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Commutators of Hardy operators in vanishing Morrey spaces

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Abstract. In this paper we study boundedness of commutators of the multi-dimensional Hardy type operators with BMO coefficients, in weighted global and/or local generalized Morrey spaces $L^{p,\varphi}_{\Pi}(\mathbf{R}^n,w)$ and vanishing local Morrey spaces $VL^{p,\varphi}_{\mathrm{loc}}(\mathbf{R}^n,w)$ defined by an almost increasing function $\varphi(r)$ and radial type weight $w(|x|)$. This study is made in the perspective of posterior applications of the weighted results to some problems in the theory of PDE. We obtain sufficient conditions, in terms of some integral inequalities imposed on φ and w, and also in terms of the Matuszewska-Orlicz indices of φ and w, for such a boundedness.

Keywords: Generalized weighted Morrey space, vanishing Morrey spaces, commutators, BMO functions, weighted Hardy inequalities, weighted Hardy operators, Bary-Stechkin classes, Matuszewska-Orlicz indices
PACS: **AMS** Mathematics Subject Classification: 42B20, 46E30

1. INTRODUCTION

The classical Morrey spaces $L^{p,\lambda}$ introduced in [16] in relation to the study of regularity properties of solutions to PDE, are well known, see for instance the books [8], [11] and references therein, see also [21] where an overview of various generalizations may be found. The generalized Morrey spaces $L^{p,\varphi}$ are obtained by replacing r^λ by a function $\varphi(r)$ in the definition of the Morrey space.

During the last decades various classical operators, such as maximal, singular and potential operators and their commutators with BMO functions were widely investigated in both classical and generalized Morrey spaces, we refer e.g. to papers [1, 2, 3, 6, 7, 17, 18, 22, 23, 31], see also references therein. Commutators with functions in a subspace of BMO, named VMO (see [30]), are useful in the study of regularity of solutions of elliptic partial differential equations of second order, see e.g. [5].

As is well known, the boundedness of commutators of various operators, such as maximal and singular operators, in Lebesgue and Morrey spaces, is of importance in applications to PDE. Such boundedness was not studied in weighted spaces. The approach we used in the papers [26], [27] for the study of maximal, singular and potential operators in weighted Morrey spaces, allows us to reduce the problem of the boundedness of commutators of such operators to the study of commutators of Hardy type operators.

In this paper we start to study the boundedness of commutators in the case of Hardy operators in weighted Morrey spaces. Note that Hardy operators in Morrey spaces were less studied in comparison with maximal, singular and potential operators. We refer for instance to the paper [26] where there were proved weighted $p \to p$ - estimates in Morrey spaces $L^{p,\lambda}$ for Hardy operators on $\mathbf{R}^1_+$ and one-dimensional singular operators on $\mathbf{R}^1$ (also on Carleson curves in the complex plane), see also an application of the latter in [13] to the study of singular integral equations in weighted Morrey spaces. In paper [27] there were given conditions for the weighted $p \to q$-boundedness of multidimensional Hardy and potential operators within the frameworks of Morrey spaces $L^{p,\lambda}(\mathbf{R}^n)$. In [19], there was applied another approach which allowed to obtain weighted estimates in a more general case. The radial type weights $w(|x-x_0|)$ admitted in [19] were generated by functions $w(r)$ from the Bary-Stechkin-type class; these weights have possible decay or growth at $r=0$ and $r=\infty$, characterized by the condition that the weight becomes almost increasing or almost decreasing after the multiplication by a power function. Such weights oscillate between two powers at the origin and infinity (with different exponents for the origin and infinity, in general), see an overview on such weights in [20].

The results for their commutators and weighted estimates obtained in this paper are new. It is planned to give a further development of the obtained weighted estimations and give their applications for PDE in another

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paper.

Note that Morrey spaces, both the classical version $L^{p,\lambda}$ with $\lambda > 0$ and the generalized one $L^{p,\varphi}$ with $\varphi(0) = 0$ are not separable. A version of Morrey space where it is possible to approximate by "nice" functions is the so called vanishing Morrey spaces $VL^{p,\lambda}(\Omega)$ introduced in [33]. This is a subspace of functions in $L^{p,\lambda}(\Omega), \Omega \subseteq \mathbf{R}^n$, which satisfy the condition

$$\lim_{r\to 0} \sup_{\substack{x\in\Omega \\ 0<\rho<r}} \frac{1}{\rho^\lambda} \int\limits_{\widetilde{B}(x,\rho)} |f(y)|^p dy = 0, \tag{1}$$

where $\widetilde{B}(x,r) = B(x,r) \cap \Omega$.

In Definition in the next section we introduce more general spaces of such a kind, the global version $VL^{p,\varphi}(\Omega)$ and the local one $VL^{p,\varphi}_{\mathrm{loc};x_0}(\Omega)$.

In this paper, within the frameworks of the global $L^{p,\varphi}(\Omega)$ and local spaces $VL^{p,\varphi}_{\mathrm{loc};0}(\Omega)$, we study the commutators

$$[a, H_w]f = aH_w(f) - H_w(af)$$

of weighted multidimensional Hardy operators

$$H_w f(x) = \frac{w(|x|)}{|x|^n} \int\limits_{|y|<|x|} \frac{f(y)dy}{w(|y|)}$$

with functions $a \in BMO$.

