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Some elliptic equations with $W_0^{1,1}$ solutions

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$W_0^{1,1}$ solutions = no Nicola, no BV (*see* [1])

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ABSTRACT

We consider some nonlinear Dirichlet problems and we study how lower order terms can give a regularizing effect on the solutions: the existence of distributional solutions with minimal properties (solutions in $W_0^{1,1}$, functional space not so usual for finding solutions of elliptic problems) or finite energy solutions, even with nonregular data.

1. Introduction

In this paper we study the regularizing effect of some lower order terms in boundary value problems of the type

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + H(x,u,\nabla u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 2$, $f(x)$ is a function with poor summability and $1 < p \leq N$ (if $p > N$, the problem has easily a solution).

Regularizing effect means that the presence of the lower order term improves the summability of the solution u and its gradient ∇u with respect to the summability of w and ∇w, where w is the solution of the boundary value problem (1.1) with $H = 0$: about this circumstance, we only recall the difference between linear and semilinear problems in two "model examples", in the particular case $p = 2$.

(A) If $H(x,s,\xi) = s|s|^{r-2}$, in [18], it is proved the existence of a weak solution u of the Dirichlet problem

$$u \in W_0^{1,2}(\Omega): \quad -\Delta u + u|u|^{r-2} = f(x)$$

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even if $f(x)$ does not belong to $L^{\frac{2N}{N+2}}(\Omega)$, but $f \in L^m(\Omega)$, with $r' \leq m < \frac{2N}{N+2}$.

(B) If $H(x,s,\xi) = s|\xi|^2$, in [11], it is proved the existence of a weak solution u of the Dirichlet problem

$$u \in W_0^{1,2}(\Omega): \quad -\Delta u + u|\nabla u|^2 = f(x)$$

even if $f(x)$ only belongs to $L^1(\Omega)$.

The main aim of this paper is the study of problems of the type (1.1) under minimal assumptions on the data, so that the existence of a distributional solutions u is proved, whereas the boundary value problem (1.1) with $H = 0$ has no solutions in $W_0^{1,1}(\Omega)$.

A second feature of this paper is the existence of solutions in $W_0^{1,1}(\Omega)$ and not in $BV(\Omega)$, even if our proof hinges on $W_0^{1,1}(\Omega)$ a priori estimates of the solutions of approximate problems.

Other elliptic problems with $W_0^{1,1}(\Omega)$ solutions are studied in [6,7,5,12].

2. Setting

In this paper we will consider differential operators with a more general (with respect to (1.1)) principal part $\mathcal{A}$, acting on $W_0^{1,p}(\Omega)$, defined by

$$\mathcal{A}(v) = -\mathrm{div}\,(a(x,v,\nabla v)), \tag{2.1}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies classical hypotheses (see [19]); namely, a is a Carathéodory function such that the following holds for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, for every $\xi \neq \eta \in \mathbb{R}^N$:

$$\begin{cases} a(x,s,\xi)\xi \geq \alpha\,|\xi|^p, \\ |a(x,s,\xi)| \leq \beta|\xi|^{p-1}, \\ [a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0, \end{cases} \tag{2.2}$$

where α, β are positive constants. Thus $\mathcal{A}$ is a pseudo-monotone (see [14]) and coercive differential operator.

Concerning the function H, we will consider polynomial or gradient depending lower order terms.

By a polynomial lower order term we mean a Carathéodory function $g(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ having the same sign of s and bounded from below by a power of $|s|$, namely

$$\begin{cases} g(x,s)\ \mathrm{sign}(s) \geq A\,|s|^{r-1}, \\ \text{for all } t > 0: \quad \sup\{|g(x,s)|,\ |s| \leq t\} = h_t(x) \in L^1(\Omega), \end{cases} \tag{2.3}$$

where $A > 0$ and $r > 1$. A model example of $g(x,s)$ is the function

$$g(x,s) = b(x)s|s|^{r-2},$$

with $b \in L^1(\Omega)$ and $b(x) \geq A > 0$ for a.e. $x \in \Omega$.

While, by a gradient depending lower order term we mean a function of the kind $g(x,u)|\nabla u|$, with $g(x,s) = As|s|^{r-2}$.

Thus, we will consider the Dirichlet problems

$$\begin{cases} -\mathrm{div}\,(a(x,u,\nabla u)) + g(x,u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{2.4}$$

and

$$\begin{cases} -\mathrm{div}\,(a(x,u,\nabla u)) + Au|u|^{r-2}|\nabla u| = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{2.5}$$

and we will show how the presence of lower order terms allows to obtain finite energy solutions, even if the source term f does not belong to $W^{-1,p'}(\Omega)$ (see Examples 3.4 and 4.2), or $W_0^{1,1}$-distributional solutions, if f satisfies suitable minimal assumptions.

