BOUNDEDNESS OF THE SOLUTIONS TO NONLINEAR SYSTEMS WITH MORREY DATA

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Abstract. We consider nonlinear elliptic systems satisfying componentwise coercivity condition. The nonlinear terms have controlled growths with respect to the solution and its gradient, while the behaviour in the independent variable is governed by functions in Morrey spaces. We firstly prove essential boundedness of the weak solution and then obtain Morrey regularity of its gradient.

1. Introduction

Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a bounded domain satisfying the (A)-condition. We are interested in boundedness and Morrey regularity of the weak solutions to nonlinear elliptic systems of the type

$$-\operatorname{div}\mathbf{A}(x,\mathbf{u},D\mathbf{u}) + \mathbf{b}(x,\mathbf{u},D\mathbf{u}) = \mathbf{f}(x), \qquad x \in \Omega \tag{1}$$

where the nonlinear terms are Carathéodory maps

$$\mathbf{A}(x,\mathbf{u},\mathbf{z}) : \Omega \times \mathbb{R}^N \times \mathbb{M}^{N\times n} \to \mathbb{R}^{N\times n},$$
$$\mathbf{b}(x,\mathbf{u},\mathbf{z}) : \Omega \times \mathbb{R}^N \times \mathbb{M}^{N\times n} \to \mathbb{R}^{N}.$$

The celebrated result of De Giorgi [5] and Nash [16] implies that any weak solution $u \in W_0^{1,2}(\Omega)$ of the linear elliptic equation $D_i(A_{ij}(x)D_ju + g_i(x)) = f(x)$ is locally Hölder continuous when $g_i \in L^p$ with $p > n$ and $f \in L^q$ with $q > n/2$, even if the coefficients are only L^∞. Unfortunately the De Giorgi-Nash result does not hold anymore if we consider a system of uniformly elliptic equations because of the lack of *Maximum principle.* This was shown by De Giorgi himself almost ten years later, constructing a counterexample [6]. Precisely, the function $\mathbf{u} = 1 - x/|x|^\gamma \in W_0^{1,2}(\mathcal{B}_1(0);\mathbb{R}^n)$ is a solution to

$$D_i(A_{ij}^{\alpha\beta}(x)D_ju^\beta(x)) = 0 \qquad \text{in } \mathcal{B}_1(0)$$

with suitably chosen coefficients $A_{ij}^{\alpha\beta} \in L^\infty(\mathcal{B}_1(0))$.

Moreover, the result of De Giorgi-Nash cannot be extended to quasilinear systems even if the coefficients are analytic functions, as it was shown by Giusti and Miranda in [10]. In order to get a maximum principle for elliptic systems we need to impose some quite restrictive structural conditions. The simplest one requires the system to be in diagonal form, or *decoupled.*

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Example 1. *Consider the operator* $\operatorname{div}(\mathbf{A}(x, D\mathbf{u})) = 0$ *in* Ω *with coefficients*

$$A_i^\alpha(x, D\mathbf{u}) = \sum_{j=1}^{n}\sum_{\beta=1}^{N} \delta_{\alpha\beta} A_{ij}^{\alpha\beta}(x) D_j u^\beta$$

where $\delta_{\alpha\beta}$ *is the Kronecker delta. Then* u^α *solves a single elliptic equation and* $\sup_\Omega u^\alpha \le \sup_{\partial\Omega} u^\alpha$, *for each* $\alpha = 1, \dots, N$.

One more example was given by Nečas and Stará in [17].

Example 2. *Consider the system* $\operatorname{div}\mathbf{A}(x, \mathbf{u}, D\mathbf{u}) = 0$ *in* Ω *that is diagonal for large values of* u^α, *that is,*

$$0 < \theta_\alpha \le u^\alpha \implies A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) = \sum_{j=1}^{n}\sum_{\beta=1}^{N} \delta_{\alpha\beta} A_{ij}^{\alpha\beta}(x, \mathbf{u}) D_j u^\beta \tag{2}$$

with bounded and elliptic $A_{ij}^{\alpha\beta}$. *It turns out that*

$$\sup_\Omega u^\alpha \le \max\left\{\theta_\alpha; \sup_{\partial\Omega} u^\alpha\right\}$$

also in this case.

The situation becomes more complicated if we consider *general nonlinear system*

$$\operatorname{div}\mathbf{A}(x, \mathbf{u}, D\mathbf{u}) = \mathbf{b}(x, \mathbf{u}, D\mathbf{u})\,. \tag{3}$$

Along with the Carathéodory conditions on the maps $\mathbf{A}(x, \mathbf{u}, \mathbf{z})$ and $\mathbf{b}(x, \mathbf{u}, \mathbf{z})$ we need to control also the growths of $\mathbf{A}$ and $\mathbf{b}$ with respect to $\mathbf{u}$ and $\mathbf{z}$. These additional *controlled growth conditions* ensure the convergence of the integrals in the definition of *weak solution* to (3) (see (13)).

