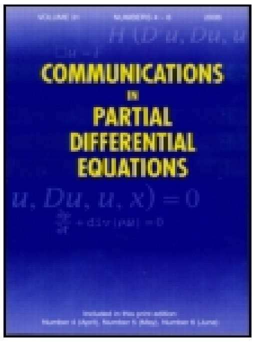

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BOUNDEDNESS OF SOLUTIONS TO VARIATIONAL PROBLEMS UNDER GENERAL GROWTH CONDITIONS

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1 Introduction and main results

The present paper deals with minimum problems of the calculus of variations and quasilinear elliptic equations in divergence form.

The minimum problems we take into account have the form

$$\begin{cases} \min \int_G F(x, v, Dv)\, dx \\ v = u_0 \qquad \text{on } \partial G. \end{cases} \tag{1.1}$$

Here G is an open subset of $\mathbf{R}^n$, whose Lebesgue measure $m(G)$ is finite; $n \geq 2$; F is a Carathéodory function from $G \times \mathbf{R} \times \mathbf{R}^n$ into $\mathbf{R}$; D stands for gradient; u_0 is a prescribed boundary datum.
Our assumptions on integrand F amount to requiring that A, B and s_0 exists such that

$$F(x, s, \xi) \geq A(|\xi|) - B(|s|) \tag{1.2}$$

$$F(x, s, 0) \leq B(|s|) \tag{1.3}$$

for $|s| \geq s_0$, $\xi \in \mathbf{R}^n$ and a.e. $x \in G$, where $|\xi|$ denotes the euclidean norm of ξ. Here, s_0 is a nonnegative number, A is a Young function, i.e. a convex increasing function from $[0, \infty)$ into $[0, \infty)$ vanishing at 0, and B is an increasing function from $[0, \infty)$ into $[0, \infty)$.

The boundary datum u_0 is assumed to be a bounded weakly differentiable function on $\mathbf{R}^n$ such that $\int A(|Du_0|)\,dx < \infty$.
The competing functions v in problem (1.1) are taken from the class $K^A_{u_0}$ defined as

$$(1.4)\quad K^A_{u_0} = \{v : v \text{ is a real-valued weakly differentiable function in } G, \int_G A(|Dv|)\,dx < \infty \text{ and the continuation of } v - u_0 \text{ by } 0 \text{ outside } G \text{ is a weakly differentiable function in } \mathbf{R}^n\}.$$

We are concerned with conditions on A and B ensuring that any minimizer of problem (1.1) is bounded in G. Our result can be stated as follows.

Theorem 1 *Let $n \geq 2$ and let A be a Young function such that*

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{n'-1} dt = \infty, \tag{1.5}$$

where $n' = n/(n-1)$, the Hölder's conjugate of n. Let A_n be the function defined by

$$A_n = A \circ H_n^{-1}, \tag{1.6}$$

where

$$H_n(r) = \left(\int_0^r \left(\frac{t}{A(t)}\right)^{n'-1} dt\right)^{1/n'} \qquad \text{for } r \geq 0. \tag{1.7}$$

(In (1.7) A is modified, if necessary, near 0 in such a way that the integral be convergent). Assume that a positive constant c exists such that

$$B(s) \leq A_n(cs) \tag{1.8}$$

for large s. Then any minimizer u of problem (1.1) is bounded.

Remarks. **1.** If the assumption (1.5) does not hold, then a theorem of [8] (see also [3]) ensures that every function from the class $K^A_{u_0}$ is bounded. Thus, every minimizer is automatically bounded in this case.

2. The function A_n defined by (1.6) plays the role of a Sobolev conjugate of A — see Theorem 3, section 2. Observe that A_n is, in fact, a Young function. Actually, A_n is the composition of A and H_n^{-1}, both of which are Young functions, the former by assumption and the latter because H_n is (strictly) concave, increases and vanishes at 0.

3. Notice that assumption (1.8) of Theorem 1, which is required to hold for some c and for large s, is not affected by the way A is (possibly) modified near 0 in definition (1.7).

4. Let us point out that, unlike most results in the theory of calculus of variations and of partial differential equations, the boundedness result of Theorem 1 does not have a corresponding a priori estimate for the maximum.

5. In the special case where A and B are powers, Theorem 1 improves classical results appearing in [6] and [7]. Indeed, choose $A(s) = s^p$ for some $p \in [1, n]$ (when $p > n$ every $u \in K^A_{u_0}$ is bounded by Sobolev's embedding theorem). If $p < n$, then $A_n(s) = s^{p^*}$, where $p^* = \dfrac{np}{n-p}$, the Sobolev conjugate of p. When $p = n$, $A_n(s)$ is equivalent near infinity to the function $e^{s^{n'}}$, in the sense that constants c_1, c_2 exist such that $A_n(c_1 s) \le e^{s^{n'}} \le A_n(c_2 s)$ for large s. Thus, Theorem 1 ensures that any minimizer of problem (1.1) is bounded provided that either $p < n$ and $B(s) \le cs^{p^*}$ or $p = n$ and $B(s) \le e^{cs^{n'}}$ for some $c > 0$ and for large s. The boundedness of minimizers of (1.1) follows from Theorem 3.2, chap. 5 of [6] or Theorem 6.2 of [7] under the stronger assumption that $B(s) \le cs^q$ for some $q < p^*$ in case $p < n$ and for any $q > 0$ in case $p = n$.

The above example can be generalised on taking into account functions $A(s)$ having the form $s^p \log^q(e + s)$, where either $p > 1$ and $q \in \mathbf{R}$ or $p = 1$ and $q \ge 0$. Theorem 1 tells us that minimizers of (1.1) are bounded if

$$B(s) \le \begin{cases} s^{p^*}(\log s)^{nq/(n-p)} & \text{if } 1 \le p < n \\ e^{s^{n/(n-1-q)}} & \text{if } p = n,\ q < n-1 \\ e^{e^{s^{n'}}} & \text{if } p = n,\ q = n-1. \end{cases}$$

When either $p > n$ or $p = n$ and $q > n - 1$, then every $u \in K^A_{u_0}$ is bounded (see Remark 1).

