JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **202,** 27–52 (1996)
ARTICLE NO. 0301

Continuity Properties of Minimizers of Integral Functionals in a Limit Case*

Vincenzo Ferone and Nicola Fusco

Dipartimento di Matematica e Applicazioni "R. Caccioppoli," Università di Napoli "Federico II," Complesso Monte S. Angelo, Via Cintia, 80126 Napoli, Italy

Submitted by Avner Friedman

Received August 1, 1994

We prove continuity of minimizers of integral functionals of the form $\int_\Omega [f_0(x,u,Du) + hDu]$, where the integrand $f_0(x,\eta,\xi)$ grows like $|\xi|^p$, with $p > 1$, and h is in a suitable Lorentz space. Continuity of solutions of nonlinear equations of the form $\operatorname{div} A(x,u,Du) = \operatorname{div} h$ is also proved, where $A(x,\eta,\xi)$ grows like $|\xi|^{p-1}$.

1. INTRODUCTION

Let $u \in W^{1,2}_{loc}(\Omega)$ be a solution of the equation

$$D_j\big(a_{ij}(x)D_i u\big) = \operatorname{div} h, \tag{1.1}$$

where $[a_{ij}]$ is a uniformly elliptic, bounded matrix. It is well known, see [DG], that if h is in $L^s(\Omega)$ with $s > n$, then u is Hölder-continuous in Ω. However, if h is only in $L^n(\Omega)$ simple examples show that u is neither continuous nor locally bounded.

A natural question is then to characterize those spaces to which h should belong in order to guarantee that u is continuous. Clearly these spaces should lay between $L^n(\Omega)$ and any $L^s(\Omega)$, with $s > n$.

A similar situation arises in considering weak solutions of the equation

$$D_j\big(a_{ij}(x)D_i u\big) = g. \tag{1.2}$$

*Work partially supported by MURST (40%).

0022-247X/96 $18.00

In this case if $g \in L^s(\Omega)$ for some $s > n/2$ then u is Hölder-continuous. But again this result does not hold in the limit case $s = n/2$.

In the study of the continuity of solutions of equations like (1.1) or (1.2) it is useful to introduce the Lorentz spaces $L^{p,q}(\Omega)$, where $p > 1$, $q > 0$ (see definition in Section 2). For these spaces the following inclusion relations hold,

$$\begin{aligned} L^r(\Omega) &\subset L^{p,q}(\Omega) \subset L^{p,p}(\Omega) \\ &\equiv L^p(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega), \end{aligned}$$

whenever $0 < q < p < r \le \infty$.

Indeed one can prove, see [A, T], that if g belongs to the Lorentz space $L^{n/2,1}(\Omega)$ and $n > 2$ any $W^{1,2}_{loc}(\Omega)$ solution of (1.2) is continuous. In the framework of Lorentz spaces this result is optimal in the sense that if $g \in L^{n/2,q}(\Omega)$, with $q > 1$, examples can be given showing that the solution u may be even unbounded. Similarly one can prove, see [F], that if $h \in L^{n,1}(\Omega)$ and $n > 2$ any weak solution of (1.1) is continuous. Also this result is optimal.

Another approach to the regularity of solutions of Eq. (1.1) is based on the observation that weak solutions of this equation are indeed local minimizers of the functional

$$\int_\Omega \left[a_{ij} D_i u D_j u - h(x) Du \right]. \tag{1.3}$$

For a more general functional of the type

$$\int_\Omega f(x, u(x), Du(x))\, dx, \tag{1.4}$$

with

$$\lambda |\xi|^p - g(x) \le f(x, \eta, \xi) \le \Lambda(|\xi|^p + |\eta|^p + g(x)), \qquad p > 1, \tag{1.5}$$

Giaquinta and Giusti, see [GG], proved that if $g(x) \in L^s(\Omega)$ with $s > n/p$, then a $W^{1,p}(\Omega)$ minimizer u (or even a Q-minimizer) is Hölder-continuous. Clearly the functional (1.3) is of the type (1.4) with f satisfying (1.5) for $p = 2$ and $g = |h|^2$. In a similar way one can study the regularity of weak solutions u of an equation of the type

$$\operatorname{div}\left(a(x)|Du|^{p-2} Du\right) = \operatorname{div} h,$$

remarking that u is also a minimizer of a functional of the type (1.4) with $g(x) = c|h(x)|^{p/(p-1)}$. In conclusion Hölder-continuity of solutions of many linear and nonlinear elliptic equations can be obtained from the corresponding property of minimizers and Q-minimizers of functionals of the type (1.4) (see [GG]).

In this paper we extend the continuity results for solutions of linear equations to the case of minimizers of the functional (1.4). As a model case covered by our Theorem 3.1, *consider the functional*

$$\int_\Omega \left[f_0(x, u(x), Du(x)) + h(x) Du \right] dx, \tag{1.6}$$

where $f_0(x, \eta, \xi)$ is a Carathéodory function such that

$$|\xi|^p \le f_0(x, \eta, \xi) \le L(1 + |\eta|^p + |\xi|^p), \qquad p > 1,$$

and $h \in L^{n/(p-1), 1/(p-1)}(\Omega)$. We then prove that if a function u in $W^{1,p}(\Omega)$ is a minimizer of the functional (1.6), *then u is continuous*.

Note that in order to deduce the continuity of the minimizer u it is not necessary to assume any differentiability of the integrand f_0. In this respect the regularity result given here is similar to the one proved in [GG]. Here too the continuity of u is obtained using only the polynomial growth of f_0 with respect to ξ and the summability assumption on $h(x)$.

However, our proof is different from the proof of the Hölder-continuity of minimizers of the functional (1.4) given by [GG]. In fact they prove that if $u \in W^{1,p}_{loc}(\Omega)$ is a minimizer of (1.4) and f satisfies (1.5) then both functions u and $-u$ satisfy the estimate

$$\int_{\{u>k\}\cap B_\rho} |Du|^p \, dx \le \frac{c}{(R-\rho)^p} \int_{\{u>k\}\cap B_R} |u-k|^p \, dx + c \int_{\{u>k\}\cap B_R} g(x) \, dx \tag{1.7}$$

for any ball $B_R \Subset \Omega$, $0 < \rho < R$, $k \ge 0$. Then, from the results of De Giorgi (see [LU]) it follows that if u and $-u$ satisfy this estimate and $g \in L^s(\Omega)$, with $s > n/p$, u is Hölder-continuous. It is not clear to us if using the De Giorgi argument one can prove that if g is only in $L^{n/p, 1/p}(\Omega)$ and u and $-u$ satisfy (1.7) then u is at least a continuous function.

Indeed, using a variational principle due to Ekeland (see Theorem 2.3) we compare the minimizer u of functional (1.6) with the minimizer v of a simpler functional to which the De Giorgi result applies. Then from the gradient estimates satisfied by v we get similar estimates (see Proposition 3.6) for u. The continuity of u is then achieved by fully exploiting the properties of Lorentz spaces (see Lemma 2.2) through a sharp version of the standard iteration arguments.

As an application of the techniques developed in Section 3 we also prove (see Section 4) the continuity of solutions of the nonlinear equation

$$\operatorname{div} A(x, u(x), Du(x)) + H(x, u) = \operatorname{div} h,$$

where $h \in L^{n/(p-1), 1/(p-1)}(\Omega)$, $p > 1$, $A(x, \eta, \xi)$ grows like $|\xi|^{p-1}$ and satisfies the usual monotonicity assumptions, and $H(x, \eta)$ grows like $|\eta|^{p-1}$. It seems to us that in this case the monotonicity of A plays the role of Ekeland principle which holds only for minimizers. We finally remark that no sign conditions on $H(x, \eta)$ are required.