In [14] Leonetti and Petricca assume *componentwise coercivity condition* on $\mathbf{A}$ and positivity of $\mathbf{b}$ for large values of u^α, that is, there exist positive constants θ_α such that

$$\theta_\alpha \le u^\alpha \quad \implies \quad \begin{cases} \nu|\mathbf{z}^\alpha|^p - M_\alpha \le \displaystyle\sum_{i=1}^{n} A_i^\alpha(x, \mathbf{u}, \mathbf{z}) z_i^\alpha \\ 0 \le b^\alpha(x, \mathbf{u}, \mathbf{z})\,. \end{cases} \tag{4}$$

Combining the *Sobolev inequality* with the *Stampacchia Lemma* [23] they get a componentwise bound of the solution, covering this way also the systems studied in [17], since (2) is a special case of (4). Let us note that getting essential boundedness of the weak solution to (1) is a starting point for a further study of its regularity in various function spaces. In [7, 18, 20] the authors obtain better integrability and Hölder regularity of the bounded solutions to quasilinear elliptic equations ($N = 1$) under controlled growth conditions on the nonlinear terms. Further this result has been extended in [22] to semilinear uniformly elliptic systems of the form

$$\operatorname{div}(\mathbf{A}(x)D\mathbf{u} + \mathbf{a}(x, \mathbf{u})) = \mathbf{b}(x, \mathbf{u}, D\mathbf{u}) \qquad \text{in } \Omega \tag{5}$$

with minimal regular assumptions on the coefficients and the underlying domain. Precisely, it is shown that if the nonlinear terms satisfy the controlled growth conditions (10) with $\varphi \in L^p(\Omega)$, $p > 2$ and $\psi \in L^q(\Omega)$, $q > \frac{2n}{n+2}$ then any bounded weak solution to (5) belongs to $W_0^{1,r}$ with $r = \min\{p, q^*\}$.

The natural question that arises is what kind of regularity of the solution to (1) we can expect if the given functions φ and ψ belong to some Morrey spaces. In the case of a single equation we count with the result of Byun and Palagachev [2]. Combining the Gehring-Giaquinta-Modica lemma, the Adams trace inequality and the Hartmann-Stampacchia maximum principle they obtain L^∞ estimate of the solution. Further, the Morrey-type estimate of the gradient permits the authors to show also Hölder regularity of the solution.

Our goal is to obtain a componentwise maximum principle for any solution of (3) supposing that the operators $\mathbf{A}$ and $\mathbf{b}$ satisfy structural conditions expressed in terms of Morrey functions. As a consequence we obtain also Morrey regularity of the gradient of $\mathbf{u}$ extending such a way the regularity results obtained in [2, 7, 14, 17, 19, 22] to nonlinear systems with Morrey data.

Recall that a real valued function $f \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ with $p \in [1,\infty)$, $\lambda \in (0,n)$, if

$$\|f\|_{p,\lambda;\Omega} = \left(\sup_{\mathcal{B}_r(x)} \frac{1}{r^\lambda} \int_{\Omega\cap\mathcal{B}_r(x)} |f(y)|^p \, dy \right)^{1/p} < \infty \tag{6}$$

where the supremum is taken over all balls $\mathcal{B}_r(x)$, $r \in (0, \operatorname{diam}\Omega]$ and $x \in \overline{\Omega}$. Working in the framework of the Morrey spaces we note that the Sobolev trace inequality is not enough anymore. For this goal we will use the following result due to Adams.

Lemma 3 (Adams Trace Inequality, [1, 4, 21])**.** *Let m be a positive Radon measure with support in Ω and such that for each ball $\mathcal{B}_\rho$ it holds*

$$m(\mathcal{B}_\rho) \le K\rho^{\tau_0}, \quad \tau_0 = \frac{s}{r}(n-r), \quad 1 < r < s < \infty, \quad r < n \tag{7}$$

with an absolute constant $K > 0$. Then

$$\left(\int_\Omega |v(x)|^s \, dm \right)^{\frac{1}{s}} \le C(n,s,r) K^{\frac{1}{s}} \left(\int_\Omega |Dv(x)|^r \, dx \right)^{\frac{1}{r}} \tag{8}$$

for each function $v \in W_0^{1,r}(\Omega)$.

In what follows we suppose that $\Omega \subset \mathbb{R}^n, n \ge 2$, is a bounded domain satisfying the (A)-condition, that is, there exists a constant $A_\Omega > 0$ such that

$$|\Omega_r(x)| \ge A_\Omega \, r^n \qquad \forall\ x \in \overline{\Omega},\ r \in (0, \operatorname{diam}\Omega] \tag{A}$$

where $\Omega_r(x) = \Omega\cap\mathcal{B}_r(x)$. It is worth noting that the (A)-condition excludes interior cusps at each point of the boundary and guarantees the validity of the Sobolev embedding theorem in $W^{1,p}(\Omega)$. This geometric property is surely satisfied when $\partial\Omega$ has the uniform interior cone property (e.g. C^1-smooth or Lipschitz continuous boundaries), but it holds also for the Reifenberg falt domains boundaries (cf. [20]).

Throughout the text the standard summation convention on the repeated indexes is adopted. The letter $C > 0$ is used for various constants and may change from one occurrence to another.

