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On the Integrability of the Jacobian under Minimal Hypotheses

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0. Introduction

For Ω a domain in $\mathbb{R}^n$ and $f: \Omega \to \mathbb{R}^n$, $f = (f^1, \ldots, f^n)$ a mapping of Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $1 \leqq p < \infty$, we denote by $Df(x): \mathbb{R}^n \to \mathbb{R}^n$ the differential and by $J(x,f) = \det Df(x)$ the Jacobian of f.

The Jacobian function occurs in many different contexts, such as the geometric theory of measure and integration, the mapping degree theory, quasiconformal analysis, nonlinear elasticity, etc. Most often the expression $J(x,f)\, dx$ serves as a volume element on Ω, which in conjunction with the formula

$$J(x,f)\, dx = df^1 \wedge \cdots \wedge df^n = d(f^1\, df^2 \wedge \cdots \wedge df^n)$$

leads, via integration by parts, to important estimates.

In order to make use of these properties it is necessary to integrate the Jacobian. The usual hypothesis ensuring this integrability has been that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. A natural question now arises: under what conditions on f is the Jacobian function locally integrable? Without any restrictions, there is no reason to expect that the degree of integrability of $J(x,f)$ is different from that of $|Df(x)|^n$. Surprisingly, just one condition, that $J(x,f)$ does not change sign in Ω, implies higher integrability of the Jacobian. STEFAN MÜLLER [MU 2] was the first to observe this phenomenon.

For notational simplicity, let us declare that $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is *orientation-preserving* if $J(x,f) \geqq 0$ almost everywhere. MÜLLER has shown that the Jacobian of an orientation-preserving mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ actually belongs to the Zygmund class $L \log L(E)$ for each compact set $E \subset \Omega$; see also [CLMS] and [T]. In its most general form, the result can be rephrased as follows

$$\int_E J(x,f) \log\left(e + \frac{J(x,f)}{J_E}\right) dx \leqq C(n,E) \int_\Omega |Df(x)|^n\, dx, \tag{0.1}$$

where J_E denotes the integral mean of the Jacobian over E. As a corollary MÜLLER has improved RESHETNYAK's theorem [R] on the weak convergence of the Jacobians. Again we frame it in a slightly more general form [I 2].

Corollary 0.1. *Let $f_j \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $j = 1, 2, \ldots$, be orientation-preserving mappings converging to f weakly in $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Then*

$$\lim_{j\to\infty} \int_\Omega \varphi(x) J(x, f_j)\, dx = \int_\Omega \varphi(x) J(x, f)\, dx \tag{0.2}$$

for every $\varphi \in \exp(\Omega)$, the closure of step functions in the norm $\| \|_{\mathrm{Exp}(\Omega)}$ (see Section 1 for the notation we use here and in the sequel).

If $\varphi \in C^1_0(\Omega)$, the condition $J(x, f_j) \geqq 0$ is not required. In the proof of this result there is hidden a weak form of the Jacobian function that we shall eventually employ [B, MU 1, BM, DMU].

Definition 0.1. For a mapping $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, with $p \geqq n^2/(n+1)$, the weak *Jacobian* is a Schwartz distribution $J_f \in \mathscr{D}'(\Omega)$ defined by the rule

$$J_f[\varphi] = -\int_\Omega f^n\, df^1 \wedge \cdots \wedge df^{n-1} \wedge d\varphi \tag{0.3}$$

for all test functions $\varphi \in C^\infty_0(\Omega)$.

The argument showing $f^n\, df^1 \wedge \cdots \wedge df^{n-1} \wedge d\varphi$ to be integrable is routine. By the Sobolev Imbedding Theorem $f^n \in L^{n^2}_{\mathrm{loc}}(\Omega)$ and $df^1 \wedge \cdots \wedge df^{n-1} \in L^{n^2/(n^2-1)}_{\mathrm{loc}}(\Omega)$, as desired.

Here is one of our main estimates:

Theorem 1. *Let $f: \mathbb{R}^n \to \mathbb{R}^n$, $f = (f^1, f^2 \ldots, f^n)$, be a mapping of Sobolev class $W^{1,n-\varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$, with $-\infty < \varepsilon \leqq 1$.* Then

$$\begin{aligned}\int_{\mathbb{R}^n} |df^1|^{-\varepsilon} J(x, f)\, dx &\leqq C(n)\,|\varepsilon|\, \|df^1\|^{1-\varepsilon}_{n-\varepsilon} \|df^2\|_{n-\varepsilon} \cdots \|df^n\|_{n-\varepsilon} \\ &\leqq C(n)\,|\varepsilon| \int_{\mathbb{R}^n} |Df(x)|^{n-\varepsilon}\, dx.\end{aligned} \tag{0.4}$$

The presence of the factor $|\varepsilon|$ in (0.4) is the essence of this inequality, since, by Hadamard's Inequality, we always have a pointwise estimate $|df^1|^{-\varepsilon} J(x, f) \leqq |df^1|^{1-\varepsilon} |df^2| \cdots |df^n|$.

The arguments establishing Theorem 1 are based on new estimates in the Hodge Decomposition, see [I 1].

Throughout what lies ahead, we are concerned mainly with the determination of the minimal conditions on f to ensure local integrability of the Jacobian.