2. PRELIMINARY RESULTS

In this section we recall the definition and some properties of Lorentz spaces. In the following Ω will always denote an open set in $\mathbb{R}^n$. The ball centered in x_0, with radius R will be denoted by $B_R(x_0)$, or simply by B_R, and the average on such a ball of an integrable function f will be denoted by

$$(f)_{x_0, R} = \fint_{B_R} f(x)\, dx$$

or just f_R.

DEFINITION. A function f belongs to the Lorentz space $L^{p,q}(\Omega)$, with $1 < p < \infty$, $0 < q \leq \infty$ if

$$[f]_{p,q} = \begin{cases} \left(\displaystyle\int_0^{+\infty} \left(f^*(s) s^{1/p} \right)^q \frac{ds}{s} \right)^{1/q} & \text{when } q > 0 \\ \displaystyle\sup_{s>0} \left(f^*(s) s^{1/p} \right) & \text{when } q = +\infty \end{cases} \tag{2.1}$$

is finite.

In (2.1) we have denoted by $f^*(s)$: $[0, +\infty[\to [0, +\infty[$ the decreasing rearrangement of f in Ω

$$f^*(s) = \sup\left\{ t \geq 0 \colon \left| \{ x \in \Omega \colon |f(x)| > t \} \right| > s \right\}.$$

Here and in the following, if f is a vector field, we say that f belongs to $L^{p,q}(\Omega)$ if $|f|$ does. Using the definition above and Hardy inequality it is possible to prove (see [BS]) that the following inclusions hold,

$$\begin{aligned} L^r(\Omega) &\subset L^{p,q}(\Omega) \subset L^{p,p}(\Omega) \\ &\equiv L^p(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega), \end{aligned}$$

whenever $0 < q < p < r \leq \infty$. Further properties of Lorentz spaces can be found in [BS]. It is easy to check that $[\cdot]_{p,q}$ is a seminorm on the linear space $L^{p,q}(\Omega)$. However, the following result (see [ON]) shows that one can define a norm on $L^{p,q}$ which is equivalent to $[f]_{p,q}$.

THEOREM 2.1. *Let us define for* $1 < p < \infty$, $0 < q \leq \infty$

$$\|f\|_{p,q} = \begin{cases} \left(\displaystyle\int_0^{+\infty} \left(f^{**}(s) s^{1/p} \right)^q \frac{ds}{s} \right)^{1/q} & \text{when } q > 0 \\ \displaystyle\sup_{s>0} \left(f^{**}(s) s^{1/p} \right) & \text{when } q = +\infty, \end{cases}$$

where for any $s > 0$

$$f^{**}(s) = \frac{1}{s} \int_0^s f^*(t)\, dt.$$

Then there exists $c_{p,q}$ *such that for any* $f \in L^{p,q}(\Omega)$

$$[f]_{p,q} \leq \|f\|_{p,q} \leq c_{p,q}[f]_{p,q}. \tag{2.2}$$

The following technical lemma will be useful at the end of next section.

LEMMA 2.2. *Let* $a > 0$, $b > 1/a$, $f \in L^{ab,a}(\Omega)$, $0 < q < b$, $d = 1/n - 1/b + 1/q$, $\tau \in (0,1)$. *For any compact subset* K *of* Ω *the series*

$$\sum_{i=1}^{\infty} (\tau^i R)^{1-dn} \left(\int_{B_{\tau^i R}(x)} |f(y)|^{aq}\, dy \right)^{1/q}, \tag{2.3}$$

where $R < \operatorname{dist}(K, \partial\Omega)$, *converges uniformly for* $x \in K$.

Proof. Denoting by ω_n the measure of the unit ball in $\mathbb{R}^n$, and using the properties of rearrangements we have

$$\int_{B_{\tau^i R}(x)} |f(y)|^{aq}\, dy \leq \int_0^{\omega_n(\tau^i R)^n} (f^*(s))^{aq}\, ds \qquad \text{for any } x \in K,$$

where f^* is the decreasing rearrangement of f in $B_R(x)$. Hence the series appearing in (2.3) is majorized by

$$\sum_{i=1}^{\infty}(\tau^i R)^n\left(\frac{1}{\left[(\tau^i R)^n\right]^{q+dq-q/n}}\int_0^{\omega_n(\tau^i R)^n}(f^*(s))^{aq}\,ds\right)^{1/q}$$

$$\le c(n,\tau,q)\sum_{i=1}^{\infty}\left[\omega_n(\tau^{i-1}R)^n-\omega_n(\tau^i R)^n\right]$$

$$\times\left(\frac{1}{\left(\omega_n(\tau^{i-1}R)^n\right)^{1+q-q/b}}\int_0^{\omega_n(\tau^i R)^n}(f^*(s))^{aq}\,ds\right)^{1/q}$$

$$\le c\sum_{i=1}^{\infty}\int_{\omega_n(\tau^i R)^n}^{\omega_n(\tau^{i-1}R)^n}F(s)\,ds,$$

where

$$F(s)=\left(\frac{1}{s^{1+q-q/b}}\int_0^s(f^*(s))^{aq}\,ds\right)^{1/q}.$$

Therefore series (2.3) is controlled by

$$c\int_0^{|B_R|}\left(\frac{1}{s^{1-q/b}}\int_0^s(f^*(t))^{aq}\,dt\right)^{1/q}\frac{ds}{s}=c\int_0^{|B_R|}\left(s^{q/b}(|f|^{aq})^{**}(s)\right)^{1/q}\frac{ds}{s}$$

$$\le c\||f|^{aq}\|^{1/q}_{L^{b/q,1/q}(B_R)}.$$

By (2.1) and (2.2) we may conclude that

$$\sum_{i=1}^{\infty}(\tau^i R)^{1-dn}\left(\int_{B_{\tau^i R}(x)}|f(y)|^{aq}\,dy\right)^{1/q}\le c\int_0^{|B_R|}\left(f^*(t)t^{1/ab}\right)^a\frac{dt}{t},$$

where f^* is from now on the rearrangement of f in Ω. Then series (2.3) converges.

The uniform convergence follows from the observation that the Nth remainder of (2.3)

$$\sum_{i=N}^{\infty}(\tau^i R)^{1-dn}\left(\int_{B_{\tau^i R}(x)}|f(y)|^{aq}\,dy\right)^{1/q}$$

can be obtained from (2.3) substituting R with $\tau^N R$. Hence it is controlled uniformly by

$$c\int_0^{|B_{\tau^n R}|} \left(f^*(t)\,t^{1/ab}\right)^a \frac{dt}{t},$$

a quantity which tends to 0 as $N \to \infty$. ∎

Remark. If one takes $q = b$ in Lemma 2.2, the series (2.3) may diverge. As an example take the function

$$f(x) = \frac{1}{\omega_n^{1/n}|x|\left(\log(1/|x|^n)\right)^{1+1/n}}$$

in the ball $B_R(0)$, with $R < 1$ small enough in order to guarantee that f is radially decreasing. Obviously,

$$f^*(s) = \frac{1}{s^{1/n}\left(\log(\omega_n/s)\right)^{1+1/n}}$$

and $f \in L^{n,1}(B_R(0))$. Then

$$\sum_{i=1}^{\infty}\left(\int_{B_{\tau^i R}(0)} |f(y)|^n\, dy\right)^{1/n} = \sum_{i=1}^{\infty}\left(\int_0^{\omega_n \tau^{in} R^n} \frac{1}{s\left(\log(\omega_n/s)\right)^{n+1}}\, ds\right)^{1/n}$$
$$= \sum_{i=1}^{\infty}\left(\frac{1}{n}\right)^{1/n}\left(\log\frac{1}{\tau^{in}R^n}\right)^{-1} = +\infty.$$

Finally we recall the following variational principle due to Ekeland, see [E, Theorem 1].