2. Maximum principle

Consider the nonlinear system

$$-D_i\big(A_i^\alpha(x,\mathbf{u},D\mathbf{u})\big) + b^\alpha(x,\mathbf{u},D\mathbf{u}) = f^\alpha(x) \ \text{ in } \Omega \tag{9}$$

where $\mathbf{A} = \{A_i^\alpha(x,\mathbf{u},\mathbf{z})\}_{i\le n}^{\alpha\le N}$ and $\mathbf{b} = (b^1(x,\mathbf{u},\mathbf{z}),\ldots,b^N(x,\mathbf{u},\mathbf{z}))$ are measurable in $x \in \Omega$ and continuous in $(\mathbf{u},\mathbf{z})$ for almost all (a.a.) $x \in \Omega$. Suppose that for each $(x, \mathbf{u}, \mathbf{z}) \in \Omega \times \mathbb{R}^N \times \mathbb{M}^{N\times n}$ the following *controlled growth conditions* hold. Namely,

$$\begin{cases} |\mathbf{A}(x,\mathbf{u},\mathbf{z})| \le \Lambda\big(\varphi(x) + |\mathbf{u}|^{\frac{\nu}{2}} + |\mathbf{z}|\big) \\ |\mathbf{b}(x,\mathbf{u},\mathbf{z})| \le \Lambda\big(\psi(x) + |\mathbf{u}|^{\nu-1} + |\mathbf{z}|^{2\frac{\nu-1}{\nu}}\big) \end{cases} \tag{10}$$

as $|\mathbf{u}|, |\mathbf{z}| \to \infty$, with some positive constant Λ. Here ν is the Sobolev conjugate of 2, that is,

$$\nu = \begin{cases} \dfrac{2n}{n-2} & \text{if } n \ge 3 \\ \text{any number } > 2 & \text{if } n = 2, \end{cases} \tag{11}$$

and the given functions φ, ψ and f^α satisfy

$$\begin{cases} \varphi \in L^{p,\lambda}(\Omega), & p > 2,\ \lambda \in (0,n),\ p + \lambda > n \\ \psi, f^\alpha \in L^{q,\mu}(\Omega), & q > \frac{\nu}{\nu-1},\ \mu \in (0,n),\ 2q + \mu > n. \end{cases} \tag{12}$$

In the particular case $n = 2$ the powers of $|\mathbf{u}|$ could be arbitrary positive numbers greater then 1, while the growth of $|\mathbf{z}|$ is strictly sub-quadratic (cf. [8, 13]).

Under a *weak solution* of (9) we mean a function $\mathbf{u} \in W^{1,2}(\Omega;\mathbb{R}^N) \cap L^\nu(\Omega;\mathbb{R}^N)$, satisfying

$$\begin{aligned} \sum_{i=1}^n \int_\Omega A_i^\alpha(x,\mathbf{u},D\mathbf{u}) D_i\phi^\alpha(x)\,dx + \int_\Omega b^\alpha(x,\mathbf{u}(x),D\mathbf{u}(x))\phi^\alpha(x)\,dx \\ = \int_\Omega f^\alpha(x)\phi^\alpha(x)\,dx \end{aligned} \tag{13}$$

for all $\phi = (\phi^1, \dots, \phi^N) \in W_0^{1,2}(\Omega;\mathbb{R}^N)$. The conditions (10)-(12) are the natural ones that ensure the convergence of the integrals in (13). Moreover, they are optimal as it is seen from the following example in the case of single equation (cf. [12, 18]).

Example 4. *The function $u(x) = |x|^{\frac{r-2}{r-1}} \in W^{1,2}(\mathcal{B}_1(0))$, with $n \ge 3$ and $\frac{n+2}{n} < r < 2$ is a solution to the equation $\Delta u = C|Du|^r$ in $\mathcal{B}_1(0)$. Note that $u \notin L^\infty(\mathcal{B}_1(0))$.*

Generally we cannot expect boundedness of the solutions to (9) unless we add some restrictions on the structure of the operator (see for example [11, 14]). For this goal we impose componentwise coercivity on A_i^α and a sign condition on b^α.

For every $\alpha \in \{1, \dots, N\}$ there exist positive constants θ_α, γ and a function φ such that for each $u^\alpha \ge \theta_\alpha$ we have

$$\begin{cases} \gamma|\mathbf{z}^\alpha|^2 - \Lambda\varphi(x)^2 \le \displaystyle\sum_{i=1}^n A_i^\alpha(x,\mathbf{u},\mathbf{z}) z_i^\alpha \\ \varphi \in L^{p,\lambda}(\Omega),\ p > 2,\ p + \lambda > n \\ 0 \le b^\alpha(x,\mathbf{u},\mathbf{z}) \quad \text{for a.a. } x \in \Omega,\ \forall\, \mathbf{z} \in \mathbb{M}^{N\times n}. \end{cases} \tag{14}$$

Theorem 5 (Maximum principle)**.** *Let Ω be* (A)*-type domain and $\mathbf{u} \in W^{1,2}(\Omega;\mathbb{R}^N) \cap L^\nu(\Omega;\mathbb{R}^N)$ be a weak solution to* (9) *under the conditions* (10)*,* (12) *and* (14) *and such that $\sup_{\partial\Omega} u^\alpha < \infty$. Then*

$$\sup_\Omega u^\alpha \le \max\{\theta_\alpha, \sup_{\partial\Omega} u^\alpha\} + M_\alpha \qquad \alpha \in \{1, \dots, N\}$$

where M_α depends on $n, p, \lambda, \Lambda, \gamma$, $\|\varphi\|_{p,\lambda;\Omega}$, $\|f^\alpha\|_{q,\mu;\Omega}$ and $|\Omega|$.

