ for $x \geqslant 2R$. Then $M_R \cdot u$ is a function with compact support and $M_R \cdot u \to u$ in $W^mE_B(E^n)$.

Assume now that u has compact support and let ρ_k be the sequence defined in Lemma 2.1 above. Define

$$u_k(x) = u^*\rho_k(x) = \int u(t)\, \rho_k(x - t)\, dt, \tag{2.1}$$

and note that since $D^\alpha u \in E_B$ for all α, $|\alpha| \leqslant m$, and $\rho_k \in C_0^\infty$, we must have

$$\begin{aligned} D^\alpha u_k(x) &= D^\alpha \int u(t)\, \rho_k(x - t) \\ &= \int u(t)\, D^\alpha \rho_k(x - t)\, dt \\ &= \int D^\alpha u(t)\, \rho_k(x - t)\, dt. \end{aligned} \tag{2.2}$$

Clearly, $u_k \in C_0^\infty$ for every k. One may then observe that for any α with $|\alpha| \leqslant m$,

$$\begin{aligned} \| D^\alpha u_k - D^\alpha u \|_B &= \| D^\alpha u^* \rho_k - D^\alpha u \|_B \\ &\to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

The theorem is proved. Q.E.D.

A second important theorem on the separability properties of the spaces $W^mE_B(\Omega)$ may be obtained by means of a generalization to Orlicz–Sobolev spaces of a theorem proved originally by Meyers and Serrin for Sobolev spaces in Ref. [8].

THEOREM 2.2. *Let Ω be a bounded domain in E^n. Then $C^\infty(\Omega)$ is dense in $W^mE_B(\Omega)$.*

Proof. It will be sufficient to show

LEMMA 2.2. *Let $u \in W^m E_B(\Omega)$. Then for every $\epsilon > 0$ there exists a function $v \in C^\infty(\Omega)$ such that $\| u - v \|_{W^m E_B} \leqslant \epsilon$.*

Proof. Let Ω_ν be the open set defined by

$$\Omega_\nu = \{t;\ t \in \Omega,\ | t | < \nu,\ \text{distance}(t, \partial\Omega) > 1/\nu\}, \tag{2.3}$$

where $\nu = 1, 2, \ldots$. One may also define Ω_0 and Ω_{-1} to be null sets. Let $\Sigma\psi = 1$ be a partition of unity on Ω such that

$$\operatorname{supp} \psi_\nu \subset \Omega_{\nu+1} - \Omega_{\nu-1}, \qquad \nu = 1, 2, \ldots \tag{2.4}$$

For every ν, $\psi_\nu u \in W^m E_B(E^n)$. Therefore by Theorem 2.1 above for every integer $\nu = 1, 2, \ldots$ there exists a C_0^∞ function K_ν such that

(i) for every α, $| \alpha | \leqslant m$

$$\| K_\nu * D^\alpha \psi_\nu u - D^\alpha \psi_\nu u \|_B \leqslant \epsilon/2^\nu$$

(ii) $\operatorname{supp} K_\nu \subset \{t;\ | t | < 1/(\nu + 1)(\nu + 2)\}$

By expression (2.4) and condition (ii) above,

$$\operatorname{supp} K_\nu * D^\alpha \psi_\nu u \subset \Omega_{\nu+2} - \Omega_{\nu-2}\,.$$

Hence the series

$$v = \sum_1^\infty K_\nu * \psi_\nu u$$

converges and defines a function $v \in C^\infty(\Omega)$.

Let k be an arbitrary nonnegative integer. Then by condition (i) and the convexity of B, one has that for every α, $| \alpha | \leqslant m$,

$$\begin{aligned}\int_{\Omega_k} B([D^\alpha v - D^\alpha u]1/\epsilon)\, dt &= \int_{\Omega_k} B\left(\sum_1^{k+1} (K_\nu * D^\alpha \psi_\nu \cdot u - D^\alpha \psi_\nu \cdot u) \frac{2^\nu}{2^\nu \epsilon}\right) dt \\ &\leqslant \sum_1^{k+1} \frac{1}{2^\nu} \int_{\Omega_k} B\left([K_\nu * D^\alpha \psi_\nu u - D^\alpha \psi_\nu u] \frac{2^\nu}{\epsilon}\right) dt \\ &\leqslant 1.\end{aligned}$$

Hence, letting $k \to \infty$, by the monotone convergence theorem

$$\int_\Omega B\left(\frac{1}{\epsilon}[D^\alpha v - D^\alpha u]\right) dt \leqslant 1,$$

for every α, $|\alpha| \leqslant m$, and hence

$$\| u - v \|_{W^m E_B} = \max_{|\alpha| \leqslant m} \| D^\alpha u - D^\alpha v \|_B \leqslant \epsilon. \qquad \text{Q.E.D.}$$

The final density theorem gives conditions under which a statement similar to that of the preceding theorem may be made for the region $\bar{\Omega}$, the closure of an open region Ω. We shall state it here together with the necessary definition for its formulation; the proof is an easy generalization of the argument for Sobolev spaces which may be found for instance in [1] or [7].

DEFINITION 2.1. Let Ω be a bounded domain and $\bar{\Omega}$ its closure. $\bar{\Omega}$ will be said to satisfy a segment condition if there exists a finite, open covering $\{\mathscr{U}_i\}$ of $\bar{\Omega}$ such that for every $\mathscr{U}_i$, if $\mathscr{U}_i \cap \partial\Omega \neq \varnothing$, then there exists a vector y^i such that for $0 < t < 1$, $x + ty^i \in \Omega$ for all $x \in \bar{\Omega} \cap \mathscr{U}_i$.

THEOREM 2.3. *Let $\bar{\Omega}$ satisfy a segment condition. Then $C^\infty(\bar{\Omega})$ is dense in $W^m E_B(\Omega)$.*

3. IMBEDDING THEOREMS FOR ORLICZ–SOBOLEV SPACES

To simplify our presentation we consider initially the spaces $W^1 L_B(\Omega)$ for arbitrary N functions B, i.e., the case $m = 1$. The general case, which follows by iteration, is treated at the end of this section.

DEFINITION 3.1. The domain $\Omega \subset E^n$ satisfies a *cone condition* if there exists a fixed cone $k_\Omega \subset E^n$, such that each point $x \in \partial\Omega$ is the vertex of a cone $k_\Omega(x) \subset \Omega$ and congruent to k_Ω.

