

Global Morrey Regularity of Strong Solutions to the Dirichlet Problem for Elliptic Equations with Discontinuous Coefficients

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Communicated by D. Sarason

Received August, 15, 1997; revised November 19, 1998

Well-posedness is proved in the space $W^{2,p,\lambda}(\Omega) \cap W_0^{1,p}(\Omega)$ for the Dirichlet problem

$$\begin{cases} \sum_{i,j}^n a_{ij}(x) D_{x_i x_j} u = f(x) \in L^{p,\lambda}(\Omega) & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if the principal coefficients $a_{ij}(x)$ of the uniformly elliptic operator belong to $VMO \cap L^\infty(\Omega)$. © 1999 Academic Press

1. INTRODUCTION

In the last thirty years a number of papers have been devoted to the study of local and global regularity properties of strong solutions to elliptic

equations with discontinuous coefficients. To be more precise, let us consider the second order equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^n a_{ij}(x) D_{x_i x_j} u = f(x) \quad \text{for almost all } x \in \Omega, \quad (1.1)$$

where \mathcal{L} is a uniformly elliptic operator over the bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$.

The regularizing properties of \mathcal{L} in Hölder spaces (i.e., $\mathcal{L}u \in C^\alpha(\bar{\Omega})$ implies $u \in C^{2+\alpha}(\bar{\Omega})$) have been well studied in the case of Hölder continuous coefficients $a_{ij}(x)$. Also, unique classical solvability of the Dirichlet problem for (1.1) has been derived in this case. (We refer the reader to [19] and the references therein.) In the case of uniformly continuous coefficients a_{ij} , an L^p -Schauder theory has been elaborated for the operator \mathcal{L} [1, 19]. In particular, $\mathcal{L}u \in L^p(\Omega)$ always implies that the strong solutions to (1.1) belong to the Sobolev space $W^{2,p}(\Omega)$ for each $p \in (1, \infty)$.

However, the situation becomes rather difficult if one tries to allow discontinuity at the principal coefficients of \mathcal{L} . In general, it is well known (cf. [23]) that arbitrary discontinuity of a_{ij} 's breaks down as the L^p -theory of \mathcal{L} , as the strong solvability of the Dirichlet problem for (1.1). A notable exception of that rule is the two-dimensional case ($\Omega \subset \mathbb{R}^2$). It was shown by Talenti ([27]) that the sole condition on measurability and boundedness of the a_{ij} 's ensure isomorphic properties of \mathcal{L} considered as a mapping from $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ into $L^2(\Omega)$. This way, to handle with the multi-dimensional case ($n \geq 3$) additional requirements on $a_{ij}(x)$ should be added to the uniform ellipticity in order \mathcal{L} to possess the regularizing property in Sobolev functional scales. In particular, if $a_{ij}(x) \in W^{1,n}(\Omega)$ (cf. [24]), or if the difference between the largest and the smallest eigenvalues of $\{a_{ij}(x)\}$ is small enough (the Cordes condition, see [7]), then $\mathcal{L}u \in L^2(\Omega)$ yields $u \in W^{2,2}(\Omega)$ and these results can be extended to $W^{2,p}(\Omega)$ for $p \in (2 - \varepsilon, 2 + \varepsilon)$ with sufficiently small ε .

Recently (see [8] for an exhaustive presentation) the Sarason class VMO of functions with vanishing mean oscillation was employed in the study of local and global Sobolev regularity of the strong solutions to (1.1). More precisely, it was proved in a series of papers by Chiarenza, Frasca, Franciosi and Longo [11, 12, 10] that if $a_{ij}(x) \in VMO \cap L^\infty(\Omega)$ and $\mathcal{L}u \in L^p(\Omega)$ then $u \in W^{2,p}(\Omega)$ for each value of p in the range $(1, \infty)$. Also, the well-posedness of the Dirichlet problem for (1.1) in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ were proved. These results were extended to the case of oblique derivative boundary condition [13, 22] as well as to quasilinear equations with VMO coefficients [25, 14]. As consequence, Hölder continuity of the strong solution or of its gradient follows, if the exponent p is sufficiently large. On the other hand, Hölder continuity can be inferred for small p also, if one

known more fine information on $\mathcal{L}u$ such as its belonging to suitable Morrey space $L^{p,\lambda}(\Omega)$.

This way, a natural problem arises to study the regularizing properties of the operator \mathcal{L} in Morrey spaces in the case of VMO principal coefficients. In [4], Caffarelli proved that each $W^{2,p}$ -viscosity solution to (1.1) lies in $C_{\text{loc}}^{1+\alpha}(\Omega)$ if $f(x)$ belongs to the Morrey space $L_{\text{loc}}^{n,n\alpha}(\Omega)$ with $\alpha \in (0, 1)$. However, it seems that the assumption on f cannot be replaced by the weaker one $f \in L_{\text{loc}}^{p,\lambda}(\Omega)$, $p < n$, $\lambda > 0$, because of the Aleksandrov–Pucci maximum principle employed in the proofs in [4]. In the paper [16], Di Fazio and Ragusa proved interior Morrey regularity for the second derivatives of $W^{2,p}$ -solutions to (1.1). Precisely, the authors showed that $D^2u \in L_{\text{loc}}^{p,\lambda}(\Omega)$ whenever $f \in L_{\text{loc}}^{p,\lambda}(\Omega)$ and $a_{ij}(x) \in VMO \cap L^\infty(\Omega)$.

