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Journal: Forum Math.
DOI: 10.1515/forum-2017-0240
Article title: Global integrability for solutions to some anisotropic problem with nonstandard growth

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Research Article

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Global integrability for solutions to some anisotropic problem with nonstandard growth

https://doi.org/10.1515/forum-2017-0240
Received November 10, 2017; revised December 3, 2017

Abstract: This paper deals with the problem

$$\begin{aligned} &u \in \mathcal{K}_{u_*,\psi}(\Omega), \\ &\forall v \in \mathcal{K}_{u_*,\psi}(\Omega) : \int_\Omega \sum_{i=1}^n [a_i(x, Du) - f^i] D_i(u - v)\,dx \leqslant \int_\Omega f(u - v)\,dx, \end{aligned}$$

where

$$\begin{cases} \mathcal{K}_{u_*,\psi}(\Omega) = \Big\{ v \in u_* + W_0^{1,(p_i)}(\Omega) : \sum_{i=1}^n a_i(x, Du) D_i v \in L^1(\Omega) \text{ and } v \geqslant \psi, \text{ a.e. } \Omega \Big\}, \\ u_* \in W^{1,(p_i)}(\Omega), \quad \theta = \max\{u_*, \psi\} \in u_* + W_0^{1,(p_i)}(\Omega), \\ f \in L^{(\bar{p}^*)'}(\Omega), \quad f^i \in L^{p_i'}(\Omega),\ i = 1, \dots, n, \end{cases}$$

and the Carathéodory functions $a_i : \Omega \times R^n \mapsto R$, $i = 1, \dots, n$, satisfy some coercivity condition. We assume that the function $\theta = \max\{u_*, \psi\}$ makes $a_i(x, D\theta)$ to be more integrable than $L^{p_i'}(\Omega)$, $i = 1, \dots, n$, and then we prove that the solution u enjoys higher integrability.

Note 1: Should we replace $\mapsto$ by $\to$ in the blue-marked formula?

Note 2: Throughout, should we replace R by $\mathbb{R}$?

Keywords: Global integrability, anisotropic problem, nonstandard growth

MSC 2010: 49N60, 35J60

Communicated by: Frank Duzaar

1 Introduction

Throughout this paper, Ω stands for a bounded domain in R^n, $n \geqslant 2$. For $p_1, \dots, p_n \in (1, +\infty)$, we let

Note 3: Red parts indicate major changes. Please check them carefully.

$$\bar{p} : \frac{1}{\bar{p}} = \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} \quad \text{and} \quad p_i' = \frac{p_i}{p_i - 1}$$

be the harmonic mean of $p_1, \dots, p_n$ and the Hölder conjugate of p_i, respectively. In this paper, we assume $\bar{p} < n$ and we introduce the Sobolev exponent $\bar{p}^* = \frac{n\bar{p}}{n-\bar{p}}$. The anisotropic Sobolev space $W^{1,(p_i)}(\Omega)$ is defined as usual by

$$W^{1,(p_i)}(\Omega) = \{ v \in W^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1, \dots, n \},$$

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and $W_0^{1,(p_i)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,(p_i)}(\Omega)$, where for every function $v \in W^{1,(p_i)}(\Omega)$,

$$\|v\|_{1,(p_i)} = \int_\Omega |v|\,dx + \sum_{i=1}^n \Big(\int_\Omega |D_i v|^{p_i}\,dx\Big)^{\frac{1}{p_i}}.$$

In this paper, we will need the anisotropic Sobolev Embedding Theorem, which can be found in [19].

Note 4: Do you comply with our proposed running title?

Lemma 1.1. *Let Ω be a bounded open subset of R^n, let $p_1, \dots, p_n$ be in $[1, +\infty)$ and let $w \in W_0^{1,(p_i)}(\Omega)$. If $\bar p < n$, then $w \in L^{\bar p^*}(\Omega)$ with*

$$\|w\|_{L^{\bar p^*}(\Omega)} \leqslant c_*\Big[\prod_{i=1}^n \|D_i w\|_{L^{p_i}(\Omega)}\Big]^{\frac{1}{n}}$$

and

$$\|w\|_{L^{\bar p^*}(\Omega)} \leqslant c_* \sum_{i=1}^n \|D_i w\|_{L^{p_i}(\Omega)},$$

where

$$c_* = \max_{1\leqslant i\leqslant n}\Big\{1 + \bar p^* \frac{p_i - 1}{p_i}\Big\}.$$

Remark 1.2. A similar embedding, suitable for functions not vanishing on the boundary, is contained in [1, Lemma 2.1].