Proof. We choose a constant $L>0$ such that $L\geq\max\{\theta_\alpha;\sup_{\partial\Omega}u^\alpha\}$ and define the set $\mathcal{A}_L^\alpha=\{x\in\Omega:\ u^\alpha(x)-L>0\}$. Then we take a vector function $\mathbf{v}$ as follows

$$v^\beta=\begin{cases}\max\{u^\alpha-L;0\} & \text{if } \beta=\alpha\\ 0 & \text{if } \beta\neq\alpha\end{cases},\qquad Dv^\beta=\begin{cases}Du^\alpha\chi_{\mathcal{A}_L^\alpha} & \text{if } \beta=\alpha\\ 0 & \text{if } \beta\neq\alpha\end{cases}.$$

It is clear that $\mathbf{v}\in W_0^{1,2}(\Omega;\mathbb{R}^N)$ and hence $\mathbf{v}\in L^\nu(\Omega;\mathbb{R}^N)$ by the Sobolev embedding. Choosing $\phi^\alpha=v^\alpha$ as a test function we obtain

$$\begin{aligned}\int_{\mathcal{A}_L^\alpha}A_i^\alpha(x,\mathbf{u},D\mathbf{u})D_iu^\alpha(x)\,dx+&\int_{\mathcal{A}_L^\alpha}b^\alpha(x,\mathbf{u},D\mathbf{u})(u^\alpha(x)-L)\,dx\\ &=\int_{\mathcal{A}_L^\alpha}f^\alpha(x)(u^\alpha(x)-L)\,dx\,.\end{aligned}$$

We start with the case $n\geq3$ when $\nu=2n/(n-2)$. Define the Radon measure dm supported in Ω by

$$dm:=(\chi_\Omega(x)+\varphi(x)^2+|f^\alpha(x)|)\,dx,$$

where χ_Ω is the characteristic function of Ω. Then by (14) we get the estimate

$$\begin{aligned}\int_{\mathcal{A}_L^\alpha}|Du^\alpha(x)|^2\,dx&\leq\frac{\Lambda}{\gamma}\int_{\mathcal{A}_L^\alpha}\varphi(x)^2\,dx+\frac{1}{\gamma}\int_{\mathcal{A}_L^\alpha}|f^\alpha(x)|(u^\alpha(x)-L)\,dx\\ &\leq\frac{\Lambda}{\gamma}\int_{\mathcal{A}_L^\alpha}(\chi_\Omega(x)+\varphi(x)^2+|f^\alpha(x)|)\,dx\\ &\quad+\frac{1}{\gamma}\int_{\mathcal{A}_L^\alpha}|f^\alpha(x)|(u^\alpha(x)-L)\,dx\\ &\leq C(\Lambda,\gamma)\big(m(\mathcal{A}_L^\alpha)+J\big).\end{aligned}\tag{15}$$

In order to estimate the integral $J=\int_{\mathcal{A}_L^\alpha}|f^\alpha(x)|(u^\alpha(x)-L)\,dx$ we make use of the Lemma 3 applied to the Radon measure $dm'=|f^\alpha(x)|dx$. Hence

$$J=\int_{\mathcal{A}_L^\alpha}(u^\alpha(x)-L)dm'\leq\left(\int_{\mathcal{A}_L^\alpha}|u^\alpha(x)-L|^{s'}\,dm'\right)^{\frac{1}{s'}}m'(\mathcal{A}_L^\alpha)^{1-\frac{1}{s'}}\,.$$

Evaluating the measure m' over a ball $\mathcal{B}_\rho$ we get

$$\begin{aligned}m'(\mathcal{B}_\rho)=\int_{\mathcal{B}_\rho}|f^\alpha(x)|\,dx&\leq C(n)\rho^{n-\frac{n-\mu}{q}}\left(\frac{1}{\rho^\mu}\int_{\mathcal{B}_\rho}|f^\alpha(x)|^q\,dx\right)^{1/q}\\ &\leq C(n)\rho^{n-\frac{n-\mu}{q}}\|f^\alpha\|_{q,\mu;\Omega}=K\rho^{n-\frac{n-\mu}{q}}\end{aligned}$$

with $K=K(n,q,\operatorname{diam}\Omega,\|f^\alpha\|_{q,\mu;\Omega})$. We apply now the Lemma 3 with $r'=2$, $\tau_0'=n-\frac{n-\mu}{q}$ and $s'=\frac{2}{n-2}\big(n-\frac{n-\mu}{q}\big)>2$, calculated via (7). Hence

$$\begin{aligned}J&\leq CK^{\frac{1}{s'}}\left(\int_{\mathcal{A}_L^\alpha}|Du^\alpha(x)|^2\,dx\right)^{\frac12}m'(\mathcal{A}_L^\alpha)^{1-\frac{1}{s'}}\\ &\leq C\left[\varepsilon\int_{\mathcal{A}_L^\alpha}|Du^\alpha(x)|^2\,dx+\frac{1}{\varepsilon}m'(\mathcal{A}_L^\alpha)^{2(1-\frac{1}{s'})}\right].\end{aligned}\tag{16}$$

Combining (15) and (16), taking ε small enough, moving the integral of the gradient on the left-hand side, and keeping in mind that $2(1-\frac{1}{s'})>1$ and $m'(\mathcal{A}_L^\alpha)\le m(\mathcal{A}_L^\alpha)$ we obtain

$$\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2\,dx \le C\big(m(\mathcal{A}_L^\alpha)+m(\mathcal{A}_L^\alpha)^{2(1-\frac{1}{s'})}\big)\le Cm(\mathcal{A}_L^\alpha) \tag{17}$$

where the constant C depends on known quantities.