6. In [9] the question of boundedness of (1.1) was considered under the assumptions (1.3) and

$$F(x, s, \xi) \ge A(|\xi|) - A(\lambda|s|) \tag{1.9}$$

for some $\lambda > 0$. Theorem 2 of that paper states that the relevant minimizers are bounded provided that the number δ defined by

$$\delta = \sup_{t>1} \liminf_{s\to+\infty} \frac{\log[A(st)/A(s)]}{\log t} \tag{1.10}$$

is strictly greater than 1 and there exists $\sigma < \delta$ such that

$$\int^{\infty} \frac{ds}{[A^{-1}(B(s))]^{1-\sigma/n}} = \infty. \tag{1.11}$$

Theorem 1 above improves this result in the following two directions. First, assumption (1.9) is more stringent than (1.2)-(1.8), since $\lim_{s\to+\infty} A_n(ks)/A(s)$

for every $k > 0$. Second, Theorem 2 of [9] cannot be applied if B grows as fast as A_n does. Actually, suppose that $B(s) = A_n(cs)$ for some positive constant c and for large s. Then $A^{-1}(B(s)) = H_n^{-1}(cs)$ for large s. On the other hand, assumption (1.10) ensures that for every $\varepsilon > 0$, $A(s) \geq s^{\delta-\varepsilon}$ if s is large (see e.g. Lemma 2 of [9]). Hence, if $\delta \leq n$ (the only case of interest), $[A^{-1}(B(s))]^{\sigma/n-1} = [H_n^{-1}(cs)]^{\sigma/n-1} \leq$ Const. $s^{(\sigma-n)/(\varepsilon-\delta+n)}$ for large s. Choosing $\varepsilon < \delta - \sigma$ shows that (1.11) cannot hold.

Now let us discuss the boundedness of solutions to boundary value problems of the type

$$(1.12) \qquad \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x,u,Du) + b(x,u,Du) = 0 & \text{in } G \\ u = u_0 & \text{on } \partial G. \end{cases}$$

Here, a_i, $i = 1, \ldots n$, and b are Carathéodory functions from $G \times \mathbf{R} \times \mathbf{R}^n$ into R satisfying growth conditions of the form

$$(1.13) \qquad \sum_{i=1}^{n} a_i(x,u,\xi)\xi_i \geq A(|\xi|) - B(|s|)$$

$$(1.14) \qquad \operatorname{sign}(s) b(x,s,\xi) \leq C(|s|) + D(|s|)E(|\xi|)$$

for $|s| \geq s_0$, $\xi \in \mathbf{R}^n$ and a.e. $x \in G$, where s_0 is a positive number, A is a Young function and B, C, D, E are increasing functions from $[0,\infty)$ into $[0,\infty)$.

We consider weak solutions to problem (1.12) from the class $K_{u_0}^A$, where u_0 is a function as above (in particular bounded). A function $u \in K_{u_0}^A$ will be called a weak solution to (1.12) if

$$(1.15) \qquad \int_G \sum_{i=1}^{n} a_i(x,u,Du) \frac{\partial \phi}{\partial x_i} - b(x,u,Du)\phi(x)dx = 0$$

for all test functions $\phi \in K_0^A$. Here, K_0^A is defined as in (1.4) with $u_0 \equiv 0$.

The next theorem gives conditions on the functions A, B, C, D, E ensuring that every weak solution to (1.12) is bounded in G. The notation $\Phi^{\sim}$ is used throughout for the Young conjugate of a function $\Phi : [0,\infty) \to [0,\infty)$. Recall that

$$\Phi^{\sim}(s) = \sup\{rs - \Phi(r) : r \geq 0\}.$$

Theorem 2 *Let n, A and A_n be as in Theorem 1. Assume that:*

i) *$A \circ E^{-1}$ is a Young function;*

ii) *constants $c > 0$ and $k > 1$ exist such that*

$$(1.16) \qquad B(s) \leq A_n(cs)$$

$$(1.17) \qquad C(s) \leq \frac{1}{s} A_n(cs)$$

$$(1.18) \qquad D(s) \leq \frac{1}{ks}(((A \circ E^{-1})^{\sim})^{-1} \circ A_n)(cs)$$

for large s.

Then any weak solution to problem (1.12) *is bounded.*

Remark 6 When $A, ..., E$ are powers (with positive exponents), $A(s) = s^p$, $B(s) = s^\alpha$, $C(s) = s^\beta$, $D(s) = s^\gamma$, $E(s) = s^q$, say, then Theorem 2 states that weak solutions to problem (1.12) are bounded provided that

$$1 \leq p < n, \; q \leq p - 1 + \frac{p}{n}, \; \alpha \leq \frac{np}{n-p}, \; \beta \leq \frac{np}{n-p} - 1, \; \gamma \leq n\frac{p-q}{n-p} - 1.$$

This result should be compared with Theorems 7.1, chap. 4, and 3.1, chap. 5 of [6], where equality is not allowed in the inequalities involving α, β and γ.

2 A Sobolev-type inequality

In this section we establish an extension of the classical Sobolev-Poincaré inequality which is a basic step in the proof of Theorems 1-2.

Theorem 3 *Let $n \geq 2$ and let A be a Young function such that*

$$(2.1) \qquad \int_0 \left(\frac{t}{A(t)}\right)^{n'-1} dt < \infty \quad \text{and} \quad \int^\infty \left(\frac{t}{A(t)}\right)^{n'-1} dt = \infty.$$

Let A_n be the Young function defined by (1.6). *Then*

$$(2.2) \qquad \int_{\mathbf{R}^n} A_n\left(\frac{|u(y)|}{8C_n^{-1/n}(\int_{\mathbf{R}^n} A(|Du|)\, dx)^{1/n}}\right) dy \leq \int_{\mathbf{R}^n} A(|Du|)dx$$

for every weakly differentiable function u on $\mathbf{R}^n$ such that $m(\{x \in \mathbf{R}^n : |u(x)| > t\}) < \infty$ for every $t > 0$ and such that $\int_{\mathbf{R}^n} A(|Du|)dx < \infty$. Here, $C_n = \pi^{n/2}/\Gamma(1 + n/2)$, the measure of the n-dimensional unit ball.