The one-dimensional case includes the version

$$H_w f(x) = x^{-1} w(x) \int\limits_0^x \frac{f(t)dt}{w(t)}$$

adjusted for the half-axis $\mathbf{R}^1_+$, so that in the sequel $\mathbf{R}^n$ with $n = 1$ may be read either as $\mathbf{R}^1$ or $\mathbf{R}^1_+$. The notation $H = H_w\Big|_{w\equiv 1}$ will be also used in the sequel.

We obtain conditions, for the weighted boundedness of the commutators of Hardy operators with BMO functions in global non-vanishing generalized Morrey spaces and show that these conditions guarantee also that the commutators act within the frameworks of the corresponding local vanishing subspaces.

The paper is organized as follows: In Section 2 we give definitions and necessary preliminaries. Section 3 contains some results on weighted Hardy-type operators in Morrey and vanishing Morrey spaces obtained before, which we present with slight modification for completeness of presentation and for convenience of the reader. In Section 4 we present the main results, namely we prove theorems on the weighted $p \to p$-boundedness of the commutators of Hardy operators with functions which satisfy some conditions of BMO-type at the origin in global Morrey spaces and in local vanishing Morrey spaces. In Appendix we collect various properties of weights from the Bary-Stechkin class which we need in this paper. Most of them may be found dispersed in various papers, for instance, in [10, 14, 25, 26, 29], but for reader's convenience we gathered them in Appendix.

2. DEFINITIONS

Let Ω be an open set in $\mathbf{R}^n$, $\Omega \subseteq \mathbf{R}^n$ and $\ell = diam\,\Omega$, $0 < \ell \le \infty$, $B(x,r) = \{y \in \mathbf{R}^n : |x-y| < r\}$ and $\widetilde{B}(x,r) = B(x,r) \cap \Omega$. Let also $\varphi(r)$ be a non-negative function on $[0,\ell]$, continuous near the origin, such that

$$\inf_{\delta<r<\ell} \varphi(r) > 0 \tag{2}$$

for every $\delta > 0$. Let also $1 \le p < \infty$.

We introduce the vanishing generalized Morrey spaces, global and local, by the definition below. We find it convenient to give a definition unique for both cases via a unifying their version, with a prescribed set $\Pi \subseteq \Omega$, where the "Morrey-type" behaviour should hold.

In the sequel $\Pi \subseteq \Omega$ is an arbitrary measurable set of points.

Definition 1. *The generalized Morrey space $L^{p,\varphi}_\Pi(\Omega)$ is defined as the space of functions $f \in L^p_{loc}(\Omega)$ with the finite norm*

$$\|f\|_{p,\varphi;\Pi} := \sup_{x\in\Pi, r>0} \left(\frac{1}{\varphi(r)} \int\limits_{\widetilde{B}(x,r)} |f(y)|^p\, dy \right)^{\frac{1}{p}} < \infty. \tag{3}$$

In the case $\Pi = \Omega$ we obtain the global Morrey space, in the case $\Pi = \{x_0\}$ we have the local Morrey space. In these two extreme cases we will also use the notation

$$L^{p,\varphi}(\Omega) := L^{p,\varphi}_\Pi(\Omega)\Big|_{\Pi=\Omega} \tag{4}$$

and

$$L^{p,\varphi}_{\mathrm{loc};x_0}(\Omega) := L^{p,\varphi}_\Pi(\Omega)\Big|_{\Pi=\{x_0\}}.$$

In the next definition we use the notation

$$\mathbf{M}_{p,\varphi}(f;x,r) := \sup_{0<\rho<r} \frac{1}{\varphi(\rho)} \int\limits_{\widetilde{B}(x,\rho)} |f(y)|^p\, dy. \tag{5}$$

Definition 2. *The generalized vanishing Morrey space $VL^{p,\varphi}_\Pi(\Omega)$ is defined as the space of functions $f \in L^{p,\varphi}_\Pi(\Omega)$ such that*

$$\lim_{r\to 0} \sup_{x\in\Pi} \mathbf{M}_{p,\varphi}(f;x,r) = 0. \tag{6}$$

This is a Banach space (closed proper subspace of $L^{p,\varphi}_{\Pi}(\Omega)$ when $\varphi(0)=0$) with respect to the norm (3).

Similarly to (4) we write

$$VL^{p,\varphi}(\Omega) := VL^{p,\varphi}_{\Pi}(\Omega)\Big|_{\Pi=\Omega} \tag{7}$$

and

$$VL^{p,\varphi}_{\mathrm{loc};x_0}(\Omega) := VL^{p,\varphi}_{\Pi}(\Omega)\Big|_{\Pi=\{x_0\}}.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0}\frac{r^n}{\varphi(r)} = 0, \tag{8}$$

and additionally

$$\sup_{0<r<\infty}\frac{r^n}{\varphi(r)} < \infty \quad \text{in the case} \quad \Omega \quad \text{is unbounded}, \tag{9}$$

which makes the spaces $VL^{p,\varphi}_{\Pi}(\Omega)$ non-trivial, because bounded functions with compact support belong then to all such spaces. Note that the condition

$$\sup_{0<r<\infty}\frac{r^n}{\varphi(r)} < \infty, \tag{10}$$

means the similar non-triviality of the corresponding non-vanishing spaces.

