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Morrey estimates for some classes of elliptic equations with a lower order term

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To Professor Carlo Sbordone on the occasion of his 70th birthday, with affection

ABSTRACT

We will browse a series of results on elliptic equations and systems, with particular nonlinear lower order terms and measure right-hand side, obtained in the last years in the papers (Cirmi and Leonardi, 2014; Cianci et al., 2017; Cirmi et al., 2014, 2017, 2018).

Namely we take into account equations whose prototypes are:

$$-\Delta u + u|Du|^2 = f$$

or

$$-\Delta u + \frac{|Du|^2}{u^\theta} = f, \qquad \theta \in]0,1[,$$

where the right-hand side f belongs to a suitable Morrey space, for instance to $L^{1,\lambda}$, $0 < \lambda \le N-2$, and we prove corresponding Morrey estimates for the gradient of a solution.

1. Introduction

It is well known from 1959, by the fundamental result of E. De Giorgi [33], that any H^1-weak solution on the elliptic equation

$$-\operatorname{div}(a(x)Du) = 0 \ \text{ in } \Omega \subset \mathbb{R}^N,$$

with measurable bounded coefficients $a(x)$, belongs to the Hölder space $C^{0,\mu}$, for some $0 < \mu < 1$.

Some years later, the seminal result by De Giorgi was extended by S. Campanato (see [18,19]) to an equation of the type

$$-\operatorname{div}(a(x)Du) = \operatorname{div} g + f \ \text{ in } \Omega \subset \mathbb{R}^N, \tag{1.1}$$

with g in the Morrey space $L^{2,\lambda}(\Omega)$ and $f \in L^{\frac{2N}{N+2}, \frac{N+2}{N}\lambda}(\Omega)$.

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Actually, with respect to the work of E. De Giorgi, the information on Hölder continuity of u is gained estimating $|Du|$ in corresponding Morrey space $L^{2,\lambda}$.

The, nowadays classical, technique of estimating $|Du|$ in a suitable Morrey space relies on the so called Campanato's decomposition. Namely, in a ball $B_R \subset\subset \Omega$, we can write

$$u = v + w$$

where $w \in H^1_0(B_R)$ is the solution of the problem

$$-\operatorname{div}(a(x)Dw) = -\operatorname{div} g + f,$$

while $v \in H^1(B_R)$ is a weak solution of the problem

$$\begin{cases} -\operatorname{div}(a(x)Dv) = 0 & \text{in } B_R \\ v = u & \text{on } \partial B_R. \end{cases}$$

Moreover, v satisfies the so called Saint-Venant principle, i.e. for $0 < \rho \leq R$ it holds

$$\int_{B_\rho} |Dv|^2\, dx \leq c \left(\frac{\rho}{R}\right)^{N-2+2\mu} \int_{B_R} |Dv|^2\, dx.$$

Lax–Milgram theorem and the aforementioned estimate yield to

$$\int_{B_\rho} |Du|^2\, dx \leq c_1 \left(\frac{\rho}{R}\right)^{N-2+2\mu} \int_{B_R} |Du|^2\, dx + c_2(\|f\|, \|g\|)\, \rho^\lambda$$

and this latter estimate, via a sophisticated iterative process (see [18,19]), induces $Du \in L^{2,\lambda}(\Omega)$. If $\lambda > N-2$, the membership of Du to $L^{2,\lambda}$ gives back the Hölder continuity of u as well.

The linear structure of the operator on the left-hand side of (1.1) is necessary in order to perform the Campanato's decomposition.

In the same period of time, G. Stampacchia [64,65], by a duality method, proved the existence of a so called very weak solution $u \in W^{1,q}_0(\Omega)$, $1 \leq q < \frac{N}{N-1}$, of the problem

$$-\operatorname{div}(a(x)Du) = f \quad \text{in } \Omega$$

under the assumption $f \in L^1(\Omega)$.

At the beginning of the eighties the famous result of G. Stampacchia has been proved also for nonlinear operators, like the p-Laplacian, by L. Boccardo et al. [11–16].

Stampacchia's global result was extended to the framework of Morrey spaces by G. Mingione [61] (see also [6,34–36,40–42,44,45,47,62,63]) and, independently, by the author in the papers [24,25] in the first decade of year 2000.

Namely, using again Campanato's decomposition, under the assumption $f \in L^{1,\lambda}$, $0 < \lambda \leq N-2$, it was proved that $Du \in L^{q,\lambda}$, $1 \leq q \leq \frac{N}{N-1} < 2$.