Remark 7 Incidentally, let us sketch some consequences of Theorem 3 in the framework of Orlicz-Sobolev spaces. Recall that the Orlicz space $L^A(\mathbf{R}^n)$ is the Banach space of measurable functions f such that there exists $\lambda > 0$ for which $\int_{\mathbf{R}^n} A(|f(x)|/\lambda)\, dx < \infty$. $L^A(\mathbf{R}^n)$ is equipped with the Luxemburg norm $\|\cdot\|_{L^A(\mathbf{R}^n)}$ defined as

$$\|f\|_{L^A(\mathbf{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbf{R}^n} A\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

The Orlicz-Sobolev space $W^{1,A}(\mathbf{R}^n)$ is defined by $W^{1,A}(\mathbf{R}^n) = \{u : u$ is a weakly differentiable function on $\mathbf{R}^n$ such that u and $|Du| \in L^A(\mathbf{R}^n)\}$.

The integral inequality (2.2) is equivalent to the following norm inequality in Orlicz spaces:

$$\|u\|_{L^{A_n}(\mathbf{R}^n)} \le \text{Const.}\|Du\|_{L^A(\mathbf{R}^n)} \tag{2.3}$$

for every weakly differentiable function u on $\mathbf{R}^n$ such that $m(\{x \in \mathbf{R}^n : |u(x)| > t\}) < \infty$ for every $t > 0$. Actually, (2.3) is a consequence of (2.2) by the very definition of Luxemburg norm, whereas (2.2) follows on replacing $A(s)$ by $A(s)/\int_{\mathbf{R}^n} A(|Du|)\, dx$ in (2.3).

Inequality (2.3) can in turn be shown to be equivalent to that established (with a different proof) in Theorem 1 of [4]. Thus, as a consequence of that theorem, inequality (2.3) is sharp, in the sense that $L^{A_n}(\mathbf{R}^n)$ cannot be replaced by any smaller Orlicz space. In particular, (2.3) improves the Sobolev inequality for Orlicz spaces contained in [5] (see also section 1 of [4]).

An obvious consequence of inequality (2.3) is the (continuous) embedding

$$W^{1,A}(\mathbf{R}^n) \subset L^{A_n}(\mathbf{R}^n).$$

In fact, a stronger information can be derived from Theorem 3. Actually, on exploiting inequality (2.2) as in the proof of Theorem 1 (Section 3) and observing that $\lim_{s\to 0} A_n(\lambda s)/A(s) = 0$ for every $\lambda > 0$, one can show that $\int_{\mathbf{R}^n} A(|u(x)|/\lambda)\, dx < \infty$ for every $\lambda > 0$ whenever $u \in W^{1,A}(\mathbf{R}^n)$. Moreover, the same argument as in the proof of Theorem 2.1 of [2] tells us that

$$\left\{f : \int_{\mathbf{R}^n} A_n\left(\frac{|f(x)|}{\lambda}\right) dx < \infty \text{ for every } \lambda > 0\right\} \subset \text{closure of } L^\infty(\mathbf{R}^n) \text{ in } L^{A_n}(\mathbf{R}^n).$$

Hence,

$$W^{1,A}(\mathbf{R}^n) \subset \text{ closure of } L^\infty(\mathbf{R}^n) \text{ in } L^{A_n}(\mathbf{R}^n),$$

a space which, in general, is strictly contained in $L^{A_n}(\mathbf{R}^n)$.

Our proof of Theorem 3 requires the following interpolation property.

Theorem 4 *Let (M_1, ν_1) and (M_2, ν_2) be positive non-atomic measure spaces and let T be a linear operator whose domain is some linear subspace of the set of ν_1-measurable functions on M_1 and whose range is contained in the set of ν_2-measurable functions on M_2. Let $p \in (1, \infty)$. Assume that T is bounded from $L^1(M_1)$ into $L^{p'}(M_2)$ with norm $\le N_0$ and from the Lorentz space $L^{p,1}(M_1)$*

into $L^\infty(M_1)$ with norm $\le N_1$. Let A be a Young function satisfying conditions (2.1) with n replaced by p, and let A_p be the Young function defined as in (1.6) with n replaced by p. Then

$$\int\limits_{M_2} A_p\left(\frac{|Tf(y)|}{8N_1\Big(\int\limits_{M_1} A(|f(x)|)\,d\nu_1(x)\Big)^{1/p}}\right) d\nu_2(y) \le 3\Big(\frac{N_0}{N_1}\Big)^{p'} \int\limits_{M_1} A(|f(x)|)\,d\nu_1(x) \tag{2.4}$$

for every ν_1-measurable function f on M_1 such that $\int\limits_{M_1} A(|f(x)|)\,d\nu_1(x) < \infty$.

Recall that, given a positive measure space (M,ν) and a real number $p \ge 1$, $L^{p,1}(M)$ is the space of those ν-measurable functions on M_1 for which the quantity

$$\|f\|_{L^{p,1}(M)} = \int\limits_0^\infty \nu(\{|f| > s\})^{1/p}\,ds$$

is finite. Henceforth, $\{|f| > s\}$ stands for $\{x \in M : |f(x)| > s\}$.

Proof of Theorem 3. Consider the linear operator T defined by

$$Tg(s) = \int\limits_s^\infty r^{-1/n'} g(r)dr \qquad \text{for } s \ge 0$$

on functions $g : [0,\infty) \to \mathbf{R}$. Minkowski's integral inequality yields

$$\|Tg\|_{L^{n'}(0,\infty)} \le \int\limits_0^\infty r^{-1/n'}|g(r)|\Big(\int\limits_0^r ds\Big)^{1/n'} dr = \int\limits_0^\infty |g(r)|dr. \tag{2.5}$$

Thus, T is bounded from $L^1(0,\infty)$ into $L^{n'}(0,\infty)$ with norm ≤ 1. On the other hand, since $r^{-1/n'}$ decreases on $(0,\infty)$, Hardy-Littlewood inequality tells us that

$$\int\limits_0^\infty r^{-1/n'}|g(r)|dr \le \int\limits_0^\infty r^{-1/n'} g^*(r)dr, \tag{2.6}$$

where g^* denotes the decreasing rearrangement of g. Recall that if (M,ν) is a positive measure space and f is a ν-measurable function on M, then f^* is the decreasing function from $(0,\infty)$ into $[0,\infty)$ equimeasurable with f. It is easily verified that the right-hand side of (2.6) equals $n\|g\|_{L^{n,1}(0,\infty)}$. Hence, T is also bounded from $L^{n,1}(0,\infty)$ into $L^\infty(0,\infty)$ with norm $\le n$. Therefore, by Theorem 4,

$$(2.7)\qquad \int_0^\infty A_n\left(\frac{|\int_s^\infty r^{-1/n'}g(r)dr|}{8n\left(\int_0^\infty A(|g(r)|)dr\right)^{1/n}}\right)ds \le \int_0^\infty A(|g(s)|)ds$$

for every g such that $\int_0^\infty A(|g(s)|)ds < \infty$.