We recall that if $g = 0$ and $f \in L^m(\Omega), m < (p^*)'$ (then f does not belong to the dual of $W_0^{1,p}(\Omega)$ and u does not belong to $W_0^{1,p}(\Omega)$), it has been proved that (Calderon–Zygmund theory for infinite energy solutions, see [9,10,12])

$$\begin{cases} \dfrac{N}{(p-1)N+1} < m < \dfrac{pN}{(p-1)N+p} \Rightarrow u \in W_0^{1,(p-1)m^\star}(\Omega), \\ m = \dfrac{N}{(p-1)N+1} \quad \text{and} \quad 1 < p < 2 - \dfrac{1}{N} \Rightarrow u \in W_0^{1,1}(\Omega). \end{cases} \tag{2.6}$$

Note that the assumption $m > \frac{N}{(p-1)N+1}$ implies $(p-1)m^\star > 1$, while $1 < p < 2 - \frac{1}{N}$ implies $\frac{N}{(p-1)N+1} > 1$.

If we consider boundary value problems with a lower order term $g(x,u)$ having the same sign of u, the existence results stated above still hold, because of the "coercivity" properties of g. Furthermore, in [13,18], it is shown that the presence of a lower order term satisfying assumption (2.3) improves, in some cases, the results concerning the summability of ∇u in (2.6).

In particular, in [18], it is proved that

$$\begin{cases} \text{if } r' \le m < (p^*)',\ r > p^*, \quad \text{there exists a weak solution } u \in W_0^{1,p}(\Omega); \\ \text{if } \dfrac{r'}{p} < m < r', \quad \text{there exists a distributional solution } u \in W_0^{1,\frac{mp}{r'}}(\Omega). \end{cases} \tag{2.7}$$

If $1 \le m < \frac{r'}{p}$, the meaning of distributional solution is lost and the definition of "entropy solution" is useful (see [2]). In this paper we complete this study (see Theorem 3.1) with the borderline case $m = \frac{r'}{p}$, which gives $W_0^{1,1}(\Omega)$ distributional solutions.

Then, we study the regularizing effect due to the gradient depending lower order term in the Dirichlet problem (2.5) and we will prove both the existence of $W_0^{1,1}(\Omega)$-solutions (see Theorem 4.1), if $f \in L^1(\Omega)$, and the existence of $W_0^{1,p}$-solutions, if $f \in L^m(\Omega)$, with $m < (p^*)'$ suitably chosen.

Concerning the assumption on p in the sequel we always assume that

$$1 < p \le 2 - \frac{1}{N}. \tag{2.8}$$

since otherwise both problems (2.4) and (2.5) have solutions in $W_0^{1,q}(\Omega)$, with $q > 1$, already in the case $f \in L^1(\Omega)$ (see [9,10,13,18] and Section 4).

3. Polynomial lower order term

Theorem 3.1. *We assume (2.2), (2.3), (2.8) and*

$$f \in L^{\frac{r'}{p}}(\Omega), \quad 1 < p \le r'. \tag{3.1}$$

Then, there exists $u \in W_0^{1,1}(\Omega)$ such that $g(x,u) \in L^1(\Omega)$, which is a distributional solution of (2.4).

Remark 3.2. Note that the summability assumption (3.1) is weaker than the summability assumption in (2.6), at least in the case $r > \frac{pN}{N-1}$, which implies $\frac{r'}{p} < \frac{N}{(p-1)N+1}$.

Remark 3.3. In [4] is given a partial result ($r = 2$).

Proof of Theorem 3.1. Let us consider the sequence of approximate Dirichlet problems

$$u_n \in W_0^{1,p}(\Omega) : \; A(u_n) + g(x, u_n) = \frac{f(x)}{1 + \frac{1}{n}|f|}. \tag{3.2}$$

The existence of a weak solution u_n is proved in [16]. Moreover every function u_n belongs to $L^\infty(\Omega)$ (see [20]).

Step 1 - For $s \in \mathbb{R}$ and $k \in \mathbb{R}^+$ we denote by $T_k(s)$ the usual truncature operator defined by

$$T_k(s) = \max(-k, \min(k, s))$$

Due to the "positivity" properties of the function g, the estimates below (where the $\|f\|_{L^1(\Omega)}$ is only used), proved in [9,2] and in [12], still hold

$$\int_\Omega |\nabla T_k(u_n)|^p \le \frac{\|f\|_{L^1(\Omega)}}{\alpha} k, \tag{3.3}$$

$$\alpha(p-1)\mathcal{S}\left[\int_\Omega \{\log(1 + |u_n|)\}^{p^*}\right]^{\frac{p}{p^*}} \le \alpha(p-1)\int_\Omega |\nabla \log(1 + |u_n|)|^p \le \int_\Omega |f|, \tag{3.4}$$

where $\mathcal{S}$ is the Sobolev constant. We point out that, as a consequence of the estimate (3.4) we have

$$\text{meas}\{k \le |u_n|\} \le \left(\frac{\|f\|_{L^1(\Omega)}}{\alpha(p-1)\mathcal{S}\,\log(1+k)^p}\right)^{\frac{p^*}{p}}; \tag{3.5}$$

since the sequence $\{\log(1 + |u_n|)\}$ is bounded in $W_0^{1,p}(\Omega)$, up to a subsequence still denoted by $\{u_n\}$, there exists a measurable function $u(x)$, such that

$$\text{the sequence } \{u_n(x)\} \quad \text{converges a.e. to } u(x). \tag{3.6}$$

Moreover, by (3.3) we have

$$\{T_k(u_n)\} \quad \text{converges a.e. and weakly in } W_0^{1,p}(\Omega) \text{ to } T_k(u), \tag{3.7}$$

Step 2 - We state the following estimate, which is a slight variation of a classic result due to H. Brezis and W. A. Strauss (see [17] and also [15,9,18]).