The general aim of the present paper is to extend the local regularity results in Morrey spaces from [4, 16] to global ones. More precisely, we will consider the Dirichlet problem

$$\begin{cases} \mathcal{L}u = f(x) & \text{almost everywhere in } \Omega, \\ u \in W^{2,p,\lambda}(\Omega) \cap W_0^{1,p}(\Omega), & p \in (1, \infty) \end{cases} \quad (1.2)$$

in the case of $VMO \cap L^\infty(\Omega)$ principal coefficients $a_{ij}(x)$ and $f \in L^{p,\lambda}(\Omega)$. As we mentioned above, problem (1.2) with $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ has been already studied in [12], and so the regularity and existence results in $W^{2,p}(\Omega)$ are known. Our goal will be to show that finer regularity of the right-hand side $f(x)$ increases the regularity of the second derivatives of the solution. Namely, if $u \in W^{2,p}(\Omega)$, $1 < p < \infty$, is a strong solution to (1.2) with $f \in L^{p,\lambda}(\Omega)$, then $u \in W^{2,p,\lambda}(\Omega)$ (the set of functions $u \in W^{2,p}(\Omega)$ such that $D^2u \in L^{p,\lambda}(\Omega)$). This way, our first result asserts boundary Morrey regularity for the second derivatives of the strong $W^{2,p}$ -solution. Indeed, that result combined with the interior Morrey regularity already derived in [16], implies immediately the global regularizing property of \mathcal{L} in Morrey spaces (Theorem 3.3). To derive the boundary regularity, we use an explicit representation formula for the second derivatives in terms of singular integral operators and commutators with Calderón–Zygmund kernels. This way, the analytic heart of our paper consists of proving boundedness in Morrey spaces of these integral operators. After that, using standard techniques from the theory of PDE's, we derive the global a priori estimate in Morrey spaces for strong solutions to (1.2). A combination of that estimate with the $W^{2,p}$ -strong solvability of (1.2) proved in [12] leads to the second result of the paper. Namely, we show well-posedness of the Dirichlet problem (1.2) in the space $W^{2,p,\lambda}(\Omega) \cap W_0^{1,p}(\Omega)$ for each $f \in L^{p,\lambda}(\Omega)$ and each $p \in (1, \infty)$ (Theorem 3.4).

Indeed, as a by-product of our global a priori estimate and the known properties of Morrey spaces, we derive Hölder continuity of the gradient

Du for suitable values of $p \in (1, \infty)$ and $\lambda \in (0, n)$ (Corollary 4.1). We believe that the possibility to have an estimate for the Hölder seminorm of the gradient under very weak assumptions on $a_{ij}(x)$ and $f(x)$ will be useful in the theory of nonlinear elliptic equations (cf. [5]).

At this end, let us note that quasilinear problems as well as the case of other (oblique derivative) boundary conditions will be treated in forthcoming papers.

2. SOME DEFINITIONS, NOTATIONS, AND PRELIMINARY RESULTS

In this section we define all function spaces needed in the sequel. Let Ω be an open and bounded set in \mathbb{R}^n with sufficiently smooth boundary $\partial\Omega$.

Throughout the paper we will denote by $B_\rho(x)$ a ball of radius ρ centered at the point x , while $Q_\rho(x)$ stands for a cube centered at x and of side length ρ . Further, we set $B_\rho^+ = B_\rho(x) \cap \{x \in \mathbb{R}^n : x_n > 0\}$.

First of all we start with the definition of Morrey spaces. Let $1 < p < \infty$ and $0 < \lambda < n$. We say that a locally integrable function $f(x)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\|f\|_{L^{p,\lambda}(\Omega)}^p \equiv \sup_{\substack{x \in \Omega \\ \rho > 0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x) \cap \Omega} |f(y)|^p dy < \infty.$$

To formulate the main hypotheses on the coefficients of elliptic operators under consideration, we shall need also the John–Nirenberg space [21] of functions with bounded mean oscillation (BMO) and the Sarason class VMO of the functions with vanishing mean oscillation ([26]). Let $f(x)$ be a locally integrable function. We say that $f(x)$ belongs to BMO if

$$\|f\|_* \equiv \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty.$$

Hereafter, f_B stands for the integral average $1/|B| \int_B f(x) dx$ of the function $f(x)$ over the set B , and B ranges in the class of balls of \mathbb{R}^n . For a function $f(x) \in BMO$ set

$$\eta(r) = \sup_{\substack{x \in \mathbb{R}^n \\ \rho \leq r}} \frac{1}{|B_\rho|} \int_{B_\rho} |f(x) - f_{B_\rho}| dx.$$

We say that $f(x) \in VMO$ if $\lim_{r \rightarrow 0^+} \eta(r) = 0$ and refer to $\eta(r)$ as the VMO -modulus of $f(x)$.

Let us note that replacing the ball B above by the intersection $B \cap \Omega$, one obtains the definitions of $BMO(\Omega)$ and $VMO(\Omega)$. Further, in view of

[2, Proposition 1.3], having a function defined on Ω that belongs to $BMO(\Omega)$ ($VMO(\Omega)$) it is possible to extend it to all \mathbb{R}^n preserving its $BMO(VMO)$ character. In the following we will use this remark without explicit reference.

We shall also use the classical Sobolev spaces

$$W^{k,p}(\Omega) = \{f(x): D^\alpha f \in L^p(\Omega), |\alpha| \leq k\}, \quad k \text{ is an integer,}$$

equipped with the norm $\|f\|_{W^{k,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|D^k f\|_{L^p(\Omega)}$. The closure of the space $C_0^\infty(\Omega)$ with respect to the norm in $W^{k,p}(\Omega)$ will be denoted, as usual, by $W_0^{k,p}(\Omega)$.