In the sequel, a non-negative function f on $[0,\ell], 0<\ell\le\infty$, is called almost increasing (almost decreasing), if there exists a constant $C(\ge 1)$ such that $f(x)\le Cf(y)$ for all $x\le y$ ($x\ge y$, respectively). Equivalently, a function f is almost increasing (almost decreasing), if it is equivalent to an increasing (decreasing, resp.) function g, i.e. $c_1 f(x)\le g(x)\le c_2 f(x), c_1>0, c_2>0$.

Definition 3. (see [9]). *A real-valued locally integrable function f on $\mathbf{R}^n$ is said to be in the space BMO($\mathbf{R}^n$) if*

$$\|f\|_* := \sup_{x\in\mathbf{R}^n, r>0}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-f_B|\,dy < \infty, \tag{11}$$

where $f_B = \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} f(y)\,dy$.

Remark 1. *As is known, the norm (11) is equivalent to the norm*

$$\|f\|_{*;p} := \sup_{x\in\Omega, r>0}\left(\frac{1}{|B(x,r)|}\int\limits_{\widetilde{B}(x,r)}|f(y)-f_B|^p\,dy\right)^{\frac{1}{p}} < \infty, \tag{12}$$

and also to the norm

$$\|f\|_{*;p} := \sup_{x\in\Omega, r>0}\inf_{c_B}\left(\frac{1}{|B(x,r)|}\int\limits_{\widetilde{B}(x,r)}|f(y)-c_B|\,dy\right)^{\frac{1}{p}} < \infty \tag{13}$$

where $0<p<\infty$. (We keep the same notation $\|f\|_{*;p}$ for both the norms.)

For Hardy operators, instead of BMO, we will use the class of functions introduced in the next definition.

Definition 4. *A real-valued locally integrable function f on $\mathbf{R}^n$ is said to be in the space $\widetilde{BMO}^p(\mathbf{R}^n;0), 1\le p<\infty$, if $\|f\|^*_{0;p} :=$*

$$\sup_{r>0\sigma\in S^{n-1}}\left(\frac{1}{|B(0,r)|}\int\limits_{B(0,r)}|f(y)-f(r\sigma)|^p\,dy\right)^{\frac{1}{p}} < \infty. \tag{14}$$

Remark 2. *One could introduce the space $BMO^p(\mathbf{R}^n;0)$ of locally integrable functions with BMO-behaviour at a single point, say $x=0$, by the condition: $\|f\|_{*;0;p} :=$*

$$\sup_{r>0}\left(\frac{1}{|B(0,r)|}\int\limits_{B(0,r)}|f(y)-f_{B(0,r)}|^p\,dy\right)^{\frac{1}{p}} < \infty. \tag{15}$$

From the known inequality

$$\frac{1}{|B|}\int\limits_B|f(y)-f_B|^p dy \le \frac{2^p}{|B|}\int\limits_B|f(y)-C|^p dy \tag{16}$$

for any constant C on the right-hand side, it follows that

$$\|f\|_{*;0;p}\le 2\|f\|^*_{0;p}, \quad \textit{so that} \quad \widetilde{BMO}(\mathbf{R}^n;0)\subseteq BMO(\mathbf{R}^n;0)$$

Definition 5. *Let $0<\ell<\infty$.*
1) By $W=W([0,\ell])$ we denote the class of continuous and positive functions φ on $(0,\ell]$ such that there exists finite or infinite limit $\lim\limits_{x\to 0}\varphi(x)$;
2) by $W_0=W_0([0,\ell])$ we denote the class of almost increasing functions $\varphi\in W$ on $(0,\ell)$;
3) by $\overline{W}=\overline{W}([0,\ell])$ we denote the class of functions $\varphi\in W$ such that $x^\mu\varphi(x)\in W_0$ for some $\mu=\mu(\varphi)\in\mathbf{R}^1$;
4) by $\underline{W}=\underline{W}([0,\ell])$ we denote the class of functions $\varphi\in W$ such that $\frac{\varphi(t)}{t^\nu}$ is almost decreasing for some $\nu\in\mathbf{R}^1$.

Definition 6. *Let $0<\ell<\infty$.*
1) By $W_\infty=W_\infty([\ell,\infty])$ we denote the class of functions φ which are continuous and positive and almost increasing on $[\ell,\infty)$ and which have the finite or infinite limit $\lim_{x\to\infty}\varphi(x)$,
2) by $\overline{W}_\infty=\overline{W}_\infty([\ell,\infty))$ we denote the class of functions $\varphi\in W_\infty$ such $x^\mu\varphi(x)\in W_\infty$ for some $\mu=\mu(\varphi)\in\mathbf{R}^1$.

By $\overline{W}(\mathbf{R}^1_+)$ we denote the set of functions on $\mathbf{R}^1_+$ whose restrictions onto $(0,1)$ are in $\overline{W}([0,1])$ and restrictions onto $[1,\infty)$ are in $\overline{W}_\infty([1,\infty))$. Similarly, the set $\underline{W}(\mathbf{R}^1_+)$ is defined.

3. SOME RESULTS ON WEIGHTED HARDY-TYPE OPERATORS IN MORREY AND VANISHING MORREY SPACES

In this Section, for completeness of presentation and for reader's convenience, we present some results which are either published or to appear.