Now, let u be as in the statement. Then u^* is locally absolutely continuous and the following Pòlya-Szegö type inequality holds

$$(2.8)\qquad \int_0^\infty A\left(-nC_n^{1/n}s^{1/n'}\frac{du^*}{ds}\right)ds \le \int_{\mathbf{R}^n} A(|Du|)dx$$

(see e.g. [1]). The equimeasurability of u and u^* and inequality (2.8) ensure that

$$(2.9)\qquad \int_0^\infty A_n\left(\frac{|u(y)|}{8C_n^{-1/n}\left(\int_{\mathbf{R}^n} A(|Du|)dx\right)^{1/n}}\right)dy$$

$$\le \int_0^\infty A_n\left(\frac{u^*(s)}{8C_n^{-1/n}\left(\int_0^\infty A\left(-nC_n^{1/n}r^{1/n'}\frac{du^*}{dr}\right)dr\right)^{1/n}}\right)ds.$$

Since $u^*(s) = \int_s^\infty -\frac{du^*}{dr}\,dr$ for $s \ge 0$, the conclusion follows from inequalities (2.7), (2.8) and (2.9). $\square$

Proof of Theorem 4. We begin by showing that Tf is well defined whenever

$$(2.10)\qquad \int_{M_1} A(|f(x)|)\,d\nu_1(x) < \infty.$$

To this purpose, it suffices to show that if we set $f_t = \operatorname{sign}(f)\min\{t,|f|\}$ and $f^t = f - f_t$ for $t > 0$, then $f^t \in L^1(M_1)$ and $f_t \in L^{p,1}(M_1)$. Since A is a Young function, then

$$(2.11)\qquad \frac{A(s)}{s}\quad \text{increases}$$

and

$$(2.12)\qquad \frac{A(s)}{s} \le \frac{dA}{ds}(s) \le \frac{A(2s)}{s}\quad \text{for } s \ge 0.$$

Thus

$$(2.13)\qquad \|f^t\|_{L^1} = \int_t^\infty \nu_1(\{|f| > s\})\,ds \le \frac{t}{A(t)} \int_t^\infty \frac{A(s)}{s} \nu_1(\{|f| > s\})\,ds$$
$$\le \frac{t}{A(t)} \int_t^\infty \frac{dA}{ds}(s)\nu_1(\{|f| > s\})\,ds = \frac{t}{A(t)} \int_{M_1} A(|f(x)|)d\nu_1(x),$$

whence $f^t \in L^1(M_1)$. On the other hand,

$$(2.14)\qquad \|f_t\|_{L^{p,1}} = \int_0^t \nu_1(\{|f| > s\})^{1/p} ds$$
$$\le \left(\int_0^t \left(\frac{r}{A(r)}\right)^{p'-1} dr\right)^{1/p'} \left(\int_0^t \frac{A(s)}{s} \nu_1(\{|f| > s\})\,ds\right)^{1/p}$$
$$\le \left(\int_0^t \left(\frac{r}{A(r)}\right)^{p'-1} dr\right)^{1/p'} \left(\int_0^\infty \frac{dA}{ds}(s)\nu_1(\{|f| > s\})\,ds\right)^{1/p}$$
$$= \left(\int_0^t \left(\frac{r}{A(r)}\right)^{p'-1} dr\right)^{1/p'} \left(\int_{M_1} A(|f(x)|)\,d\nu_1(x)\right)^{1/p},$$

whence $f_t \in L^{p,1}(M_1)$.

Let us now prove inequality (2.4). Set

$$(2.15)\qquad \overline{A}_p(s) = \left(\frac{A(H_p^{-1}(s))}{H_p^{-1}(s)} s\right)^{p'} \quad \text{for } s \ge 0,$$

where H_p is the function defined as in (1.7) with n replaced by p. Let f be any function satisfying (2.10). Then, if k is any positive number, the following chain of inequalities holds

$$(2.16)\qquad \int_{M_2} A_p\left(\frac{|Tf(y)|}{8k}\right) d\nu_2(y) = \int_0^\infty \frac{dA_p}{dt}(t)\nu_2(\{|Tf| > 8kt\})\,dt$$
$$\le \int_0^\infty \frac{A_p(2t)}{t}\nu_2(\{|Tf| > 8kt\})\,dt \le p' \int_0^\infty \frac{\overline{A}_p(4t)}{t}\nu_2(\{|Tf| > 8kt\})\,dt$$
$$= -\int_0^\infty \frac{\overline{A}_p(t)}{t^{p'}} \frac{d}{dt}\left(p' \int_t^\infty \nu_2(\{|Tf| > 2ks\})s^{p'-1}ds\right) dt.$$

Observe that the first inequality in (2.16) is due to (2.12) with A replaced by A_p, whereas the second inequality is a consequence of Lemma 1 below. An integration by parts yields

$$(2.17)\qquad -\int_0^\infty \frac{\overline{A}_p(t)}{t^{p'}}\frac{d}{dt}\Big(p'\int_t^\infty \nu_2(\{|Tf|>2ks\})s^{p'-1}\,ds\Big)dt$$
$$= \Big(-\frac{\overline{A}_p(t)}{t^{p'}}p'\int_t^\infty \nu_2(\{|Tf|>2ks\})s^{p'-1}ds\Big)\Big|_{t=0}^{t=\infty}$$
$$+\int_0^\infty \frac{d}{dt}\Big(\frac{\overline{A}_p(t)}{t^{p'}}\Big)\Big(p'\int_t^\infty \nu_2(\{|Tf|>2ks\})s^{p'-1}\,ds\Big)dt.$$