If $f \in L^m(\Omega)$ with $m \ge 1$, then we have, for $k \ge 0$

$$\int_{\{k \le |u_n|\}} |g(x, u_n)||h(u_n)|^{m-1} \le \frac{1}{A^{\frac{m}{m'}}} \int_{\{k \le |u_n|\}} |f(x)|^m, \tag{3.8}$$

where $h(t) = t|t|^{r-2}$. Let $1 < m < \infty$. Define

$$\psi_{k,\delta}(t) = \begin{cases} 0, & \text{if } 0 \le t \le k, \\ \frac{1}{\delta}(t-k), & \text{if } k < t < k+\delta, \\ 1, & \text{if } t \ge k+\delta, \\ \psi_{k,\delta}(-t) = -\psi_{k,\delta}(t). & \end{cases}$$

Since $u_n \in L^\infty(\Omega)$ the function $v = |h(u_n)|^{m-1}\,\psi_{k,\delta}(u_n)$ belongs to $W_0^{1,p}(\Omega)$ and can be used as a test function in the weak formulation of (3.2). Then we apply the Hölder inequality and we have, dropping two positive terms,

$$\begin{cases} \displaystyle\int_{k+\delta\le|u_n|} |g(x,u_n)||h(u_n)|^{m-1} \le \int_{\{k\le|u_n|\}} |f(x)||h(u_n)|^{m-1} \\ \displaystyle\quad \le \left[\int_{\{k\le|u_n|\}} |f(x)|^m\right]^{\frac{1}{m}} \left[\int_{\{k\le|u_n|\}} |h(u_n)|^m\right]^{\frac{1}{m'}} \\ \displaystyle\quad \le \left[\int_{\{k\le|u_n|\}} |f(x)|^m\right]^{\frac{1}{m}} \left[\frac{1}{A}\int_{\{k\le|u_n|\}} |g(x,u_n)||h(u_n)|^{m-1}\right]^{\frac{1}{m'}}, \end{cases}$$

Thanks to the Fatou lemma, letting $\delta \to 0$ we obtain (3.8).

If $m = 1$, we use as a test function $\psi_{k,\delta}(u_n)$ in (3.2). Dropping a positive term we have

$$\int_{k+\delta\le|u_n|} |g(x,u_n)| \le \int_{\{k\le|u_n|\}} |f(x)|.$$

The Fatou Lemma (as $\delta \to 0$) easily implies

$$\int_{\{k\le|u_n|\}} |g(x,u_n)| \le \int_{\{k\le|u_n|\}} |f(x)| \tag{3.9}$$

which gives (3.8) with $m = 1$.

As a consequence of the estimate (3.8) and the assumption (2.3) we obtain

$$\int_{\{k\le|u_n|\}} |u_n|^{m(r-1)} \le \left[\frac{1}{A}\right]^{\frac{m}{m'}} \int_{\{k\le|u_n|\}} |f|^m \tag{3.10}$$

which in turn implies

$$\int_\Omega |u_n|^{m(r-1)} \le \left[\frac{\|f\|_m}{A^{\frac{1}{m'}}}\right]^m. \tag{3.11}$$

Step 3 - Now we recall that, if $f \in L^m(\Omega), m \ge 1$, then

$$\{g(x,u_n)\} \quad \text{converges in } L^1(\Omega) \text{ to } g(x,u). \tag{3.12}$$

As a matter of the fact by (3.6) and the assumptions on g the sequence $\{g(x,u_n)\}$ converges a.e. Let us prove its equiintegrability.

Let $\varepsilon > 0$. Taking into account the absolute continuity of the Lebesgue integral, there exists $\delta(\varepsilon) > 0$ such that, for every measurable subset E, with $|E| \le \delta(\varepsilon)$ (where $|E|$ denotes the Lebesgue measure of E), we have

$$\int_E |f(x)| \le \varepsilon.$$

Now, due to (3.5),there exists k_0 such that, for $k \ge k_0$

$$\big|\{k \le |u_n|\}\big| \le \delta, \quad \text{uniformly with respect to } n, \tag{3.13}$$

which implies, for $k \ge k_0$,

$$\int_{\{k\le|u_n|\}} |f(x)| \le \varepsilon, \quad \text{uniformly with respect to } n.$$

Let E be a measurable subset of Ω. Then, thanks to hypothesis (2.3) and estimate (3.9), for any $k > k_0$, we have

$$\int_E |g(x,u_n)| \le \int_E h_k(x) + \int_{\{k\le|u_n|\}} |f(x)| \le \int_E h_k(x) + \varepsilon$$

uniformly with respect to n. At last, the equi-integrability of $\{g(x,u_n)\}$ is a consequence of the assumption $h_k \in L^1(\Omega)$ for any $k>0$ and the convergence (3.12) easily follows by applying the Vitali Theorem.