On belongness of some classes of radial functions to Morrey and vanishing Morrey spaces

In this subsection we present sufficient conditions for radial type functions $u(|x-x_0|)$, $x_0 \in \Omega$, to belong to generalized vanishing Morrey spaces. These conditions were given in [28], conditions of belongness of radial functions to non-vanishing Morrey space were given in [19], Proposition 3.3 and in [28] in a bit different form. We will use the notions of the Matuszewska-Orlicz indices and Zygmund-Bary-Stechkin classes, the definitions and properties of which are collected in Appendix. In the sequel we suppose that

$$u \in \mathbf{Z}^{-\frac{n}{p}}([0,\ell]), \text{ if } \ell < \infty \text{ and } u \in \mathbf{Z}^{-\frac{n}{p},-\frac{n}{p}}(\mathbf{R}^1_+), \text{if } \ell = \infty. \tag{17}$$

see (48). Recall that

$$\int_0^r u^p(t)t^{n-1}\,dt \sim r^n u^p(r), \quad 0 < r < \ell \tag{18}$$

under this assumption, see Lemma 4. Note that (17) is equivalent to the inequalities

$$pm(u) + n > 0, \quad pm_\infty(u) + n > 0 \tag{19}$$

in terms of the indices, where the second inequality is to be used only in the case $\ell = \infty$, see (51).

Lemma 1. *Let $\ell = diam\,\Omega \leq \infty$, $\varphi(r)$ satisfy conditions (2) and (10), and let $u \in \underline{W}([0,\ell])$ and fulfill the assumption (17). The condition*

$$\sup_{0<r<\ell} \frac{r^n u^p(r)}{\varphi(r)} < \infty, \tag{20}$$

is sufficient for the function $f(x) := u(|x-x_0|)$ to belong to $L^{p,\varphi}(\Omega)$. It is also necessary when Ω is bounded or $\Omega = \mathbf{R}^n$. Consequently, condition (20) is also necessary for f to belong to $L^{p,\varphi}(\Omega)$ in these cases. The condition (20) is sufficient also for $f \in L^{p,\varphi}(\Omega)$, in the following two cases:
i) $u(r)$ is bounded,
ii) $M(u) < 0$.

Let Ω be bounded. In terms of the indices of the functions u and φ, the conditions for f to belong to $L^{p,\varphi}_{loc;x_0}(\Omega)$ or $L^{p,\varphi}(\Omega)$ have the forms

$$M(\varphi) - pm(u) < n \quad \textit{(sufficient conditions)}, \tag{21}$$

$$m(\varphi) - pM(u) \leq n \quad \textit{(necessary conditions)} \tag{22}$$

under the assumption that $u, \frac{1}{\varphi} \in \overline{W}([0,\ell])$.

Remark 3. *The necessity of the condition (20) for unbounded domains different from $\mathbf{R}^n$ in general depends on the geometry of Ω at infinity. We do not touch this case.*

Corollary 1. *Let Ω be bounded. A power function $|x-x_0|^\gamma$ belongs to the local space $L^{p,\varphi}_{loc;x_0}(\Omega)$ or global space $L^{p,\varphi}(\Omega)$, if and only if*

$$n + \gamma p > 0 \quad \textit{and} \quad \sup_{r>0} \frac{r^{n+\gamma p}}{\varphi(r)} < \infty. \tag{23}$$

Lemma 2. *Let a radial function $f(x) = u(|x-x_0|)$ be in the local Morrey space $L^{p,\varphi}_{loc;x_0}(\Omega)$ or global Morrey space $L^{p,\varphi}(\Omega)$. It belongs to its subspace $VL^{p,\varphi}_{loc;x_0}(\Omega)$ or $VL^{p,\varphi}(\Omega)$, respectively, if and only if*

$$\lim_{r\to 0} \frac{1}{\varphi(r)} \int_0^r u^p(t)t^{n-1}\,dt = 0.$$

In particular, when Ω is bounded, a power function $|x-x_0|^\gamma$ belongs to $VL^{p,\varphi}(\Omega)$ and $VL^{p,\varphi}_{loc,x_0}(\Omega)$, if and only if

$$n + \gamma p > 0 \quad \textit{and} \quad \lim_{r\to 0} \frac{r^{n+\gamma p}}{\varphi(r)} = 0.$$

Corollary 2. *Let φ be a nonnegative measurable function. Then*

$$L^\infty(\Omega) \subseteq VL^{p,\varphi}(\Omega), \tag{24}$$

if and only if (8) holds. In the case Ω is bounded and $\varphi \in \overline{W}([0,\ell])$, the condition $m(\varphi) \leq n$ is necessary for (24) and $M(\varphi) < n$ is sufficient.

Weighted Hardy operators in generalized global Morrey and local vanishing Morrey spaces

The $L^p \to L^q$-boundedness of the multidimensional Hardy operators within the frameworks of Lebesgue spaces (the case $\varphi \equiv 1$) with $1 < p < \infty$ and $0 < q < \infty$ is known, see for instance, [12], p. 54.

We call attention of the reader to the fact that, in contrast to the case of Lebesgue spaces, Hardy-type inequalities in Morrey spaces different from Lebesgue spaces (i.e. in the case $\varphi(0) = 0$) admit the value $p = 1$. We need the following two recent results.