The primary result is the following consequence of Theorem 1.

Theorem 2. *Let $B \subset 3B$ be given concentric balls in $\mathbb{R}^n$ and let $f: 3B \to \mathbb{R}^n$ be an orientation-preserving mapping of class $\bigcap_{1 \leqq s < n} W^{1,s}(3B, \mathbb{R}^n)$, such that*

$$\sup_{1 \leqq s < n} (n - s) \int_{3B} |Df(x)|^s\, dx < \infty. \tag{0.5}$$

Then $J(\cdot, f) \in L^1_{\mathrm{loc}}(3B)$ *and the following uniform estimate holds*:

$$\text{(0.6)}\qquad \begin{aligned} &\fint_B J(x,f)\,dx \\ &\leqq C(n)\left(\fint_{3B} |Df(x)|^{\frac{n^2}{n+1}}\,dx\right)^{\frac{n+1}{n}} + C(n)\limsup_{s\nearrow n}\,(n-s)\fint_{3B}|Df(x)|^s\,dx \\ &\leqq C(n)\sup_{1\leqq s<n}\left[(n-s)\fint_{3B}|Df(x)|^s\,dx\right]^{\frac{n}{s}}. \end{aligned}$$

Two classes of functions for which condition (0.5) is satisfied will be of particular interest. The first class, denoted by weak-$W^{1,n}(3B, \mathbb{R}^n)$, consists of mappings with $|Df| \in$ weak-$L^n(3B)$. However, the emphasis will be on the second class, called the Orlicz-Sobolev space $D^n\log^{-1}D(\Omega, \mathbb{R}^n)$; see the next section for definitions. The point is that smooth mappings are dense in $D^n\log^{-1}D(\Omega, \mathbb{R}^n)$.

In the latter case Theorem 2 reduces to the following result which should be regarded as dual to that of Müller.

Theorem 3. *Let* $f : \Omega \to \mathbb{R}^n$ *be an orientation-preserving mapping of the Orlicz-Sobolev class* $D^n\log^{-1}D(\Omega, \mathbb{R}^n)$. *Then the Jacobian of f is locally integrable. Moreover, for each compact subset* $E \subset \Omega$, *the following estimate holds*

$$\text{(0.7)}\qquad \int_E J(x,f)\,dx \leqq C(n,E)\int_\Omega \frac{|Df(x)|^n\,dx}{\log\left(e + \dfrac{|Df(x)|}{|Df|_\Omega}\right)},$$

where $|Df|_\Omega$ *denotes the integral mean of* $|Df|$ *over* Ω.

Another result of this paper identifies the weak Jacobian J_f as the pointwise Jacobian $J(\cdot, f)$.

Theorem 4. *Let* $f : \Omega \to \mathbb{R}^n$ *be as in Theorem 3. Then*

$$\text{(0.8)}\qquad \int_\Omega \varphi(x)J(x,f)\,dx = -\int_\Omega f^n\,df^1 \wedge \cdots \wedge df^{n-1} \wedge d\varphi$$

for all test functions $\varphi \in C_0^\infty(\Omega)$.

As a result, we can now assert:

Corollary 1. *Let f and* f_j, $j = 1, 2, \ldots$, *be orientation-preserving mappings of the Orlicz-Sobolev class* $D^n\log^{-1}D(\Omega, \mathbb{R}^n)$. *Suppose that* $\lim f_j = f$ *weakly in* $W^{1,p}(\Omega, \mathbb{R}^n)$ *with some* $p > n^2/(n+1)$. *Then*

$$\text{(0.9)}\qquad \lim_{j\to\infty}\int_\Omega \varphi(x)J(x,f_j)\,dx = \int_\Omega \varphi(x)J(x,f)\,dx$$

for every $\varphi \in C_0^1(\Omega)$.

The question to face next is: What are really the minimal assumptions on $f:\Omega\to\mathbb{R}^n$ which guarantee that $J(x,f)\in L^1_{\mathrm{loc}}(\Omega)$? Theorem 2 is certainly the best known to date. On the one hand, in the light of the two special cases, weak-$W^{1,n}(\Omega,\mathbb{R}^n)$ and $D^n\log^{-1}D(\Omega,\mathbb{R}^n)$, it is not obvious how to formulate minimal conditions, at least in terms of the familiar function spaces. On the other hand, our example, see Section 6, illustrates undoubtedly that Theorem 3 does not leave much room for improvement. There is an example, due to J. Ball & F. Murat [BM, counterexample 7.4] which shows that Theorem 4 and Corollary 1 fail for mappings $f\in$ weak-$W^{1,n}(\Omega,\mathbb{R}^n)$. It remains to argue whether the above results stay valid if the condition $J(x,f)\geqq 0$ a.e. is replaced by $J_f[\varphi]\geqq 0$ for all non-negative test functions $\varphi\in C_0^\infty(\Omega)$.