From (2.16) and (2.17) we get

$$(2.18)\quad \int_{M_2} A_p\Big(\frac{|Tf(y)|}{8k}\Big)\,d\nu_2(y) \le \limsup_{t\to 0^+}\frac{\overline{A}_p(t)}{t^{p'}}p'\int_t^\infty \nu_2(\{|Tf|>2ks\})s^{p'-1}ds$$
$$+\int_0^\infty \frac{d}{dt}\Big(\frac{\overline{A}_p(t)}{t^{p'}}\Big)\Big(p'\int_t^\infty \nu_2(\{|Tf|>2ks\})s^{p'-1}\,ds\Big)dt.$$

Now, since $\frac{d}{dt}(\overline{A}_p(t)/t^{p'}) \ge 0$ and since $\nu_2(\{|Tf|>2ks\}) = \nu_2(\{|T(f_\lambda+f^\lambda)|>2ks\}) \le \nu_2(\{|Tf_\lambda|>ks\})+\nu_2(\{|Tf^\lambda|>ks\})$ for every $\lambda>0$, then

$$(2.19)\quad \int_{M_2} A_p\Big(\frac{|Tf(y)|}{8k}\Big)d\nu_2(y) \le \limsup_{t\to 0^+}\frac{\overline{A}_p(t)}{t^{p'}}p'\int_t^\infty \nu_2(\{|Tf_{\lambda(t)}|>ks\})s^{p'-1}ds$$
$$+\limsup_{t\to 0^+}\frac{\overline{A}_p(t)}{t^{p'}}p'\int_t^\infty \nu_2(\{|Tf^{\lambda(t)}|>ks\})s^{p'-1}ds$$
$$+\int_0^\infty \frac{d}{dt}\Big(\frac{\overline{A}_p(t)}{t^{p'}}\Big)\Big(p'\int_t^\infty \nu_2(\{|Tf_{\lambda(t)}|>ks\})s^{p'-1}ds\Big)dt$$
$$+\int_0^\infty \frac{d}{dt}\Big(\frac{\overline{A}_p(t)}{t^{p'}}\Big)\Big(p'\int_t^\infty \nu_2(\{|Tf^{\lambda(t)}|>ks\})s^{p'-1}ds\Big)dt.$$

Here we have set

$$(2.20)\qquad \lambda(t) = H_p^{-1}(t) \qquad \text{for } t\ge 0.$$

Call I_1, J_1, I_2, J_2, the terms on the right hand side of (2.19), respectively. First, let us consider the terms I_i, $i=1,2$. Owing to the fact that T is bounded from $L^{p,1}$ into L^∞ with norm $\le N_1$ and to inequality (2.14) with t replaced by $\lambda(t)$ we get

$$(2.21)\qquad \|Tf_{\lambda(t)}\|_{L^\infty} \le N_1\|f_{\lambda(t)}\|_{L^{p,1}}$$
$$\le N_1 H_p(\lambda(t))\Big(\int_{M_1} A(|f(x)|)d\nu_1(x)\Big)^{1/p} = N_1 t\Big(\int_{M_1} A(|f(x)|)d\nu_1(x)\Big)^{1/p}.$$

Hence, $\nu_2(\{|Tf_{\lambda(t)}| > ks\}) = 0$ provided that $s \geq t$ and

$$k = N_1\Big(\int\limits_{M_1} A(|f(x)|)\,d\nu_1(x)\Big)^{1/p}. \tag{2.22}$$

Thus,

$$I_1 = I_2 = 0 \tag{2.23}$$

if k is given by (2.22).

Let us now take into account the terms J_i, $i = 1, 2$. Since T is bounded from L^1 into $L^{p'}$ with norm $\leq N_0$, one has

$$p'\int\limits_t^\infty \nu_2(\{|Tf^{\lambda(t)}| > ks\})s^{p'-1}\,ds \leq \Big(\frac{N_0}{k}\Big)^{p'}\Big(\int\limits_{\lambda(t)}^\infty \nu_1(\{|f| > s\})\,ds\Big)^{p'}. \tag{2.24}$$

From (2.13) with t replaced by $\lambda(t)$ we get

$$\begin{aligned} &\frac{\overline{A}_p(t)}{t^{p'}}\Big(\int\limits_{\lambda(t)}^\infty \nu_1(\{|f| > s\})ds\Big)^{p'} \\ &\leq \frac{\overline{A}_p(t)}{t^{p'}}\Big(\frac{\lambda(t)}{A(\lambda(t))}\Big)^{p'}\Big(\int\limits_{M_1} A(|f(x)|)d\nu_1(x)\Big)^{p'} = \Big(\int\limits_{M_1} A(|f(x)|)d\nu_1(x)\Big)^{p'}. \end{aligned} \tag{2.25}$$

Clearly, (2.24) and (2.25) imply that

$$J_1 \leq \Big(\frac{N_0}{k}\Big)^{p'}\Big(\int\limits_{M_1} A(|f(x)|)d\nu_1(x)\Big)^{p'}. \tag{2.26}$$

Finally, let us estimate the term J_2. On making use of (2.24), integrating by parts and then making use of (2.25) one obtains

$$\begin{aligned} J_2 &\leq \Big(\frac{N_0}{k}\Big)^{p'}\int\limits_0^\infty \frac{d}{dt}\Big(\frac{\overline{A}_p(t)}{t^{p'}}\Big)\Big(\int\limits_{\lambda(t)}^\infty \nu_1(\{|f| > s\})ds\Big)^{p'}dt \\ &= \Big(\frac{N_0}{k}\Big)^{p'}\Big\{\Big(\frac{\overline{A}_p(t)}{t^{p'}}\Big(\int\limits_{\lambda(t)}^\infty \nu_1(\{|f| > s\})ds\Big)^{p'}\Big)\Big|_{t=0}^{t=\infty} \\ &\quad - \int\limits_0^\infty \frac{\overline{A}_p(t)}{t^{p'}}\frac{d}{dt}\Big(\int\limits_{\lambda(t)}^\infty \nu_1(\{|f| > s\})ds\Big)^{p'}dt\Big\} \\ &\leq \Big(\frac{N_0}{k}\Big)^{p'}\Big\{\Big(\int\limits_{M_1} A(|f(x)|)d\nu_1(x)\Big)^{p'} \\ &\quad + p'\int\limits_0^\infty \frac{\overline{A}_p(t)}{t^{p'}}\Big(\int\limits_{\lambda(t)}^\infty \nu_1(\{|f| > s\})ds\Big)^{p'-1}\nu_1(\{|f| > \lambda(t)\})\frac{d\lambda}{dt}(t)dt\Big\}. \end{aligned} \tag{2.27}$$