THEOREM 2.3. *Let (V, d) be a complete metric space, $\mathscr{F}\colon V \to]-\infty, +\infty]$ a lower semicontinuous functional such that $\inf_V \mathscr{F}$ is finite. Let $\epsilon > 0$ and $u \in V$ such that*

$$\mathscr{F}(u) \le \inf_V \mathscr{F} + \epsilon.$$

Then there exists $v \in V$ such that

(i) $d(u, v) \le 1$;

(ii) $\mathscr{F}(v) \le \mathscr{F}(u)$;

(iii) *v minimizes the functional $\mathscr{G}(w) = \mathscr{F}(w) + \epsilon d(v, w)$.*

3. VARIATIONAL MINIMA

In this section we consider integral functionals of the type

$$\mathscr{F}(u) = \mathscr{F}(u, \Omega) = \int_\Omega f(x, u(x), Du(x))\, dx, \tag{3.1}$$

where $u \in W^{1,p}(\Omega)$, $1 < p < n$, and the integrand $f(x, \eta, \xi)$ satisfies the following assumptions:

(i) $f(x, \eta, \xi)$: $\Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function which can be written as

$$f(x, \eta, \xi) = f_0(x, \eta, \xi) + f_1(x, \eta, \xi);$$

(ii) f_0 satisfies the growth condition

$$|\xi|^p \le f_0(x, \eta, \xi) \le L\big(1 + |\eta|^p + |\xi|^p\big), \qquad p > 1;$$

(iii) there exists $g(x) \in L^{p/(p-1)}(\Omega)$, $g(x) \ge 0$ such that

$$\big|f_1(x, \eta, \xi)\big| \le g(x)|\xi|.$$

Our aim is to prove the following

THEOREM 3.1. *If $u \in W^{1,p}(\Omega)$ is a minimizer of functional* (3.1) *and f satisfies* (i)–(iii), *with $g(x) \in L^{n/(p-1), 1/(p-1)}(\Omega)$, then u is continuous.*

In order to achieve the proof of this theorem we need to recall some well known results concerning Q-minima.

DEFINITION. Let $h(x) \in L^1(\Omega)$, $h(x) \ge 0$, and $Q \ge 1$. We say that $u \in W^{1,p}(\Omega)$, $p \ge 1$, is a Q-minimizer of the functional

$$\mathscr{G}(v) = \mathscr{G}(v, \Omega) = \int_\Omega \big(|Dv|^p + |v|^p + h(x)\big)\, dx \tag{3.2}$$

if for any test function $\phi \in W_0^{1,p}(\Omega)$

$$\mathscr{G}(v, \operatorname{supp} \phi) \le Q\mathscr{G}(v + \phi, \operatorname{supp} \phi)$$

where, as usual, $\operatorname{supp} \phi$ denotes the support of ϕ.

The following result is contained in [GG, Theorem 3.1].

THEOREM 3.2. *If $u \in W^{1,p}(\Omega)$ is a Q-minimizer of functional* (3.2), *with $p > 1$, and $h \in L^s(\Omega)$, $s > 1$, then there exists q, with $1 < q/p \le s$ and a constant $c_1 > 0$ such that for any $B_R \subset \Omega$, $R \le 1$,*

$$\left(⨍_{B_{R/2}} (|Du|^q + |u|^q)\right)^{1/q} \le c_1\left[\left(⨍_{B_R} (|Du|^p + |u|^p)\right)^{1/p} + \left(⨍_{B_R} h^{q/p}\right)^{1/q}\right].$$

A simple consequence of this theorem is

COROLLARY 2.3. *If $u \in W^{1,p}(\Omega)$ is a minimizer of functional $\mathscr{F}(u)$ and f satisfies* (i)–(iii), $g(x) \in L^{s/(p-1)}(\Omega)$, $s > p$, *then there exists $p < q < s$ and a constant $c_2 > 0$ such that for any $B_R \subset \Omega$, $R \le 1$,*

$$\left(\fint_{B_{R/2}} (|Du|^q + |u|^q) \right)^{1/q} \le c_2 \left[\left(\fint_{B_R} (|Du|^p + |u|^p) \right)^{1/p} + \left(\fint_{B_R} (g^{q/(p-1)} + 1) \right)^{1/q} \right].$$

Proof. It is enough to remark that if u is a minimizer of $\mathscr{F}(u)$, then u is a Q-minimizer, with $Q \equiv Q(L)$, of functional (3.2), where $h(x) = g^{p/(p-1)}(x) + 1$. ∎

The next result is well known as the Caccioppoli inequality (see, for example, [G]).

THEOREM 3.4. *If $u \in W^{1,p}(\Omega)$, is a Q-minimizer of functional $\mathscr{G}(u)$, and $h \in L^1(\Omega)$, then for any $B_R \subset \Omega$*

$$\int_{B_{R/2}} |Du|^p \le c \left[\int_{B_R} \frac{|u - u_R|^p}{R^p} + \int_{B_R} |u|^p + \int_{B_R} h \right], \tag{3.3}$$

where c depends only on Q, n, p.

In the following we need a suitable version of a classical result due to De Giorgi [DG] (see also [GG, Theorems 4.1 and 4.2]) for Q-minimizers of the functional

$$\mathscr{G}(v, B_R(x_0)) = \int_{B_R(x_0)} (|Dv|^p + |v|^p + k^p)\, dx, \tag{3.4}$$

where $k \ge 0$ is a constant.

First of all we can make the following simple observation.

Remark 3.1. If u is a Q-minimizer of functional (3.4) in $B_R(x_0)$, $R \le 1$, the function $\tilde{u}(y) = u(x_0 + Ry)/kR$ is a $\tilde{Q}$-minimizer of the functional

$$v \to \int_{B_1(0)} (|Dv|^p + |v|^p + 1)\, dx,$$

where $\tilde{Q}$ depends only on Q, p, n. In fact, by the Q-minimality of u, it follows that for any function $\tilde{\varphi}(y)$ with compact support on $B_1(0)$

$$\begin{aligned}\int_{\operatorname{supp}\tilde{\varphi}} \big(|D\tilde{u}|^p + R^p|\tilde{u}|^p + 1\big)\,dx \\ \le Q\int_{\operatorname{supp}\tilde{\varphi}} \Big(|D\tilde{u} + D\tilde{\varphi}|^p + R^p|\tilde{u} + \tilde{\varphi}|^p + 1\Big)\,dx.\end{aligned}$$

Since, by the Poincaré inequality on B_1,

$$\begin{aligned}\int_{\operatorname{supp}\tilde{\varphi}} |\tilde{u}|^p\,dx &\le 2^{p-1}\int_{\operatorname{supp}\tilde{\varphi}} |\tilde{u} + \tilde{\varphi}|^p\,dx + 2^{p-1}\int_{\operatorname{supp}\tilde{\varphi}} |\tilde{\varphi}|^p\,dx \\ &\le 2^{p-1}\int_{\operatorname{supp}\tilde{\varphi}} |\tilde{u} + \tilde{\varphi}|^p\,dx + c(n,p) \\ &\quad\times\left[\int_{\operatorname{supp}\tilde{\varphi}} |D\tilde{u} + D\tilde{\varphi}|^p\,dx + \int_{\operatorname{supp}\tilde{\varphi}} |D\tilde{u}|^p\,dx\right], \qquad (3.5)\end{aligned}$$

from the above inequality, using the assumption $R \le 1$, we have

$$\begin{aligned}&\int_{\operatorname{supp}\tilde{\varphi}} \left(|D\tilde{u}|^p + \frac{1}{2c}|\tilde{u}|^p + 1\right)dx \\ &\quad\le Q\int_{\operatorname{supp}\tilde{\varphi}} \Big(|D\tilde{u} + D\tilde{\varphi}|^p + |\tilde{u} + \tilde{\varphi}|^p + 1\Big)\,dx \\ &\qquad + \frac{2^{p-2}}{c}\int_{\operatorname{supp}\tilde{\varphi}} |\tilde{u} + \tilde{\varphi}|^p\,dx + \frac{1}{2}\int_{\operatorname{supp}\tilde{\varphi}} |D\tilde{u} + D\tilde{\varphi}|^p\,dx \\ &\qquad + \frac{1}{2}\int_{\operatorname{supp}\tilde{\varphi}} |D\tilde{u}|^p\,dx.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\operatorname{supp}\tilde{\varphi}} \left(\frac{1}{2}|D\tilde{u}|^p + \frac{1}{2c}|\tilde{u}|^p + 1\right)dx \\ &\quad\le \left(Q + \frac{2^{p-2}}{c} + \frac{1}{2}\right)\int_{\operatorname{supp}\tilde{\varphi}} \Big(|D\tilde{u} + D\tilde{\varphi}|^p + |\tilde{u} + \tilde{\varphi}|^p + 1\Big)\,dx,\end{aligned}$$

which proves the assertion.