Remark 1.3. Lemma 1.1 implies, for $w \in W_0^{1,(p_i)}(\Omega)$ and $\bar p < n$,

$$\|w\|_{L^{\bar p^*}(\Omega)} \leqslant c_*\Big[\prod_{i=1}^n \Big(\int_\Omega |D_i w|^{p_i}\,dx\Big)^{\frac{1}{p_i}}\Big]^{\frac{1}{n}} \leqslant c_*\Big[\prod_{i=1}^n \Big(\sum_{j=1}^n \int_\Omega |D_j w|^{p_j}\,dx\Big)^{\frac{1}{p_i}}\Big]^{\frac{1}{n}} = c_*\Big[\sum_{j=1}^n \int_\Omega |D_j w|^{p_j}\,dx\Big]^{\frac{1}{\bar p}}.$$

We will use this result in the proof of the main theorem.

For every $k > 0$, let $T_k : R \mapsto R$ be the truncation function of level k, that is,

$$T_k(s) = \begin{cases} s & \text{if } |s| \leqslant k, \\ k \cdot \operatorname{sign}(s) & \text{if } |s| > k. \end{cases}$$

For every $\lambda > 0$, the weak Lebesgue space, known also as the Marcinkiewicz space, $L^\lambda_{\text{weak}}(\Omega)$, is defined as the set of all measurable functions $v : \Omega \mapsto R$ such that

$$\sup_{s>0} s^\lambda \big|\{|v| \geqslant s\}\big| < +\infty,$$

where $|E|$ is the Lebesgue measure of E. Note that if $f \in L^\lambda_{\text{weak}}(\Omega)$ for some $\lambda > 1$, then $f \in L^\tau(\Omega)$ for every $1 \leqslant \tau < \lambda$.

Let $a_i(x, z) : \Omega \times R^n \mapsto R$, $i = 1, 2, \dots, n$, be measurable with respect to x and continuous with respect to z, and let it satisfy the following coercivity condition: there exist a function $g : \Omega \mapsto [0, +\infty)$ and a constant $\nu > 0$ such that the inequality

$$\nu \sum_{i=1}^n |z_i - \tilde z_i|^{p_i} - g(x) \leqslant \sum_{i=1}^n (a_i(x, z) - a_i(x, \tilde z))(z_i - \tilde z_i) \tag{1.1}$$

holds true for almost every $x \in \Omega$ and all $z, \tilde z \in R^n$.

Let $u \in W^{1,(p_i)}(\Omega)$. Let u_*, ψ be two functions such that $u_* \in W^{1,(p_i)}(\Omega)$, $\theta = \max\{u_*, \psi\} \in u_* + W_0^{1,(p_i)}(\Omega)$ and

$$\sum_{i=1}^n a_i(x, Du) D_i\theta \in L^1(\Omega). \tag{1.2}$$

We introduce the set

$$\mathcal{K}_{u_*,\psi}(\Omega) = \Big\{v \in u_* + W_0^{1,(p_i)}(\Omega) : \sum_{i=1}^n a_i(x, Du) D_i v \in L^1(\Omega) \text{ and } v \geqslant \psi,\ \text{a.e. } \Omega\Big\}.$$

Let $f \in L^{(\bar{p}^*)'}(\Omega)$ and $f^i \in L^{p_i'}(\Omega)$, $i = 1, \dots, n$. Consider the following problem:

$$u \in \mathcal{K}_{u_*,\psi}(\Omega), \tag{1.3}$$

$$\forall v \in \mathcal{K}_{u_*,\psi}(\Omega) : \int_\Omega \sum_{i=1}^n [a_i(x, Du) - f^i] D_i(u - v)\,dx \leqslant \int_\Omega f(u - v)\,dx. \tag{1.4}$$

By assumption (1.3) and $\theta \in u_* + W_0^{1,(p_i)}(\Omega)$, we have $u - \theta \in W_0^{1,(p_i)}(\Omega)$ for a solution of problem (1.3)–(1.4); Lemma 1.1 shows that $u - \theta \in L^{\bar{p}^*}(\Omega)$. Now we ask the following question: if $\theta = \max\{u_*, \psi\}$ makes $a_i(x, D\theta)$ more integrable than $L^{p_i'}(\Omega)$ requires, does the solution u of problem (1.3)–(1.4) enjoy higher integrability? The answer is positive and we prove the following result.