Step 4 - Let us prove the a.e. convergence of $\{\nabla u_n\}$. First of all, we will prove that $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$. Recall that $m=\frac{r'}{p}$ and $m(r-1)=\frac{r}{p}$, so that (3.10) can be rewritten as

$$\int_{\{k\le |u_n|\}} |u_n|^{\frac{r}{p}} \le \left[\frac{1}{A}\right]^{\frac{m}{m'}} \int_{\{k\le |u_n|\}} |f|^{\frac{r'}{p}}. \tag{3.14}$$

Note that $(r-1)(m-1)<1$, since $p>1$. We use as a test function in (3.2) $[(\delta+|u_n|)^{(r-1)(m-1)}-\delta^{(r-1)(m-1)}]\operatorname{sign}(u_n)$, we drop some positive terms and we have,

$$\alpha(r-1)(m-1)\int_\Omega \frac{|\nabla u_n|^p}{(\delta+|u_n|)^{1-(r-1)(m-1)}} \le \|f\|_m \left[\int_\Omega (\delta+|u_n|)^{(r-1)m}\right]^{\frac{1}{m'}}.$$

Here the joint use of the Fatou Lemma on the left hand side and the Lebesgue Theorem in right hand side, as $\delta\to 0$, gives, thanks to (3.14)

$$\alpha(r-1)(m-1)\int_\Omega \frac{|\nabla u_n|^p}{|u_n|^{1-(r-1)(m-1)}} \le \|f\|_m \left[\int_\Omega |u_n|^{(r-1)m}\right]^{\frac{1}{m'}} \le C_f.$$

Now we use the Hölder inequality with exponents p and $\frac{p}{p-1}$ in

$$\int_{\{k\le |u_n|\}} |\nabla u_n| = \int_{\{k\le |u_n|\}} \frac{|\nabla u_n|}{|u_n|^{[1-(r-1)(m-1)]\frac{1}{p}}}\, |u_n|^{[1-(r-1)(m-1)]\frac{1}{p}}$$

and thanks to the last inequality we get

$$\begin{cases} \displaystyle\int_{\{k\le |u_n|\}} |\nabla u_n| \le \tilde{C}_f \left[\int_{\{k\le |u_n|\}} |u_n|^{[1-(r-1)(m-1)]\frac{1}{p-1}}\right]^{\frac{1}{p'}} \\ \displaystyle\qquad = \tilde{C}_f \left[\int_{\{k\le |u_n|\}} |u_n|^{\frac{r}{p}}\right]^{\frac{1}{p'}} \le C_{f,A}\left[\int_{\{k\le |u_n|\}} |f|^{\frac{r'}{p}}\right]^{\frac{1}{p'}}. \end{cases} \tag{3.15}$$

We notice that the estimate (3.15), with $k=0$, implies that the sequence $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$; moreover, the convergence (3.12) says that $\{u_n\}$ is the sequence of solutions of the homogeneous Dirichlet problem

$$A(u_n) = y_n, \tag{3.16}$$

with $\{y_n\}$ compact in $L^1(\Omega)$. Thus, we can use Lemma A.1 and we deduce that (up to a subsequence)

$$\nabla u_n(x) \quad \text{converges a.e. to } \nabla u(x). \tag{3.17}$$

Thanks to (3.15) and to (3.5) for every measurable subset $E\subset\Omega$ we have

$$\begin{cases} \displaystyle\int_E |\nabla u_n| \le \int_E |\nabla T_k(u_n)| + \int_{\{k\le |u_n|\}} |\nabla u_n| \\ \displaystyle\quad \le \left[\int_\Omega |\nabla T_k(u_n)|^p\right]^{\frac{1}{p}} \operatorname{meas}(E)^{1-\frac{1}{p}} + (C_f)^{\frac{1}{p}} \left[\int_{\{k\le |u_n|\}} |u_n|^{\frac{r}{p}}\right]^{\frac{1}{p'}} \\ \displaystyle\quad \le \left[\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right]^{\frac{1}{p}} \operatorname{meas}(E)^{1-\frac{1}{p}} + C_{f,A}\left[\int_{\{k\le |u_n|\}} |f|^{\frac{r'}{p}}\right]^{\frac{1}{p'}}. \end{cases}$$

Thus, the sequence $\{\nabla u_n\}$ is equi-integrable. By Vitali's Theorem we obtain

$$\nabla u_n \quad \text{converges to } \nabla u \text{ in } (L^1(\Omega))^N. \tag{3.18}$$