Finally, we set $W^{k,p,\lambda}(\Omega)$ for the Banach space of functions belonging to $W^{k,p}(\Omega)$ and having k th order derivative lying in the Morrey space $L^{p,\lambda}(\Omega)$. A natural norm in that space is

$$\|f\|_{W^{k,p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + \|D^k f\|_{L^{p,\lambda}(\Omega)}.$$

DEFINITION 2.1. Let $f(x)$ be a locally integrable function in \mathbb{R}^n we define the maximal function and the sharp maximal function of $f(x)$ respectively as

$$\begin{aligned} Mf(x) &= \sup_{r>0} \frac{1}{|Q_r(x)|} \int_{Q_r(x)} |f(y)| dy, \\ f^\#(x) &= \sup_{r>0} \frac{1}{|Q_r(x)|} \int_{Q_r(x)} |f(y) - f_{Q_r}| dy, \end{aligned} \quad (2.1)$$

where $Q_r(x)$ ranges in the class of all cubes centered in $x \in \mathbb{R}^n$ with side length $r > 0$.

Let $f(x)$ be a locally integrable function defined in the half space $\mathbb{R}_+^n \equiv \mathbb{R}^n \cap \{x_n > 0\}$. Following [20], we may define the local maximal function and the local sharp maximal function of the function f respectively as did in (2.1) but now the cubes are centered at $x \in \mathbb{R}_+^n$ and it is assumed $f(x)$ to be extended identically zero in the half-space $\{x_n < 0\}$. It is a known result that the maximal function is bounded in the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$. The same is true, with the same proof, if the function is considered only in the half-space. Namely we have

THEOREM 2.2. For any $1 < p < \infty$ and any $0 < \lambda < n$, the local maximal operator is bounded. Precisely, there exists a constant c independent of $f(x)$ such that

$$\|Mf\|_{L^{p,\lambda}(\mathbb{R}_+^n)} \leq c \|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)}.$$

Proof. Let $x_0 \in \mathbb{R}_+^n$ and $\rho > 0$. Now, using the inequality in [17, Lemma 1, p. 111], we have

$$\begin{aligned} \int_{Q_\rho(x_0) \cap \mathbb{R}_+^n} |Mf(x)|^p dx &= \int_{\mathbb{R}^n} |Mf(x)|^p \chi_{Q_\rho(x_0) \cap \mathbb{R}_+^n} dx \\ &\leq c \int_{\mathbb{R}^n} |f(x)|^p M\chi_{Q_\rho(x_0) \cap \mathbb{R}_+^n} dx \end{aligned}$$

with $\chi_{Q_\rho(x_0) \cap \mathbb{R}_+^n}$ being the characteristic function of the respective set. Following the lines of [9, Theorem 1, p. 275]), we obtain

$$\int_{Q_\rho(x_0) \cap \mathbb{R}_+^n} |Mf(x)|^p dx \leq c\rho^\lambda \|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)}^p$$

and the theorem is proved. ■

In a similar manner it is possible to prove the following theorem about the local sharp maximal function (for the global result we refer the reader to [15]).

THEOREM 2.3. *For any $1 < p < \infty$ and any $0 < \lambda < n$, there exists a constant c independent of $f(x)$ such that*

$$\|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)} \leq c \|f^\#\|_{L^{p,\lambda}(\mathbb{R}_+^n)}.$$

Proof. Let $x_0 \in \mathbb{R}_+^n$, $r > 0$ be arbitrary and take a point $\bar{x}_0 \in \mathbb{R}_+^n$ such that $Q_r^+(x_0) \equiv Q_r(x_0) \cap \mathbb{R}_+^n \subset Q_{2r}(\bar{x}_0)$. Denote by $\chi_{Q_{2r}(\bar{x}_0)}$ the characteristic function of the cube $Q_{2r}(\bar{x}_0)$. Let $\gamma \in (\lambda/n, 1)$. Then, by means of the known properties of the maximal function, we obtain

$$\begin{aligned} \int_{Q_r^+(x_0)} |f(x)|^p dx &\leq \int_{\mathbb{R}^n} |f(x)|^p \chi_{Q_{2r}(\bar{x}_0)} dx \leq \int_{\mathbb{R}^n} |f(x)|^p M\chi_{Q_{2r}(\bar{x}_0)} dx \\ &\leq \int_{\mathbb{R}^n} |Mf(x)|^p (M\chi_{Q_{2r}(\bar{x}_0)})^\gamma dx. \end{aligned}$$

To proceed further, we employ the Fefferman–Stein inequality between the maximal and the sharp maximal functions (cf. [18, p. 410]), with weight $(M\chi_{Q_{2r}(\bar{x}_0)})^\gamma$. Thus,

$$\begin{aligned} \int_{\mathbb{R}^n} |Mf(x)|^p (M\chi_{Q_{2r}(\bar{x}_0)})^\gamma dx \\ \leq c(n, p) \int_{\mathbb{R}^n} |f^\#(x)|^p (M\chi_{Q_{2r}(\bar{x}_0)})^\gamma dx \end{aligned}$$

$$\begin{aligned}
&\leq c(n, p) \left\{ \int_{Q_{2r}(\bar{x}_0)} |f^\#(x)|^p dx \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \int_{Q_{2^{k+1}r}(\bar{x}_0) \setminus Q_{2^k r}(\bar{x}_0)} |f^\#(x)|^p \left(\frac{r}{|\bar{x}_0 - x| - r} \right)^{ny} dx \right\} \\
&\leq c(n, p, \lambda, \gamma) r^\lambda \|f^\#\|_{L^{p, \lambda}(\mathbb{R}_+^n)}^p \sum_{k=0}^{\infty} 2^{k(\lambda - n\gamma)} \\
&\leq c(n, p, \lambda, \gamma) \frac{1}{1 - 2^{\lambda - n\gamma}} r^\lambda \|f^\#\|_{L^{p, \lambda}(\mathbb{R}_+^n)}^p
\end{aligned}$$

(recall $\lambda < n\gamma$), and the result follows. \blacksquare

Our strategy in deriving boundary Morrey regularity of solutions to the Dirichlet problem is based on an explicit representation of solution's second-order derivatives. That representation formula is a sum of singular integrals of two types. The first one is a sum of singular integral operators and commutators with Calderon-Zygmund kernels (see [12, 16]). The boundedness in Morrey spaces of that types of operators has been proved in [16] and we present the respective result here (Theorem 2.4). On the other hand, the second type operators appearing in the representation formula are less singular and they due to the specific boundary condition. The boundedness of these operators will be proved below (Theorems 2.5 and 2.6).