To complete the estimate (15) we will use once again the Lemma 3. It is immediate that $m(\mathcal{B}_\rho)$ of a ball $\mathcal{B}_\rho \Subset \Omega$ is

$$\begin{aligned} m(\mathcal{B}_\rho) &= \int_{\mathcal{B}_\rho} \big(\chi_\Omega(x)+\varphi(x)^2+|f^\alpha(x)|\big)\,dx \\ &\le C(n)\rho^n + \rho^{n-\frac{2(n-\lambda)}{p}}\|\varphi\|^2_{p,\lambda;\Omega} + \rho^{n-\frac{n-\mu}{q}}\|f^\alpha\|_{q,\mu;\Omega} \le K\rho^{\tau_0} \end{aligned} \tag{18}$$

with $K=K(n,p,q,\operatorname{diam}\Omega,\|\varphi\|_{p,\lambda;\Omega},\|f^\alpha\|_{q,\mu;\Omega})$ and

$$\tau_0=\min\left\{n-\frac{2(n-\lambda)}{p};n-\frac{n-\mu}{q}\right\}>n-2.$$

Applying (8) with $r=2<n$ and calculating s from (7) we get

$$\begin{aligned} \int_{\mathcal{A}_L^\alpha}(u^\alpha(x)-L)\,dm &\le \left(\int_{\mathcal{A}_L^\alpha}|u^\alpha(x)-L|^s\,dm\right)^{\frac{1}{s}} m(\mathcal{A}_L^\alpha)^{1-\frac{1}{s}} \\ &\le CK^{\frac{1}{s}}\left(\int_{\mathcal{A}_L^\alpha}|Du^\alpha(x)|^2\,dx\right)^{\frac12} m(\mathcal{A}_L^\alpha)^{1-\frac{1}{s}} \\ &\le C(n,p,K,\gamma,\Lambda)m(\mathcal{A}_L^\alpha)^{1+\frac12-\frac1s} \end{aligned} \tag{19}$$

with $s=\min\left\{\frac{2np-4(n-\lambda)}{p(n-2)};\frac{2nq-2(n-\mu)}{q(n-2)}\right\}>2.$

A similar bound holds also in the case $n=2$. In fact, for any ball $\mathcal{B}_\rho\subset\mathbb{R}^2$ we have

$$m(\mathcal{B}_\rho)\le C\rho^2+\rho^{2-\frac{2(2-\lambda)}{p}}\|\varphi\|^2_{p,\lambda;\Omega}+\rho^{2-\frac{2-\mu}{q}}\|f^\alpha\|_{q,\mu;\Omega}\le K\rho^{\tau_0}$$

with $\tau_0=\min\left\{2-\frac{2(2-\lambda)}{p};2-\frac{2-\mu}{q}\right\}>0$. Choosing $s=2$ we calculate r from (7)

$$r=\max\left\{\frac{2p}{2p-2+\lambda};\frac{4q}{4q-2+\mu}\right\}\in(1,2).$$

Then by the Hölder and the Adams trace inequalities we obtain

$$\begin{aligned} \int_{\mathcal{A}_L^\alpha}(u^\alpha(x)-L)\,dm &\le \left(\int_{\mathcal{A}_L^\alpha}(u^\alpha(x)-L)^2\,dm\right)^{\frac12} m(\mathcal{A}_L^\alpha)^{\frac12} \\ &\le CK^{\frac12}\left(\int_{\mathcal{A}_L^\alpha}|Du^\alpha(x)|^r\,dx\right)^{\frac1r} m(\mathcal{A}_L^\alpha)^{\frac12} \\ &\le CK^{\frac12}\left(\int_{\mathcal{A}_L^\alpha}|Du^\alpha(x)|^2\,dx\right)^{\frac12}\left(\int_{\mathcal{A}_L^\alpha}\chi_\Omega(x)\,dx\right)^{\frac1r-\frac12} m(\mathcal{A}_L^\alpha)^{\frac12} \\ &= CK^{\frac12}\left(\int_{\mathcal{A}_L^\alpha}|Du^\alpha(x)|^2\,dx\right)^{\frac12} m(\mathcal{A}_L^\alpha)^{\frac1r}\,. \end{aligned} \tag{20}$$

In order to estimate the integral in the last term we go back to (15). Consider again the Radon measure $dm' = |f^\alpha(x)|dx$ and calculate $m'(\mathcal{B}_\rho) \le K\rho^{2-\frac{2-\mu}{q}}$. Then choosing $s' = 2$ we get $r' = \frac{4q}{4q-(2-\mu)} \in (1,2)$ from (7). This way, the Lemma 3 and the Hölder inequality give

$$\begin{aligned} J &\le \left(\int_{\mathcal{A}_L^\alpha} |u^\alpha(x) - L|^2 \, dm' \right)^{\frac{1}{2}} m'(\mathcal{A}_L^\alpha)^{\frac{1}{2}} \\ &\le CK^{\frac{1}{2}} \left(\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^{r'} \, dx \right)^{\frac{1}{r'}} m'(\mathcal{A}_L^\alpha)^{\frac{1}{2}} \\ &\le CK^{\frac{1}{2}} \left(\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 \, dx \right)^{\frac{1}{2}} m(\mathcal{A}_L^\alpha)^{\frac{1}{r'}-\frac{1}{2}} m(\mathcal{A}_L^\alpha)^{\frac{1}{2}} \\ &\le C \left[\varepsilon \int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 \, dx + \frac{1}{\varepsilon} m(\mathcal{A}_L^\alpha)^{\frac{2}{r'}} \right]. \end{aligned} \tag{21}$$