That our main imbedding theorem, Theorem 3.2 below, holds for domains satisfying cone conditions is a consequence of part of the Sobolev imbedding theorem. For this reason and for comparison purposes, a statement of the latter follows. We will use arrows in the sequel to indicate continuous imbeddings.

THEOREM 3.1 (Sobolev imbedding theorem). *Let the domain Ω satisfy a cone condition. Then*

(a) *if $p < n$, $W^1 L_p(\Omega) \to L_{p^*}(\Omega)$ where $p^* = np/(n-p)$*

(b) *if $p > n$, $W^1 L_p(\Omega) \to L_\infty(\Omega) \cap C(\Omega)$.*

By $C(\Omega)$ we mean the space of functions continuous in Ω. The norm in $L_\infty(\Omega) \cap C(\Omega)$ is the sup norm over Ω. For a proof of Theorem 3.1, see, e.g., Ref. [1]. We will generalize the cone condition in the following way:

DEFINITION 3.2. A domain $\Omega \subset E^n$ is *admissible* if the conclusion of Theorem 3.1 holds in the case $p = 1$, i.e., $W^1L_1(\Omega) \to L_{n/(n-1)}(\Omega)$.

In fact, we will see below that Ω is admissible if and only if the conclusion of Theorem 3.1 holds for all p.

The exponent $p^* = np/(n - p)$ is usually called the Sobolev conjugate of p. We extend this notion to Orlicz spaces. Associated with any N function B, we define a function

$$g_B(t) = \frac{B^{-1}(t)}{t^{1+1/n}}, \qquad t \geqslant 0. \tag{3.1}$$

Let $\{B\}$ denote an equivalence class of N functions, i.e., $B_1, B_2 \in \{B\}$ implies $B_1 \sim B_2$, and suppose that the members of $\{B\}$ satisfy

$$\int_1^\infty g_B(t)\, dt = \infty. \tag{3.2}$$

The *Sobolev conjugate class* $\{B^*\}$ of $\{B\}$ is then defined as the class generated by the N-function B^* given by

$$(B^*)^{-1}(|x|) = \int_0^{|x|} \frac{B^{-1}(t)}{t^{1+1/n}}\, dt = \int_0^{|x|} g_B(t)\, dt, \tag{3.3}$$

where $B \in \{B\}$ is so chosen that

$$\int_0^1 g_B(t)\, dt < \infty. \tag{3.4}$$

It is clear that from any class, $\{B\}$, an N function B satisfying the expression (3.4) may be chosen. In the sequel we will always assume that the expression (3.4) is satisfied.[1] Note that for $B(t) = |t|^p$, $1 < p < n$, we have

$$g_B(t) = t^{1/n-1/p-1}, \qquad B^*(t) = \frac{|t|^{p^*}}{p^*} \tag{3.5}$$

[1] Note, however, that in the case of unbounded Ω equivalent N functions may define different Orlicz spaces and hence the inequality (3.4) must be assumed to hold.

and that for $B(t) = |t|^n$, we have

$$g_B(t) = t^{-1}, \qquad B^*(t) \sim e^{|t|} - |t| - 1. \tag{3.6}$$

Our main imbedding theorem is then

THEOREM 3.2. *Let Ω be a bounded, admissible domain in E^n. Then*

(a) *if* $\int_1^\infty g_B(t)\,dt = \infty$, $W^1L_B(\Omega) \to L_{B^*}(\Omega)$,

(b) *if* $\int_1^\infty g_B(t)\,dt < \infty$, $W^1L_B(\Omega) \to L_\infty(\Omega) \cap C(\Omega)$.

For the proof of Theorem 3.2 we invoke the following two lemmas: The first is a fairly well-known calculus lemma.

LEMMA 3.1. *Let u be a strongly differentiable function on a domain $\Omega \subset E^n$ (i.e. $u \in W_1^{1,\mathrm{loc}}(\Omega)$) and g a uniformly, Lipshitz continuous function on E. Then the composite function $f = g \circ u$ is strongly differentiable and the chain rule holds, i.e.,*

$$Df = g' \cdot Du \qquad a.e.\ (\Omega) \tag{3.7}$$

Lemma 3.1 is crucial to our proof and is proved, e.g., in Ref. [9]. The second lemma is an elementary, interpolation result.

LEMMA 3.2. *Let f, g be continuous, non-decreasing functions on an interval $(0, N)$, $0 < N \leqslant \infty$ and suppose $\lim_{t\to N} f/g = \infty$. Then for arbitrary $\epsilon > 0$, there exists a constant K depending on ϵ such that*

$$g(t) \leqslant \epsilon f(t) + K \tag{3.8}$$

Proof. There exists $N_0 \in (0, N)$ such that $t > N_0$ implies $g(t) \leqslant \epsilon f(t)$. Define $K = \sup_{(0,N_0)} g$. Q.E.D.

Proof of Theorem 3.2(a). The function B^* defined by Eq. (3.3) clearly satisfies the differential equation

$$B^{-1}(y)y' = y^{1+1/n}, \tag{3.9}$$

and hence by the expression (1.6), the inequality

$$y' \leqslant y^{1/n}\bar{B}^{-1}(y). \tag{3.10}$$

Let $C = (B^*)^{1-1/n}$. Then from the inequality (3.10)

$$C' \leqslant \frac{n-1}{n}\,\bar{B}^{-1}(C^{n/(n-1)}). \tag{3.11}$$

We will prove now that $W^1L_B(\Omega) \to L_{B^*}(\Omega)$ for any C satisfying the inequality (3.11) with $B^* = C^{n/(n-1)}$. In fact, a trivial extension of our proof will show that the imbedding will hold for any C satisfying

$$C' \leqslant a + b\bar{B}^{-1}(C^{n/(n-1)}), \tag{3.12}$$

where a is a continuous mapping from $L_B(\Omega)$ to $L_{\bar{B}}(\Omega)$, (e.g., $a =$ constant) and b is a constant.