Theorem 1. *Let $1 \le p < \infty$ and φ and w satisfy the conditions*

$$\sup_{0<r<\infty} \frac{r^n}{\varphi(r)} < \infty, \quad w \in \overline{W}(\mathbf{R}^1_+),\ w(2t) \le cw(t),$$

$$\frac{\varphi^{\frac{1}{p}}}{w} \in \underline{W}(\mathbf{R}^1_+). \tag{25}$$

The operator H_w is bounded in the space $L^{p,\varphi}_{\Pi}(\mathbf{R}^n)$ for every choice of the set Π, if $\sup\limits_{r>0} \mathbf{W}(r) < \infty$, where

$$\mathbf{W}(r) := \frac{1}{\varphi(r)} \int_0^r \varphi(\rho) \left(1 + \frac{1}{V(\rho)} \int_0^\rho \frac{V(t)}{t}\, dt\right)^p \frac{d\rho}{\rho} \tag{26}$$

and $V(t) = \frac{t^{\frac{n}{p'}} \varphi^{\frac{1}{p}}(t)}{w(t)}$. Under this condition,

$$\|H^\alpha_w f\|_{L^{p,\varphi}_\Pi} \le C \sup_{r>0} \mathbf{W}^{\frac{1}{p}}(r) \|f\|_{L^{p,\varphi}_\Pi} \tag{27}$$

Theorem 1 was proved in [19] in a more general $p \to q$-setting. We gave its formulation here for the case $q = p$ for our goals. Note also that in [19] the formulation was given for the cases $\Pi = \mathbf{R}^n$ and $\Pi = \{0\}$, see Theorems 4.1 and 4.2 in [19], but it holds for any Π, since the proof in [19] was beside on the pointwise estimate via the norm of a function in $L^{p,\varphi}_{\mathrm{loc},0}$, see the inequality (4.4) in [19].

Theorem 2. *Let $1 \le p < \infty$ and the conditions in (25) be satisfied and*

$$\lim_{r\to 0} \frac{r^n}{\varphi(r)} = 0. \tag{28}$$

Then the operator H^α_w is bounded in the space $VL^{p,\varphi}_{loc;0}(\mathbf{R}^n)$, if $\sup\limits_{r>0} \mathbf{W}_s(r) < \infty$, and then

$$\mathbf{M}_{q,\varphi}(H_w f; 0, r) \le C\mathbf{M}(f; 0, r) \sup_{r>0} \mathbf{W}(r). \tag{29}$$

Theorem 2 was proved in [28], also in the $p \to q$-setting, we gave here its formulation for the case $q = p$.

The following corollary gives sufficient conditions for the boundedness of the operator H_w in terms of the Matuszewska-Orlicz indices of the function φ and the weight w.

Corollary 3. *Let $1 \le p < \infty$, the conditions in (25) be satisfied. The operator H_w is bounded in the space $L^{p,\varphi}_\Pi(\mathbf{R}^n)$ for any choice of the set Π, if*

$$\min\{m(V), m_\infty(V)\} > 0 \ \text{ and } \ \min\{m(\varphi), m_\infty(\varphi)\} > 0. \tag{30}$$

It is also bounded in the space $L^{p,\varphi}_{loc;0}(\mathbf{R}^n)$ under the above conditions, if additionally (28) holds. The condition $\min\{m(V), m_\infty(V)\} > 0$ is guaranteed by the inequalities

$$M(w) < \frac{n}{p'} + \frac{m(\varphi)}{p}, \quad M_\infty(w) < \frac{n}{p'} + \frac{m_\infty(\varphi)}{p}.$$

In the case of the classical (localized) Morrey space, i.e. $\varphi(r) = r^\lambda, 0 < \lambda < n$, and the power weight $w(r) = r^\mu$, the conditions (30) reduce to

$$1 \le p < \infty, \quad \mu < \frac{n}{p'} + \frac{\lambda}{p}; \tag{31}$$

conditions (31) are also necessary for the operator H_w to be bounded in the space $VL^{p,\lambda}_{loc;0}(\mathbf{R}^n)$.

4. MAIN RESULTS

In this Section we prove main result (see Theorem 3) on the boundedness of the commutator

$$[a, H_w]f = aH_w(f) - H_w(af) \tag{32}$$

with functions $a \in \widetilde{BMO}^s(\mathbf{R}^n; 0)$, for the weighted Hardy operator

$$H_w f(x) = |x|^{-n} w(|x|) \int_{|y|<|x|} \frac{f(y)dy}{w(|y|)}, \tag{33}$$

with quasi-monotone weights. We also obtain a theorem on the corresponding boundedness in vanishing local generalized Morrey space $VL^{p,\varphi}_{\mathrm{loc};x_0}(\Omega)$. The obtained conditions on the weight functions are given both in terms of integral conditions and also in terms of indices of the parameters of spaces, i.e. φ and p.

Theorem 3. *Let $1 \le p < \infty, a \in \widetilde{BMO}^s(\mathbf{R}^n; 0)$ for some $s \ge p'$ and φ and w satisfy the conditions of Theorem 1. Then commutator (32) is bounded in $L^{p,\varphi}_\Pi(\mathbf{R}^n)$ for every $\Pi \subseteq \mathbf{R}^n$, if*

$$\sup_{r>0} \mathbf{W}_s(r) < \infty, \tag{34}$$

where

$$\mathbf{W}_s(r) := \frac{1}{\varphi(r)} \int_0^r \varphi(\rho) \left(1 + \frac{1}{V_s(\rho)} \int_0^\rho \frac{V_s(t)}{t}\, dt\right) \frac{d\rho}{\rho} \tag{35}$$

and $V_s(t) = \frac{t^{\frac{n(p-s')}{p}} \varphi^{\frac{s'}{p}}(t)}{w^p(t)}$. Under this condition

$$\|[a, H_w]f\|_{L^{p,\varphi}_\Pi} \le C\|a\|^*_{0;s} \|f\|_{L^{p,\varphi}_\Pi}. \tag{36}$$

where C does not depend on a and f. The operator H_w is also bounded in $VL^{p,\varphi}_{loc;0}(\mathbf{R}^n)$ under the above conditions, if additionally (28) holds.