THEOREM 2.4 [16, Theorem 2.3]. *Let D be an open subset of \mathbb{R}^n , $f \in L^{p, \lambda}(D)$, $1 < p < \infty$, $0 < \lambda < n$, $a \in VMO \cap L^\infty(\mathbb{R}^n)$. Let $k(x, z)$ be a Calderón-Zygmund kernel in the z variable for almost all $x \in D$ such that*

$$\max_{|\alpha| \leq 2n} \left\| \frac{\partial^\alpha}{\partial z^\alpha} k(x, z) \right\|_{L^\infty(D \times \Sigma)} = M < +\infty,$$

where Σ denotes the surfaces of the unit sphere in \mathbb{R}^n . For any $\varepsilon > 0$, we set

$$K_\varepsilon f(x) = \int_{|x-y| > \varepsilon, x \in D} k(x, x-y) f(y) dy,$$

$$C_\varepsilon(a, f)(x) = \int_{|x-y| > \varepsilon, x \in D} k(x, x-y)(a(x) - a(y)) f(y) dy.$$

Then there exist $Kf, C(a, f) \in L^{p, \lambda}(D)$ such that

$$\lim_{\varepsilon \rightarrow 0} \|K_\varepsilon f - Kf\|_{L^{p, \lambda}(D)} = \lim_{\varepsilon \rightarrow 0} \|C_\varepsilon(a, f) - C(a, f)\|_{L^{p, \lambda}(D)} = 0$$

and moreover, there is a positive constant $c = c(n, p, \lambda, M)$ such that

$$\|Kf\|_{L^{p, \lambda}(D)} \leq c \|f\|_{L^{p, \lambda}(D)}, \quad \|C(a, f)\|_{L^{p, \lambda}(D)} \leq c \|a\|_* \|f\|_{L^{p, \lambda}(D)}.$$

THEOREM 2.5. *Let $x \in \mathbb{R}_+^n$ and*

$$\tilde{K}f(x) = \int_{\mathbb{R}_+^n} \frac{f(y)}{|\tilde{x} - y|^n} dy, \quad \tilde{x} \equiv (x_1, \dots, x_{n-1}, -x_n).$$

Then there exists a constant c independent of $f(x)$, such that

$$\|\tilde{K}f\|_{L^{p, \lambda}(\mathbb{R}_+^n)} \leq c \|f\|_{L^{p, \lambda}(\mathbb{R}_+^n)}.$$

Proof. Let $x_0 \in \mathbb{R}_+^n$ and $r > 0$ be arbitrary. Then, for $x \in \mathbb{R}_+^n$ we have

$$f(x) = f(x) \chi_{B_{2r}^+(x_0)} + \sum_{k=1}^{\infty} f(x) \chi_{B_{2^{k+1}r}^+(x_0) \setminus B_{2^k r}^+(x_0)} \equiv \sum_{k=0}^{\infty} f_j(x).$$

By virtue of [12, Theorem 2.5], one has

$$\begin{aligned} \int_{B_r^+(x_0)} |\tilde{K}f_0(x)|^p dx &\leq \|\tilde{K}f_0\|_{L^p(\mathbb{R}_+^n)}^p \leq c(n, p) \|f_0\|_{L^p(\mathbb{R}_+^n)}^p \\ &= c(n, p) \int_{B_{2r}^+(x_0)} |f(x)|^p dx \leq c(n, p, \lambda) r^\lambda \|f\|_{L^{p, \lambda}(\mathbb{R}_+^n)}^p. \end{aligned} \quad (2.2)$$

Let $k \geq 1$ and $x \in B_r^+(x_0)$. For every $y \in B_{2^{k+1}r}^+(x_0) \setminus B_{2^k r}^+(x_0)$ we have

$$|\tilde{x} - y| \geq |x - y| \geq (2^k - 1)r \geq 2^{k-1}r.$$

Therefore,

$$\begin{aligned} |\tilde{K}f_k(x)|^p &\leq \left(\int_{2^k r \leq |x_0 - y| \leq 2^{k+1}r} \frac{|f(y)|}{|\tilde{x} - y|^n} dy \right)^p \\ &\leq \left(2^{n(1-k)} r^{-n} \int_{|x_0 - y| \leq 2^{k+1}r} |f(y)| dy \right)^p \\ &\leq c(n) 2^{np(1-k)} r^{-np} 2^{(k+1)p/p'} r^{np/p'} \int_{|x_0 - y| \leq 2^{k+1}r} |f(y)|^p dy \end{aligned}$$

as consequence of the Hölder inequality ($1/p + 1/p' = 1$). In other words

$$\begin{aligned} |\tilde{K}f_k(x)|^p &\leq c(n) 2^{2np-n-kn} r^{-n} \int_{|x_0-y| \leq 2^{k+1}r} |f(y)|^p dy \\ &\leq c(n) 2^{2np-n-kn} r^{-n} 2^{\lambda+k\lambda} r^\lambda \|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)}^p \\ &\leq c(n, p, \lambda) 2^{k(\lambda-n)} r^{\lambda-n} \|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)}^p. \end{aligned}$$

Hence

$$\int_{B_r^+(x_0)} |\tilde{K}f_k(x)|^p dx \leq c(n, p, \lambda) 2^{k(\lambda-n)} r^\lambda \|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)}^p.$$