We have

$$(2.28)\int_0^\infty \frac{\overline{A}_p(t)}{t^{p'}}\Big(\int_{\lambda(t)}^\infty \nu_1(\{|f|>s\})ds\Big)^{p'-1}\nu_1(\{|f|>\lambda(t)\})\frac{d\lambda}{dt}(t)dt$$
$$=\int_0^\infty \frac{\overline{A}_p(\lambda^{-1}(r))}{\lambda^{-1}(r)^{p'}}\Big(\int_r^\infty \nu_1(\{|f|>s\})ds\Big)^{p'-1}\nu_1(\{|f|>r\})dr$$
$$\leq\int_0^\infty \frac{\overline{A}_p(\lambda^{-1}(r))}{\lambda^{-1}(r)^{p'}}\Big(\frac{r}{A(r)}\Big)^{p'-1}\Big(\int_r^\infty \frac{A(s)}{s}\nu_1(\{|f|>s\})ds\Big)^{p'-1}\nu_1(\{|f|>r\})dr$$
$$=\int_0^\infty \frac{A(r)}{r}\Big(\int_r^\infty \frac{A(s)}{s}\nu_1(\{|f|>s\})ds\Big)^{p'-1}\nu_1(\{|f|>r\})dr$$
$$=\frac{1}{p'}\Big(\int_0^\infty \frac{A(s)}{s}\nu_1(\{|f|>s\})ds\Big)^{p'}\leq\frac{1}{p'}\Big(\int_{M_1} A(|f(x)|)d\nu_1(x)\Big)^{p'}.$$

From (2.27) and (2.28) we obtain

$$J_2\leq 2\Big(\frac{N_0}{k}\Big)^{p'}\Big(\int_{M_1} A(|f(x)|)d\nu_1(x)\Big)^{p'}. \tag{2.29}$$

Combining (2.19), (2.23), (2.26) and (2.29) tells us that

$$\int_{M_2} A_p\Big(\frac{|Tf(y)|}{8k}\Big)d\nu_2(y)\leq 3\Big(\frac{N_0}{k}\Big)^{p'}\Big(\int_{M_1} A(|f(x)|)d\nu_1(x)\Big)^{p'},$$

provided that k is given by (2.22). Hence, the conclusion follows. □

Lemma 1 *Let $p\in(1,\infty)$ and let A be a Young function satisfying* (2.1) *with n replaced by p. Let A_p be the function defined as in* (1.6) *with n replaced by p and let $\overline{A}_p$ be the function defined by* (2.15). *Then*

$$\frac{1}{p'}A_p(s/2)\leq\overline{A}_p(s)\leq\frac{1}{p'}A_p(2s)\quad \textit{for } s\geq 0. \tag{2.30}$$

Proof. Consider the auxiliary function $\widehat{A}_p$ defined by

$$\widehat{A}_p(s)=\int_0^s \frac{\overline{A}_p(r)}{r}dr\quad\text{for } s\geq 0. \tag{2.31}$$

Since $\overline{A}_p(r)/r$ increases, we have

$$\overline{A}_p(s/2)\leq\widehat{A}_p(s)\leq\overline{A}_p(s)\quad\text{for } s\geq 0. \tag{2.32}$$

After the change of variable $t=H_p^{-1}(r)$ on the right-hand side of (2.31) one has

$$\widehat{A}_p(s) = \frac{1}{p'} \int_0^{H_p^{-1}(s)} \frac{A(t)}{t} dt \quad \text{for } s \geq 0. \tag{2.33}$$

Inasmuch as $A(t)/t$ increases, equation (2.33) implies that

$$\frac{1}{p'} A(H_p^{-1}(s)/2) \leq \widehat{A}_p(s) \leq \frac{1}{p'} A(H_p^{-1}(s)) \quad \text{for } s \geq 0. \tag{2.34}$$

Since H_p^{-1} is a Young function (see Remark 2),

$$cH_p^{-1}(s) \leq H_p^{-1}(cs) \quad \text{for } s \geq 0 \text{ and } c \geq 1. \tag{2.35}$$

Inequalities (2.30) follow from (2.32), (2.34) and (2.35). ◻

3 Proofs of Theorems 1-2

Proof of Theorem 1. Assume, by contradiction, that $\sup |u| = \infty$. Let $t > 0$ and set $v(x) = \text{sign}(u) \min\{t, |u(x)|\}$. Clearly, $v \in K_{u_0}^A$ provided that $t > t_0$, where $t_0 = \sup |u_0|$. Therefore,

$$\int_G F(x, u, Du)\, dx \leq \int_G F(x, v, Dv)\, dx. \tag{3.1}$$

Hence,

$$\int_{\{|u|>t\}} F(x, u, Du)\, dx \leq \int_{\{|u|>t\}} F(x, t\text{sign}(u), 0)\, dx \tag{3.2}$$

if $t > t_0$. This is the point where assumption (1.8) plays its role. Without loss of generality, we may suppose that (1.8) holds for $s \geq s_0$. Moreover, for simplicity, we assume that the integral in definition (1.7) is convergent, so that there is no need to modify A near 0. Thus, from (3.2) one infers that

$$\int_{\{|u|>t\}} A(|Du|)\, dx \leq \int_{\{|u|>t\}} A_n(c|u|)\, dx + A_n(ct)\mu(t) \tag{3.3}$$

if $t > \max\{t_0, s_0\}$. Here, $\mu(t) = m(\{|u| > t\})$, the distribution function of u.