In some respects estimates (0.1) and (0.7) determine the limitations on the integrability theory of the Jacobian function. It would be possible at this moment to interpolate between these two limits. Indeed it has been already communicated to us by H. Brezis, N. Fusco and C. Sbordone that

$$\int_E J(x,f)\log^{1-\alpha}\left(e+\frac{J(x,f)}{J_E}\right)dx\leqq C(n,E)\int_\Omega\frac{|Df(x)|^n\,dx}{\log^\alpha\left(e+\frac{|Df(x)|}{|Df|_\Omega}\right)}$$

for $0\leqq\alpha\leqq 1$, where $f:\Omega\to\mathbb{R}^n$ is an orientation-preserving mapping of Orlicz-Sobolev class $D^n\log^{-\alpha}D(\Omega,\mathbb{R}^n)$.

As a closing remark, although we do not pursue the matter here, our results relate directly to the existence and regularity problems in non-linear elasticity [AF, B, BM, BU, CD, D, DM, DG, G, GMS, M, R, T] and quasiregular mappings [BI, GE, I3].

1. Notation and properties of some function spaces

In the Introduction we have been relatively inexact with the notation and definitions. This section is devoted to giving more precise information about some function spaces.

Let E be a measurable subset of $\mathbb{R}^n$. The usual Lebesgue space $L^p(E)$ is equipped with the norm

$$\|g\|_{p,E}=\left(\int_E|g(x)|^p\,dx\right)^{\frac{1}{p}},\quad 1\leqq p<\infty.$$

The average of a function $g\in L^1(E)$ is denoted by

$$g_E=\fint_E g(x)\,dx=\frac{1}{|E|}\int_E g(x)\,dx,$$

provided $|E|$ (the Lebesgue measure of E) is positive and finite.

A measurable function g on $E\subset\mathbb{R}^n$ is said to be of Zygmund class $L\log L(E)$ if

$$\int_E|g(x)|\log\left(e+\frac{|g(x)|}{|g|_E}\right)dx<\infty. \tag{1.1}$$

It is not readily apparent that the above integral defines an order-preserving norm in $L \log L(E)$. Actually, the triangle inequality is fairly non-trivial; see [IK]. In connection with Corollary 0.1 we should mention that the dual space, denoted by $\operatorname{Exp}(E)$, consists of functions g such that

$$\|g\|_{\operatorname{Exp}(E)} = \inf_{k>0} \frac{1}{k} \int_E e^{k|g(x)|}\, dx < \infty. \tag{1.2}$$

We shall now distinguish three classes of functions which interpolate between $L^p(E)$ and $\bigcap_{1 \leqq s < p} L^s(E)$.

The Marcinkiewicz class, denoted by weak-$L^p(E)$, consists of functions g on $E \subset \mathbb{R}^n$ such that

$$M_p(g) = \left[\frac{1}{|E|} \sup_{t>0} t^p g_*(t)\right]^{\frac{1}{p}} < \infty, \tag{1.3}$$

where $g_*(t) = |\{x \in E; |g(x)| > t\}|$ denotes the distribution function of g. Recall that

$$\int_E |g(x)|^s\, dx = s \int_0^\infty t^{s-1} g_*(t)\, dt < \infty \tag{1.4}$$

for all $1 \leqq s < p$.

A measurable function g on the set $E \subset \mathbb{R}^n$, $n \geqq 2$, belongs to the Orlicz space $L^n \log^{-1} L(E)$ if

$$\int_E \frac{|g(x)|^n\, dx}{\log(e + |g(x)|)} < \infty. \tag{1.5}$$

We should observe that the corresponding Young function $\Phi: \mathbb{R}^+ \to \mathbb{R}^+$, $\Phi(t) = t^n \log^{-1}(e+t)$, is strictly increasing, convex and satisfies $\lim_{t \to 0} t^{-1} \Phi(t) = 0$, $\lim_{t\to\infty} t^{-1}\Phi(t) = \infty$. More importantly, $\Phi(2t) \leqq 2^n \Phi(t)$. The last inequality, known as the Δ_2-condition, implies that the step functions are dense in the norm topology of $L^n \log^{-1} L(E)$. We shall make use of the Luxemburg norm

$$\|g\|_\Phi = \inf\left\{k > 0;\ \fint_E \Phi\left(\frac{|g(x)|}{k}\right) dx \leqq 1\right\}. \tag{1.6}$$

There is an advantage in using the following integral expression instead of $\|g\|_\Phi$

$$[g]_E = \left[\fint_E \frac{|g(x)|^n\, dx}{\log\left(e + \frac{|g(x)|}{|g|_E}\right)}\right]^{\frac{1}{n}}. \tag{1.7}$$

This is not a norm, but compares well with the Luxemburg norm, see Lemma 1.2. For more properties of Orlicz spaces we refer to the recent book by Rao & Ren [RR].

The third class of functions on $E \subset \mathbb{R}^n$, denoted by $L^{n)}(E)$, consists of all functions $g \in \bigcap_{1 \leqq s < n} L^s(E)$ such that

$$\|g\|_{n),E} = \sup_{1 \leqq s < n} \left[(n-s) \fint_E |g(x)|^s \, ds \right]^{\frac{1}{s}} < \infty. \tag{1.8}$$

This is a norm in $L^{n)}(E)$ which makes $L^{n)}(E)$ a Banach space. We introduce the following quantity

$$(g)_{n,E} = \limsup_{s \nearrow n} \left[(n-s) \fint_E |g(x)|^s \, dx \right]^{\frac{1}{s}} < \infty \tag{1.9}$$

for $g \in L^{n)}(E)$.