Note 5: Do you agree with the new phrasing which improves the line breaks?

Note 6: Throughout, we wrote problem (1.3)–(1.4) in this form with a dash.

Theorem 1.4. *Let $\sigma > 1$. Assume that $g \in L^\sigma(\Omega)$, $f \in L^{(\bar{p}^*)'\sigma}(\Omega)$ and $f^i \in L^{\sigma p_i'}(\Omega)$ for every $i = 1, \dots, n$. Let $\theta = \max\{u_*, \psi\}$ be such that $a_i(x, D\theta) \in L^{p_i'\sigma}(\Omega)$, $i = 1, \dots, n$. If u satisfies (1.3) and (1.4), then the following assertions hold:*

(i) *If $\sigma < \frac{n}{\bar{p}}$, then*

$$u - \theta \in L_{\mathrm{weak}}^{\frac{n\bar{p}\sigma}{n-\bar{p}\sigma}}(\Omega).$$

Note 7: Throughout, we displayed some formulas.

(ii) *If $\sigma = \frac{n}{\bar{p}}$, then there exists $\alpha > 0$ such that $e^{\alpha|u-\theta|} \in L^1(\Omega)$.*

(iii) *If $\sigma > \frac{n}{\bar{p}}$, then $u - \theta \in L^\infty(\Omega)$.*

In Theorem 1.4, we do not assume that the Carathéodory functions $a_i(x, z)$, $i = 1, \dots, n$, satisfy any controllable growth condition, which is standard to derive regularity results. Leonetti and Petricca [14] considered the variational integral

$$\int_\Omega f(x, Du(x))\,dx \tag{1.5}$$

with nonstandard growth condition, and obtained a regularity result for minimizers of (1.5). Kovalevsky [12] studied the variational problem (1.3)–(1.4) under some coercivity and growth conditions, and obtained some regularity results. We should mention that Gorban and Kovalevsky [10] obtained some boundedness results for a class of degenerate anisotropic elliptic second-order variational inequalities. For some other results related to anisotropic elliptic equations, anisotropic integral functionals and anisotropic variational inequalities, we refer to [6–9, 11, 13, 15, 16, 18].

In the present paper, we consider the variational problem (1.3)–(1.4), which is closely related to the equation

$$-\sum_{i=1}^n D_i(a_i(x, Du(x))) = -\sum_{i=1}^n D_i f^i(x) + f(x).$$

Following the idea of [14], we derive Theorem 1.4, which shows that if $\theta = \max\{u_*, \psi\}$ makes $a_i(x, D\theta)$ to be more integrable than $L^{p_i'}(\Omega)$ requires, then the solution of problem (1.3)–(1.4) is more integrable.

We now give some examples for which the control from below is different from the control from above.

Example 1.5. We start from functionals

$$\int_\Omega f(x, Du(x))\,dx$$

Note 8: Throughout, we deleted all unused labels.

and we take $a_i(x, z) = \frac{\partial f}{\partial z_i}(x, z)$. An interesting choice is

$$f_{\max}(z) = |z|^2 + [\max\{z_n; 0\}]^q,$$

where $2 < q$; see [17]. This gives the functional

$$\mathcal{F}_{\max}(u) = \int_\Omega (|Du|^2 + [\max\{D_n u; 0\}]^q)\,dx.$$

Minimizers make the energy $\mathcal{F}_{\max}(u)$ finite. Then

$$f_{\max}(Du) \in L^1(\Omega). \tag{1.6}$$

When $a_i(z) = \frac{\partial f_{\max}}{\partial z_i}(z)$, we have

$$0 \leqslant 2f_{\max}(z) \leqslant \sum_{i=1}^{n} a_i(z)z_i \leqslant qf_{\max}(z),$$

so (1.6) is equivalent to

$$\sum_{i=1}^{n} a_i(Du)D_iu \in L^1(\Omega).$$

Example 1.6. Another interesting choice is

$$f_{2,q}(x,z) = |z|^2 + a(x)|z|^q,$$

where $2 < q$ and $0 \leqslant a(x) \leqslant A$; see [3–5, 20, 21]. This choice gives the functional

$$\mathcal{F}_{2,q}(u) = \int_{\Omega} (|Du|^2 + a(x)|Du|^q)\,dx.$$

Minimizers make the energy $\mathcal{F}_{2,q}(u)$ finite. Then

$$f_{2,q}(x, Du) \in L^1(\Omega). \tag{1.7}$$

When $a_i(x,z) = \frac{\partial f_{2,q}}{\partial z_i}(x,z)$, we have

$$0 \leqslant 2f_{2,q}(x,z) \leqslant \sum_{i=1}^{n} a_i(x,z)z_i \leqslant qf_{2,q}(x,z),$$