Unifying (15) and (21), taking ε small enough and keeping in mind that $\frac{2}{r'} > 1$, we get

$$\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 \, dx \le Cm(\mathcal{A}_L^\alpha)$$

where the constant depends on the same quantities as in (17). Then the estimate (20) becomes

$$\int_{\mathcal{A}_L^\alpha} (u^\alpha(x) - L) \, dm \le Cm(\mathcal{A}_L^\alpha)^{1+\frac{1}{r}-\frac{1}{2}}. \tag{22}$$

Unifying the estimates (19) and (22) we obtain

$$\int_{\mathcal{A}_L^\alpha} (u^\alpha(x) - L) \, dm \le Cm(\mathcal{A}_L^\alpha)^{1+\sigma_0} \tag{23}$$

where

$$\sigma_0 = \begin{cases} \dfrac{1}{2} - \dfrac{1}{s} = \max\Big\{ \dfrac{p+\lambda-n}{np-2(n-\lambda)}; \dfrac{\mu+2q-n}{2nq-2(n-\mu)} \Big\} & \text{if } n > 2 \\ \dfrac{1}{r} - \dfrac{1}{2} = \min\Big\{ \dfrac{p+\lambda-2}{2p}; \dfrac{2q+\mu-2}{4q} \Big\} & \text{if } n = 2\,. \end{cases}$$

Suppose now that $m(\mathcal{A}_L^\alpha) > 0$, otherwise $\sup_\Omega u^\alpha(x) \le L$. For any $L_1 > L$ we have $\mathcal{A}_{L_1}^\alpha \subset \mathcal{A}_L^\alpha$ and therefore (23) yields

$$\begin{aligned} (L_1 - L)m(\mathcal{A}_{L_1}^\alpha) &\le \int_{\mathcal{A}_{L_1}^\alpha} (u^\alpha(x) - L) \, dm \\ &\le \int_{\mathcal{A}_L^\alpha} (u^\alpha(x) - L) \, dm \le Cm(\mathcal{A}_L^\alpha)^{1+\sigma_0}\,. \end{aligned}$$

Hence

$$m(\mathcal{A}_{L_1}^\alpha) \le \frac{C}{L_1 - L} m(\mathcal{A}_L^\alpha)^{1+\sigma_0}\,.$$

In order to estimate the measure of the set $\mathcal{A}_L^\alpha$ we will apply the following Maximum Principle due to Stampacchia [23, Lemma 4.1].

Lemma 6. *Let $\Theta : [L_0, \infty) \to [0, \infty)$ be a decreasing function. Assume that there exist $c, a \in (0, \infty)$ and $b \in (1, \infty)$ such that*

$$L_1 > L \geq L_0 \implies \Theta(L_1) \leq \frac{c}{(L_1 - L)^a} (\Theta(L))^b.$$

Then

$$\Theta(L_0 + d) = 0 \quad \text{where} \quad d = \left[c\Theta(L_0)^{b-1} 2^{\frac{ab}{b-1}}\right]^{\frac{1}{a}}.$$

The application of the Lemma 6 to the function $\Theta(L) = m(\mathcal{A}_L^\alpha)$ with $a = 1$, $b = 1 + \sigma_0$ and $L_0 = \max\{\theta_\alpha, \sup_{\partial\Omega} u^\alpha\}$ yields

$$m(\mathcal{A}^\alpha_{L_0 + d_\alpha}) = 0 \qquad \text{where} \qquad d_\alpha \leq Cm(\Omega)^{\sigma_0} 2^{1 + \frac{1}{\sigma_0}}. \tag{24}$$

The last assertion means that for each $\alpha = 1, \dots, N$ there exists a constant M_α depending on n, p, λ, q, μ, γ, Λ, $|\Omega|$, $\|\varphi\|_{p,\lambda;\Omega}$ and $\|f^\alpha\|_{q,\mu;\Omega}$ such that

$$\sup_\Omega u^\alpha < \max\{\theta_\alpha; \sup_{\partial\Omega} u^\alpha\} + M_\alpha \tag{25}$$

and this completes the proof of Theorem 5 □

3. The Dirichlet Problem

We study the boundedness and the Morrey regularity of the weak solutions to the following Dirichlet problem

$$\begin{cases} -\mathrm{div}\mathbf{A}(x, \mathbf{u}(x) D\mathbf{u}(x)) + \mathbf{b}(x, \mathbf{u}, D\mathbf{u}) = \mathbf{f}(x) & x \in \Omega \\ \mathbf{u}(x) = 0 & x \in \partial\Omega \end{cases} \tag{26}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$.