The next two lemmas establish a relation between the above three classes of functions.

Lemma 1.1. *The Marcinkiewicz class* weak-$L^n(E)$ *is contained in* $L^{n)}(E)$, *and the following uniform bounds hold:*

$$(g)_{n,E} \leqq 2M_n(g), \tag{1.10}$$

$$\|g\|_{n),E} \leqq nM_n(g) \tag{1.11}$$

for all $g \in$ weak-$L^n(E)$.

Proof. If $1 \leqq s < n$, for each $a > 0$, one can split the integral in the right-hand side of (1.4) to obtain

$$\begin{aligned} \fint_E |g(x)|^s \, dx &= s \int_0^a t^{s-1} g_*(t) \, dt + s \int_a^\infty t^{s-1} g_*(t) \, dt \\ &\leqq s|E| \int_0^a t^{s-1} \, dt + \frac{sa^{s-n}}{n-s} |E| \, M_n^n(g). \end{aligned}$$

The second integral has been estimated by the inequality $g_*(t) \leqq |E| \, t^{-n} M_n^n(g)$, which is a direct consequence of the definition of the constant $M_n(g)$ (see (1.3)). Setting $a = M_n(g)$ we arrive at the homogeneous inequality

$$(n-s) \fint_E |g(x)|^s \, dx \leqq n[M_n(g)]^s.$$

Now estimates (1.10) and (1.11) follow because $n^{1/n} < 2$ and $n^{1/s} \leqq n$ for $1 \leqq s < n$.

The second lemma deals with the class $L^n \log^{-1} L(E)$.

Lemma 1.2. *The Orlicz space* $L^n \log^{-1} L(E)$ *is contained in* $L^{n)}(E)$, *and the following uniform bounds hold:*

$$(g)_{n,E} \leqq 2[g]_E, \tag{1.12}$$

$$\|g\|_{n),E} \leqq 4n[g]_E, \tag{1.13}$$

$$[g]_E \leqq e\|g\|_\Phi \tag{1.14}$$

for all $g \in L^n \log^{-1} L(E)$.

Proof. We need two elementary inequalities:

$$\log^{\lambda}(e+t) \leqq \lambda^{\lambda}(e+t), \tag{1.15}$$

$$\min\{1, \lambda\} \leqq \frac{\log(e+\lambda t)}{\log(e+t)} \leqq \max\{1, \lambda\} \tag{1.16}$$

for t and λ non-negative. These are left as an exercise.

It follows from the definition of the Luxemburg norm $\|g\| = \|g\|_{\Phi}$ that

$$\|g\| = \left[\fint_E \frac{|g(x)|^n \, dx}{\log\left(e + \frac{|g(x)|}{\|g\|}\right)} \right]^{\frac{1}{n}}. \tag{1.17}$$

Let $1 \leqq s < n$. By the Hölder Inequality and by (1.15), for each $k > 0$, we can write

$$\begin{aligned}
\fint_E |g|^s &\leqq \left[\fint_E \frac{|g|^n}{\log(e+k|g|)} \right]^{\frac{s}{n}} \left[\fint_E \log^{\frac{s}{n-s}}(e+k|g|) \right]^{\frac{n-s}{n}} \\
&\leqq \left[\frac{s}{n-s} \fint_E \frac{|g|^n}{\log(e+k|g|)} \right]^{\frac{s}{n}} \left[\fint_E (e+k|g|) \right]^{\frac{n-s}{n}}.
\end{aligned}$$

Hence, we immediately obtain

$$\left[(n-s) \fint_E |g|^s \right]^{\frac{1}{s}} \leqq (n-s)^{\frac{n-s}{ns}} s^{\frac{1}{n}} (e+k|g|_E)^{\frac{n-s}{ns}} \left[\fint_E \frac{|g|^n}{\log(e+k|g|)} \right]^{\frac{1}{n}}. \tag{1.18}$$

Letting $k = 1/|g|_E$ yields (1.12) and (1.13). Of course, one must do some arithmetic to obtain the constants 2 and $4n$, respectively. In order to prove (1.14), we put $k = \|g\|^{-1}$ and $s = 1$ in (1.18). This yields

$$|g|_E \leqq \left(e + \frac{|g|_E}{\|g\|}\right)^{\frac{n-1}{n}} \|g\|.$$

Hence, by a routine calculation, we infer that

$$|g|_E \leqq en\|g\| \leqq e^n \|g\|.$$

Finally, by inequality (1.16) and by identity (1.17), we conclude that

$$\begin{aligned}
[g]_E^n &= \fint_E |g|^n \log^{-1}\left(e + \frac{|g|}{|g|_E}\right) \\
&\leqq \max\left\{1, \frac{|g|_E}{\|g\|}\right\} \fint_E |g|^n \log^{-1}\left(e + \frac{|g|}{\|g\|}\right) \leqq e^n \|g\|^n,
\end{aligned}$$

as desired.