On the other hand, Mingione's result in [61] opened also the way to further extensions and, in particular, to the theory of fractional differentiability of Du. In the quoted paper the following fundamental observation is made: *Morrey regularity of Du leads to its fractional differentiability.*

In the last few years, Morrey estimates of the gradient of a solution and Mingione's differentiability theory have been extended also to operators with lower order terms and measure right-hand side.

In the papers [20–23,27] there have been taken into account equations whose prototypes are:

$$-\Delta u + u|Du|^2 = f \tag{1.2}$$

or

$$-\Delta u+\frac{|Du|^2}{u^\theta}=f,\qquad \theta\in]0,1[,\tag{1.3}$$

where the right-hand side f belongs to a suitable Morrey space, for instance to $L^{1,\lambda}$, $0<\lambda\le N-2$, and Morrey estimates of Du have been proved.

Of course the linear structure of the operators is lost and so a new technique for dealing with these problems had to be developed.

Here we show also that Mingione's differentiability theory (see also the works of Kristensen and Mingione [41,42]) can be transferred to operators of the type (1.2) and (1.3). This is part of a new set of estimates, including potential estimates, as for instance in [44,45,47], that would be very interesting to extend in the present setting too. In this sense the reader can refer, besides to the cited papers, also to [7,46,48] for further useful remarks, observations and developments.

Actually in papers [20,22,23,27] it has even shown more, that is, for equations of the type (1.2) or (1.3), the lack of Morrey regularity of the gradient prevents its fractional differentiability (see examples below).

The systematic study of equations of the type (1.2) or (1.3) with measure right-hand sides was initiated by Boccardo, Arcoya et al. [4,5,8,10,29]. The aim was to show that the presence of the lower order term guaranteed more regularity (for instance H^1-regularity) on a solution with respect to the same operator but without the lower order term like in (1.1).

For operators with a ("bad") nonlinear lower order term the aforementioned decomposition procedure cannot be performed and thus to avoid such an obstruction for obtaining the necessary local estimates one has to adopt an alternative method as the one originally used in [20,27].

The new method of proof involves a certain delicate and tricky sophisticated set of a priori estimates and truncation methods, combining integral estimates and Morrey type density conditions.

Roughly speaking the equation is tested with the product $\eta^2 u$, where η is the standard cut-off function and the information on the gradient is inferred directly from the operator without splitting the solution. Then a special iteration procedure starts, which relies on a refinement of Sobolev–Morrey embedding theorem (see Theorem 2.3) contained in [24,25].

This paper is organized as follows. In Section 3 we deal with operator (1.2), while in Section 4 we take into account operator (1.3).

2. Notations, functions spaces and auxiliary tools

In $\mathbb{R}^N$ $(N\ge 3)$, with generic point $x=(x_1,x_2,\dots,x_N)$, we shall denote by Ω a bounded open nonempty set and $\mathrm{C}^{0,1}$-boundary $\partial\Omega$ and diameter d_Ω.

For $R>0$ and $x^0\in\mathbb{R}^N$ we define

$$B_R(x^0)=\{x\in\mathbb{R}^N:|x-x^0|<R\}.$$

Now, let us define the functional spaces we will use. We use a modification of the usual definitions, essentially equivalent, to simplify the treatment in the following.

Definition 2.1 (*Morrey Space*). *Let $q\ge 1$ and $\lambda\in]0,N[$. By $\mathrm{L}^{q,\lambda}(\Omega)$ we denote the space of all functions $u\in\mathrm{L}^q(\Omega)$ such that*

$$\|u\|_{\mathrm{L}^{q,\lambda}(\Omega)}=\sup_{\substack{x^0\in\Omega\\0<R\le d_\Omega}}\left\{R^{-\lambda}\int_{B_R(x^0)\cap\Omega}|u(x)|^q dx\right\}^{1/q}$$

is finite. $\mathrm{L}^{q,\lambda}(\Omega)$ equipped with the above norm is a Banach space.

We will often use the short notation $\|\cdot\|_q$ and $\|\cdot\|_{q,\lambda}$ instead of $\|\cdot\|_{\mathrm{L}^q(\Omega)}$ and $\|\cdot\|_{\mathrm{L}^{q,\lambda}(\Omega)}$, respectively.