Proof. The proof is based on the following pointwise estimation of the commutator via Hardy operator $[a,H_w]$ of the function $|f|^{s'}$ with the weight $w^{s'}$:

$$|[a,H_w]f(x)| \le C\|a\|^*_{0;s}\left(H_{w^{s'}}(|f|^{s'})\right)^{\frac{1}{s'}}. \tag{37}$$

To prove (37), we apply the Hölder inequality with the exponent s in the integral:

$$[a,H_w]f(x) = \frac{w(|x|)}{|x|^n}\int\limits_{|y|<|x|}[a(x)-a(y)]\frac{f(y)dy}{w(|y|)}$$

and obtain

$$|[a,H_w]f(x)| \le \left(\frac{1}{|x|^n}\int\limits_{B(0,|x|)}|a(x)-a(y)|^s\,dy\right)^{\frac{1}{s}}$$

$$\times\left(\frac{w^{s'}(|x|)}{|x|^n}\int\limits_{B(0,|x|)}\frac{|f(y)|^{s'}dy}{w^{s'}(|y|)}dy\right)^{\frac{1}{s'}}$$

which proves (37).

Applying (37), we have

$$\|[a,H_w]f\|_{L^{p,\varphi}_{\Pi}(\mathbf{R}^n)} \le C\|a\|^*_{0;s}\cdot\left\|\left(H_{w^{s'}}(|f|^{s'})\right)^{\frac{1}{s'}}\right\|_{L^{p,\varphi}_{\Pi}(\mathbf{R}^n)}$$

$$= C\|a\|^*_{0;s}\cdot\left(\left\|H_{w^{s'}}(|f|^{s'})\right\|_{L^{\frac{p}{s'},\varphi}_{\Pi}(\mathbf{R}^n)}\right)^{\frac{1}{s'}},$$

which enables us to apply Theorem 1 with p replaced by $\frac{p}{s'}>1$ and w by $w^{s'}$. This transforms the condition of Theorem 1 on $\mathbf{W}_s(r)$ to the conditions (34)-(35). Then the conditions (34)-(35) and Theorem 1 yield

$$\|[a,H_w]f\|_{L^{p,\varphi}_{\Pi}(\mathbf{R}^n)} \le C\|a\|^*_{0;s}\cdot\left(\left\||f|^{s'}\right\|_{L^{\frac{p}{s'},\varphi}_{\Pi}(\mathbf{R}^n)}\right)^{\frac{1}{s'}}$$

$$= C\|a\|^*_{0;s}\cdot\left\||f|^{s'}\right\|_{L^{p,\varphi}_{\Pi}(\mathbf{R}^n)},,$$

which proves the theorem in the case of the spaces $L^{p,\varphi}_{\Pi}(\mathbf{R}^n)$. The proof for the space $VL^{p,\varphi}_{\mathrm{loc};0}(\mathbf{R}^n)$ is obtained in a similar way from the estimate (37) with application of Theorem 2.

From the proof of Theorem 3 and Corollary 3 we obtain the following Corollary in terms of the Matuszewska-Orlicz indices.

Corollary 4. *Let $1\le p<\infty, a\in \widetilde{BMO}^s(\mathbf{R}^n;0)$ for some $s\ge p'$ and the conditions in (25) be satisfied. Then the commutator $[a,H_w]$ is bounded in the space $L^{p,\varphi}_{\Pi}(\mathbf{R}^n)$ and also in $VL^{p,\varphi}_{loc;0}(\mathbf{R}^n)$ when (28) additionally holds, if*

$$\min\{m(\varphi),m_\infty(\varphi)\}>0, \tag{38}$$

$$M(w) < n\left(\frac{1}{s'}-\frac{1}{p}\right)+\frac{m(\varphi)}{p} \quad \textit{and}$$

$$M_\infty(w) < n\left(\frac{1}{s'}-\frac{1}{p}\right)+\frac{m_\infty(\varphi)}{p}. \tag{39}$$

In the case of the classical (localized) Morrey space, i.e. $\varphi(r)=r^\lambda, 0<\lambda<n$, and the power weight $w(r)=r^\mu$, the conditions of the boundedness of the commutator $[a,H_w]$ in both the spaces $L^{p,\lambda}_{\Pi}(\mathbf{R}^n)$ and $VL^{p,\lambda}_{loc;0}(\mathbf{R}^n)$ reduce to

$$1\le p<\infty, \quad \mu < n\left(\frac{1}{s'}-\frac{1}{p}\right)+\frac{\lambda}{p}. \tag{40}$$

Recall that the Hardy operator H_w itself is bounded in the considered spaces when instead of the conditions on w in (38) we have $M(w)<\frac{n}{p'}+\frac{m(\varphi)}{p}$, $M_\infty(w)<\frac{n}{p'}+\frac{m_\infty(\varphi)}{p}$. This means that the least possible restriction on the weight w in (38) should coincide with the case $\frac{1}{s'}-\frac{1}{p}=\frac{1}{p'}$, i.e $s=\infty$. This choice of s is inadmissible for us, since it would then mean that $a\in L^\infty(\mathbf{R}^n)$ and yield the triviality of the result for the commutator. Though we can take s arbitrarily large, taking into account that the interval for $M(w)$ and $M_\infty(w)$ are open. This yields the consequence of the above corollary given below. Note that weakening restrictions on the indices $M(w)$ and $M_\infty(w)$ of the weight by taking s larger, we simultaneously impose stronger condition on the function a, since the norm $\|a\|^*_{0;s}$ is an increasing function of s: $\|a\|^*_{0;s_1}\le\|a\|^*_{0;s_2}$ for $s_1\le s_2$.