so (1.7) is equivalent to

$$\sum_{i=1}^{n} a_i(x,Du)D_iu \in L^1(\Omega).$$

Example 1.7. Finally, let us take

$$f_{2,\log}(x,z) = |z|^2 + a(x)|z|^2\ln(e+|z|),$$

where $0 \leqslant a(x) \leqslant A$; see [2]. This choice gives the functional

$$\mathcal{F}_{2,\log}(u) = \int_{\Omega} (|Du|^2 + a(x)|Du|^2\ln(e+|Du|))\,dx.$$

Minimizers make the energy $\mathcal{F}_{2,\log}(u)$ finite. Then

$$f_{2,\log}(x, Du) \in L^1(\Omega). \tag{1.8}$$

When $a_i(x,z) = \frac{\partial f_{2,\log}}{\partial z_i}(x,z)$, we have

$$0 \leqslant 2f_{2,\log}(x,z) \leqslant \sum_{i=1}^{n} a_i(x,z)z_i \leqslant 3f_{2,\log}(x,z),$$

so (1.8) is equivalent to

$$\sum_{i=1}^{n} a_i(x,Du)D_iu \in L^1(\Omega).$$

Remark 1.8. The solution u verifies (1.3) which requires

$$\sum_{i=1}^{n} a_i(x,Du)D_iu \in L^1(\Omega). \tag{1.9}$$

The test function v satisfies (1.4); the first part of (1.4) requires

$$\sum_{i=1}^{n} a_i(x,Du)D_iv \in L^1(\Omega). \tag{1.10}$$

When $a_i(x, z)$ verifies the control from above

$$\sum_{i=1}^{n} |a_i(x, z)|^{p_i/(p_i-1)} \leqslant c \sum_{i=1}^{n} |z_i|^{p_i} + g_2(x)$$

with $g_2 \in L^1(\Omega)$, then the standard assumption $u, v \in W^{1,(p_i)}(\Omega)$ guarantees both (1.9) and (1.10).

2 Proof of Theorem 1.4

In order to prove Theorem 1.4, we need a preliminary lemma, which can be found in [12].

Lemma 2.1. *Let $w \in W_0^{1,(p_i)}(\Omega)$, and let $M > 0$, $\gamma > 0$ and $k_0 \geqslant 0$. Let for every $k > k_0$,*

$$\int_{\{|w|\geqslant k\}} \sum_{i=1}^{n} |D_i w|^{p_i}\, dx \leqslant M\big|\{|w| \geqslant k\}\big|^{\frac{\gamma \bar p}{\bar p^*}}.$$

Then the following assertions hold:
(i) *If $\gamma < 1$, then*

$$w \in L_{\mathrm{weak}}^{\frac{\bar p^*}{1-\gamma}}(\Omega).$$

(ii) *If $\gamma = 1$, then there exists $\alpha > 0$ such that $e^{\alpha|w|} \in L^1(\Omega)$.*
(iii) *If $\gamma > 1$, then $w \in L^\infty(\Omega)$.*

We want to use Lemma 2.1 with $w = u - \theta$. To this end, we define a test function v as

$$v = \theta + T_k(u-\theta) = \begin{cases} \theta + k & \text{if } u - \theta \geqslant k, \\ u & \text{if } |u-\theta| < k, \\ \theta - k & \text{if } u - \theta \leqslant -k. \end{cases} \tag{2.1}$$

For $u \in \mathcal{K}_{u_*,\psi}(\Omega)$, we have to show that $v \in \mathcal{K}_{u_*,\psi}(\Omega)$. In fact, it is obvious that $v \in W^{1,(p_i)}(\Omega)$. In order to prove $v \in u_* + W_0^{1,(p_i)}(\Omega)$, we notice that $\theta = \max\{u_*, \psi\} = u_* = u$ on $\partial\Omega$. This together with (2.1) implies $v = u = u_*$ on $\partial\Omega$. In order to prove $\sum_{i=1}^n a_i(x, Du)D_i v \in L^1(\Omega)$, we notice that $Dv = Du$ on $\{|u-\theta| < k\}$ and $Dv = D\theta$ on $\{|u-\theta| \geqslant k\}$, thus $\sum_{i=1}^n a_i(x, Du)D_i v \in L^1(\Omega)$ is guaranteed by $\sum_{i=1}^n a_i(x, Du)D_i u \in L^1(\Omega)$ and (1.2). In order to prove $v \geqslant \psi$ a.e., we notice that in the first case of (2.1), $v = \theta + k \geqslant \theta \geqslant \psi$, in the second case of (2.1), $v = u \geqslant \psi$, and in the third case of (2.1), $v = \theta - k \geqslant u \geqslant \psi$.