A combination of the last inequality with (2.2) yields

$$\begin{aligned} \int_{B_r^+(x_0)} |\tilde{K}f(x)|^p dx &\leq \sum_{k=0}^{\infty} \int_{B_r^+(x_0)} |\tilde{K}f_k(x)|^p dx \\ &\leq c(n, p, \lambda) 2^{k(\lambda-n)} r^\lambda \|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)}^p \sum_{k=0}^{\infty} 2^{k(\lambda-n)} \\ &= c(n, p, \lambda) \frac{1}{1-2^{\lambda-n}} r^\lambda \|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)}^p. \end{aligned}$$

This completes the proof of Theorem 2.5. ■

THEOREM 2.6. *Let $f \in L^{p,\lambda}(\mathbb{R}_+^n)$, $1 < p < \infty$, $0 < \lambda < n$, $a \in VMO \cap L^\infty(\mathbb{R}_+^n)$. For $x \in \mathbb{R}_+^n$, define*

$$\tilde{C}(a, f)(x) = \int_{\mathbb{R}_+^n} \frac{|a(x) - a(y)|}{|\tilde{x} - y|^n} f(y) dy. \quad (2.3)$$

The commutator defined by (2.3) is bounded from $L^{p,\lambda}(\mathbb{R}_+^n)$ into itself. Precisely, there exists a constant c independent of $a(x)$ and $f(x)$ such that

$$\|\tilde{C}(a, f)\|_{L^{p,\lambda}(\mathbb{R}_+^n)} \leq c \|a\|_* \|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)}. \quad (2.4)$$

Proof. It is known that, for almost all $x \in \mathbb{R}_+^n$, $1 < r < \infty$, (see [3, Theorem 2.1]) one has

$$|\tilde{C}(a, f)^\#(x)| \leq c \|a\|_* \{ (M(\tilde{K}(|f|))^r)^{1/r}(x) + (M(|f|)^r)^{1/r}(x) \}. \quad (2.5)$$

Let $x_0 \in \mathbb{R}_+^n$, $\rho > 0$, $f \in L^{p,\lambda}(\mathbb{R}^n)$, $1 < r < p < \infty$. Using Theorems 2.2, 2.3, and 2.5, we get

$$\begin{aligned}
\int_{B_\rho(x_0) \cap \mathbb{R}_+^n} |M(\tilde{K}(|f|)^r)^{p/r}(x)| dx &\leq \rho^\lambda \|M(\tilde{K}(|f|^r))^{1/r}\|_{L^{p,\lambda}(\mathbb{R}_+^n)}^p \\
&\leq c \rho^\lambda \|(\tilde{K}(|f|^r))^{1/r}\|_{L^{p,\lambda}(\mathbb{R}_+^n)}^p \\
&= c \rho^\lambda \|\tilde{K}(|f|^r)\|_{L^{p/r,\lambda}(\mathbb{R}_+^n)}^{p/r} \\
&\leq c \rho^\lambda \| |f|^r \|_{L^{p/r,\lambda}(\mathbb{R}_+^n)}^{p/r} \\
&= c \rho^\lambda \|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)}^p.
\end{aligned}$$

The estimate of the second term in (2.5) is even simpler, so

$$\|C(a, f)^\# \|_{L^{p,\lambda}(\mathbb{R}_+^n)} \leq c \|a\|_* \|f\|_{L^{p,\lambda}(\mathbb{R}_+^n)}.$$

Thus the estimate (2.4) follows by virtue of the boundedness of the sharp maximal operator (Theorem 2.3). ■

3. THE DIRICHLET PROBLEM

Let Ω be an open bounded set in \mathbb{R}^n with a $C^{1,1}$ smooth boundary and consider the second order differential operator

$$L = a_{ij}(x) D_{ij}, \quad D_{ij} \equiv \frac{\partial^2}{\partial x_i \partial x_j}.$$

(Here we have adopted the usual-summation convention on repeated indices.)

In the sequel, we will need the following regularity and ellipticity assumptions on the coefficients of \mathcal{L} :

$$\begin{cases} a_{ij}(x) \in VMO(\Omega), & a_{ij}(x) = a_{ji}(x), \\ \exists \kappa > 0: \kappa^{-1} |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \kappa |\xi|^2 & \forall \xi \in \mathbb{R}^n, \text{ a.a. } x \in \Omega. \end{cases} \quad (3.1)$$

Set η_{ij} for the VMO -modulus of the function $a_{ij}(x)$ and let $\eta(r) = (\sum_{i,j=1}^n \eta_{ij}^2)^{1/2}$. We denote by $\Gamma(x, t)$ the normalized fundamental solution of \mathcal{L} , i.e.,

$$\begin{aligned}
\Gamma(x, \xi) &= \frac{1}{n(2-n) \omega_n \sqrt{\det\{a_{ij}(x)\}}} \left(\sum_{i,j=1}^n A_{ij}(x) \xi_i \xi_j \right)^{(2-n)/2} \\
&\text{for a.a. } x \text{ and all } \xi \in \mathbb{R}^n \setminus \{0\},
\end{aligned}$$

where $A_{ij}(x)$ stand for the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\dots,n}$, and ω_n is the measure of the unit ball in \mathbb{R}^n . We set also

$$\Gamma_i(x, \xi) = \frac{\partial}{\partial \xi_i} \Gamma(x, \xi),$$

$$\Gamma_{ij}(x, \xi) = \frac{\partial}{\partial \xi_i \partial \xi_j} \Gamma(x, \xi),$$

$$M = \max_{i, j=1, \dots, n} \max_{|\alpha| \leq 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(x, \xi)}{\partial \xi^\alpha} \right\|_{L^\infty(\Omega \times \Sigma)}.$$

It is well known that $\Gamma_{ij}(x, \xi)$ are Calderón–Zygmund kernels in the ξ variable.