THEOREM 3.5. *If* $u \in W^{1,p}(B_R(x_0))$, $R \le 1$, $p > 1$, *is a* Q*-minimizer of the functional* (3.4), *then there exists* $\alpha > 0$, $c_3 > 0$ *depending only on* p, n, Q *such that if* $0 < \rho < R$

$$\int_{B_\rho(x_0)} (|Du|^p + |u|^p) \le c_3 \left(\frac{\rho}{R} \right)^{n-p+p\alpha} \left[\int_{B_R(x_0)} (|Du|^p + |u|^p + k^p) \right]. \tag{3.6}$$

Proof. If u is a Q-minimizer of (3.4) in $B_R(x_0)$, then

$$\sup_{B_{R/2}} |u|^p \le c \left[\left(\fint_{B_R} |u|^p \right) + k^p R^p \right]. \tag{3.7}$$

In fact, if $\tilde{u}$ is a $\tilde{Q}$-minimizer in $B_1(0)$ of the functional $\int_{B_1} (|Dv|^p + |u|^p + 1)$, it is well known that $\tilde{u}$ is locally bounded. Moreover (see [GE, Theorem 7.4] or [LU, Theorem 5.3, Chap. 2])

$$\sup_{B_{1/2}} |\tilde{u}|^p \le c \left[\left(\fint_{B_1} |\tilde{u}|^p \right) + 1 \right].$$

The estimate (3.7) then follows using Remark 3.1 and rescaling.

On the other hand, if $\tilde{u} \in W^{1,p}(B_1(0))$ is a $\tilde{Q}$-minimizer in $B_1(0)$ of the functional $\int_{B_1} (|Dv|^p + |v|^p + 1)$ and $\sup_{B_{1/2}} |\tilde{u}| \le 1$, then $\tilde{u}$ is locally Hölder-continuous in $B_{1/2}$ for some exponent $\alpha > 0$. More precisely (see [GE, Theorem 7.7] or [LU, Theorem 6.1, Chap. 2]) for every $0 < \rho < 1/2$

$$\int_{B_\rho} |D\tilde{u}|^p \le c \rho^{n-p+p\alpha} \left[\int_{B_{1/2}} |D\tilde{u}|^p + 1 \right]. \tag{3.8}$$

If $\sup_{B_{1/2}} |\tilde{u}| = M > 1$ the function $\tilde{u}/M$ is obviously a $\tilde{Q}$-minimizer of the functional $\int_{B_1} (|Dv|^p + |v|^p + 1/M^p)$ and a fortiori a $(\tilde{Q} + 1)$-minimizer of the functional $\int_{B_1} (|Dv|^p + |v|^p + 1)$. Since $\sup_{B_{1/2}} |\tilde{u}/M| = 1$, from (3.8) we get that for any $0 < \rho < 1/2$

$$\int_{B_\rho} |D\tilde{u}|^p \le c \rho^{n-p+p\alpha} \left[\int_{B_{1/2}} |D\tilde{u}|^p + \sup_{B_{1/2}} |\tilde{u}|^p \right].$$

Hence, from estimate (3.7) we obtain

$$\int_{B_\rho} |D\tilde{u}| \le c \rho^{n-p+p\alpha} \left[\int_{B_1} (|D\tilde{u}|^p + |\tilde{u}|^p + 1) \right],$$

for any $0 < \rho < 1/2$.

If u is a Q-minimizer on $B_R(x_0)$ of the functional $\mathscr{G}$, using again the fact that the function $\tilde{u}(y) = u(x_0 + Ry)/kR$ is a $\tilde{Q}$-minimizer on $B_1(0)$ of the functional $\int_{B_1}(|Dv|^p + |v|^p + 1)$, from the estimate above we get, after rescaling,

$$\int_{B_\rho}|Du|^p \le c\left(\frac{\rho}{R}\right)^{n-p+p\alpha}\int_{B_R}\left[|Du|^p + \frac{|u|^p}{R^p} + k^p\right],$$

for any $0 < \rho < R/2$. Since $u - u_R$ is a Q'-minimizer of functional $\int_{B_R}(|Dv|^p + |v|^p + |u_R|^p + k^p)$, for some $Q' = c(p)Q$, applying the estimate above to $u - u_R$ and the Poincaré inequality, we have

$$\begin{aligned}\int_{B_\rho}|Du|^p &\le c\left(\frac{\rho}{R}\right)^{n-p+p\alpha}\int_{B_R}\left[|Du|^p + \frac{|u-u_R|^p}{R^p} + |u_R|^p + k^p\right]\\ &\le c\left(\frac{\rho}{R}\right)^{n-p+p\alpha}\int_{B_R}[|Du|^p + |u|^p + k^p],\end{aligned}$$

for any $0 < \rho < R/2$.

Finally, by (3.7), if $0 < \rho < R/2$, and since $R \le 1$,

$$\begin{aligned}\int_{B_\rho}|u|^p \le \rho^n \sup_{B_{R/2}}|u|^p &\le c\rho^n\left[\fint_{B_R}|u|^p + k^pR^p\right]\\ &\le c\left(\frac{\rho}{R}\right)^n\left[\int_{B_R}|u|^p + k^pR^n\right],\end{aligned}$$

which implies

$$\int_{B_\rho}(|Du|^p + |u|^p) \le c\left(\frac{\rho}{R}\right)^{n-p+p\alpha}\int_{B_R}[|Du|^p + |u|^p + k^p]. \qquad \blacksquare$$

PROPOSITION 3.6. *If $u \in W^{1,p}(\Omega)$ is a minimizer of functional* (3.1) *and f satisfies* (i)–(iii), *with $g(x) \in L^{n/(p-1)}(\Omega)$, for any $\varepsilon > 0$ and $B_\rho \subset B_R \subset \Omega$, $R \le 1$, the following estimate holds*:

$$\begin{aligned}\int_{B_\rho}(|Du|^p + |u|^p) \le c_4&\left(\left(\frac{\rho}{R}\right)^{n-p+p\sigma} + \varepsilon + R^p\right)\int_{B_R}(|Du|^p + |u|^p)\\ &+ c_\varepsilon R^{n(1-p/q)}\left(\int_{B_R}\left(g^{q/(p-1)} + 1\right)\right)^{p/q},\end{aligned} \tag{3.9}$$

for some $c_4 > 0$, $\sigma > 0$, $p < q < n$ (*depending only on* n, L, p) *and* c_ε (*depending only on* n, L, p, ε).

Proof. Let us fix $B_R \subset \Omega$, $R \leq 1$, and consider the functional

$$\mathscr{F}_0(w, B_R) = \int_{B_R} f_0(x, w(x), Dw(x))\, dx,$$

with $w \in V$, where $V = u + W_0^{1,1}(B_R)$. As it will be clear in the rest of the proof, the use of V, instead of the "natural" space $u + W_0^{1,p}(B_R)$, is needed in order to apply Ekeland's lemma.