We let $u \in W^1L_B(\Omega)$ and assume at first that u is bounded. Let

$$k = \| u \|_{L_{B^*}(\Omega)} = \inf \left\{ s; \int_\Omega B^*(u/s)\, dx \leqslant 1 \right\}.$$

Since u is bounded, $f(s) = \int_\Omega B^*(u/s)\, dx$ is a continuous function of s with $f(0) = \infty$, $f(\infty) = 0$. Hence we have, in fact,

$$\int_\Omega B^*(u/k)\, dx = 1. \tag{3.13}$$

Let $g = C(u/k)$. By Lemma 3.1, $g \in W^1L_1(\Omega)$, and since Ω is admissible

$$\| g \|_{n/(n-1)} \leqslant \gamma(\| Dg \|_1 + \| g \|_1) \tag{3.14}$$

where the constant γ is independent of u. Therefore,

$$\left\{ \int_\Omega B^*(u/k)\, dx \right\}^{1-1/n} \leqslant \frac{\gamma}{k} \int_\Omega C'(u/k) \mid Du \mid dx + \gamma \int_\Omega C(u/k)\, dx.$$

Applying Hölder's inequality (1.7) to the first term on the right side and using Eq. (3.13), we have

$$1 \leqslant \frac{2\gamma}{k} \| C'(u/k) \|_{\bar{B}} \| Du \|_B + \gamma \int_\Omega C(u/k)\, dx. \tag{3.15}$$

Now

$$\begin{aligned}
\| C'(u/k) \|_{\bar{B}} &\leqslant \frac{n-1}{n} \| \bar{B}^{-1}B^*(u/k) \|_{\bar{B}} \qquad \text{(by (3.11))} \\
&= \frac{n-1}{n} \inf \left\{ s; \int_\Omega \bar{B} \left\{ \frac{1}{s} \bar{B}^{-1}B^*(u/k) \right\} dx \leqslant 1 \right\} \\
&\leqslant \frac{n-1}{n} \qquad \text{(by (3.13))}.
\end{aligned}$$

Also applying Lemma 3.2 to the functions $B^*(t)/t$, $C(t)/t$ for $t = |u|/k$, $\epsilon = 1/2\gamma$, $N = \infty$, we obtain

$$\int_\Omega C(u/k)\,dx \leqslant \frac{1}{2\gamma}\int_\Omega B^*(u/k)\,dx + K\int_\Omega |u|/k\,dx$$
$$= \frac{1}{2\gamma} + \frac{K}{k}\|u\|_1 \qquad \text{(by (3.13))}.$$

Hence on substituting in the expression (3.15), we obtain

$$1 \leqslant \frac{2\gamma}{k}\frac{(n-1)}{n}\|Du\|_B + \frac{1}{2} + \frac{K}{k}\|u\|_1,$$

and hence

$$k \leqslant \frac{4\gamma(n-1)}{n}\|Du\|_B + 2K\|u\|_1. \tag{3.16}$$

To extend the estimate (3.16) to arbitrary $u \in W^1L_B(\Omega)$ we define the sequence $\{u_m\}$, $m = 1, 2,\ldots$ by

$$u_m = \begin{cases} u & \text{for } |u| \leqslant m \\ m \operatorname{sign} u & \text{for } |u| \geqslant m. \end{cases}$$

Clearly, by Lemma 3.1, $u_m \in W^1L_B(\Omega) \cap L_\infty(\Omega)$, and hence by the expression (3.16)

$$\begin{aligned} k_m = \|u_m\|_{B^*} &\leqslant C(n, |\Omega|)\,\|u_m\|_{W'L_B(\Omega)} \\ &\leqslant C(n, |\Omega|)\,\|u\|_{W'L_B(\Omega)}, \end{aligned} \tag{3.17}$$

where C is a constant depending on n and $|\Omega|$. Since the sequence $\{k_n\}$ is also nondecreasing, it converges by Eq. (3.17). Let $k = \lim k_m$. By Fatou's lemma,

$$\int_\Omega B^*(u/k)\,dx \leqslant \lim \int_\Omega B^*(u_m/k_m) = 1.$$

Hence $u \in L_{B^*}(\Omega)$ and by Eq. (3.17)

$$\|u\|_{B^*} \leqslant C(n, |\Omega|)\,\|u\|_{W'L_B(\Omega)}. \qquad \text{Q.E.D.} \tag{3.18}$$

Proof of Theorem 3.2(b). Theorem 3.2(b) follows in a manner similar to Theorem 3.2(a), if we widen our concept of N function to include the class of Young functions (see [16]). Let $A(t)$ be a positive, convex function on an interval $(0, N)$ with $A(0) = 0$, $A(N) = \infty$. We extend the domain of A to $[0, \infty)$ by defining $A(t) = \infty$ for $t \geqslant N$. Using the Luxemburg norm with the function A we may define an equivalent norm on $L_\infty(\Omega)$, viz.,

$$\| u \|_A = \inf \left\{ k; \int_\Omega A(| u |/k)\, dx \leqslant 1 \right\}. \tag{3.19}$$

Clearly,

$$\frac{1}{N} \sup_\Omega u \leqslant \| u \|_A \leqslant A^{-1} \left(\frac{1}{| \Omega |} \right) \sup_\Omega u. \tag{3.20}$$

To prove Theorem 3.2(b), we let

$$N = \int_0^\infty g_B(t)\, dt.$$

Then the function B^* given by its inverse

$$(B^*)^{-1}(x) = \int_0^x g_B(t)\, dt \tag{3.3}$$

defines a function of the above type on the interval $(0, N)$. The proof of Theorem 3.2(a) extends automatically to B^* and we obtain once more

$$\| u \|_{B^*} \leqslant C(| \Omega |, n) \| u \|_{W^1 L_B(\Omega)}\,, \tag{3.18}$$

and hence by the expression (3.20),

$$\sup_\Omega | u | \leqslant C(| \Omega |, n) \| u \|_{W^1 L_B(\Omega)} \int_0^\infty \frac{B^{-1}(t)}{t^{1+1/n}}\, dt. \tag{3.21}$$

To establish the continuity of $u \in W^1 L_B(\Omega)$, let $y \in \Omega$ and $\delta > 0$ satisfy

$$\Omega_{2\delta}(y) = \{x; | x_i - y_i | < 2\delta\} \subset \Omega.$$

Then for $x \in \Omega_\delta(y)$ and h satisfying $\sup | h_i | < \delta$, we apply the expression (3.21) to the function

$$v_h(x) = u(x + h) - u(x).$$

We obtain, therefore,

$$\begin{aligned} | v_h(y)| &\leqslant \text{const} \| u(x+h) - u(x)\|_{W^1L_B(\Omega_\delta)} \\ &\to 0 \quad \text{as} \quad h \to 0. \end{aligned}$$

Hence $u \in C(\Omega)$. Note that by u continuous, we strictly mean that u is equivalent in measure to a continuous function. The continuity of the imbedding of Theorem 3.2(b) is then guaranteed by the expression (3.21). Q.E.D.