Theorem 7 (Essential Boundedness of the Solution)**.** *Let $\mathbf{u} \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ be a solution to* (26) *and assume* (A), (10), *and* (12). *Suppose in addition that*

$$\begin{cases} \gamma |\mathbf{z}^\alpha|^2 - \Lambda \varphi(x)^2 \leq \displaystyle\sum_{i=1}^n A_i^\alpha(x, \mathbf{u}, \mathbf{z}) z_i^\alpha \\ \varphi(x) \in L^{p,\lambda}(\Omega),\ p > 2,\ \lambda \in (0, n),\ \lambda + p > n, \\ 0 \leq b^\alpha(x, \mathbf{u}, \mathbf{z}) \,\mathrm{sign}\, u^\alpha(x) \end{cases} \tag{27}$$

for $|u^\alpha| \geq \theta_\alpha > 0$. Then there exists a constant M depending on known quantities such that

$$\|\mathbf{u}\|_{\infty;\Omega} \leq M.$$

Proof. Take a positive constant L such that $L \geq \theta_\alpha$ and consider the set $\bar{\mathcal{A}}_L^\alpha = \{x \in \Omega : u^\alpha(x) + L < 0\}$. Then the Theorem 5 applied to $-u^\alpha$ gives

$$\inf_\Omega u^\alpha > -\theta_\alpha - M_\alpha. \tag{28}$$

Unifying (25) and (28) we get boundedness of $\|u^\alpha\|_{\infty;\Omega}$ for each $\alpha = 1, \dots, N$. Then

$$\|\mathbf{u}\|_{\infty;\Omega} = \max_{1 \leq \alpha \leq N} \|u^\alpha\|_{\infty;\Omega} =: M < \infty.$$

□

Theorem 8 (Morrey regularity of the gradient)**.** *Let Ω be a bounded* (A)*-type domain in $\mathbb{R}^n, n \geq 3$, and $\mathbf{u} \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to* (26) *under the assumptions* (10)*,* (12)*, and* (27)*. Then $D\mathbf{u} \in L^{2,n-2}(\Omega, \mathbb{R}^N)$ and*

$$\int_{\Omega_\rho(x)} |Du(y)|^2 \, dy \leq C\rho^{n-2} \qquad \forall x \in \Omega, \ \rho \in (0, \operatorname{diam} \Omega] \tag{29}$$

with a constant depending on known quantities.

Proof. Fix $x_0 \in \Omega$ and $\rho > 0$ be such that $\mathcal{B}_\rho(x_0) \subset \mathcal{B}_{2\rho}(x_0) \Subset \Omega$, $\rho > 1$. Define a cut-off function $\zeta(x) \in C^1(\mathbb{R}^n)$

$$\zeta(x) = \begin{cases} 1 & x \in \mathcal{B}_\rho(x_0), \\ 0 & x \notin \mathcal{B}_{2\rho}(x_0), \end{cases} \qquad |D\zeta| \leq \frac{C}{\rho}\,.$$

For any fixed α take $\phi^\alpha(x) = e^{u^\alpha(x)}\zeta(x)^2$ as a test function in (13) to get

$$\begin{aligned} \sum_{i=1}^n \int_\Omega & A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) e^{u^\alpha(x)} D_i u^\alpha(x) \zeta(x)^2 \, dx \\ &= \int_\Omega f^\alpha(x) e^{u^\alpha(x)} 2\zeta(x) D_i\zeta(x) \, dx \\ &\quad - \sum_{i=1}^n \int_\Omega A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) e^{u^\alpha(x)} 2\zeta(x) D_i \zeta(x) \, dx \\ &\quad - \int_\Omega b^\alpha(x, \mathbf{u}, D\mathbf{u}) e^{u^\alpha(x)} \zeta(x)^2 \, dx\,. \end{aligned}$$

The left-hand side can be estimated by (27) while for the right-hand side we use (10) and (12)

$$\begin{aligned} \sum_{i=1}^n & e^{-M} \int_\Omega \big(\gamma |Du^\alpha(x)|^2 - \Lambda\varphi(x)^2\big)\zeta(x)^2 \, dx \\ &\leq 2e^M \int_\Omega |f^\alpha(x)|\zeta(x)|D\zeta(x)| \, dx \\ &\quad + 2n\Lambda e^M \int_\Omega \big(\varphi(x) + |\mathbf{u}|^{\frac{n}{n-2}} + |D\mathbf{u}|\big)\zeta(x)|D\zeta| \, dx \\ &\quad + \Lambda e^M \int_\Omega \big(\psi(x) + |\mathbf{u}|^{\frac{n+2}{n-2}} + |D\mathbf{u}|^{\frac{n+2}{n}}\big)\zeta(x)^2 \, dx. \end{aligned}$$

To proceed further, we use the Young inequality $ab \leq \varepsilon a^p + \frac{b^{p/(p-1)}}{\varepsilon^{1/(p-1)}}$, whence