Let Ω be a domain in $\mathbb{R}^n$. We shall consider the Orlicz-Sobolev space $D^n \log^{-1} D(\Omega, \mathbb{R}^n)$ consisting of all functions $f : \Omega \to \mathbb{R}^n$ whose distributional differential $Df : \Omega \to GL(n)$ has norm $|Df|$ belonging to the Orlicz space $L^n \log^{-1} L(\Omega)$. The class of such functions may be given a norm

$$\| f \|_{D^n \log^{-1} D(\Omega, \mathbb{R}^n)} = \left(\int_\Omega | f |^{\frac{n^2}{n+1}} \right)^{\frac{n+1}{n^2}} + \| |Df| \|_\Phi . \tag{1.19}$$

Basic properties of the Orlicz-Sobolev spaces have been established by DONALDSON & TRUDINGER [DT]. We infer from [DT] the following density lemma analogous to that of MEYERS & SERRIN for Sobolev spaces.

Lemma 1.3. *The class* $C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ *is dense in* $D^n \log^{-1} D(\mathbb{R}^n, \mathbb{R}^n)$.

This result can be obtained by a standard convolution technique, which requires the following property of the translation operator

$$\lim_{y \to 0} \| g(x+y) - g(x) \|_\Phi = 0 .$$

The latter can easily be verified, since the step functions are dense in $L^n \log^{-1} L(\mathbb{R}^n)$.

2. Hodge decomposition

In this section we formulate a technical lemma which will play an important role later. Let $\Omega \subset \mathbb{R}^n$ be a regular domain, for instance a Lipschitz domain. In our applications Ω will be the whole of $\mathbb{R}^n$. We begin by looking at the familiar Poisson equation in Ω:

$$\operatorname{div} \nabla u = \operatorname{div} \omega , \tag{2.1}$$

where u is an unknown function of Sobolev class $W_0^{1,r}(\Omega)$, $1 < r < \infty$, and $\omega = (\omega^1, \dots, \omega^n)$ is a given vector field in $L^r(\Omega, \mathbb{R}^n)$. The existence and uniqueness of the solution folllows by variational principles. Therefore we have the Hodge decomposition of ω

$$\omega = \eta + \nabla u , \tag{2.2}$$

where η is a divergence-free vector field of class $L^r(\Omega, \mathbb{R}^n)$, that is, $\operatorname{div} \eta = 0$. The gradient of u can be given an explicit integral representation in terms of ω. Employing the Calderón-Zygmund theory of singular integrals yields the following estimate

$$\| \nabla u \|_{r,\Omega} + \| \eta \|_{r,\Omega} \leqq C(n, r) \| \omega \|_{r,\Omega} . \tag{2.3}$$

Notice that, by the uniqueness of the decomposition, $u = 0$ whenever $\operatorname{div} \omega = 0$. Similarly, $\eta = 0$ if ω is the gradient of a function from $W_0^{1,r}(\Omega)$. We would therefore expect the following stability property of the Hodge decomposition under non-linear perturbations of ω.

Lemma 2.1. *Let ω be a vector field of class $L^{r(1-\varepsilon)}(\Omega, \mathbb{R}^n)$ with $r > 1$ and $-\infty < \varepsilon < 1 - \frac{1}{r}$. Consider the Hodge decomposition*

$$|\omega|^{-\varepsilon}\omega = \eta + \nabla u \tag{2.4}$$

with $u \in W_0^{1,r}(\Omega)$ and $\operatorname{div}\eta = 0$. The following estimates hold:
(i) *If $\omega = \nabla v$ is a gradient field with $v \in W_0^{1,r(1-\varepsilon)}(\Omega)$, then*

$$\|\eta\|_{r,\Omega} \leqq C(n, r)|\varepsilon| \, \|\omega\|_{r(1-\varepsilon)}^{1-\varepsilon}.$$

(ii) *If ω is divergence-free, then*

$$\|\nabla u\|_{r,\Omega} \leqq C(n, r)|\varepsilon| \, \|\omega\|_{r(1-\varepsilon)}^{1-\varepsilon}.$$

We shall apply this result only for $\Omega = \mathbb{R}^n$ and for some $r \geqq \frac{7}{4}$. In this case Lemma 2.1 is just a rephrasing of what was shown in [I1]; see Theorem 8.2. The stronger statement will be proved in the forthcoming paper [IS].

3. Proof of Theorem 1

In this section we are dealing with mapping $f = (f^1, f^2, \ldots, f^n)$ of Sobolev class $W^{1,n-\varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$, $-\infty < \varepsilon \leqq 1$, without any assumption on the sign of the Jacobian $J(x, f)\,dx = df^1 \wedge \cdots \wedge df^n$. As a preliminary step we recall that

$$\big||df^1|^{-\varepsilon} J(x, f)\big| \leqq |df^1|^{1-\varepsilon}|df^2| \cdots |df^n|.$$

Thus (0.4) always holds with constant equal to 1 in place of $C(n)|\varepsilon|$. It may therefore be assumed that $|\varepsilon| < \frac{3}{7}$. With this assumption we may apply Lemma 2.1 with $r = \dfrac{n-\varepsilon}{1-\varepsilon} \geqq \dfrac{7}{4}$ and $\Omega = \mathbb{R}^n$. Accordingly, we consider the Hodge decomposition

$$|\nabla f^1|^{-\varepsilon}\nabla f^1 = \eta + \nabla u \tag{3.1}$$

with $u \in W^{1,r}(\mathbb{R}^n)$ and $\operatorname{div}\eta = 0$. Therefore

$$\|\eta\|_{\frac{n-\varepsilon}{1-\varepsilon}} \leqq C(n)|\varepsilon| \, \|\nabla f^1\|_{n-\varepsilon}^{1-\varepsilon}. \tag{3.2}$$