In the remainder of this section we deal with some extensions and refinements of Theorem 3.2. We remark here that imbedding theorems for Orlicz–Sobolev spaces appear to have been first considered by Dankert [2] who used a potential representation but imposed very stringent conditions on the N functions involved. The potential estimates of O'Neill [11] yield Theorem 3.2(a) for the case where Ω satisfies a cone condition and $\tilde{B}$ satisfies a Δ_2 condition. The first imbedding result for arbitrary N-functions, B, was obtained by Donaldson [3] who established Theorem 3.2(a) for a class of smooth domains. His method, although motivated by the differential equation (3.9), proceeds by an approximation technique and is considerably different from the above proof. The present proof of Theorem 3.2(a) and its extensions to Theorem 3.2(b) and the results following in this section are due to the second author.

An immediate generalization of Theorem 3.2 follows by replacing the exponent $n/(n-1)$ in the definition of admissibility by an exponent ν satisfying $1 \leqslant \nu \leqslant n/(n-1)$. Defining $B_\nu{}^*$ by

$$(B_\nu{}^*)^{-1}(x) = \int_0^x \frac{B^{-1}(t)}{t^{2-1/\nu}}, \tag{3.22}$$

we obtain from the proof of Theorem 3.2.

THEOREM 3.3. *Let Ω be bounded and let it satisfy $W^1L(\Omega) \to L_\nu(\Omega)$ for some ν, $1 \leqslant \nu \leqslant n/(n-1)$. Then*

(a) *if $\int_1^\infty B^{-1}(t)/(t^{2-1/\nu})\,dt = \infty$, $W^1L_B(\Omega) \to L_{B_\nu^*}(\Omega)$,*
(b) *if $\int_1^\infty B^{-1}(t)/(t^{2-1/\nu})\,dt < \infty$, $W^1L_B(\Omega) \to L_\infty(\Omega) \cap C(\Omega)$.*

Next, we define a further Orlicz–Sobolev space, $W_0{}^mL_B(\Omega)$ to be the closure of $C_0{}^\infty$ in $W^mL_B(\Omega)$. We have $W_0{}^mL_B(\Omega) \subset W^mE_B(\Omega)$ with equality holding if $\Omega = E^n$ (Theorem 2.1). For the spaces $W_0{}^1L_B(\Omega)$, as with the Sobolev spaces $W_p{}^1(\Omega)$, no restrictions are necessary on the domains Ω for the imbedding results to hold. In fact, let the constant $\gamma(n)$ satisfy

$$\| u \|_{n/(n-1)} \leqslant \gamma(n) \| Du \|_1 \tag{3.23}$$

for all $u \in C_0^\infty(E^n)$. In the proof of Theorem 3.2 applied to $u \in C_0^\infty(\Omega)$, Lemmas 3.1 and 3.2 are not required. We readily deduce

THEOREM 3.4. *Let Ω be a domain in E^n. Then*

(a) *if* $\int_1^\infty g_B(t)\,dt = \infty$, $W_0{}^1L_B(\Omega) \to L_{B^*}(\Omega)$ *and for any* $u \in W_0{}^1L_B(\Omega)$

$$\| u \|_{B^*} \leqslant \frac{2\gamma(n-1)}{n} \| Du \|_B \tag{3.24}$$

(b) *if* $\int_1^\infty g_B(t)\,dt < \infty$, $W_0{}^1L_B(\Omega) \to C(\bar{\Omega})$ *and for any* $u \in W_0{}^1L_B(\Omega)$

$$\sup_\Omega | u | \leqslant \frac{2\gamma(n-1)}{n} \| Du \|_B \int_0^\infty g_B(t)\,dt \tag{3.25}$$

The example (3.4) shows that the Sobolev theorem, Theorem 3.1, is a special case of Theorem 3.2. There are two refinements of the Sobolev theorem pertaining to the cases $p = n$, $p > n$, respectively, which we discuss now in the light of our results. First, in the case $p = n$, Theorem 3.2 yields $W^1L_n(\Omega) \to L_{B^*}(\Omega)$, where $B^* = e^{|t|} - |t| - 1$. This result is not as sharp as the imbedding theorem of Trudinger [14], $W^1L_n(\Omega) \to L_C(\Omega)$, where $C = e^{|t|^{n/(n-1)}} - 1$, derived using a potential representation.[2] Hence Theorems 3.2 and 3.4 are not optimal in the sense that for any N function B, $L_{B^*}(\Omega)$ is not necessarily the smallest Orlicz space possible in the conclusions. Using the method of [14], our results can be improved in the range $B \succ |t|^q$ for all $q < n$. However, the details being messy are not presented here. It is our contention though that the results are optimal in the cases $B \prec |t|^q$ for some $q < n$.

Imbeddings into Spaces of Continuous Functions

In the case $p > n$ of Theorem 3.1, we have for strongly Lipshitz domains the result of Morrey [1], $W^1L_p(\Omega) \to C^\alpha(\bar{\Omega})$, $\alpha = 1 - n/p$,

[2] The result in [14] has recently been shown to be optimal. The proof will appear in a forthcoming note by J. Hempel, G. R. Morris and N. S. Trudinger.

where $C^\alpha(\bar{\Omega})$ denotes the space of functions, uniformly Hölder continuous in $\bar{\Omega}$ with exponent α and

$$\| u \|_{C^\alpha(\bar{\Omega})} = \sup_{\Omega} | u | + \sup_{x,y \in \Omega} \frac{| u(x) - u(y)|}{| x - y |^\alpha}. \tag{3.26}$$

By a *strongly Lipshitz* domain Ω we will understand a domain Ω, where each $x \in \partial\Omega$ has a neighbourhood $\mathscr{U}_x$ such that in some coordinate system, with origin at x, $\Omega \cap \mathscr{U}_x$ is represented in $\mathscr{U}$ by $\xi_n < F(\xi')$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$ with F a Lipshitz continuous function.