Corollary 5. *Let $1\le p<\infty$ and the conditions in (25) be satisfied. Then the commutator $[a,H_w]$ is bounded in the space $L^{p,\varphi}_{\Pi}(\mathbf{R}^n)$ and also in $VL^{p,\varphi}_{loc;0}(\mathbf{R}^n)$ when (28) additionally holds, if*

$$\min\{m(\varphi),m_\infty(\varphi)\}>0 \quad \textit{and} \quad M(w)<\frac{n}{p'}+\frac{m(\varphi)}{p},$$

$$M_\infty(w)<\frac{n}{p'}+\frac{m_\infty(\varphi)}{p}. \tag{41}$$

and $a\in\widetilde{BMO}^s(\mathbf{R}^n;0)$ with s sufficiently large, namely, $s\ge p'$ and $\frac{n}{s}<\frac{n}{p'}+\frac{\min\{m(\varphi),m_\infty(\varphi)\}}{p}-$ $\max\{M(w),M_\infty(w)\}$

5. APPENDIX: MATUSZEWSKA-ORLICZ (MO) TYPE INDICES

ZBS-classes and MO-indices of weights at the origin

In this subsection we assume that $\ell < \infty$.

We say that a function φ belongs to a Zygmund class $\mathbf{Z}^\beta$, $\beta \in \mathbf{R}^1$, if $\varphi \in \overline{W}([0,\ell])$ and

$$\int_0^x \frac{\varphi(t)}{t^{1+\beta}} dt \le c\frac{\varphi(x)}{x^\beta}, \quad x \in (0,\ell),$$

and to a Zygmund class $\mathbf{Z}_\gamma$, $\gamma \in \mathbf{R}^1$, if $\varphi \in \underline{W}([0,\ell])$ and

$$\int_x^\ell \frac{\varphi(t)}{t^{1+\gamma}} dt \le c\frac{\varphi(x)}{x^\gamma}, \quad x \in (0,\ell).$$

We also denote $\Phi_\gamma^\beta := \mathbf{Z}^\beta \bigcap \mathbf{Z}_\gamma$, the latter class being also known as Bary-Stechkin-Zygmund class [4].

It is known that the property of a function to be almost increasing or almost decreasing after the multiplication (division) by a power function is closely related to the notion of the so called Matuszewska-Orlicz indices. We refer e.g. to [14], [15], [20], [24], [25], for the properties of the indices of such a type. For a function $\varphi \in \overline{W}$, the numbers $m(\varphi) = \sup_{0<x<1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x} = \lim_{x\to 0} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x}$ and $M(\varphi) = \sup_{x>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x} = \lim_{x\to \infty} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x}$ are known as *the Matuszewska-Orlicz type lower and upper indices* of the function $\varphi(r)$.

The following statement is known, see [10], Theorems 3.1, 3.2 and 3.5. (In the formulation of Theorem 4 in [10] it was supposed that $\beta \ge 0, \gamma > 0$ and $\varphi \in W_0$. It is evidently true also for $\varphi \in \overline{W}$ and all $\beta, \gamma \in \mathbf{R}^1$).

Theorem 4. *Let $\varphi \in \overline{W}$ and $\beta,\gamma \in \mathbf{R}^1$. Then $\varphi \in \mathbf{Z}^\beta \Longleftrightarrow m(\varphi) > \beta$ and $\varphi \in \mathbf{Z}_\gamma \Longleftrightarrow M(\varphi) < \gamma$. Besides this*

$$m(\varphi) = \sup\left\{\mu > 0 : \ \frac{\varphi(x)}{x^\mu} \quad \text{is almost increasing}\right\}, \tag{42}$$

$$M(\varphi) = \inf\left\{\nu > 0 : \ \frac{\varphi(x)}{x^\nu} \quad \text{is almost decreasing}\right\}. \tag{43}$$

We define the following subclass in $\overline{W}_0$:

$$\overline{W}_{0,b} = \left\{\varphi \in \overline{W}_0 : \frac{\varphi(t)}{t^b} \text{ is almost increasing}\right\}, \ b \in \mathbf{R}^1.$$

ZBS-classes and MO-indices of weights at infinity

Following [29], Subsection 2.2, we introduce the following definitions:

Let $-\infty < \alpha < \beta < \infty$. We put $\Psi_\alpha^\beta := \widehat{\mathbf{Z}}^\beta \cap \widehat{\mathbf{Z}}_\alpha$, where $\widehat{\mathbf{Z}}^\beta$ is the class of functions $\varphi \in \overline{W}_\infty$ satisfying the condition

$$\int_x^\infty \left(\frac{x}{t}\right)^\beta \frac{\varphi(t)\,dt}{t} \le c\varphi(x), \quad x \in (\ell,\infty), \tag{44}$$

and $\widehat{\mathbf{Z}}_\alpha$ is the class of functions $\varphi \in W([\ell,\infty))$ satisfying the condition

$$\int_\ell^x \left(\frac{x}{t}\right)^\alpha \frac{\varphi(t)\,dt}{t} \le c\varphi(x), \quad x \in (\ell,\infty) \tag{45}$$

where $c = c(\varphi) > 0$ does not depend on $x \in [\ell,\infty)$.