$$\begin{aligned} \int_\Omega |f^\alpha(x)|\zeta(x)|D\zeta(x)| \, dx &\leq \frac{1}{2}\int_\Omega |f^\alpha(x)|^2\zeta(x)^2 \, dx + \frac{1}{2}\int_\Omega |D\zeta(x)|^2 \, dx \\ \int_\Omega \varphi(x)\zeta(x)|D\zeta(x)| \, dx &\leq \frac{1}{2}\int_\Omega \varphi(x)^2\zeta(x)^2 \, dx + \frac{1}{2}\int_\Omega |D\zeta(x)|^2 \, dx \\ \int_\Omega |\mathbf{u}|^{\frac{n}{n-2}}\zeta(x)|D\zeta(x)| \, dx &\leq \frac{1}{2} M^{\frac{n}{n-2}} \left(\int_\Omega \zeta(x)^2 \, dx + \int_\Omega |D\zeta(x)|^2 \, dx\right) \\ \int_\Omega |D\mathbf{u}|\zeta(x)|D\zeta| \, dx &\leq \varepsilon \int_\Omega |D\mathbf{u}|^2\zeta(x)^2 \, dx + \frac{1}{\varepsilon}\int_\Omega |D\zeta(x)|^2 \, dx \\ \int_\Omega |D\mathbf{u}|^{\frac{n+2}{n}}\zeta(x)^2 \, dx &\leq \varepsilon \int_\Omega |D\mathbf{u}|^2\zeta(x)^2 \, dx + \varepsilon^{-\frac{n+2}{n-2}} \int_\Omega \zeta(x)^2 \, dx\,. \end{aligned}$$

Unifying the above estimates we get

$$\begin{aligned}
&\int_\Omega |Du^\alpha(x)|^2\zeta(x)^2\,dx\\
&\qquad \le C\int_\Omega (1+|f^\alpha(x)|+\psi(x)+\varphi(x)^2)\zeta(x)^2\,dx\\
&\qquad\quad + C\int_\Omega |D\zeta(x)|^2\,dx+\varepsilon C\int_\Omega |D\mathbf{u}(x)|^2\zeta(x)^2\,dx
\end{aligned}\tag{30}$$

with constants depending on n,Λ,γ,M, and ε. Summing up (30) over α from 1 to N, fixing ε small enough and moving the last term to the left-hand side we obtain

$$\begin{aligned}
\int_\Omega |D\mathbf{u}(x)|^2\zeta(x)^2\,dx &\le C\int_\Omega (1+\psi(x)+\varphi(x)^2)\zeta(x)^2\,dx\\
&+ C\sum_{\alpha=1}^N\int_\Omega |f^\alpha(x)|\zeta(x)^2\,dx + C\int_\Omega |D\zeta(x)|^2\,dx\,.
\end{aligned}\tag{31}$$

Then, by the definition of ζ and by (12) we have

$$\begin{aligned}
&\int_{\mathcal{B}_{2\rho}} \big(1+\psi(x)+\varphi(x)^2\big)\,dx \le C\big[\rho^n+\rho^{n-\frac{n-\mu}{q}}\|\psi\|_{q,\mu;\Omega}\\
&\qquad\qquad + \rho^{n-\frac{2(n-\lambda)}{p}}\|\varphi\|^2_{p,\lambda;\Omega}\big]\\
&\sum_{\alpha=1}^N\int_{\mathcal{B}_{2\rho}} |f^\alpha(x)|\,dx \le C\rho^{n-\frac{n-\mu}{q}}\|\mathbf{f}\|_{q,\mu;\Omega}\\
&\int_{\mathcal{B}_{2\rho}} |D\zeta(x)|^2\,dx \le C\rho^{n-2}\,.
\end{aligned}$$

Hence

$$\int_{\mathcal{B}_\rho} |D\mathbf{u}|^2\,dx \le C\rho^{\lambda_0}\tag{32}$$

with $\lambda_0=\min\{n-2,n-\frac{2(n-\lambda)}{p},n-\frac{n-\mu}{q}\}=n-2$ and the constant depends on known quantities.

Let $\mathcal{B}_\rho(x_0)\cap\partial\Omega\neq\emptyset$. Then we extend u^α and the given functions f^α,φ, and ψ as zero in Ω^c and consider the test functions

$$\phi^\alpha(x)=(e^{|u^\alpha(x)|}-1)\zeta(x)^2\operatorname{sign}u^\alpha(x)$$

where $\zeta(x)$ is the cut-off function defined above. Thus (13) gives

$$\begin{aligned}
&\sum_{i=1}^n\int_\Omega A_i^\alpha(x,\mathbf{u},D\mathbf{u})e^{|u^\alpha(x)|}D_iu^\alpha(x)\zeta(x)^2\,dx\\
&\quad=\int_\Omega f^\alpha(x)(e^{|u^\alpha(x)|}-1)\zeta(x)^2\operatorname{sign}u^\alpha(x)\,dx\\
&\quad-\sum_{i=1}^n\int_\Omega A_i^\alpha(x,\mathbf{u},D\mathbf{u})(e^{|u^\alpha(x)|}-1)2\zeta(x)D_i\zeta(x)\operatorname{sign}u^\alpha(x)\,dx\\
&\quad-\int_\Omega b^\alpha(x,\mathbf{u},D\mathbf{u})(e^{|u^\alpha(x)|}-1)\zeta(x)^2\operatorname{sign}u^\alpha(x)\,dx\,.
\end{aligned}$$

Hence the conditions (10) and (27) give

$$\gamma \int_\Omega |Du^\alpha(x)|^2 \zeta(x)^2\, dx \le \Lambda e^M \int_\Omega \varphi(x)^2\zeta(x)^2\, dx$$
$$+ e^M \int_\Omega |f^\alpha(x)|\zeta(x)^2\, dx$$
$$+ 2n\Lambda e^M \int_\Omega (\varphi(x) + |\mathbf{u}|^{\frac{n}{n-2}} + |D\mathbf{u}|)\zeta(x)|D\zeta(x)|\, dx$$

and to get the desired estimate (29) we argue as above. □

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