An analogous refinement of Theorem 3.2(b) then follows for strongly Lipshitz domains using essentially the same method as Morrey [9]. We will instead derive the generalization here directly from a more general result of Spanne [13].

THEOREM 3.5 (Ref. [13]). *Let Q_0 be a cube in E^n, $u \in L_1(Q_0)$ and suppose that for every parallel subcube Q of Q_0,*

$$\int_Q | u - u_Q | \, dx \leqslant | Q | \, \rho(| Q |^{1/n}), \tag{3.27}$$

where $u_Q = | Q |^{-1} \int_Q u \, dx$ and ρ is a positive, nondecreasing function. Then if $\int_0^\delta \rho/t \, dt < \infty$ for some $\delta > 0$, $u \in C(Q_0)$ and the modulus of continuity estimate

$$| u(x) - u(y)| \leqslant \text{const} \int_0^r \rho/t \, dt, \qquad r = | x - y |, \qquad x, y \in Q_0 \tag{3.28}$$

holds.

As a corollary of Theorem 3.5, we then have

THEOREM 3.6. *Let Ω be a strongly Lipshitz domain, and let $u \in W^1L_B(\Omega)$ with $\int_1^\infty g_B(t) dt < \infty$. Then for any $x, y \in \Omega$*

$$| u(x) - u(y)| \leqslant K \| u \|_{W^1L_B(\Omega)} \int_{\rho^{-n}}^\infty g_B(t) \, dt, \qquad \rho = | x - y | \tag{3.29}$$

where the constant K depends on n and Ω.

Proof. Since Ω is strongly Lipshitz, it is sufficient to consider the case where Ω is a cube. Using the Hölder inequality (1.7), we have

$$\begin{aligned} \int_\Omega | Du | \, dx &\leqslant 2 \| Du \|_B \| 1 \|_{\bar{B}} \\ &= 2 \| Du \|_B \, | \Omega | \, B^{-1}\left(\frac{1}{| \Omega |}\right). \end{aligned}$$

By the Poincaré inequality for $W^1L_1(\Omega)$ [12],

$$\int_\Omega |u - u_\Omega| \, dx \leqslant \text{const} \, |\Omega|^{1/n} \int_\Omega |Du| \, dx. \tag{3.30}$$

We may therefore obtain, for any parallel subcube Ω',

$$\int_{\Omega'} |u - u_{\Omega'}| \, dx \leqslant \text{const} \, |\Omega'|^{1+1/n} B^{-1}\left(\frac{1}{|\Omega|}\right) \| Du \|_B .$$

Therefore, applying Theorem 3.5, we have

$$\begin{aligned} |u(x) - u(y)| &\leqslant \text{const} \, \| Du \|_B \int_0^\rho B^{-1}(t^{-n}) \, dt, \qquad \rho = |x - y \\ &\leqslant \text{const} \, \| Du \|_B \int_{\rho^{-n}}^\infty \frac{B^{-1}(s)}{s^{1+1/n}} \, ds \end{aligned}$$

by substituting $s = t^{-n}$. Q.E.D.

We may express Theorem 3.6 in the form of a continuous imbedding into a class of continuous functions. Let $\mu(t)$ denote an increasing, continuous function of $t \geqslant 0$ with $\mu(0) = 0$. We define the space $C_\mu(\bar{\Omega})$ to be the set of functions, continuous in Ω and satisfying

$$\| u \|_\mu = \| u \|_{C_\mu(\bar\Omega)} = \sup_\Omega |u| + \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{\mu(|x - y|)} < \infty \tag{3.31}$$

The space $C_\mu(\bar{\Omega})$ is a Banach space under the norm (3.31). From Theorems 3.2(b) and 3.6, we have, therefore,

COROLLARY 3.1. *Let Ω be strongly Lipshitz, $\int_1^\infty g_B(t)\, dt < \infty$. Then $W^1L_B(\Omega) \to C_\mu(\bar{\Omega})$ where $\mu(t) = \int_{t^{-n}}^\infty g_B(s)\, ds$.*

Compactness of the Imbeddings

We begin with two criteria for compactness.

LEMMA 3.3. *Let A_1, A_2 be N functions with $A_1 \ll A_2$. Let $\| u \|_{A_2} \leqslant K \| u \|_{W^1L_B}$. Then $B_1 W^1L_B$, the unit ball in W^1L_B is compact in L_{A_1}, provided $|\Omega| < \infty$.*

Proof. Since $\mathscr{N} = B_1 W^1 L_B$ is compact in L_1, it is compact in measure. Hence, by the hypothesis and the criterion for convergence mentioned in the introduction, $\mathscr{N}$ is compact in L_{A_1}. Q.E.D.

The second criterion refers to compactness in the C_μ spaces. We say that two functions μ_1, μ_2 of the type defined above satisfy the relation $\mu_1 \ll \mu_2$ if

$$\mu_2(t) = \mu_1(t)\,\mu(t), \qquad t > 0,$$

where μ is a function of the same type. We then have

LEMMA 3.4. *$B_1 C_{\mu_2}$, the unit ball in C_{μ_2}, is compact in C_{μ_1} if $\mu_1 \ll \mu_2$.*

Proof. We have

$$\begin{aligned}
\| u \|_{\mu_1} &= \sup_{\Omega} | u | + \sup_{x,y\in\Omega} \frac{| u(x) - u(y)|}{\mu_2(| x - y |)}\, \mu(| x - y |) \\
&\leqslant \sup_{\Omega} | u | + \mu(\epsilon) \| u \|_{\mu_2} + \frac{2}{\mu_2(\epsilon)} \sup_{\Omega} | u |, \qquad \text{for any } \epsilon > 0 \\
&= \left\{ 1 + \frac{2}{\mu_2(\epsilon)} \right\} \sup_{\Omega} | u | + \mu(\epsilon) \| u \|_{\mu_2};
\end{aligned}$$

and since $C_{\mu_2} \to C$ is compact by the Ascoli theorem, the result follows from the above interpolation inequality. Q.E.D.