Let us fix $1 > \delta > 0$ and chose $u_\delta \in V$ such that

$$\mathscr{F}_0(u_\delta, B_R) \leq \inf_{w \in V} \mathscr{F}_0(w, B_R) + \delta R^n.$$

By assumption (ii) on f we have

$$\int_{B_R} |Du_\delta|^p \leq 2L \int_{B_R} (1 + |u|^p + |Du|^p). \tag{3.10}$$

By the minimality of u we have also

$$\begin{aligned}
\mathscr{F}_0(u, B_R) &= \mathscr{F}(u, B_R) - \int_{B_R} f_1(x, u, Du) \\
&\leq \mathscr{F}_0(u_\delta, B_R) + \int_{B_R} \left(f_1(x, u_\delta, Du_\delta) - f_1(x, u, Du) \right) \\
&\leq \inf_{w \in V} \mathscr{F}_0(w, B_R) + H(R) + \delta R^n,
\end{aligned}$$

where, by (iii) and (3.10),

$$H(R) = \mu \int_{B_R} (|Du|^p + |u|^p) + c_\mu \int_{B_R} (g^{p/(p-1)} + 1), \tag{3.11}$$

with $1 > \mu > 0$ to be chosen later. Letting δ go to zero, we have

$$\mathscr{F}_0(u, B_R) \leq \inf_{w \in V} \mathscr{F}_0(w, B_R) + H(R).$$

The functional $\mathscr{F}_0(w, B_R)$ is lower semicontinuous with respect to the topology induced on V by the distance

$$d(u_1, u_2) = (H(R))^{-1/p} R^{-n(1-1/p)} \int_{B_R} |Du_1 - Du_2|.$$

Therefore, Theorem 2.3 implies that a function $v \in V$ exists such that

$$\begin{cases} \int_{B_R} |Du - Dv| \leq \left(H(R) \right)^{1/p} R^{n(1-1/p)} \\ \mathscr{F}_0(v, B_R) \leq \mathscr{F}_0(u, B_R) \\ v \text{ minimizes the functional: } \mathscr{F}_0(w, B_R) + \left(\dfrac{H(R)}{R^n} \right)^{(p-1)/p} \int_{B_R} |Dw - Dv|. \end{cases} \tag{3.12}$$

Actually $v \in u + W_0^{1,p}(B_R)$ and it is a Q-minimizer (with Q depending only on L) of the functional

$$w \to \int_{B_R} \left(|Dw|^p + |w|^p + \frac{H(R)}{R^n} + 1 \right).$$

In fact, if $\varphi \in W_0^{1,p}(B_R)$, by the minimality of v, we have

$$\begin{aligned} \mathscr{F}_0(v, \operatorname{supp} \varphi) &\leq \mathscr{F}_0(v + \varphi, \operatorname{supp} \varphi) + \left(\frac{H(R)}{R^n} \right)^{1-1/p} \int_{\operatorname{supp} \varphi} |D\varphi| \\ &\leq \mathscr{F}_0(v + \varphi, \operatorname{supp} \varphi) + \frac{1}{2^p} \int_{\operatorname{supp} \varphi} |D\varphi|^p + c \frac{H(R)}{R^n} |\operatorname{supp} \varphi| \\ &\leq \mathscr{F}_0(v + \varphi, \operatorname{supp} \varphi) + \frac{1}{2} \int_{\operatorname{supp} \varphi} |Dv|^p + \frac{1}{2} \int_{\operatorname{supp} \varphi} |Dv + D\varphi|^p \\ &\qquad + c \frac{H(R)}{R^n} |\operatorname{supp} \varphi|, \end{aligned}$$

hence, by (ii),

$$\begin{aligned} \int_{\operatorname{supp} \varphi} |Dv|^p \leq (2L + 1) \int_{\operatorname{supp} \varphi} \left(1 + |v + \varphi|^p + |Dv + D\varphi|^p \right) \\ + c \frac{H(R)}{R^n} |\operatorname{supp} \varphi|. \end{aligned} \tag{3.13}$$

Arguing as in the proof of (3.5) and using the assumption $R \leq 1$, we have

$$\begin{aligned} \int_{\operatorname{supp} \varphi} |v|^p \leq 2^{p-1} \int_{\operatorname{supp} \varphi} |v + \varphi|^p + c(n, p) \\ \times \left[\int_{\operatorname{supp} \varphi} |Dv + D\varphi|^p + \int_{\operatorname{supp} \varphi} |Dv|^p \right]. \end{aligned}$$

From this estimate and (3.13) one proves easily the Q-minimality of v.

Using Theorem 3.2 we get for some $p < q < n$,

$$\left(\fint_{B_{R/2}} |Dv|^q\right)^{1/q} \le c_1\left[\left(\fint_{B_R} (|Dv|^p + |v|^p)\right)^{1/p} + \left(1 + \frac{H(R)}{R^n}\right)^{1/p}\right], \tag{3.14}$$

where q can be taken equal to the one appearing in Corollary 3.3.

Now Theorem 3.5 gives that for any $0 < \rho < R$,

$$\int_{B_\rho} (|Dv|^p + |v|^p) \le c\left(\frac{\rho}{R}\right)^{n-p+p\alpha}\left[\int_{B_R} (|Dv|^p + |v|^p) + H(R) + R^n\right]. \tag{3.15}$$

On the other hand

$$\left(\fint_{B_{R/2}} |Du - Dv|^p\right)^{1/p} \le \left(\fint_{B_{R/2}} |Du - Dv|^q\right)^{\vartheta/q}\left(\fint_{B_{R/2}} |Du - Dv|\right)^{1-\vartheta}.$$

where $0 < \vartheta < 1$ is such that $\vartheta/q + 1 - \vartheta = 1/p$.

From the first inequality in (3.12), (3.14), and Corollary 3.3 we have, for some $0 < \varepsilon < 1$,

$$\begin{aligned}
&\left(\fint_{B_{R/2}} |Du - Dv|^p\right)^{1/p} \\
&\quad \le \varepsilon\left[\left(\fint_{B_{R/2}} |Du|^q\right)^{1/q} + \left(\fint_{B_{R/2}} |Dv|^q\right)^{1/q}\right] \\
&\qquad + c_\varepsilon \fint_{B_{R/2}} |Du - Dv| \\
&\quad \le \varepsilon c\left[\left(\fint_{B_R} (|Du|^p + |u|^p)\right)^{1/p} + \left(\fint_{B_R} \left(g^{q/(p-1)} + 1\right)\right)^{1/q}\right. \\
&\qquad\qquad \left. + \left(\fint_{B_R} (|Dv|^p + |v|^p)\right)^{1/p} + \left(1 + \frac{H(R)}{R^n}\right)^{1/p}\right] \\
&\qquad + c_\varepsilon\left(\frac{H(R)}{R^n}\right)^{1/p}.
\end{aligned}$$

Using the Poincaré inequality in B_R, the assumption $R \leq 1$, and the second inequality in (3.12), the above inequality gives

$$\left(\fint_{B_{R/2}} |Du - Dv|^p\right)^{1/p} \leq \varepsilon c\left[\left(\fint_{B_R} (|Du|^p + |u|^p)\right)^{1/p} + \left(\fint_{B_R} (g^{q/(p-1)} + 1)\right)^{1/q} + \left(\frac{H(R)}{R^n}\right)^{1/p}\right] + c_\varepsilon \left(\frac{H(R)}{R^n}\right)^{1/p}.$$

Hence, raising to the power p both sides of the previous inequality and getting rid of the averages, we get

$$\int_{B_{R/2}} |Du - Dv|^p \leq c\varepsilon^p \left[\int_{B_R} (|Du|^p + |u|^p) + R^{n(1-p/q)}\left(\int_{B_R} (g^{q/(p-1)} + 1)\right)^{p/q}\right] + c_\varepsilon H(R).$$