We shall need the following result concerning interior Morrey regularity, which is proved in [16]:

THEOREM 3.1. *Let (3.1) be true, $1 < p < \infty$, $0 < \lambda < n$. Suppose $u \in W_{\text{loc}}^{2,p}(\Omega)$ and $\mathcal{L}u \in L_{\text{loc}}^{p,\lambda}(\Omega)$.*

Then, for any $\Omega' \subset \Omega'' \subset \Omega$, we have $D_{ij}u \in L^{p,\lambda}(\Omega')$ and there exists a constant $c = c(n, \kappa, p, \lambda, \text{dist}(\Omega', \partial\Omega''), M, \eta)$ such that

$$\|D_{ij}u\|_{L^{p,\lambda}(\Omega')} \leq c(\|u\|_{L^{p,\lambda}(\Omega'')} + \|\mathcal{L}u\|_{L^{p,\lambda}(\Omega'')}).$$

In the following theorem we prove the main result of the paper which concerns the boundary regularity in Morrey spaces for the second derivatives of solutions to the Dirichlet problem for the operator \mathcal{L} . Define $W_{\gamma_0}^{2,p}(B_r^+)$ to be the closure in $W^{2,p}$ of the subspace $C_{\gamma_0} = \{u: u \text{ is the restriction to } B_r^+ = \{x = (x_1, \dots, x_n) \equiv (x', x_n) \in \mathbb{R}^n: x_n > 0\} \text{ of a function belonging to } C_0^\infty(B_r), u(x', 0) = 0\}$.

THEOREM 3.2. *Let (3.1) be true, $1 < p < \infty$, $0 < \lambda < n$. There exist constants $c = c(n, \kappa, p, \lambda, M, \partial\Omega)$ and $\rho_0 \in (0, r)$ such that for every $\rho < \rho_0$ and every $u \in W_{\gamma_0}^{2,p}(B_r^+)$ satisfying $\mathcal{L}u \in L^{p,\lambda}(B_r^+)$ and $D_{ij}u \in L^{p,\lambda}(B_r^+)$, one has*

$$\|D_{ij}u\|_{L^{p,\lambda}(B_\rho^+)} \leq c \|\mathcal{L}u\|_{L^{p,\lambda}(B_\rho^+)}. \quad (3.2)$$

Proof. We recall the representation formula for second derivatives of functions in $W_{\gamma_0}^{2,p}(B_r^+)$ (see [12]).

$$\begin{aligned} D_{ij}u(x) &= \text{P.V.} \int_{B_r^+} \Gamma_{ij}(x, x-y) \\ &\quad \times \left\{ \sum_{h,k=1}^n (a_{hk}(x) - a_{hk}(y)) D_{hk}u(y) + \mathcal{L}u(y) \right\} dy \\ &\quad + \mathcal{L}u(x) \int_{|\xi|=1} \Gamma_i(x, \xi) \xi_j d\sigma_\xi + I_{ij}(x), \end{aligned} \quad (3.3)$$

where we have set

$$I_{ij}(x) = \int_{B_r^+} \Gamma_{ij}(x, T(x) - y) \left\{ \sum_{h,k=1}^n (a_{hk}(x) - a_{hk}(y)) D_{hk} u(y) + \mathcal{L}u(y) \right\} dy$$

for $1 \leq i, j < n$;

$$\begin{aligned} I_{in}(x) &= I_{ni}(x) \\ &= \int_{B_r^+} \left(\sum_{j=1}^n \Gamma_{ij}(x, T(x) - y) A_j(x) \right) \\ &\quad \times \left\{ \sum_{h,k=1}^n (a_{hk}(x) - a_{hk}(y)) D_{hk} u(y) + \mathcal{L}u(y) \right\} dy \end{aligned}$$

for $1 \leq i < n$;

$$\begin{aligned} I_{nn}(x) &= \int_{B_r^+} \left(\sum_{i,j=1}^n \Gamma_{ij}(x, T(x) - y) A_i(x) A_j(x) \right) \\ &\quad \times \left\{ \sum_{h,k=1}^n (a_{hk}(x) - a_{hk}(y)) D_{hk} u(y) + \mathcal{L}u(y) \right\} dy. \end{aligned}$$

Further,

$$T(x, y) = x - \frac{2x_n}{a_{nn}(y)} \mathbf{a}_n(y), \quad T(x) \equiv T(x, x),$$

with $\mathbf{a}_n(y) = (a_{in}(y))_{i=1, \dots, n}$ being the last row (column) of the matrix $\mathbf{a}(y) = \{a_{ij}(y)\}_{i,j=1, \dots, n}$, and

$$A(x) = (A_1(x), \dots, A_n(x)) = T(e_n, x) \equiv T((0, \dots, 0, 1), x).$$

In order to derive the estimate (3.2), we will take the $L^{p, \lambda}$ -norms of the both sides of (3.3). So, let us remark that

- (i) the first integral appearing in (3.3) is a Principal Value one and to estimate it in $L^{p, \lambda}$ is necessary to use Theorem 2.4;
- (ii) $\int_{|\xi|=1} \Gamma_i(x, \xi) \xi_j d\sigma_\xi \in L^\infty(B_r^+)$ with a bound independent of r ;
- (iii) the integrals appearing in the definition of I_{ij} are Lebesgue integrals. Although they are not singular, they cannot be treated by the help of Theorem 2.4.