We now have, as a corollary of Theorems 3.2(a) and 3.6 and of the above lemmas,

THEOREM 3.7. *Let Ω be a bounded, admissible domain in E^n. Then*

(a) *if $\int_1^\infty g_B(t)\, dt = \infty$, the imbedding $W^1L_B(\Omega) \to L_C(\Omega)$ is compact for any $C \ll B^*$.*

(b) *if $\int_1^\infty g_B(t)\, dt < \infty$ and Ω is strongly Lipshitz, the imbedding $W^1L_B(\Omega) \to C_\mu(\Omega)$ is compact for any $\mu(t) \ll \int_{t^{-n}}^\infty g_B(t)\, dt$.*

Traces on Hyperplanes

Let Ω satisfy a cone-condition and Ω_k denote the intersection of Ω with a k-dimensional hyperplane in E^n. We then have the following imbeddings for Sobolev spaces (see [1] or [10]) $W^1L_p(\Omega) \to L_{kp/(n-p)}(\Omega_k)$ for $n \geqslant k > n - p$ ($k \geqslant n - p$ if $p = 1$). We extend this result to Orlicz–Sobolev spaces in the following way:

THEOREM 3.8. *Let Ω be bounded and let it satisfy a cone condition; then if $\int_1^\infty g_B(t)\, dt = \infty$ and $n - p < k \leqslant n$, where $p \geqslant 1$ is such that $B(t^{1/p})$ is an N function, we have the imbedding $W^1L_B(\Omega) \to L_{(B^*)^{k/n}}(\Omega_k)$. For $p = 1$, we may let $k = n - 1$.*

Proof. We use an argument similar to the proof of Theorem 3.2. Let $u \in W^1L_B(\Omega)$ be bounded, and suppose that

$$l = \| u \|_{L_{(B^*)^{k/n}}(\Omega_k)} \geqslant \| u \|_{L_{B^*}(\Omega)} . \tag{3.32}$$

Let $C = (B^*)^{1/p-1/n} = (B^*)^{1/p^*}$. Apply the above imbeddings to the function $C(u/l)$. We obtain

$$\begin{aligned}\left\| C\left(\frac{u}{l}\right)\right\|_{kp/(n-p)(\Omega_k)} &\leqslant \gamma \left\{\left\| DC\left(\frac{u}{l}\right)\right\|_{p,\Omega} + \left\| C\left(\frac{u}{l}\right)\right\|_{p,\Omega}\right\} \\ &= \frac{\gamma}{l}\left\| C'\left(\frac{u}{l}\right) | Du | \right\|_p + \gamma \left\| C\left(\frac{u}{l}\right)\right\|_p .\end{aligned}$$

From Eq. (3.32) and the definition of C, we have, therefore,

$$1 \leqslant \frac{\gamma}{l}\left\| C'\left(\frac{u}{l}\right) | Du | \right\|_p + \gamma \left\| C\left(\frac{u}{l}\right)\right\|_p$$

Applying the Hölder inequality (1.7), we obtain

$$1 \leqslant \frac{2\gamma}{l}\left\|\left\{C'\left(\frac{u}{l}\right)\right\}^p\right\|_D^{1/p} \| Du \|_B + \gamma \left\| C\left(\frac{u}{l}\right)\right\|_p , \tag{3.33}$$

where $\bar{D}(t) = B(t^{1/p})$.

Now, since $C = (B^*)^{1/p^*}$, we have

$$\begin{aligned} C' &= \frac{1}{p^*} (B^*)^{1/p^*-1}(B^*)' \\ &= \frac{1}{p^*} (B^*)^{1/p} B^{-1}(B^*) \end{aligned}$$

by Eq. (3.9), and so

$$\begin{aligned} (C')^p &= \left(\frac{1}{p^*}\right)^p B^* \cdot B^{-1}(B^*)^p \\ &= \left(\frac{1}{p^*}\right)^p B^* \cdot \bar{D}^{-1}(B^*) && \text{by the definition of } D \qquad (3.34) \\ &\leqslant \left(\frac{1}{p^*}\right)^p D^{-1}(B^*) && \text{by the expression (1.6).} \end{aligned}$$

Therefore, from Eqs. (3.34) and (3.32), we obtain

$$\left\|\left\{C'\left(\frac{u}{l}\right)\right\}^p\right\|_D \leqslant \left(\frac{1}{p^*}\right)^p .$$

Next, using Lemma 3.2, we have

$$\int \left\{C\left(\frac{u}{l}\right)\right\}^p dx \leqslant \frac{1}{2\gamma}\int \left\{B^*\left(\frac{u}{l}\right)\right\}^{k/n} dx + K/l^p \int |u|^p;$$

and hence we obtain from the expression (3.33)

$$l \leqslant \frac{4\gamma}{p^*}\| Du \|_B + 2K \| u \|_p . \tag{3.35}$$

From the expression (3.35) and Theorem 3.2(a), then follows the estimate

$$l \leqslant C(|\Omega|, n, p) \| u \|_{W^1L_B} \tag{3.36}$$

for any l, not necessarily satisfying the restriction (3.32).

For unbounded u, consideration of the sequence u_m, as in the proof of Theorem 3.2, yields the estimate (3.36) again. The theorem is thus proved. Q.E.D.

As a corollary of the proof of Theorem 3.8, we note that the cone condition may be relaxed in the following way:

COROLLARY 3.2. *The imbeddings of Theorem* 3.8 *continue to hold if in the hypotheses we assume* $W^1L_p(\Omega) \to L_{kp/(n-p)}(\Omega_k)$, *instead of* Ω *satisfying a cone condition.*

From Lemma 3.3 also follows

COROLLARY 3.3. *Under the hypotheses of Theorem* 3.8, *the imbeddings* $W^1L_B(\Omega) \to L_C(\Omega_k)$ *are compact for any* $C \ll (B^*)^{k/n}$.

The Imbedding Theorem for $W^mL_B(\Omega)$

For an N function B, we define a sequence of N functions C_ν, $\nu = 1, 2,\ldots$ by the formulas

$$\begin{cases} C_\nu^{-1}(x) = \displaystyle\int_0^x \frac{C_{\nu-1}^{-1}(t)}{t^{1+1/n}}\, dt, \\ C_0(x) = B(x). \end{cases} \tag{3.37}$$

We assume that $\int_0^1 C_\nu^{-1}/(t^{1+1/n})\, dt < \infty$, by replacing C_ν if necessary by an equivalent N function. We obtain in this way a finite sequence of N functions $C_0, C_1, \ldots, C_q$, where $q < n+1$ is such that $\int_1^\infty C_{q-1}^{-1}(t)/(t^{1+1/n}) = \infty$, but $\int_1^\infty C_q^{-1}(t)/(t^{1+1/n})\, dt < \infty$. Let us denote this integer q by $q(B)$.