The indices $m_\infty(\varphi)$ and $M_\infty(\varphi)$ responsible for the behavior of functions $\varphi \in \Psi_\alpha^\beta([\ell,\infty))$ at infinity are introduced in the way similar to $m(\varphi)$ and $M(\varphi)$: $m_\infty(\varphi) = \sup_{x>1} \frac{\ln\left[\liminf\limits_{h\to\infty} \frac{\varphi(xh)}{\varphi(h)}\right]}{\ln\ x}$, $M_\infty(\varphi) = \inf_{x>1} \frac{\ln\left[\limsup\limits_{h\to\infty} \frac{\varphi(xh)}{\varphi(h)}\right]}{\ln\ x}$.

Properties of functions in the class $\Psi_\alpha^\beta([\ell,\infty))$ are easily derived from those of functions in $\Phi_\beta^\alpha([0,\ell])$ because of the following equivalence

$$\varphi \in \Psi_\alpha^\beta([\ell,\infty)) \iff \varphi_* \in \Phi_{-\alpha}^{-\beta}([0,\ell^*]), \tag{46}$$

where $\varphi_*(t) = \varphi\left(\frac{1}{t}\right)$ and $\ell_* = \frac{1}{\ell}$. We have

$$m_\infty(\varphi) = -M(\varphi_*),\ M_\infty(\varphi) = -m(\varphi_*). \tag{47}$$

We say that a continuous function φ in $(0,\infty)$ is in the class $\overline{W}_{0,\infty}(\mathbf{R}_+^1)$, if its restriction to $(0,1)$ belongs to $\overline{W}([0,1])$ and its restriction to $(1,\infty)$ belongs to $\overline{W}_\infty([1,\infty])$. For functions in $\overline{W}_{0,\infty}(\mathbf{R}_+^1)$ the notation

$$\mathbf{Z}^{\beta_0,\beta_\infty}(\mathbf{R}_+^1) = \mathbf{Z}^{\beta_0}([0,1]) \cap \mathbf{Z}^{\beta_\infty}([1,\infty)), \tag{48}$$

$$\mathbf{Z}_{\gamma_0,\gamma_\infty}(\mathbf{R}_+^1) = \mathbf{Z}_{\gamma_0}([0,1]) \cap \mathbf{Z}_{\gamma_\infty}([1,\infty)) \tag{49}$$

has an obvious meaning (note that in (48) we use $\mathbf{Z}^{\beta_\infty}([1,\infty))$ and $\mathbf{Z}_{\gamma_\infty}([1,\infty))$, not $\widehat{\mathbf{Z}}^{\beta_\infty}([1,\infty))$ and $\widehat{\mathbf{Z}}_{\gamma_\infty}([1,\infty))$). In the case where the indices coincide, i.e. $\beta_0 = \beta_\infty := \beta$, we will simply write $\mathbf{Z}^\beta(\mathbf{R}_+^1)$ and similarly for $\mathbf{Z}_\gamma(\mathbf{R}_+^1)$. We also denote

$$\Phi_\gamma^\beta(\mathbf{R}_+^1) := \mathbf{Z}^\beta(\mathbf{R}_+^1) \cap \mathbf{Z}_\gamma(\mathbf{R}_+^1). \tag{50}$$

Making use of Theorem 4 for $\Phi_\beta^\alpha([0,1])$ and relations (47), one easily arrives at the following statement.

Lemma 3. *Let $\varphi \in \overline{W}(\mathbf{R}_+^1)$. Then*

$$\varphi \in \mathbf{Z}^{\beta_0,\beta_\infty}(\mathbf{R}_+^1) \iff m(\varphi) > \beta_0,\ m_\infty(\varphi) > \beta_\infty \tag{51}$$

and

$$\varphi \in \mathbf{Z}_{\gamma_0,\gamma_\infty}(\mathbf{R}^1_+) \quad \Longleftrightarrow \quad M(\varphi) < \gamma_0,\ M_\infty(\varphi) < \gamma_\infty. \tag{52}$$

Lemma 4. *Let $0 < \ell \leq \infty$ and $\varphi \in \overline{W}[0,\ell] \cap \underline{W}[0,\ell]$, that is, φ has both finite indices $m(\varphi)$ and $M(\varphi)$ (and also finite indices $m_\infty(\varphi)$ and $M_\infty(\varphi)$) in the case $\ell = \infty$. Then the inequality*

$$\int_0^x \frac{\varphi(t)}{t^{1+\beta}} dt \leq c \frac{\varphi(x)}{x^\beta}, \quad x \in (0,\ell), \tag{53}$$

implies the inverse inequality

$$\frac{\varphi(x)}{x^\beta} \leq c \int_0^x \frac{\varphi(t)}{t^{1+\beta}} dt, \quad x \in (0,\ell).$$

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