Moreover, (3.17) implies that $a(x, u_n(x), \nabla u_n(x))$ converges a.e. and, since (note that $p-1<1$)

$$\int_E |a(x, u_n(x), \nabla u_n(x))| \le \beta \int_E |\nabla u_n(x)|^{p-1} \le C_1 |E|^{\frac{2-p}{p-1}},$$

the Vitali Theorem yields

$$a(x, u_n, \nabla u_n) \quad \text{converges to } a(x, u, \nabla u) \text{ in } (L^1(\Omega))^N. \tag{3.19}$$

Finally, it is possible to pass to the limit in the weak formulation of (3.2) and we prove (recall (3.12), (3.18), (3.19)) that u is a distributional solution of (2.4):

$$\begin{cases} u \in W_0^{1,1}(\Omega),\ g(x,u) \in L^1(\Omega): \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla\varphi + \int_\Omega g(x,u)\varphi = \int_\Omega f(x)\varphi, \quad \square \\ \forall\, \varphi \in C_c^1(\Omega). \end{cases}$$

The following example provides a finite energy solution of a boundary value problem with polynomial order term and non-smooth datum.

Example 3.4. Here, we consider the boundary value problem (2.4) in radial coordinates with $\Omega = B(0,1), p = 2, g(x,s) = s|s|^{r-2}$ and $A = -\Delta$.

Let $r > 2^*$, so that the interval $(\frac{2}{r-2}, \frac{N-2}{2})$ is not empty. We choose $\gamma \in (\frac{2}{r-2}, \frac{N-2}{2})$. The function $w(|x|) = |x|^{-\gamma} - 1$ is positive and belongs to $W_0^{1,2}(\Omega)$, since $\gamma < \frac{N-2}{2}$. Moreover w is a solution of (2.4) with

$$f(|x|) = (N-2-\gamma)\gamma\, |x|^{-(\gamma+2)} + (|x|^{-\gamma} - 1)^{r-1}.$$

If we write

$$f(|x|) = \gamma(N-2-\gamma)\, |x|^{-(\gamma+2)} + |x|^{-\gamma(r-1)} - [(|x|^{-\gamma})^{r-1} + (|x|^{-\gamma} - 1)^{r-1}],$$

we note that the second term is the most singular (since $r > 2 + \frac{2}{\gamma}$) and it belongs to the Marcinkiewicz space $M^{\frac{N}{\gamma(r-1)}}(\Omega)$. Then $f \notin L^{\frac{2N}{N+2}}(\Omega)$ if $\frac{N}{\gamma(r-1)} < \frac{2N}{N+2}$, which is true since we have $\frac{N+2}{2(r-1)} < \frac{N-2}{2}$ (thanks to the assumption $r > 2^*$). Thus, $w \in W_0^{1,2}(\Omega)$, even if $f \notin L^{\frac{2N}{N+2}}(\Omega)$.

Moreover we note that, in this example, the assumption $r' < m$ means $\frac{N}{\gamma(r-1)} > r'$, which is equivalent to $\gamma < \frac{N}{r}$; with our assumption we have $\gamma < \frac{N-2}{2} < \frac{N}{r}$.

4. Gradient depending lower order terms

Theorem 4.1. *We assume (2.2) and (2.8). Let $f \in L^1(\Omega)$ and*

$$1 < r \le \frac{N-1}{N(p-1)}. \tag{4.1}$$

Then, there exists $u \in W_0^{1,1}(\Omega) \cap L^{r1^*}(\Omega)$ such that

$$u|u|^{r-2}|\nabla u| \in L^1(\Omega),$$

which is a distributional solution of the boundary value problem (2.5).

Proof of Theorem 4.1. We consider the Dirichlet problems

$$u_n \in W_0^{1,p}(\Omega): \quad -\mathrm{div}\,(a(x, u_n, \nabla u_n)) + A\, u_n |u_n|^{r-2} |\nabla u_n| = \frac{f(x)}{1 + \frac{1}{n}|f(x)|}. \tag{4.2}$$

The existence of a weak solution u_n is proved in [16]. Moreover every function u_n belongs to $L^\infty(\Omega)$ (see [20]). We follow the approach already used in [11]: we take $T_k(u_n)$ as a test function, then we have

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p + A\, k^r \int_{\{k\le |u_n|\}} |\nabla u_n| \le k \|f\|_{L^1(\Omega)}, \tag{4.3}$$

which implies

$$\begin{cases} \displaystyle\int_{\{|u_n|\le k\}} |\nabla u_n| \le |\Omega|^{1-\frac{1}{p}} \left(\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}}, \\ \displaystyle\int_{\{k\le |u_n|\}} |\nabla u_n| \le \frac{\|f\|_{L^1(\Omega)}}{A\, k^{r-1}}, \end{cases}$$

and (with the simple choice $k = 1$)

$$\int_\Omega |\nabla u_n| \le |\Omega|^{1-\frac{1}{p}} \left(\frac{\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}} + \frac{\|f\|_{L^1(\Omega)}}{A}. \tag{4.4}$$