We define $C^{m,\mu}(\bar{\Omega})$ to be the space of functions m times continuously differentiable having derivatives $D^\alpha u$, $|\alpha| = m$ in the spaces $C^\mu(\bar{\Omega})$, defined by Eq. (3.31) with

$$\| u \|_{C^{m,\mu}(\bar{\Omega})} = \sup_{|\alpha| \leqslant m, x \in \Omega} | D^\alpha u(x) | + \sup_{|\alpha| = m} \| D^\alpha u \|_{C^\mu(\bar{\Omega})} . \tag{3.38}$$

We can now state

THEOREM 3.9. *Let Ω be a bounded, admissible domain in E^n. Then*

(a) *If $m \leqslant q(B)$, $W^m L_B(\Omega) \to L_{C_m}(\Omega)$ and the imbedding $W^m L_B(\Omega) \to L_C(\Omega)$ is compact for any $C \ll C_m$.*

(b) *If $m > q(B)$, $W^m L_B(\Omega) \to L_\infty(\Omega) \cap C(\Omega)$.*

(c) *If $m < q(B)$ and Ω is strongly Lipshitz, then $W^m L_B(\Omega) \to C^{m-q-1,\mu}(\bar{\Omega})$, where $\mu(t) = \int_{t^{-n}}^\infty C_q^{-1}(s)/(s^{1+1/n})\, ds$ and the imbedding $W^m L_B(\Omega) \to C^{m-q-1}(\bar{\Omega})$ is compact.*

Theorem 3.9 is an immediate consequence of Theorems 3.2, 3.6 and 3.7. To conclude this section, we note that if we replace $W^m L_B(\Omega)$ by $W^m E_B(\Omega)$ in Theorem 3.9, we obtain, in part (a), an imbedding $W^m E_B(\Omega) \to E_{C_m}(\Omega)$. This follows since the bounded functions are dense in $W^m E_B(\Omega)$, (in fact the sequence u_m, defined earlier in this section, converges to u for any u in $W^1 E_B(\Omega)$).

4. COMPLEMENTARITY FOR ORLICZ–SOBOLEV SPACES

It is the purpose of this section to show that basically similar kinds of relationships hold for linear functionals on Orlicz–Sobolev spaces as hold for Orlicz spaces themselves. We first make a definition.

DEFINITION 4.1. Let X and Y be two Banach spaces such that $X \subset Y^*$ and $Y \subset X^*$. Then we shall say that X and Y are a *complementary pair* if there exist closed subspaces $X_0 \subset X$ and $Y_0 \subset Y$ such that $X_0^* = Y$ and $Y_0^* = X$.

The following proposition follows immediately from standard theorems of functional analysis.

PROPOSITION 4.1. *Let X and Y be a complementary pair with $X_0 \subset X = Y_0^*$ and $Y_0 \subset Y = X_0^*$. Then the unit ball of X is compact in the weak* topology given by Y_0, and similarly for the unit ball of Y.*

For many purposes it is useful to know a method by which, given a complementary pair $X = Y_0{}^*$ and $Y = X_0{}^*$, and a closed subspace $E_0 \subset X_0$, one may construct spaces $E \subset X$ and $F = E_0{}^* \subset Y$ such that E and F form a complementary pair.

PROPOSITION 4.2. *Let E_0 be a closed subspace of X_0, where $X = Y_0{}^*$ and $Y = X_0{}^*$ form a complementary pair. Then there exists a complementary pair E and F such that $E_0 \subset E = F_0{}^* \subset X$ and $F_0 \subset F = E_0{}^*$. Furthermore, the equality*

$$E = wk^*cl\ E_0 = F_0{}^* = (E_0{}^\perp)^\perp \cap X \textit{ holds.}$$

Proof. Let $F = E_0{}^*$. Since $E_0 \subset X_0$, $F = X_0{}^*/E_0{}^\perp = Y/E_0{}^\perp$. Let $F_0 = \overline{(Y_0/E_0{}^\perp)}$, and finally let $E = F_0{}^*$. Clearly, $E \subset F_0{}^* = E \subset X$.

To show that $F_0{}^* = wk^*cl\ E_0$, it is clear that $wk^*cl\ E_0 \subset F_0{}^*$. Let $E_1 = wk^*cl\ E_0$. The inclusion $E_0 \subset E_1 \subset E = F_0{}^*$ then holds. If $v_0 \in E$ and $v_0 \notin E_1$, then since E_1 is wk^* closed, there must exist $f_0 + E_0{}^\perp \in \overline{(Y_0/E_0{}^\perp)}$ such that $(v_0, f_0 + E_0{}^\perp) = 1$ and $(v, f_0 + E_0{}^\perp) = 0$ for every $v \in E_1$. But since $E_0 \subset E_1$, $(v, f_0 + E_0{}^\perp) = 0$ for every $v \in E_0{}^\perp$; hence $f_0 \in E_0{}^\perp$. Therefore, $(v_0, f_0 + E_0{}^\perp) = 0$ since $v_0 \in F_0{}^*$. This is a contradiction.

To show that $(E_0{}^\perp)^\perp \cap X = F_0{}^*$, it is sufficient to show that $F_0{}^* \subset (E_0{}^\perp)^\perp \cap X$ since clearly $(E_0{}^\perp)^\perp \cap X \subset F_0{}^*$. But let $v_0 \in F_0{}^*$. Then v_0 defines a linear functional on the equivalence classes $f_0 + E_0{}^\perp$. Therefore, for every $e_0{}^\perp \in E_0{}^\perp$, $(v_0, e_0{}^\perp) = 0$. This implies $v_0 \in (E_0{}^\perp)^\perp$. Clearly, $v_0 \in X$ since $F_0{}^* \subset X$. Q.E.D.

In the case of Orlicz–Sobolev spaces, one may regard $W^mL_B(\Omega)$ in a natural way as a subspace of the product space $\prod^N L_B$ where N is the number of multi-indices α, $|\alpha| \leqslant m$. We obtain automatically from this imbedding that the spaces $W^mL_B(\Omega)$ are reflexive if B and $\bar{B}$ satisfy the Δ_2 condition, i.e., when the spaces $L_B(\Omega)$ are reflexive. We examine the situation, in general, in the light of the preceding proposition.