Now, let us set $k(t) = \frac{1}{8} C_n^{1/n} \Big(\int_{\{|u|>t\}} A(|Du|)\, dx \Big)^{-1/n}$. Clearly, $k(t)$ is an increasing function of t which tends to infinity as t goes to infinity. Let t_1 be such that

$$k(t) \geq 2c \qquad \text{if } t \geq t_1. \tag{3.4}$$

The convexity of A_n and inequality (3.4) ensure that

$$(3.5)\quad \int_{\{|u|>t\}} A_n(c|u|)\,dx \le \frac{1}{2}\int_{\{|u|>t\}} A_n(2c(|u|-t))\,dx + \frac{1}{2}A_n(2ct)\mu(t) \le \frac{1}{2}\int_{\{|u|>t\}} A_n(k(t)(|u|-t))\,dx + \frac{1}{2}A_n(k(t_1)t)\mu(t)$$

if $t \ge t_1$. On the other hand, by Theorem 3 we have

$$(3.6)\qquad \int_{\{|u|>t\}} A_n(k(t)(|u|-t))\,dx \le \int_{\{|u|>t\}} A(|Du|)\,dx$$

if $t > t_0$. Combining (3.3), (3.5) and (3.6) and using the fact that A_n is a Young function yield

$$(3.7)\qquad \int_{\{|u|>t\}} A(|Du|)\,dx \le [A_n(k(t_1)t) + 2A_n(ct)]\mu(t) \le A_n(kt)\mu(t)$$

for $t > t_2$, where $k = 3\max\{k(t_1), c\}$ and $t_2 = \max\{s_0, t_0, t_1\}$.

Jensen' s inequality tells us that

$$(3.8)\qquad A\Big(\frac{1}{\mu(t)}\int_{\{|u|>t\}} |Du|\,dx\Big) \le \frac{1}{\mu(t)}\int_{\{|u|>t\}} A(|Du|)\,dx.$$

Moreover, the coarea formula and the standard isoperimetric inequality in $\mathbf{R}^n$ imply that

$$(3.9)\qquad nC_n^{1/n}\int_t^\infty \mu(\tau)^{1/n'}\,d\tau \le \int_{\{|u|>t\}} |Du|\,dx$$

if $t > t_0$ (see e.g. the proof of Theorem 1 in [9]). From (3.7), (3.8) and (3.9) we deduce that

$$(3.10)\qquad \frac{nC_n^{1/n}}{A^{-1}(A_n(kt))} \le \frac{\mu(t)}{\int_t^\infty \mu(\tau)^{1/n'}\,d\tau}$$

if $t > t_2$. On raising both sides of (3.10) to the power $1/n'$ and integrating the resulting inequality between any number $s > t_2$ and ∞, we obtain

$$(3.11)\qquad \frac{C_n^{1/n'}}{n}\Big(\int_s^\infty \frac{dt}{[A^{-1}(A_n(kt))]^{1/n'}}\Big)^n \le \int_s^\infty \mu(t)^{1/n'}dt.$$

This is already a contradiction in case the integral on the left-hand side of (3.11) diverges. If, on the contrary, the relevant integral converges, one can conclude as follows.

Inequalities (3.5)-(3.6) ensure that $\int_G A_n(\lambda|u|)\,dx < \infty$ for every $\lambda > 0$.

Thus, for every $\lambda > 0$ the function

$$\omega_\lambda(t) = \int_{\{|u|>t\}} A_n(\lambda|u|)\,dx$$

is well-defined for $t \geq 0$ and

$$\lim_{t\to+\infty} \omega_\lambda(t) = 0. \tag{3.12}$$

Obviously, $\omega_\lambda(t) \geq A_n(\lambda t)\mu(t)$ for $t \geq 0$. Hence,

$$\int_s^\infty \frac{\omega_\lambda(t)^{1/n'}}{A_n(\lambda t)^{1/n'}}dt \geq \int_s^\infty \mu(t)^{1/n'}dt \tag{3.13}$$

for every $\lambda > 0$ and $s \geq 0$. Notice that, since we are assuming that the integral on the left-hand side of (3.11) converges, then the integral on the left-hand side of (3.13) is a fortiori convergent. Equation (3.12) implies, via Hôpital' s rule, that

$$\lim_{s\to+\infty} \int_s^\infty \frac{\omega_\lambda(t)^{1/n'}}{A_n(\lambda t)^{1/n'}}\,dt\Big(\int_s^\infty \frac{1}{A_n(\lambda t)^{1/n'}}\,dt\Big)^{-1} = 0 \tag{3.14}$$

for every $\lambda > 0$. Combining (3.11), (3.13) and (3.14) leads to a contradiction, owing to Lemma 2 below. $\square$

Proof of Theorem 2. We argue by contradiction and suppose that u is unbounded. Let us call t_0 the sup of $|u_0|$. On choosing $\phi = \text{sign}(u)\max\{|u| - t, 0\}$, with $t > t_0$, as test function in (1.15) we have

$$\int_{\{|u|>t\}} \sum_{i=1}^n a_i(x,u,Du)\frac{\partial u}{\partial x_i} - b(x,u,Du)sign(u)(|u|-t)\,dx = 0. \tag{3.15}$$

As in the proof of Theorem 1, we assume for simplicity that the integral in definition (1.7) is convergent. Thanks to inequalities (1.16), (1.17) and (1.18), which, without loss of generality, may be assumed to hold for $s \geq s_0$, equation (3.15) implies that

$$\begin{aligned} &\int_{\{|u|>t\}} A(|Du|)dx \\ &\leq 2\int_{\{|u|>t\}} A_n(c|u|)dx + \int_{\{|u|>t\}} \frac{1}{k}E(|Du|)((A\circ E^{-1})^{\sim})^{-1}(A_n(c|u|))\,dx \end{aligned} \tag{3.16}$$

for $t > \max\{t_0, s_0\}$. We have

$$(3.17)\quad \begin{aligned}&\frac{1}{k}E(|Du|)((A\circ E^{-1})^{\sim})^{-1}\circ A_n(c|u|)\\ &\le A\circ E^{-1}(E(|Du|)/k)+(A\circ E^{-1})^{\sim}\circ((A\circ E^{-1})^{\sim})^{-1}\circ A_n(c|u|)\\ &\le \frac{1}{k}A(|Du|)+A_n(c|u|).\end{aligned}$$