It is not hard to see that the operators I_{ij} are bounded. Indeed, using the geometric properties of T , it is easy to show that

$$c_1 |\tilde{x} - y| \leq |T(x) - y| \tag{3.4}$$

for some positive constant c_1 (see [12, Lemma 3.1]), and then Theorems 2.5 and 2.6 yield

$$\begin{aligned} \|I_{ij}\|_{L^{p,\lambda}(B_r^+)} &\leq c \sum_{h,k=1}^n \|\tilde{C}(a_{hk}, D_{hk}u)\|_{L^{p,\lambda}(B_r^+)} + \|\tilde{K}(\mathcal{L}u)\|_{L^{p,\lambda}(B_r^+)} \\ &\leq c \left(\sum_{h,k=1}^n \|a_{hk}\|_* \|D_{hk}u\|_{L^{p,\lambda}(B_r^+)} + \|\mathcal{L}u\|_{L^{p,\lambda}(B_r^+)} \right). \end{aligned}$$

Therefore,

$$\|D_{ij}u\|_{L^{p,\lambda}(B_r^+)} \leq c(\eta(r) \|D_{ij}u\|_{L^{p,\lambda}(B_r^+)} + \|\mathcal{L}u\|_{L^{p,\lambda}(B_r^+)}).$$

This way, in view of the *VMO* assumption on the coefficients $a_{ij}(x)$, it is possible to choose ρ_0 so small that $c\eta(\rho_0) = 1/2$ and then

$$\|D_{ij}u\|_{L^{p,\lambda}(B_\rho^+)} \leq c \|\mathcal{L}u\|_{L^{p,\lambda}(B_\rho^+)} \quad \text{for each } \rho < \rho_0. \quad \blacksquare$$

The next regularity result refines what was proved in Theorem 3.2. In fact, we are able to remove the assumption $D_{ij}u \in L^{p,\lambda}$.

THEOREM 3.3. *Let (3.1) be true, $1 < p < \infty$ and $0 < \lambda < n$. Assume further that $f \in L^{p,\lambda}(\Omega)$ and $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ are such that $\mathcal{L}u = f$ a.e. in Ω .*

Then $D_{ij}u \in L^{p,\lambda}(\Omega)$ and moreover, there exists a constant $c = c(n, \kappa, p, \lambda, M, \partial\Omega)$ such that

$$\|D_{ij}u\|_{L^{p,\lambda}(\Omega)} \leq c(\|u\|_{L^{p,\lambda}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}). \quad (3.5)$$

Proof. We start with deriving a local version of (3.5) near the boundary. So, let B_ρ^+ be a half ball with a radius ρ to be chosen later. For an arbitrary function $f \in L^{p,\lambda}(B_\rho^+)$, we set

$$S_{ijhk}(f) \equiv \text{P.V.} \int_{B_\rho^+} \Gamma_{ij}(x, x-y)(a_{hk}(x) - a_{hk}(y)) f(y) dy$$

and

$$\tilde{S}_{ijhk}(f)(x) = \int_{B_\rho^+} \Gamma_{ij}(x, T(x)-y)(a_{hk}(x) - a_{hk}(y)) f(y) dy$$

for $1 \leq i, j < n, 1 \leq h, k \leq n$;

$$\tilde{S}_{inhk}(f)(x) = \int_{B_\rho^+} \left(\sum_{j=1}^n \Gamma_{ij}(x, T(x)-y) A_j(x) \right) (a_{hk}(x) - a_{hk}(y)) f(y) dy$$

for $1 \leq i < n, 1 \leq h, k \leq n$;

$$\begin{aligned} \tilde{S}_{mnhk}(f)(x) = & \int_{B_\rho^+} \left(\sum_{i,j=1}^n \Gamma_{ij}(x, T(x) - y) A_i(x) A_j(x) \right) \\ & \times (a_{hk}(x) - a_{hk}(y)) f(y) dy \end{aligned}$$

for $1 \leq h, k \leq n$.

In view of Theorems 2.4, 2.5 and (3.4), one has

$$\begin{aligned} & \sum_{i,j,h,k=1}^n \|S_{ijhk}(f) + \tilde{S}_{ijhk}(f)\|_{L^{p,\lambda}(B_\rho^+)} \\ & \leq c \sum_{h,k=1}^n \|a_{hk}\|_* \|f\|_{L^{p,\lambda}(B_\rho^+)} \leq c \sum_{h,k=1}^n \eta_{hk} \|f\|_{L^{p,\lambda}(B_\rho^+)} \\ & \leq c\eta \|f\|_{L^{p,\lambda}(B_\rho^+)} \quad \forall f \in L^{p,\lambda}(B_\rho^+). \end{aligned}$$

Therefore, taking into account $a_{ij} \in VMO$, it is possible to choose $\rho > 0$ such that

$$\sum_{i,j,h,k=1}^n \|S_{ijhk} + \tilde{S}_{ijhk}\| < 1. \quad (3.6)$$

Now, for the given $u \in W_{\gamma_0}^{2,p}$ (recall $\mathcal{L}u \in L^{p,\lambda}(B_\rho^+)$) we set

$$\begin{aligned} h_{ij}(x) = & \text{P.V.} \int_{B_\rho^+} \Gamma_{ij}(x, x - y) \mathcal{L}u(y) dy \\ & + \mathcal{L}u(x) \int_{|\xi|=1} \Gamma_i(x, \xi) \xi_j d\sigma_\xi + \tilde{I}_{ij}(x), \end{aligned}$$

where

$$\tilde{I}_{ij}(x) = \int_{B_\rho^+} \Gamma_{ij}(x, T(x) - y) \mathcal{L}u(y) dy$$

for $1 \leq i, j < n$;

$$\tilde{I}_{in}(x) = \tilde{I}_{ni}(x) = \int_{B_\rho^+} \left(\sum_{j=1}^n \Gamma_{ij}(x, T(x) - y) A_j(x) \right) \mathcal{L}u(y) dy$$

for $1 \leq i < n$;

$$I_{nn}(x) = \int_{B_\rho^+} \left(\sum_{i,j=1}^n \Gamma_{ij}(x, T(x) - y) A_i(x) A_j(x) \right) \mathcal{L}u(y) dy.$$

Having in mind (3.4) and Theorems 2.4 and 2.5, we conclude immediately that $h_{ij} \in L^{p, \lambda}(B_\rho^+)$.