If $0 < \rho < R/2$, from this inequality and (3.15) we get, recalling (3.11) and (3.12) again,

$$\begin{aligned}
\int_{B_\rho} (|Du|^p + |u|^p) &\leq 2^{p-1}\int_{B_\rho} (|Dv|^p + |v|^p) + 2^{p-1}\int_{B_{R/2}} |Du - Dv|^p \\
&\quad + 2^{p-1}\int_{B_{R/2}} |u - v|^p \\
&\leq c\left(\frac{\rho}{R}\right)^{n-p+p\alpha}\int_{B_R} (|Dv|^p + |v|^p) + 2^{p-1}\int_{B_{R/2}} |Du - Dv|^p \\
&\quad + 2^{p-1}\int_{B_R} |u - v|^p + c(R^n + H(R)) \\
&\leq c\left(\frac{\rho}{R}\right)^{n-p+p\alpha}\int_{B_R} (|Du|^p + |u|^p) + 2^{p-1}\int_{B_{R/2}} |Du - Dv|^p \\
&\quad + c\int_{B_R} |u - v|^p + c(R^n + H(R))
\end{aligned}$$

$$\leq c\left(\frac{\rho}{R}\right)^{n-p+p\alpha}\int_{B_R}(|Du|^p+|u|^p)+2^{p-1}\int_{B_{R/2}}|Du-Dv|^p$$

$$+cR^p\int_{B_R}(|Du|^p+|Dv|^p)+c(R^n+H(R))$$

$$\leq c\left(\left(\frac{\rho}{R}\right)^{n-p+p\alpha}+\varepsilon^p+R^p\right)\int_{B_R}(|Du|^p+|u|^p)$$

$$+c_\varepsilon R^{n(1-p/q)}\left(\int_{B_R}(g^{q/(p-1)}+1)\right)^{p/q},$$

choosing $\mu\equiv\mu(\varepsilon)$ small enough. ∎

Using a standard iteration argument, very similar to the one given in the proof of Lemma 2.1, Chap. 3 of [G], we obtain:

PROPOSITION 3.7. *Under the assumptions of Proposition* 3.6 *there exists* $R_0>0$ *such that for any* $B_R\subset\Omega$, $0<\rho<R\leq R_0$,

$$\int_{B_\rho}(|Du|^p+|u|^p)\leq c_5\left[\left(\frac{\rho}{R}\right)^{n-p+p\sigma'}\int_{B_R}(|Du|^p+|u|^p)\right.$$

$$\left.+\rho^{n(1-p/q)}\left(\int_{B_R}(g^{q/(p-1)}+1)\right)^{p/q}\right],\quad(3.16)$$

where $1>\sigma'>0$ *and* $c_5>0$ *are constants independent of* R.

Proof. Let us put $\alpha=n-p+\sigma p$, $\beta=n(1-p/q)$, $\tau=\rho/R$, and

$$\varphi(\rho)=\int_{B_\rho}(|Du|^p+|u|^p).$$

Then (3.9) can be written as

$$\varphi(\tau R)\leq c_4(\tau^\alpha+\varepsilon+R^p)\varphi(R)+c_\varepsilon R^\beta F(R),$$

where

$$F(R)=\left(\int_{B_R}(g^{q/(p-1)}+1)\right)^{p/q}.$$

Choose γ such that

$$n-p<\gamma<\alpha,$$

and $0<\tau<1$ such that $2c_4\tau^\alpha<\tau^\gamma$.

If $\varepsilon > 0$, R_0 are chosen such that $\varepsilon + R_0^p < \tau^\alpha$, we have

$$\varphi(\tau R) \leq \tau^\gamma \varphi(R) + c(\tau) R^\beta F(R),$$

for $0 < R \leq R_0$. Iterating, we get

$$\varphi(\tau^{k+1} R) \leq \tau^{\gamma(k+1)} \varphi(R) + c\tau^{k\beta} R^\beta \sum_{j=0}^{k} \tau^{j(\gamma-\beta)} F(\tau^{k-j} R)$$

$$\leq \tau^{\gamma(k+1)} \phi(R) + c \frac{\tau^{k\beta} R^\beta}{1 - \tau^{(\gamma-\beta)}} F(R).$$

Since for any $0 < \rho < R$, there exists $k \geq 1$ such that

$$\tau^k R \leq \rho < \tau^{k-1} R,$$

from the above inequality the assertion follows immediately. ∎

As in Proposition 3.7, it is possible to prove the following:

PROPOSITION 3.8. *Under the assumptions of Proposition* 3.6 *there exists* $R_0 > 0$ *such that for any* $B_R \subset \Omega$, $0 < \rho < R \leq R_0$.

$$\int_{B_\rho} (|Du|^p + |u|^p) \leq c_6 \left[\left(\frac{\rho}{R} \right)^{n-p+p\sigma'} \int_{B_R} (|Du|^p + |u|^p) + \rho^{n(1-p/n)} \left(\int_{B_R} \left(g^{n/(p-1)} + 1 \right) \right)^{p/n} \right], \quad (3.17)$$

where $1 > \sigma' > 0$ *and* $c_6 > 0$ *are constants independent of* R.

Remark 3.2. If (ii) is replaced by the growth condition

$$|\xi|^p \leq f_0(x, \eta, \xi) \leq L(1 + |\xi|^p), \qquad p > 1,$$

it is easy to check that in both sides of (3.16) and (3.17) the terms involving $|u|^p$ can be dropped.

Before proving Theorem 3.1 we want to stress that, under the hypotheses of Proposition 3.8, estimate (3.17) already contains interesting information on the regularity of u. In fact we have:

THEOREM 3.9. *If* $u \in W^{1,p}(\Omega)$ *is a minimizer of function* (3.1) *and* f *satisfies* (i)–(iii), *with* $g(x) \in L^{n/(p-1)}(\Omega)$, *then* u *is locally VMO in* Ω.

Proof. We remark that if u minimizes the functional in (3.1), then u is a Q-minimizer of the functional in (3.2), with $h(x) = g^{p/(p-1)}(x) + 1$.

Then, using Proposition 3.8, the Poincaré inequality, and (3.3), for any $0 < \rho < R \le R_0$ and $B_{2R} \subset \Omega$, we have

$$\begin{aligned}\int_{B_\rho} \frac{|u - u_\rho|^p}{\rho^p} &\le c\left(\frac{\rho}{R}\right)^{n-p+p\sigma'}\left[\int_{B_{2R}} \frac{|u - u_{2R}|^p}{R^p} + \int_{B_{2R}} |u|^p\right] \\ &\quad + c\left(\frac{\rho}{R}\right)^{n-p+p\sigma'}\int_{B_{2R}} \left(g^{p/(p-1)} + 1\right) \\ &\quad + c\rho^{n(1-p/n)}\left(\int_{B_{2R}} \left(g^{n/(p-1)} + 1\right)\right)^{p/n} \\ &\le c_7\left(\frac{\rho}{R}\right)^{n-p+p\sigma'}\left[\int_{B_{2R}} \frac{|u - u_{2R}|^p}{R^p} + \int_{B_{2R}} |u|^p\right] \\ &\quad + c\rho^{n(1-p/n)}\left(\int_{B_{2R}} \left(g^{n/(p-1)} + 1\right)\right)^{p/n}.\end{aligned}$$

From this inequality it follows that, if $0 < \rho < R \le R_0$,

$$\begin{aligned}\fint_{B_\rho} |u - u_\rho|^p &\le c\left(\frac{\rho}{R}\right)^{p\sigma'}\left[\fint_{B_R} |u - u_R|^p + \frac{1}{R^{n-p}}\int_{B_R} |u|^p\right] \\ &\quad + c\left(\int_{B_R} \left(g^{n/(p-1)} + 1\right)\right)^{p/n}. \qquad (3.18)\end{aligned}$$

It follows that for any x

$$\limsup_{\rho \to 0} \fint_{B_\rho(x)} |u - u_{x,\rho}|^p \le c\left(\int_{B_R(x)} \left(g^{n/(p-1)} + 1\right)\right)^{p/n}.$$

Clearly this inequality holds uniformly in compact subsets of Ω. The result follows letting R go to zero.