Substituting v into (1.4) and noticing

$$D_i(u - v) = (D_i u - D_i\theta) \cdot 1_{\{|u-\theta|\geqslant k\}} \quad \text{and} \quad u - v = 0 \text{ in } \{|u-\theta| < k\}, \tag{2.2}$$

we arrive at

$$\int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} [a_i(x, Du) - f^i](D_i u - D_i\theta)\, dx \leqslant \int_{\{|u-\theta|\geqslant k\}} f(u - v)\, dx,$$

which together with (1.1) implies

$$\begin{aligned}
&\int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} |D_i u - D_i\theta|^{p_i}\, dx \\
&\leqslant \frac{1}{\nu}\Big[\int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} (a_i(x, Du) - a_i(x, D\theta))(D_i u - D_i\theta)\, dx + \int_{\{|u-\theta|\geqslant k\}} g dx\Big] \\
&\leqslant \frac{1}{\nu}\Big[\int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} f^i(D_i u - D_i\theta)\, dx + \int_{\{|u-\theta|\geqslant k\}} f(u-v)\, dx - \int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} a_i(x, D\theta)(D_i u - D_i\theta)\, dx + \int_{\{|u-\theta|\geqslant k\}} g dx\Big] \\
&= \frac{1}{\nu}(I_1 + I_2 + I_3 + I_4).
\end{aligned} \tag{2.3}$$

Our nearest goal is to estimate $|I_i|$, $i = 1, 2, 3, 4$.

For any $\varepsilon > 0$, the Young and Hölder inequalities yield

$$
\begin{aligned}
|I_1| &= \Big| \int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} f^i (D_i u - D_i \theta)\, dx \Big| \\
&\leqslant C(\varepsilon) \int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} |f^i|^{p_i'}\, dx + \varepsilon \int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} |D_i u - D_i\theta|^{p_i}\, dx \\
&\leqslant C(\varepsilon) \sum_{i=1}^{n} \Big(\int_{\{|u-\theta|\geqslant k\}} |f^i|^{p_i'\sigma}\, dx \Big)^{\frac{1}{\sigma}} \big|\{|u-\theta| \geqslant k\}\big|^{1-\frac{1}{\sigma}} + \varepsilon \int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} |D_i u - D_i\theta|^{p_i}\, dx.
\end{aligned} \tag{2.4}
$$

Using Remark 1.3, the Hölder and Young inequalities and (2.2), we estimate $|I_2|$ as follows:

$$
\begin{aligned}
|I_2| &= \Big| \int_{\{|u-\theta|\geqslant k\}} f(u-v)\, dx \Big| \\
&\leqslant \Big(\int_{\{|u-\theta|\geqslant k\}} |f|^{(\bar p^*)'}\, dx \Big)^{\frac{1}{(\bar p^*)'}} \Big(\int_\Omega |u-v|^{\bar p^*}\, dx \Big)^{\frac{1}{\bar p^*}} \\
&\leqslant c_* \|f\|_{L^{(\bar p^*)'\sigma}(\Omega)} \big|\{|u-\theta| \geqslant k\}\big|^{\frac{\sigma-1}{\sigma(\bar p^*)'}} \Big(\sum_{i=1}^{n} \int_\Omega |D_i(u-v)|^{p_i}\, dx \Big)^{\frac{1}{\bar p}} \\
&= c_* \|f\|_{L^{(\bar p^*)'\sigma}(\Omega)} \big|\{|u-\theta| \geqslant k\}\big|^{\frac{\sigma-1}{\sigma(\bar p^*)'}} \Big(\sum_{i=1}^{n} \int_{\{|u-\theta|\geqslant k\}} |D_i u - D_i\theta|^{p_i}\, dx \Big)^{\frac{1}{\bar p}} \\
&\leqslant c_* C(\varepsilon) \|f\|^{(\bar p)'}_{L^{(\bar p^*)'\sigma}(\Omega)} \big|\{|u-\theta| \geqslant k\}\big|^{\frac{(\sigma-1)(\bar p)'}{\sigma(\bar p^*)'}} + c_*\varepsilon \int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} |D_i u - D_i\theta|^{p_i}\, dx.
\end{aligned}
$$