For notational simplicity it is advantageous to use differential forms instead of vector fields. Thus a vector field η in $\mathbb{R}^n$ will be identified with a differential form of degree 1 and the gradient ∇u with du. Clearly

$$df^2 \wedge df^3 \wedge \cdots \wedge df^n \in L^{\frac{n-\varepsilon}{n-1}}(\mathbb{R}^n).$$

Since $\frac{n-\varepsilon}{1-\varepsilon}$ and $\frac{n-\varepsilon}{n-1}$ are Hölder conjugate exponents, by Stokes' Theorem via an approximation argument we find that

(3.3) $$\int_{\mathbb{R}^n} du \wedge df^2 \wedge \cdots \wedge df^n = \int_{\mathbb{R}^n} d(udf^2 \wedge \cdots \wedge df^n) = 0.$$

Now, equation (3.1), Hölder's Inequality and Hadamard's Inequality yield

$$\begin{aligned}
&\int_{\mathbb{R}^n} |df^1|^{-\varepsilon} df^1 \wedge df^2 \wedge \cdots \wedge df^n \\
&\qquad = \int_{\mathbb{R}^n} \eta \wedge df^2 \wedge \cdots \wedge df^n \\
&\qquad \leqq \|\eta\|_{\frac{n-\varepsilon}{1-\varepsilon}} \|df^2 \wedge \cdots \wedge df^n\|_{\frac{n-\varepsilon}{n-1}} \\
&\qquad \leqq C(n)|\varepsilon| \, \|df^1\|^{1-\varepsilon}_{n-\varepsilon} \|df^2\|_{n-\varepsilon} \cdots \|df^n\|_{n-\varepsilon}.
\end{aligned}$$

In the last step we have used (3.2). This completes the proof of Theorem 1.

A glance over this proof reveals that there exist more general inequalities for integrals involving minors of the differential $Df(x)$ of arbitrary order. For this, however, one has to appeal to the Hodge decomposition for differential forms of higher degree; see Theorem 8.2 in [I 1]. This generalization and further results will be presented in [IL].

4. Proofs of Theorem 2 and Theorem 3

From now on we shall consider only orientation-preserving mappings. Let $B = B(a, r) \subset B(a, 2r) = 2B \subset B(a, 3r) = 3B$ be given concentric balls in $\mathbb{R}^n$. Let $\varphi \in C_0^\infty(2B)$ and $\psi \in C_0^\infty(3B)$ be cut-off functions such that

(i) $0 \leqq \varphi \leqq 1$, $\varphi = 1$ on B, $|\nabla\varphi| \leqq \frac{C(n)}{r}$,

(ii) $0 \leqq \psi \leqq 1$, $\psi = 1$ on $2B$, $|\nabla\psi| \leqq \frac{C(n)}{r}$.

We shall examine an auxilliary mapping $F \in W^{1,n-\varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$ with compact support, defined by

(4.1) $$F = (\psi f^1, \ldots, \psi f^{n-1}, \varphi f^n).$$

It is straightforward to see that

(4.2) $$\begin{aligned}
&\varphi|df^1|^{-\varepsilon} df^1 \wedge \cdots \wedge df^{n-1} \wedge df^n \\
&\qquad = |dF^1|^{-\varepsilon} dF^1 \wedge \cdots \wedge dF^n - f^n |df^1|^{-\varepsilon} df^1 \wedge \cdots \wedge df^{n-1} \wedge d\varphi.
\end{aligned}$$

Applying Theorem 1 to the mapping F we find that

(4.3) $$\begin{aligned}
&\int_B \varphi|\nabla f^1|^{-\varepsilon} J(x, f)\, dx \\
&\qquad \leqq \int_{3B} |\nabla\varphi|\,|f|\,|Df|^{n-1-\varepsilon} + C(n)|\varepsilon| \int_{3B} |DF|^{n-\varepsilon}.
\end{aligned}$$

In view of conditions (i) and (ii) we have an easy pointwise estimate $|DF| \leqq C(n)\, r^{-1}\, |f| + C(n)\,|Df|$. Therefore,

$$\int_{3B} |DF|^{n-\varepsilon} \leqq C(n) \cdot r^{\varepsilon-n} \int_{3B} |f|^{n-\varepsilon} + C(n) \int_{3B} |Df|^{n-\varepsilon}. \tag{4.4}$$

Notice that Df is not affected when a constant vector is added to f; thus we may assume that the mean of f over the ball $3B$ is equal to zero. This justifies the application of the Poincaré Inequality to the first integral in the right-hand side of (4.4). Hence

$$\int_{3B} |DF|^{n-\varepsilon} \leqq C(n) \int_{3B} |Df|^{n-\varepsilon}. \tag{4.5}$$