Then the sequence $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$. Thus, up to a subsequence still denoted by $\{u_n\}$, the sequence $\{u_n\}$ converges to some function u strongly in $L^\rho(\Omega)$, $\rho < \frac{N}{N-1}$, and almost everywhere in Ω. Furthermore

$$\text{meas}\{k < |u_n|\} \le \frac{C_1}{k^{1^*}}. \tag{4.5}$$

Let $\psi_{k,\delta}(s)$ be the function already used in the previous section; choosing $\psi_{k,\delta}(u_n)$ as test function in (4.2) and dropping a positive term we have

$$A \int_{\{k+\delta<|u_n|\}} |\psi_{k,\delta}(u_n)||u_n|^{r-1}|\nabla u_n| = \int_{\{k\le|u_n|\}} |f(x)|.$$

Letting $\delta \to 0$ we deduce

$$A \int_{\{k\le|u_n|\}} |u_n|^{r-1}|\nabla u_n| \le \int_{\{k\le|u_n|\}} |f(x)|, \tag{4.6}$$

which also gives

$$A \int_\Omega |u_n|^{r-1}|\nabla u_n| \le \int_\Omega |f(x)|. \tag{4.7}$$

We notice that, the inequality (4.7) and the Sobolev inequality get

$$\frac{A}{r} \mathcal{S} \left[\int_\Omega |u_n|^{1^* r}\right]^{\frac{1}{1^*}} \le \int_\Omega |f(x)| \tag{4.8}$$

which in turn gives $u \in L^{1^* r}(\Omega)$. Moreover, since the lower order term is bounded in $L^1(\Omega)$,

$$-\operatorname{div}(a(x, u_n, \nabla u_n)) \text{ is bounded in } L^1(\Omega) \tag{4.9}$$

as in the proof of Theorem 3.1 we can again use Lemma A.1 to deduce that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega.$$

In order to pass to the limit in the approximate problems (4.2), first of all we have to prove the equi-integrability of the lower order term.

Let E be a measurable subset of Ω, by the Hölder inequality, (4.3) and (4.6) we have

$$\begin{aligned}\int_E |u_n|^{r-1}|\nabla u_n| &\le k^{r-1}\int_E |\nabla T_k(u_n)| + \int_{k<|u_n|} |u_n|^{r-1}|\nabla u_n| \\ &\le k^{r-1}\Big(\|f\|_{L^1(\Omega)}\frac{k}{\alpha}\Big)^{\frac{1}{p}}|E|^{\frac{1}{p'}} + \int_{\{k\le|u_n|\}}|f(x)|,\end{aligned}$$

and working as in the proof of Theorem 3.1 we obtain

$$\lim_{|E|\to 0}\int_E |u_n|^{r-1}|\nabla u_n| \le \varepsilon, \quad \text{uniformly with respect to } n.$$

Thanks to the Vitali theorem we conclude that

$$u_n|u_n|^{r-2}|\nabla u_n| \quad \text{converges in } L^1(\Omega) \text{ to } u|u|^{r-2}|\nabla u|. \tag{4.10}$$

Then, for every measurable subset $E\subset\Omega$, we have

$$\left[\begin{aligned}\int_E|\nabla u_n| &\le \int_E|\nabla T_k(u_n)| + \int_{\{k\le|u_n|\}}|\nabla u_n| \\ &\le \left[\int_\Omega|\nabla T_k(u_n)|^p\right]^{\frac{1}{p}}\operatorname{meas}(E)^{1-\frac{1}{p}} + \frac{\|f\|_{L^1(\Omega)}}{Ak^{r-1}} \\ &\le \left[\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right]^{\frac{1}{p}}\operatorname{meas}(E)^{1-\frac{1}{p}} + \frac{\|f\|_{L^1(\Omega)}}{Ak^{r-1}}.\end{aligned}\right.$$

Thus the sequence $\{\nabla u_n\}$ is equi-integrable and we get

$$\nabla u_n \quad \text{converges to } \nabla u \text{ in } (L^1(\Omega))^N. \tag{4.11}$$

Since $p-1<1$ as in the proof of (3.19) we deduce that

$$a(x,u_n,\nabla u_n) \quad \text{converges to } a(x,u,\nabla u) \text{ in } (L^1(\Omega))^N.$$

Then it is possible to pass to the limit in the weak formulation of (4.2) and we prove (recall (4.11), (4.10)) that u is a distributional solution of (2.5), that is

$$\begin{cases} u\in W_0^{1,1}(\Omega),\ u|u|^{r-2}|\nabla u|\in L^1(\Omega): \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla\varphi + \int_\Omega u|u|^{r-2}|\nabla u|\varphi = \int_\Omega f(x)\varphi, \quad \square \\ \forall\,\varphi\in C_c^1(\Omega).\end{cases}$$

Next, as in the previous section, we provide a Dirichlet problem with gradient depending lower order term, non smooth data and finite energy solutions.