Observe that the first inequality in (3.17) is a consequence of the very definition of Young conjugate, and that the second one holds because $A\circ E^{-1}$ is a Young function and $k>1$ by assumption. From inequalities (3.16)-(3.17), on using the fact that A_n is a Young function and $3k/(k-1)>1$, one obtains

$$(3.18)\qquad \int_{\{|u|>t\}} A(|Du|)\,dx \le \int_{\{|u|>t\}} A_n\Big(\frac{3ck}{k-1}|u|\Big)\,dx$$

for $t>\max\{t_0,s_0\}$. Note that, apart from a missing term, inequality (3.18) is analogous to inequality (3.3) appearing in the proof of Theorem 1. Therefore, the conclusion follows in exactly the same way as Theorem 1 follows from (3.3). The details are omitted for brevity. □

Lemma 2 *Let $p\in(1,\infty)$ and let A be a Young function satisfying (2.1) with n replaced by p. Let A_p be the Young function defined as in (1.6) with n replaced by p. If $k>2$, then*

$$(3.19)\quad \begin{aligned}&\int_s^\infty \frac{dt}{A_p(kt)^{1/p'}}\\ &\le \frac{2}{pk}(p')^{p-1-1/p'}\Big(\frac{k}{k-2}\Big)^{p-1}\Big(\int_s^\infty \frac{dt}{[A^{-1}(A_p(t))]^{1/p'}}\Big)^{p} \quad \textit{for } s>0.\end{aligned}$$

Proof. In the case where $\int^\infty[A^{-1}(A_p(t))]^{-1/p'}\,dt=\infty$ there is nothing to prove. Thus, we may assume that the last mentioned integral is finite. Let $\overline{A}_p$ be the function defined by (2.15). By Lemma 1, it suffices to prove that

$$(3.20)\quad \begin{aligned}&\int_s^\infty \frac{dt}{\overline{A}_p(kt/2)^{1/p'}}\\ &\le \frac{2}{pk}(p')^{p-1}\Big(\frac{k}{k-2}\Big)^{p-1}\Big(\int_s^\infty \frac{dt}{[A^{-1}(A_p(t))]^{1/p'}}\Big)^{p} \quad \text{for } s>0.\end{aligned}$$

A change of variables shows that (3.20) can be written as

$$(3.20)\qquad \int_{H_p^{-1}(ks/2)}^\infty H_p(r)^{-p'}\Big(\frac{r}{A(r)}\Big)^{p'}dr$$

$$\leq \frac{1}{p}\Big(\frac{k}{k-2}\Big)^{p-1}\Big(\int_{H_p^{-1}(s)}^{\infty} r^{-1/p'} H_p(r)^{1-p'}\Big(\frac{r}{A(r)}\Big)^{p'-1} dr\Big)^{p}$$

for $s > 0$, where H_p is the function defined as in (1.7) with n replaced by p. Owing to (2.35) with $c = k/2$, inequality (3.20) will follow if we show that an analogous inequality, with lower limits of integration $H_p^{-1}(ks/2)$ and $H_p^{-1}(s)$ replaced by $kz/2$ and z, respectively, holds for every $z \geq 0$. Since the function $H_p(r)^{-p'}\Big(\frac{r}{A(r)}\Big)^{p'}$ decreases on $(0, \infty)$, the inequality in question is a consequence of Lemma 3 below. □

Lemma 3 *Let $p \in (1, \infty)$ and let $h > 1$. If g is any decreasing function from $(0, \infty)$ into $[0, \infty)$, then*

$$\int_{hs}^{\infty} g(r)dr \leq \frac{1}{p}\Big(\frac{h}{h-1}\Big)^{p-1}\Big(\int_{s}^{\infty} r^{-1/p'} g(r)^{1/p}\, dr\Big)^{p} \quad \textit{for } s > 0. \tag{3.21}$$

Proof. We shall show that

$$\Big(\int_{hs}^{\infty} \psi(r)^p r^{p-1} dr\Big)^{1/p} \leq \Big(\frac{1}{p}\Big)^{1/p}\Big(\frac{h}{h-1}\Big)^{1/p'}\int_{s}^{\infty} \psi(r)dr \text{ for } s > 0 \tag{3.22}$$

for every decreasing function $\psi : (0, \infty) \to [0, \infty)$. Clearly, (3.21) follows from (3.22) on choosing $\psi(r) = g(r)^{1/p} r^{-1/p'}$. In order to establish (3.22), observe that, since ψ is nonnegative and decreasing, then

$$(r - s)\psi(r) \leq \int_{s}^{r} \psi(t)\, dt \qquad \text{if } r \geq s. \tag{3.23}$$

Thus, since we are assuming that $h > 1$,

$$\begin{aligned} &\Big(\int_{hs}^{\infty} \psi(r)^p r^{p-1} dr\Big)^{1/p} \\ &\leq \Big(\int_{hs}^{\infty}\Big(\frac{r}{r-s}\Big)^{p-1}\Big(\int_{s}^{r}\psi(t)dt\Big)^{p-1}\psi(r)dr\Big)^{1/p} \text{ for } s > 0. \end{aligned} \tag{3.24}$$

Inasmuch as $r/(r - s)$ is a decreasing function of r in $[hs, \infty)$, the right-hand side of (3.24) does not exceed

$$\Big(\frac{h}{h-1}\Big)^{1/p'}\Big(\int_{hs}^{\infty}\Big(\int_{s}^{r}\psi(t)dt\Big)^{p-1}\psi(r)dr\Big)^{1/p}.$$

The last expression equals

$$\left(\frac{1}{p}\right)^{1/p}\left(\frac{h}{h-1}\right)^{1/p'}\left(\left(\int_s^{\infty}\psi(t)dt\right)^p-\left(\int_s^{hs}\psi(t)dt\right)^p\right)^{1/p}.$$

Hence, (3.22) follows. □

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