By conditions (i) and Hölder's Inequality we find that

$$\int_{3B} |\nabla\varphi|\,|f|\,|Df|^{n-1-\varepsilon} \leqq \frac{C(n)}{r}\left[\int_{3B} |f|^{\frac{n(n-\varepsilon)}{1+\varepsilon}}\right]^{\frac{1+\varepsilon}{n(n-\varepsilon)}} \left[\int_{3B} |Df|^{\frac{n(n-\varepsilon)}{n+1}}\right]^{\frac{(n+1)(n-1-\varepsilon)}{n(n-\varepsilon)}}.$$

Then, by the Poincaré-Sobolev Inequality with $p = \dfrac{n(n-\varepsilon)}{n+1} < n$, we can write

$$\left(\int_{3B} |f|^{\frac{n(n-\varepsilon)}{1+\varepsilon}}\right)^{\frac{1+\varepsilon}{n(n-\varepsilon)}} = \left(\int_{3B} |f|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \leqq C_p(n)\left(\int_{3B} |Df|^p\right)^{\frac{1}{p}}.$$

Therefore,

$$\int_{3B} |\nabla\varphi|\,|f|\,|Df|^{n-1-\varepsilon} \leqq \frac{C(n)}{r}\left(\int_{3B} |Df|^{\frac{n(n-\varepsilon)}{n+1}}\right)^{\frac{n+1}{n}}.$$

This, together with (4.5) reduces inequality (4.3) to

$$\fint_{B} |\nabla f^1|^{-\varepsilon} J(x,f)\, dx \leqq C(n)\left(\fint_{3B} |Df|^{\frac{n(n-\varepsilon)}{n+1}}\right)^{\frac{n+1}{n}} + C(n)\,|\varepsilon| \fint_{3B} |Df|^{n-\varepsilon}. \tag{4.6}$$

All that remains is to examine the limit as ε decreases to zero. Theorem 2 follows routinely from Fatou's Lemma.

Finally, inequality (0.6) together with (1.7), (1.8) and (1.13) implies (0.7), proving Theorem 3.

5. The weak Jacobian

In this section we shall prove formula (0.8) in Theorem 4. We may assume that the test function $\varphi \in C_0^\infty(\Omega)$ is non-negative. As in the proof of Theorem 2, we shall consider an auxilliary test function $\psi \in C_0^\infty(\Omega)$, equal to unity on the support of φ. Thus, the mapping $F=(\psi f^1,\dots,\psi f^{n-1},\varphi f^n)$ has compact support and belongs to the Orlicz-Sobolev space $D^n\log^{-1}D(\Omega,\mathbb{R}^n)$. Clearly, for $0<\varepsilon<1$ and for an arbitrary $H\in C_0^\infty(\Omega)$ we can write

$$
\begin{aligned}
(5.1)\qquad \int_\Omega |df^1|^{-\varepsilon}df^1\wedge\cdots\wedge df^{n-1}\wedge d(\varphi f^n)
&= \int_\Omega |dF^1|^{-\varepsilon}dF^1\wedge\cdots\wedge dF^{n-1}\wedge d(F^n-H)\\
&\quad+\int_\Omega |dF^1|^{-\varepsilon}dF^1\wedge\cdots\wedge dF^{n-1}\wedge dH.
\end{aligned}
$$

In order to estimate the first integral in the right-hand side of (5.1) we apply Theorem 1 to the mapping $(F^1,F^2,\dots,F^{n-1},F^n-H)$. After this, we appeal to (1.8) and (1.14) in Lemma 1.2, to obtain

$$
\begin{aligned}
&\left|\int_\Omega |dF^1|^{-\varepsilon}dF^1\wedge\cdots\wedge dF^{n-1}\wedge d(F^n-H)\right|\\
&\quad\leqq C(n)|\varepsilon|\,\|dF^1\|_{n-\varepsilon}^{1-\varepsilon}\,\|dF^2\|_{n-\varepsilon}\cdots\|dF^{n-1}\|_{n-\varepsilon}\|d(F^n-H)\|_{n-\varepsilon}\\
&\quad\leqq C(n)\left[\varepsilon\int_\Omega |DF|^{n-\varepsilon}\right]^{\frac{n-1-\varepsilon}{n-\varepsilon}}\left[\varepsilon\int_\Omega |\nabla F^n-\nabla H|^{n-\varepsilon}\right]^{\frac{1}{n-\varepsilon}}\\
&\quad\leqq C(n)|\Omega|\,\||DF|\|_{n),\Omega}^{n-1-\varepsilon}\,\||\nabla F^n-\nabla H|\|_{n),\Omega}\\
&\quad\leqq C(n)|\Omega|\,\||DF|\|_{\Phi}^{n-1-\varepsilon}\,\||DF^n-\nabla H|\|_{\Phi}.
\end{aligned}
$$

Recall that the symbol $\|\ \|_\Phi$ stands for the Luxemburg norm in the Orlicz space $L^n\log^{-1}L(\Omega)$.