Now take a $w = \{w_{ij}\}_{i, j=1, \dots, n} \in [L^{p, \lambda}(B_\rho^+)]^{n^2}$, and define the operator

$$\mathcal{U}: [L^{p, \lambda}(B_\rho^+)]^{n^2} \rightarrow [L^{p, \lambda}(B_\rho^+)]^{n^2}$$

by the setting

$$\mathcal{U}w \equiv \{(\mathcal{U}w)_{ij}\}_{i, j=1, \dots, n} = \left(\sum_{h, k=1}^n (S_{ijhk} + \tilde{S}_{ijhk}) w_{ij} + h_{ij}(x) \right)_{i, j=1, \dots, n}.$$

It is a simple matter to see that \mathcal{U} is a contraction mapping by virtue of (3.6). Therefore, there exists a unique fixed point $\bar{w} = \{\bar{w}_{ij}\}_{i, j=1, \dots, n} \in [L^{p, \lambda}(B_\rho^+)]^{n^2}$ of \mathcal{U} . On the other hand, it follows from (3.3) that also the Hessian $D^2u = \{D_{ij}u\}_{i, j=1, \dots, n} \in [L^p(B_\rho^+)]^{n^2}$ is a fixed point of \mathcal{U} . By the inclusion properties of Lebesgue and Morrey spaces we get that the fixed point is the same in the both cases. Therefore, $D^2u \equiv \bar{w}$, $D_{ij}u \in L^{p, \lambda}(B_\rho^+)$ and moreover, the estimate (3.2) holds for the second derivatives $D_{ij}u$.

Finally, standard covering and flattening arguments combined with the interior estimate (Theorem 3.1) and with the boundary estimate (3.2) yield the bound (3.5). ■

Now we can prove well-posedness of the Dirichlet problem (1.2) in the Morrey space $W^{2, p, \lambda}(\Omega)$.

THEOREM 3.4. *Let (3.1) be true, $1 < p < \infty$ and $0 < \lambda < n$.*

Then for every $f \in L^{p, \lambda}(\Omega)$ there exists a unique solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}u = f(x) & \text{a.e. in } \Omega, \\ u \in W^{2, p, \lambda}(\Omega) \cap W_0^{1, p}(\Omega). \end{cases} \quad (3.7)$$

Moreover, there is a constant $c = c(n, \kappa, p, \lambda, M, \eta, \Omega)$ such that

$$\|D_{ij}u\|_{L^{p, \lambda}(\Omega)} \leq c \|f\|_{L^{p, \lambda}(\Omega)}. \quad (3.8)$$

Proof. Since $L^{p, \lambda}(\Omega) \subset L^p(\Omega)$, existence and uniqueness of strong solution $u \in W^{2, p}(\Omega)$ to (3.7) is already known (cf. [12, Theorem 4.3]). On the other hand, Theorem 3.3 asserts $u \in W^{2, p, \lambda}(\Omega) \cap W_0^{1, p}(\Omega)$.

To show continuous dependence of the solution on the right hand side (the bound (3.8)), let us note that the linear operator

$$\mathcal{L}: W^{2, p, \lambda}(\Omega) \cap W_0^{1, p}(\Omega) \rightarrow L^{p, \lambda}(\Omega)$$

is a continuous one. In fact,

$$\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega)} \leq c \|D_{ij}u\|_{L^{p,\lambda}(\Omega)}$$

since the coefficients $a_{ij}(x)$ are bounded (cf. (3.1)). Moreover that operator is injective and surjective mapping as it was shown above. Therefore, the classical theorem of S. Banach implies continuity of the inverse operator, that is, exactly (3.8). ■

Remark 3.5. The results obtained above can be applied in the study of the nonhomogeneous problems (3.7). In fact, let $f \in L^{p,\lambda}(\Omega)$ and $\varphi \in W^{2,p,\lambda}(\Omega)$ and consider the Dirichlet problem

$$\begin{cases} \mathcal{L}u = f(x) & \text{a.e. in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \quad u - \varphi \in W_0^{1,p}(\Omega). \end{cases} \quad (3.9)$$

Obviously, $\mathcal{L}\varphi \in L^{p,\lambda}(\Omega)$ and therefore, the difference $u(x) - \varphi(x)$ will solve the *homogeneous* Dirichlet problem (3.7). This way, the strong solutions to (3.9) will belong to the space $W^{2,p,\lambda}(\Omega)$.

4. CONCLUDING REMARKS

1. The results presented here can be applied in studying Morrey regularity of the strong solutions to (3.9) for general elliptic operators

$$\mathcal{L} \equiv a_{ij}(x) D_{ij} + b_i(x) D_i + c(x)$$

with $a_{ij} \in VMO \cap L^\infty(\Omega)$ and the lower order coefficients $b_i(x)$ and $c(x)$ owning suitable integrability properties. We refer the reader to [28, 29] for the $W^{2,p}$ -regularity results concerning the problem (3.9) (see [22] also).

2. An immediate consequence of Theorem 3.3 and the known properties of Morrey spaces for suitable p and λ (cf. [6]) is the next global Hölder regularity for the gradient Du of the strong solutions to (3.9), which generalizes the result of Caffarelli [4].

COROLLARY 4.1. *Let $u \in W^{2,p}(\Omega)$ be a strong solution to (3.9) with $f \in L^{p,\lambda}(\Omega)$ and $\varphi \in W^{2,p,\lambda}(\Omega)$. Suppose $n - p < \lambda < n$.*

Then the gradient Du is a Hölder continuous function on $\bar{\Omega}$ with exponent $\alpha = 1 - (n - \lambda)/p$.

ACKNOWLEDGMENTS

We dedicate this work to the memory of Filippo Chiarenza, who died prematurely in July 1996. He was our teacher and very warm friend and he suggested we deal with the topic. The

authors are grateful to Michele Frasca and Antonino Maugeri for the very useful discussions and fruitful collaboration. We are indebted to the anonymous referee for the criticism and the very useful remarks and suggestions.

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