Remark 3.3. Notice that, if f satisfies (i)–(iii) with $p = n$, $g \in L^{n/(n-1)}(\Omega)$, we can still conclude as in Theorem 3.9 that u is locally VMO in Ω and (3.18) holds. In fact, Theorem 3.5 and Propositions 3.6, 3.7, 3.8, hold true with the same proof also in this case.

Now we are in position to prove Theorem 3.1.

Proof of Theorem 3.1. Using Proposition 3.7 and arguing as in the proof of (3.18), we get, for $0 < \rho < R \leq R_0$,

$$\fint_{B_\rho} |u - u_\rho|^p \leq c\left(\frac{\rho}{R}\right)^{p\sigma'}\left[\fint_{B_R} |u - u_R|^p + \frac{1}{R^{n-p}}\int_{B_R} |u|^p\right] + c\rho^{p(1-n/q)}\left(\int_{B_R} \left(g^{q/(p-1)} + 1\right)\right)^{p/q}. \tag{3.19}$$

In view of Theorem 3.9 it is not restrictive to suppose $u \in BMO(\Omega)$. Hence we may estimate

$$\frac{1}{R^{n-p}}\int_{B_R} |u|^p \leq c_\gamma R^{\gamma p},$$

for any $0 < \gamma < 1$.

Let us fix γ, then if $0 < \tau < 1/2$, $R \leq R_0$, $B_R \subset \Omega$, estimate (3.19) can be written in the form

$$\fint_{B_{\tau R}} |u - u_{\tau R}|^p \leq c_7 \tau^{p\sigma'} \fint_{B_R} |u - u_R|^p + cR^{\gamma p} + c(\tau R)^{p(1-n/q)}\left(\int_{B_R} \left(g^{q/(p-1)} + 1\right)\right)^{p/q}. \tag{3.20}$$

Now let us fix τ_0 such that $c_7 \tau_0^{p\sigma'} \leq 1/2^p$ and let us define

$$a_i = \left(\fint_{B_{\tau_0^i R}(x)} |u - u_{\tau_0^i R}|^p\right)^{1/p}.$$

From Theorem 3.9, a_i goes to 0 as i goes to $+\infty$, uniformly on compact subsets of Ω.

Then (3.20) implies

$$\begin{aligned} a_{i+1} &\leq \frac{1}{2}a_i + \hat{c}\tau_0^{i\gamma}R^\gamma + c(\tau_0)\left(\tau_0^i R\right)^{1-n/q}\left(\int_{B_{\tau_0^i R}(x)} \left(g^{q/(p-1)} + 1\right)\right)^{1/q} \\ &\equiv \frac{1}{2}a_i + \hat{c}\tau_0^{i\gamma}R^\gamma + c(\tau_0)b_i. \end{aligned}$$

From this we have immediately

$$\sum_{i=k+1}^{\infty} a_i \le a_k + 2\hat{c}R^{\gamma} \sum_{i=k}^{\infty} \tau_0^{i\gamma} + 2c(\tau_0) \sum_{i=k}^{\infty} b_i.$$

Using Lemma 2.2, with $a = 1/(p-1)$ and $b = n$, we have that the series generated by b_i converges uniformly. Since also a_k goes to zero uniformly, the series

$$\sum_{i=1}^{\infty} \left(\fint_{B_{\tau_0^i R}(x)} |u - u_{\tau_0^i R}|^p \right)^{1/p}$$

converges uniformly on compact subsets of Ω.

Since

$$|u_{x,\tau_0^{i+1}R} - u_{x,\tau_0^i R}| \le \fint_{B_{\tau_0^{i+1}R}(x)} |u - u_{x,\tau_0^i R}| \le c \left(\fint_{B_{\tau_0^i R}(x)} |u - u_{x,\tau_0^i R}|^p \right)^{1/p},$$

the sequence

$$U_i(x) = \fint_{B_{\tau_0^i R}(x)} u(y)\, dy$$

converges uniformly. Being $U_i(x)$ continuous and converging almost everywhere to the precise representative of $u(x)$, the continuity of $u(x)$ follows. ∎

4. SOLUTIONS OF NONLINEAR EQUATIONS

Some of the technical tools developed in the previous sections turn out to be useful also in order to study continuity of solutions of nonlinear equations. Namely, once one gets in this new framework an estimate like (3.9), it is still possible to use Lemma 2.2 and Proposition 3.8 to prove a result similar to Theorem 3.1. However, in the case of equations, there is no counterpart of the Ekeland variational principle. Hence, to get (3.9), we need to make a monotonicity assumption that we did not have in the case of variational minima. Again, as in the previous section, we will refer to a model case.

Let us consider a solution $u \in W^{1,p}(\Omega)$, $1 < p < n$, of the nonlinear equation

$$\operatorname{div}(A(x,u,Du)) + H(x,u) = \operatorname{div} f, \tag{4.1}$$

where f is in the Lorentz space $L^{n/(p-1),1/(p-1)}(\Omega)$, the vector field

$A(x,\eta,\xi)$ satisfies the following assumptions:

(I) $\quad A(x,\eta,\xi)\colon \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is Carathéodory and

$$|A(x,\eta,\xi)|\le L\big(1+|\xi|^{p-1}\big);$$

(II)

$$\begin{cases} (A(x,\eta,\xi_1)-A(x,\eta,\xi_2),\xi_1-\xi_2)\ge \nu|\xi_1-\xi_2|^p & \text{if } p\ge 2\\ (A(x,\eta,\xi_1)-A(x,\eta,\xi_2),\xi_1-\xi_2) & \\ \quad\ge \nu|\xi_1-\xi_2|^2\big(|\xi_1|^2+|\xi_2|^2\big)^{(p-2)/2} & \text{if } 1<p<2; \end{cases}$$

and $H(x,\eta)$ satisfies

(III) $$|H(x,\eta)|\le L\big(1+|\eta|^{p-1}\big).$$

We are going to prove the following:

THEOREM 4.1. *Suppose $u\in W^{1,p}(\Omega)$, $1<p<n$, is a solution of* (4.1), *under the assumptions* (I)–(III), *and f is in $L^{n/(p-1),1/(p-1)}(\Omega)$. Then u is continuous.*

Proof. Using standard existence results for monotone operators (see, e.g., Theorem 2.8 of Chap. 2 in [L]), one can easily check that under the assumptions on $A(x,\eta,\xi)$ and $H(x,\eta)$, there exists $1>R_0>0$, depending only on L,ν,p,n, such that, if $R<R_0$, the problem

$$\begin{cases} \operatorname{div}(A(x,u,Dv))+H(x,v)=0 & \text{in } B_R\\ v=u & \text{on } \partial B_R, \end{cases} \tag{4.2}$$

admits a solution $v\in W^{1,p}(B_R)$. Furthermore (see Theorem 2.1 in [GG]) v is a Q-minimizer in B_R, with Q depending only on L,ν,p,n, of the functional

$$w\to\int_{B_R}(|Dw|^p+|w|^p+1).$$

Then v is Hölder-continuous and, see Theorem 3.5,

$$\int_{B_\rho}(|Dv|^p+|v|^p)\le c\left(\frac{\rho}{R}\right)^{n-p+p\alpha}\int_{B_R}(|Dv|^p+|v|^p+1), \tag{4.3}$$

for any $0 < \rho < R$. Moreover from Eqs. (4.1) and (4.2) it follows that

$$\int_{B_R} (A(x,u,Du) - A(x,u,Dv), Du - Dv)$$
$$= \int_{B_R} (H(x,u) - H(x,v))(u - v) + \int_{B_R} (f, Du - Dv).$$