We now return to identity (5.1) where we let ε go to zero. Since $H\in C_0^\infty(\Omega)$, there is no difficulty with the convergence of the second term in the right-hand side of (5.1). Its limit is equal to zero, because $\int_\Omega dF^1\wedge\cdots\wedge dF^{n-1}\wedge dH=(-1)^{n-1}\int_\Omega d(H\,dF^1\wedge\cdots\wedge dF^{n-1})=0$. Therefore

$$
\lim_{\varepsilon\to0}\left|\int_\Omega |df^1|^{-\varepsilon}\,df^1\wedge\cdots\wedge df^{n-1}\wedge d(\varphi f^n)\right|
\leqq C(n)|\Omega|\,\||DF|\|_\Phi^{n-1}\,\||\nabla F^n-\nabla H|\|_\Phi,
$$

for every $H\in C_0^\infty(\Omega)$. According to Lemma 1.3, with an appropriate choice of H, the norm $\||\nabla F^n-\nabla H|\|_\Phi$ can be made as small as one likes. Thus the above limit equals zero as well. This, after splitting the integral, is equivalent to

$$
\lim_{\varepsilon\to0}\left[\int_\Omega \varphi|df^1|^{-\varepsilon}\,df^1\wedge\cdots\wedge df^n+\int_\Omega f^n|df^1|^{-\varepsilon}\,df^1\wedge\cdots\wedge df^{n-1}\wedge d\varphi\right]=0.
$$

Clearly, the second term converges to $\int_\Omega f^n\,df^1\wedge\cdots\wedge df^{n-1}\wedge d\varphi$. It remains to show that the first term above converges to $\int_\Omega\varphi(x)J(x,f)\,dx$. Recall that we have already established the local integrability of the Jacobian. Therefore, the following pointwise estimate

$$|df^1|^{-\varepsilon} J(x,f) \leqq (1 + |df^1|^{-1}) J(x,f)$$
$$\leqq |Df|^{n-1} + J(x,f) \in L^1_{\mathrm{loc}}(\Omega)$$

justifies the use of Lebesgue's Dominated Convergence Theorem. In conclusion,

$$\int_\Omega \varphi(x) J(x,f)\, dx = -\int_\Omega f^n\, df^1 \wedge \cdots \wedge df^{n-1} \wedge d\varphi$$

for every $\varphi \in C_0^\infty(\Omega)$, as desired.

Corollary 1 is now immediate; compare with [D].

6. An example

We consider the radial mapping

$$f(x) = \frac{x}{|x|} \log^{\frac{1}{n}} \frac{1}{|x|} \tag{6.1}$$

in the ball $B = B\left(0, e^{-\frac{1}{n}}\right)$.

For notational simplicity we introduce the function $\varphi(r) = \frac{1}{r} \log^{\frac{1}{n}} \frac{1}{r}$, with $0 < r < e^{-\frac{1}{n}}$. Then a rather elementary computation shows that

$$Df(x) = \varphi(|x|)\, Id + \frac{x \otimes x}{|x|}\, \varphi'(|x|), \tag{6.2}$$

where the tensor product $x \otimes x$ is an $n \times n$-matrix whose ij-entry equals $x_i x_j$, $i, j = 1, 2, \ldots, n$.

We find that $\varphi + r\varphi' = -\dfrac{\varphi}{n \log \frac{1}{r}}$ and $|\varphi + r\varphi'| \leqq \varphi$ for $r < e^{-\frac{1}{n}}$.

Moreover, it follows from (6.2) that

$$\det Df(x) = \varphi^{n-1}(\varphi + |x|\, \varphi') = \frac{-1}{n|x|^n}.$$

Thus the integral $\int_{|x|<e^{-1/n}} J(x,f)\, dx$ is obviously divergent.

On the other hand $|Df(x)|^2 = \mathrm{Trace}\,(Df)^t (Df) = n\varphi^2 + 2|x|\, \varphi\varphi' + |x|^2 (\varphi')^2 = (n-1)\, \varphi^2 + (\varphi + |x|\, \varphi')^2$. Hence

$$(n-1)^{\frac{n}{2}} \varphi^n(|x|) \leqq |Df(x)|^n \leqq n^{\frac{n}{2}} \varphi^n(|x|)$$

or, equivalently,

$$(n-1)^{\frac{n}{2}} |x|^{-n} \log \frac{1}{|x|} \leqq |Df(x)|^n \leqq n^{\frac{n}{2}} |x|^{-n} \log \frac{1}{|x|}.$$

From this we readily deduce that

$$\frac{c(n)}{|x|^n} \leqq \frac{|Df(x)|^n}{\log\left(e + \dfrac{|Df(x)|}{|Df|_B}\right)} \leqq \frac{C(n)}{|x|^n},$$

which shows that $f \in \text{weak} - D^n \log^{-1} D(B, \mathbb{R}^n)$. Notice that the Jacobian of f is negative. However, with the aid of a reflection about an $(n-1)$-dimensional hyperplane we easily modify this example to obtain an orientation-preserving mapping. Then we may conclude with the following statement.

Proposition 6.1. *There exists a mapping $f: B \to \mathbb{R}^n$ of a ball $B \subset \mathbb{R}^n$ such that*

(i) $|Df|^n \log^{-1}\left(e + \dfrac{|Df|}{|Df|_B}\right) \in \text{weak} - L^1(B)$,

(ii) *$J(x, f)$ is positive everywhere, but fails to be locally integrable.*

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