There exists $k_0 \geqslant 0$ such that for every $k > k_0$,

$$\big|\{|u-\theta| \geqslant k\}\big| \leqslant 1.$$

This together with the fact

$$\frac{(\bar p)'}{(\bar p^*)'} = \frac{n\bar p - n + \bar p}{n(\bar p - 1)} > 1$$

implies for $k > k_0$,

$$\big|\{|u-\theta| \geqslant k\}\big|^{\frac{(\sigma-1)(\bar p)'}{\sigma(\bar p^*)'}} \leqslant \big|\{|u-\theta| \geqslant k\}\big|^{1-\frac{1}{\sigma}}.$$

Thus for $k > k_0$,

$$|I_2| \leqslant c_* C(\varepsilon) \|f\|^{(\bar p)'}_{L^{(\bar p^*)'\sigma}(\Omega)} \big|\{|u-\theta| \geqslant k\}\big|^{1-\frac{1}{\sigma}} + c_*\varepsilon \sum_{i=1}^{n} \int_{\{|u-\theta|\geqslant k\}} |D_i(u-\theta)|^{p_i}\, dx. \tag{2.5}$$

Using the Young and Hölder inequalities, we estimate $|I_3|$ and I_4 as follows:

$$
\begin{aligned}
|I_3| &= \Big| - \int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} a_i(x, D\theta)(D_i u - D_i\theta)\, dx \Big| \\
&\leqslant C(\varepsilon) \sum_{i=1}^{n} \int_{\{|u-\theta|\geqslant k\}} |a_i(x, D\theta)|^{p_i'}\, dx + \varepsilon \sum_{i=1}^{n} \int_{\{|u-\theta|\geqslant k\}} |D_i u - D_i\theta|^{p_i}\, dx \\
&\leqslant C(\varepsilon) \sum_{i=1}^{n} \Big(\int_{\{|u-\theta|\geqslant k\}} |a_i(x, D\theta)|^{p_i'\sigma}\, dx \Big)^{\frac{1}{\sigma}} \big|\{|u-\theta| \geqslant k\}\big|^{1-\frac{1}{\sigma}} + \varepsilon \sum_{i=1}^{n} \int_{\{|u-\theta|\geqslant k\}} |D_i u - D_i\theta|^{p_i}\, dx
\end{aligned} \tag{2.6}
$$

and

$$I_4 = \int_{\{|u-\theta|\geqslant k\}} g dx \leqslant \Big(\int_{\{|u-\theta|\geqslant k\}} g^\sigma\, dx \Big)^{\frac{1}{\sigma}} \big|\{|u-\theta| \geqslant k\}\big|^{1-\frac{1}{\sigma}}. \tag{2.7}$$

Substituting (2.4)–(2.7) into (2.3), and taking ε small enough such that

$$\frac{1}{\nu}(2 + c_*)\varepsilon < 1,$$

we then derive for $k > k_0$ that

$$\int_{\{|u-\theta|\geqslant k\}} \sum_{i=1}^{n} |D_i u - D_i\theta|^{p_i}\, dx \leqslant C\big|\{|u - \theta| \geqslant k\}\big|^{1-\frac{1}{\sigma}},$$

where C is a constant depending only on $\nu, n, p_1, \dots, p_n, \|f^i\|_{p_i'\sigma}, \|f\|_{(\bar{p}^*)'\sigma}, \|a_i(x, D\theta)\|_{p_i'\sigma}$ and $\|g\|_\sigma$.

Now we define γ by

$$1 - \frac{1}{\sigma} = \frac{\gamma\bar{p}}{\bar{p}^*}.$$

Since $u - \theta \in W_0^{1,(p_i)}(\Omega)$, we can apply Lemma 2.1 with $\gamma = \frac{n(\sigma-1)}{\sigma(n-\bar{p})}$. This ends the proof of Theorem 1.4.

Acknowledgment: Both authors would like to thank the referee for valuable suggestions and comments.

Funding: The first author was supported by NSFC (grant no. 11371050) and NSF of Hebei Province (grant no. A2015201149).

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Note 9: The bibliography has been cross-checked and updated with MathSciNet. Since entries may be mismatched, please carefully double-check *each* entry.