PROPOSITION 4.3. *The spaces $\prod L_B$ and $\prod L_{\bar{B}}$ form a complementary pair.*

This proposition is clear from the observation that L_B and $L_{\bar{B}}$ form a complementary pair.

We now show that the Orlicz–Sobolev spaces W^mL_B and W^mE_B

may be realized through the process of Proposition 4.2, as subspaces of $\prod L_B$ with $E_0 = W^m E_B$, $E = W^m L_B$.[3]

THEOREM 4.1. *The following characterization holds*

$$W^m L_B = wk^*cl\ W^m E_B = \left\{\prod E_{\bar{B}}/[W^m E_B]^{\perp\perp}\right\}^* = [W^m E_B]^{\perp\perp} \cap \prod L_B .$$

Proof. The equality of the last three terms follows if one carries through the process of Proposition 4.2, setting $E_0 = W^m E_B$. To show the first equality, we need to show first that $E \subset W^m L_B$ and second that $W^m L_B \subset E$, where E is defined as in Proposition 4.2. For the first containment, let $f = (f_1, ..., f_m) \in E$. Then by the equalities of Proposition 4.2, there exists a sequence $f_k = (f_1{}^k, ..., f_m{}^k) \in E_0 = W^m E_B$ such that $f_k \to f$ in the weak* topology. Hence, in particular, since one may take $g \in E_{\bar{B}}$ of the form $(0, 0, ..., g_s, 0, ...)$ it follows that $(f_s{}^k - f_s, g^s) \to 0$ for every $|s| \leqslant m$ and every $g^s \in E_{\bar{B}}$. Therefore, $f_s{}^k \to f_s \in L_B$. To show that $f \in W^m L_B$ it is sufficient to show that for every s, $|s| \leqslant m$, $f_s = D^s f_1$. This is equivalent to the statement that for every $\rho \in C_0{}^\infty(\Omega)$

$$\int_\Omega f_s \rho\, dx = \int_\Omega f_1 D^s \rho\, dx.$$

But since $f_s{}^k \to f_s$ and $f_s{}^k = D^s f^k$, one may take

$$\left| \int_\Omega (f_s - D^s f_1)\rho\, dx \right| \leqslant \left| \int_\Omega (f_s - f_s{}^k)\rho\, dt \right| + \left| \int_\Omega (f_1 - f_1{}^k) D^s \rho\, dt \right| \to 0$$

Therefore, $E \subset W^m L_B$.

To show that $W^m L_B \subset E$, a lemma will be necessary.

LEMMA 4.1. *Let A be an N-function, and suppose $w \in E_A$. Then there exists an N-function $A_0 \gg A$ such that $w \in L_{A_0}$.*

Proof. Let $\Omega_i = \{t, \Omega; i \leqslant |w(t)| < i + 1\}$. Since $w \in E_A$, $\int_\Omega A[(k+1)\, w(t)]\, dt < \infty$ for every integer k. Furthermore, it is clear that

$$\int_\Omega A[(k+1)\, w(t)]\, dt = \sum_{i=0}^{\infty} \int_{\Omega_i} A[(k+1)\, w(t)]\, dt < \infty.$$

[3] Theorem 4.1 has been employed by Donaldson [4] to extend some aspects of monotone operator theory for reflexive Banach spaces to the nonreflexive Orlicz–Sobolev spaces.

Since the series above converges for every k, there exists n_k such that

$$\sum_{i=n_k}^{\infty} \int_{\Omega_i} A[(k+1)w]\, dt < 2^{-k}.$$

Clearly, one may choose the sequence n_k so that n_k increases and $n_k \to \infty$ as $k \to \infty$. Let $n_0 = 0$. Define an N function H by $dH(t)/dt = k+1$ for $x \in [n_k, n_{k+1})$. Let $A_0 = A \circ H$. Clearly, $A_0 \gg A$. Furthermore,

$$\begin{aligned}
\int_\Omega A_0(w)\, dx &= \int_\Omega A[H(w)]\, dx \\
&= \sum_{i=0}^{n_1} \int_{\Omega_i} A[H(w)]\, dx + \sum_{k=1}^{\infty} \sum_{\nu=n_k}^{i=n_k+1} \int_{\Omega_i} A[H(w)]\, dx \\
&\leqslant \sum_{i=0}^{n_1} \int_{\Omega_i} A[H(w)]\, dx + \sum_{k=1}^{\infty} \sum_{i=n_k}^{i=n_k+1} \int_{\Omega_i} A[(k+1)w]\, dx \\
&\leqslant \sum_{i=0}^{n_1} \int_{\Omega_i} A[H(w)]\, dt + \sum_{k=1}^{\infty} 2^{-k} \\
&< \infty
\end{aligned}$$

since on $\bigcup_{i=0}^{n_1} \Omega_i$, the function w is bounded from the definition of Ω_i. Thus $w \in L_{A_0}$. Q.E.D.

To complete the proof of the theorem, suppose there exists $f \in W^m L_B - E$. Since E is closed in the $\prod E_{\bar{B}}$ topology, there exists $g \in \prod E_{\bar{B}}$ such that $(f, g) = 1$ and $(W^m E_B, g) = 0$. Since $g \in \prod E_{\bar{B}}$, every component $g^s \in E_{\bar{B}}$. Since the set of g^s is finite, one may construct by the lemma an N function $\bar{B}_0$ such that $g^s \in L_{\bar{B}_0} \subset E_{\bar{B}}$ for every s, and hence $g \in \prod L_{\bar{B}_0}$. Since $\bar{B}_0 \gg \bar{B}$, $B \gg B_0$ and $\prod L_B \subset \prod E_{B_0}$. Since $f \in W^m L_B$, $f \in W^m E_{B_0}$, and there exists, therefore, a sequence of C^∞ functions f_k such that $\| f_k - f \|_{W^m L_{B_0}} \to 0$. So

$$|(f_k - f, g)| \leqslant \| f_k - f \|_{W^m L_{B_0}} \| g \|_{L_{\bar{B}_0}} \to 0$$

But $f_k \in C^\infty(\Omega) \subset W^m E_B$. Hence by the above assumption $|(f_k - f, g)| = |(f, g)| \neq 0$. This is a contradiction. Q.E.D.

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