Example 4.2. Let $\Omega = B(0,1), r > \frac{2^*}{2}$ and $\gamma = \frac{N}{2r}$. Then the function

$$u(x) = |x|^{-\gamma} - 1$$

is a positive weak solution of the problem (see also Example 3.4)

$$u\in W_0^{1,2}(\Omega):\ -\Delta u + u^{r-1}|\nabla u| = f(x),$$

with

$$f(x) = \gamma\left[(N-\gamma-2)\,|x|^{-(\gamma+2)} + |x|^{-(\gamma+1)}\left(|x|^{-\gamma}-1\right)^{r-1}\right]$$

and $f\notin L^{(2^*)'}(\Omega)$.

In the spirit of the previous example, we will show how to gain the regularity of ∇u by increasing the summability of f and, in some cases, we will obtain solutions with the best summability, that is $\nabla u \in L^p(\Omega)$.

As a matter of the fact, if $f \in L^m(\Omega)$, with $1 < m < (p^\star)'$ the estimate (4.8) on u_n can be improved and this allows us to improve the estimate of ∇u_n. Let $\gamma > 0$ and let us take as test function in the weak formulation of problem (4.2) $|u_n|^\gamma \mathrm{sign}(u_n)$. We obtain, dropping the first term,

$$C_1 \left[\int_\Omega |u_n|^{1^*(\gamma+r)} \right]^{\frac{1}{1^*}} \leq \|f\|_{L^m(\Omega)} \left[\int_\Omega |u_n|^{m'\gamma} \right]^{\frac{1}{m'}}$$

With the choice $\gamma = \frac{rm^*}{m'}$, we obtain $1^*(\gamma + r) = m'\gamma = rm^*$; that is

$$C_1 \left[\int_\Omega |u_n|^{rm^*} \right]^{\frac{1}{m^*}} \leq \|f\|_{L^m(\Omega)}. \tag{4.12}$$

Testing the weak formulation of problem (4.2) with u_n and dropping a positive term we get

$$\alpha \int_\Omega |\nabla u_n|^p \leq \|f\|_{L^m(\Omega)} \left[\int_\Omega |u_n|^{m'} \right]^{\frac{1}{m'}}.$$

Here we note that $m' \leq m^* r$, if we assume $\frac{(r+1)N}{rN+1} \leq m$; so that the assumption of this inequality implies (recall the estimate (4.12)) that the right hand side is bounded, which, in turn, will imply the existence of finite energy solutions as stated by the following theorem, since we can pass to the limit in the principal part and in the lower order term using the same arguments as in the proof of Theorem 4.1.

Theorem 4.3. *Let $1 < p < N$ and $r > 1$. Assume that (2.2), (2.3) hold and $f \in L^m(\Omega)$. If*

$$\frac{(r+1)N}{rN+1} \leq m < (p^\star)', \quad r > \frac{p^*}{p'},$$

then, there exists a solution $u \in W_0^{1,p}(\Omega)$ of problem (2.5).

If $1 \leq m \leq \frac{(r+1)N}{rN+1}$ working as in [13,18] it is possible to prove the existence of distributional solutions of problem (2.5) belonging to a Sobolev space $W_0^{1,q}(\Omega)$, with $q = q(r,m) > 1$.

Remark 4.4. We point out that gradient depending lower order terms are more regularizing than the polynomial ones. As a matter of the fact, the boundary value problem (2.5) has $W_0^{1,1}$-solutions even if $f \in L^1(\Omega)$ and $W_0^{1,p}$-solutions with less summable data (note that $\frac{N(r+1)}{Nr+1} < r'$).

Appendix

In order to have a selfcontained paper, we give a sketch of the proof of the following lemma, which is a slight variation of the main lemma of [3] (see also [8,12]); sketch means that we only write the parts which are different.

Lemma A.1. *Let $\{u_n\}$ be a sequence of solutions of the Dirichlet problems*

$$u_n \in W_0^{1,p}(\Omega): \quad -\operatorname{div}(a(x, u_n, \nabla u_n)) = y_n(x), \tag{A.1}$$

with $p > 1, \|y_n\|_{L^1(\Omega)} \leq L$. Assume (2.2) and that

$$\|u_n\|_{W_0^{1,1}(\Omega)} \leq M. \tag{A.2}$$

Then ∇u_n converges (up to a subsequence) a.e. to ∇u.

Proof. In this proof, many times we extract a subsequence $\{u_{n_k}\}$, but we still note $\{u_n\}$ the subsequence.