If $p \geq 2$, from (II) and (III) we get, for any $\varepsilon > 0$,

$$\int_{B_R} |Du - Dv|^p \leq c_\varepsilon \int_{B_R} |u - v|^p + \varepsilon \int_{B_R} (|u|^p + |v|^p + 1) + c \int_{B_R} |f|^{p/(p-1)}$$
$$\leq c_\varepsilon R^p \int_{B_R} |Du - Dv|^p + c\varepsilon \int_{B_R} |u|^p + c \int_{B_R} \left(|f|^{p/(p-1)} + 1\right).$$

Hence there exists $R_1 \leq R_0$ (R_1 depends also on ε) such that, if $R \leq R_1$,

$$\int_{B_R} |Du - Dv|^p \leq c\varepsilon \int_{B_R} |u|^p + c \int_{B_R} \left(|f|^{p/(p-1)} + 1\right).$$

Combining the above inequality and (4.3) we have, for any $0 < \rho < R \leq R_1$,

$$\int_{B_\rho} (|Du|^p + |u|^p)$$
$$\leq c\left[\left(\frac{\rho}{R}\right)^{n-p+p\alpha} + \varepsilon\right] \int_{B_R} (|Du|^p + |u|^p) + c \int_{B_R} \left(|f|^{p/(p-1)} + 1\right). \tag{4.4}$$

A similar estimate can be obtained also in $1 < p < 2$. In fact, subtracting Eq. (4.2) from (4.1), we have

$$\int_{B_R} |Du - Dv|^2 \left(|Du|^2 + |Dv|^2\right)^{(p-2)/2}$$
$$\leq c_\varepsilon \int_{B_R} |u - v|^p + \varepsilon \int_{B_R} (|u|^p + |v|^p + 1)$$
$$+ \varepsilon \int_{B_R} (|Du|^p + |Dv|^p) + c_\varepsilon \int_{B_R} |f|^{p/(p-1)}$$
$$\leq c_\varepsilon R^p \int_{B_R} |Du - Dv|^p + c\varepsilon \int_{B_R} (|Du|^p + |u|^p + 1) + c_\varepsilon \int_{B_R} |f|^{p/(p-1)}.$$

Taking into account the fact that, if $1 < p < 2$, the following inequality holds (see [AF, Lemma 2.2]),

$$|\xi_1|^p \le c(p,n)\Big[|\xi_2|^p + |\xi_1 - \xi_2|^2\big(|\xi_1|^2 + |\xi_2|^2\big)^{(p-2)/2}\Big], \qquad \forall \xi_1, \xi_2 \in \mathbf{R}^n,$$

we obtain again (4.4), for $0 < \rho < R \le R_1$, with R_1 depending on ε.

Inequality (4.4) is analogous to (3.9) with $q = p$ (actually it is simpler). Then the same argument used to prove Propositions 3.7 and 3.8 shows that there exists $0 < R_2 \le R_1$, $R_2 = R_2(L, \nu, p, n)$, such that, for any $0 < \rho < R \le R_2$,

$$\int_{B_\rho} (|Du|^p + |u|^p) \le c\left(\frac{\rho}{R}\right)^{n-p+p\alpha'} \int_{B_R} (|Du|^p + |u|^p) + c\int_{B_R}\big(|f|^{p/(p-1)} + 1\big), \tag{4.5}$$

and

$$\int_{B_\rho} (|Du|^p + |u|^p) \le c\left(\frac{\rho}{R}\right)^{n-p+p\alpha'} \int_{B_R} (|Du|^p + |u|^p) + c\rho^{n-p}\left(\int_{B_R}\big(|f|^{n/(p-1)} + 1\big)\right)^{p/n}, \tag{4.6}$$

for some $0 < \alpha' < \alpha$. The conclusion then follows as in the proof of Theorem 3.1. ∎

If instead of a solution of (4.1) we consider a solution $u \in W^{1,p}(\Omega)$ of the equation

$$\operatorname{div}(A(x, u, Du)) + H(x, u) = g + \operatorname{div} f, \tag{4.7}$$

with $f \in L^{n/(p-1), 1/(p-1)}\Omega)$, $g \in L^{n/p, 1/(p-1)}(\Omega)$, an argument similar to the one used to prove (4.5) gives

$$\int_{B_\rho} (|Du|^p + |u|^p) \le c\left(\frac{\rho}{R}\right)^{n-p+p\alpha'} \int_{B_R} (|Du|^p + |u|^p) + c\int_{B_R}\big(|f|^{p/(p-1)} + 1\big) + c\left(\int_{B_R}|g|^{p^*/(p^*-1)}\right)^{((p^*-1)/p^*)(p/(p-1))},$$

where, as usual, p^* denotes the Sobolev exponent of p. Similarly one gets, instead of (4.6),

$$\int_{B_\rho}(|Du|^p + |u|^p) \le c\left(\frac{\rho}{R}\right)^{n-p+p\alpha'}\int_{B_R}(|Du|^p + |u|^p) + c\rho^{n-p}\left(\int_{B_R}\left(|f|^{n/(p-1)} + 1\right)\right)^{p/n} + c\rho^{n-p}\left(\int_{B_R}|g|^{n/p}\right)^{p^2/n(p-1)}. \tag{4.8}$$

Hence, using Lemma 2.2 with $a = 1/(p-1)$, $b = n(p-1)/p$, and $q = p^*(p-1)/(p^*-1)$, it is possible to obtain the following:

THEOREM 4.2. *Suppose $u \in W^{1,p}(\Omega)$, $1 < p < n$, is a solution of* (4.7), *under the assumptions* (I)–(III), *and $f \in L^{n/(p-1),1/(p-1)}(\Omega)$, $g \in L^{n/p,1/(p-1)}(\Omega)$. Then u is continuous.*

Finally, we remark that, in order to obtain (4.8), it is enough to assume $f \in L^{n/(p-1)}(\Omega)$ and $g \in L^{n/p}(\Omega)$. Hence, as in Theorem 3.9, we have:

THEOREM 4.3. *Suppose $u \in W^{1,p}(\Omega)$, $1 < p < n$, is a solution of* (4.7), *under the assumptions* (I)–(III), *and $f \in L^{n/(p-1)}(\Omega)$, $g \in L^{n/p}(\Omega)$. Then u is locally VMO in Ω.*

REFERENCES

[AF] E. Acerbi and N. Fusco, Regularity for minimizers of non-quadratic functionals: The case $1 < p < 2$, *J. Math. Anal. Appl.* **140** (1989), 115–135.

[A] A. Alvino, Formule di maggiorazione e regolarizzazione per soluzioni di equazioni ellittiche del secondo ordine in un caso limite, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **52** (1977), 335–340.

[BS] C. Bennet and R. Sharpley, Interpolation of operators, *in* "Pure Appl. Math.," Vol. 129, Academic Press, San Diego, 1988.

[DG] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (3)* **3** (1957), 25–43.

[E] I. Ekeland, Nonconvex minimization problems, *Bull. Amer. Math. Soc.* **1** (1979), 443–474.

[F] V. Ferone, Estimates and regularity for solutions of elliptic equations in a limit case, *Boll. Un. Mat. Ital.* (7) **8**-B (1994), 257–270.

[G] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems, *in* "Ann. Math. Studies," Vol. 105, Princeton Univ. Press, Princeton, 1983.

[GG] M. Giaquinta and E. Giusti, Quasi-minima, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), 79–107.

[GE] E. Giusti, ''Metodi diretti nel calcolo delle variazioni,'' Unione Matematica Italiana, Bologna, 1994.

[L] J. L. Lions, ''Quelques méthodes de résolution des problémes aux limites non linéaires,'' Dunod, Paris, 1969.

[LU] O. Ladyženskaya and N. Ural'ceva, ''Linear and Quasilinear Elliptic Equations,'' Academic Press, New York, 1968.

[ON] R. O'Neil, Integral transforms and tensor products on Orlicz spaces and $L(p,q)$ spaces, *J. Analyse Math.* **21** (1968), 1–276.

[T] G. Talenti, Elliptic equations and rearrangements, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **3** (1976), 697–718.