As a consequence of the bound (A.2), we have that

$$u_n(x) \quad \text{converges to } u(x) \text{ almost everywhere.}$$

Moreover (3.3) gives

$$\int_\Omega |\nabla T_k(u_n)|^p \le \frac{L}{\alpha}\,k,$$

so that

$$\nabla T_k(u_n) \quad \text{converges weakly to } \nabla T_k(u) \text{ in } W_0^{1,p}(\Omega).$$

Let $0 < \theta < \frac{1}{p}$ and $k > 0$. Consider

$$I_{\Omega,n} = \int_\Omega \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta$$

We shall prove that the previous integral converges to zero. Indeed, it is equal to

$$\int_{C_k} \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta + \int_{A_k} \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta$$
$$= I_{C_k,n} + I_{A_k,n},$$

where

$$C_k = \{x \in \Omega : |u(x)| \le k\}, \qquad A_k = \{x \in \Omega : |u(x)| > k\}.$$

Here we repeat the proof of [3] (see also [8,12]) and we use the same notations (we denote by $\omega_i(k)$ quantities such that $\lim_{k\to\infty} \omega_i(k) = 0$). The only difference is that we will only use the following inequality

$$\left|\int_\Omega y_n\ T_j[u_n - T_k(u)]\right| \le j\,L, \quad \forall\, j > 0.$$

so that we have the following estimate

$$\limsup_{n\to\infty}[I_{C_k,n} + I_{A_k,n}] \le \omega_1(k) + (j\,L)^\theta + \omega_2(k), \quad \forall\, j > 0.$$

Therefore

$$\int_\Omega \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta \to 0,$$

that is

$$\|\{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta\|_{L^1(\Omega)} \to 0,$$

which implies

$$\{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta \to 0 \quad \text{almost everywhere,}$$

and also (since θ is positive)

$$\{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\} \to 0 \quad \text{almost everywhere.}$$

In [19], it is proved that, under our assumptions on the function $a(x,s,\xi)$, the previous limit implies that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{almost everywhere.} \quad \square$$

References

[1] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, in: Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez, An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Sci. Norm. Super. Pisa 22 (1995) 241–273.

[3] L. Boccardo, Some nonlinear Dirichlet problems in L^1 involving lower order terms in divergence form, in: Progress in Elliptic and Parabolic Partial Differential Equations, Capri, 1994, in: Pitman Res. Notes Math. Ser., vol. 350, Longman, Harlow, 1996, pp. 43–57.

[4] L. Boccardo, The effect of a linear term in some nonlinear elliptic equations with singular data, in: Recent Trends in Nonlinear Partial Differential Equations. II. Stationary Problems, in: Contemp. Math., vol. 595, Amer. Math. Soc, 2013, pp. 55–61.

[5] L. Boccardo, R. Cirmi, $W_0^{1,1}$ solutions of some unilateral problems, Nonlinear Anal. 121 (2015) 447–457.

[6] L. Boccardo, G. Croce, L. Orsina, $W_0^{1,1}$ minima of non coercive functionals, Atti Accad. Naz. Lincei 22 (2011) 513–523.

[7] L. Boccardo, G. Croce, L. Orsina, Nonlinear degenerate elliptic problems with $W_0^{1,1}$ solutions, Manuscripta Math. 137 (2012) 419–439.

[8] L. Boccardo, A. Dall'Aglio, T. Gallouët, L. Orsina, Nonlinear parabolic equations with measure data, J. Funct. Anal 147 (1997) 237–258.

[9] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, J. Funct. Anal. 87 (1989) 149–169.

[10] L. Boccardo, T. Gallouët, Nonlinear elliptic equations with right hand side measures, Comm. Partial Differential Equations 17 (1992) 641–655.

[11] L. Boccardo, T. Gallouët, Strongly nonlinear elliptic equations having natural growth terms and L^1 data, Nonlinear Anal. 19 (1992) 573–579.

[12] L. Boccardo, T. Gallouët, $W_0^{1,1}(\Omega)$ solutions in some borderline cases of Calderon–Zygmund theory, J. Differential Equations 253 (2012) 2698–2714.

[13] L. Boccardo, T. Gallouët, J.L. Vazquez, Nonlinear elliptic equations in R^N without growth restrictions on the data, J. Differential Equations 105 (1993) 334–363.

[14] H. Brezis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, Ann. Inst. Fourier (Grenoble) 18 (1968) 115–175.

[15] H. Brezis, Une équation semilinéaire avec conditions aux limites dans L^1; unpublished, 1972.

[16] H. Brezis, F.E. Browder, Some properties of higher order Sobolev spaces, J. Math. Pures Appl. 61 (1982) 245–259.

[17] H. Brezis, W.A. Strauss, Semi-linear second-order elliptic equations in L^1, J. Math. Soc. Japan 25 (1973) 565–590.

[18] G.R. Cirmi, Regularity of the solutions to nonlinear elliptic equations with a lower order term, Nonlinear Anal. 25 (1995) 569–580.

[19] J. Leray, J.L. Lions, Quelques résultats de Višik sur les problèmes elliptiques semi-linéaires par les méthodes de Minty et Browder, Bull. Soc. Math. France 93 (1965) 97–107.

[20] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble) 15 (1965) 189–258.