Remark 2.2. Recall that

(i) $L^{q,\lambda} \not\subset L^{q+\varepsilon}$, $\forall \varepsilon > 0$;
(ii) if $\lambda > N-p$ then $L^{1,\lambda} \subset W^{-1,p}$.

Theorem 2.3 (*Sobolev–Morrey Embedding Theorem*)**.** *Let $\Omega \subset \mathbb{R}^N$, $u \in W_0^{1,p}(\Omega)$ be such that $Du \in L^{p,\sigma}_{loc}(\Omega)$, with $\sigma < N-p$.*

Then $u \in L^{p_\sigma,\sigma}_{loc}(\Omega)$ with $\frac{1}{p_\sigma} = \frac{1}{p} - \frac{1}{N-\sigma}$.

Moreover, for any $H \subset\subset \Omega$, there exists a positive constant c, depending on N, p, σ, Ω, H, such that

$$\|u\|_{L^{p_\sigma,\sigma}(H)} \le c\|Du\|_{L^{p,\sigma}(H)}\,.$$

Remark 1. The previous theorem can be stated in a global form under the assumption that $Du \in L^{p,\sigma}(\Omega)$.

Now, we recall some basic facts about fractional order Sobolev spaces.

Definition 2.4 (*Fractional Sobolev Space*)**.** *Let $t \in]0,1]$ and $q \ge 1$. $\mathrm{W}^{t,q}(\Omega)$ is the space of all functions $u \in L^q(\Omega)$ such that*

$$\|u\|_{\mathrm{W}^{t,q}(\Omega)} = \|u\|_q + [u]_{t,q,\Omega} < +\infty$$

where

$$[u]_{t,q,\Omega} = \begin{cases} \left(\int_\Omega\int_\Omega \dfrac{|u(x)-u(y)|^q}{|x-y|^{N+tq}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{q}} & \text{if } t<1 \\ \|\mathrm{D}u\|_q & \text{if } t=1. \end{cases}$$

Here $\mathrm{D}u$ represents the gradient of the function u i.e.

$$\mathrm{D}u \equiv \left(\frac{\partial u}{\partial x_i}\right)_{i=1,\dots,N} \equiv (D_i\,u)_{i=1,\dots,N}.$$

The following result is a well-known Sobolev's embedding theorem in the case of fractional space.

Theorem 2.5 (*Fractional Sobolev Embedding*)**.** *Let Ω be a domain of $\mathbb{R}^N$ with $\mathrm{C}^{0,1}$ boundary, $q \ge 1$ and $t \in]0,1]$ such that $tq < N$. Then*

$$\mathrm{W}^{t,q}(\Omega) \subset \mathrm{L}^{\frac{Nq}{N-tq}}(\Omega)$$

with continuous embedding.

3. Operators with natural growth lower order term

We start considering the model system of equations

$$\begin{cases} -\mathrm{div}\,[(s^2+|Du|^2)^{\frac{p-2}{2}}Du] + u|Du|^p = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where $p \in [2, N[$, $u : \Omega \to \mathbb{R}^n$ $(n \geq 1)$ is the unknown vector, $s \geq 0$ is a constant and $f \in L^1(\Omega, \mathbb{R}^n)$ is a vector-valued function belonging to a suitable Morrey space.

It is well known that the presence of lower order terms can have a regularizing effect on the solutions. There is an extensive literature about Dirichlet problems with lower order terms having quadratic growth with respect to the gradient (see [9,17]) and also for higher order equations whose coefficients satisfy a strengthened ellipticity condition (see [21] and see also [30–32] for more details concerning such kind of high-order equation).

The existence of a weak solution with finite energy (that is $u \in W_0^{1,p}(\Omega, \mathbb{R}^n)$) for systems whose prototype is (3.1) has been proved by A. Bensoussan and L. Boccardo in [8] assuming that the main part of the operator satisfies the so-called "Landes condition" (see [49]), which amounts to a sort of diagonal structure of the system, and that the lower order term verifies a sign (or angle) condition (see below for the precise statements of the assumptions).

G. Mingione (see [61]) has investigated the differentiability properties of the distributional solutions of a nonlinear elliptic equation $(n = 1)$ of the type

$$- \operatorname{div} \left[(s^2 + |Du|^2)^{\frac{p-2}{2}} Du\right] = \mu$$

where s is a nonnegative constant, $p > 1$ and μ is a signed Radon measure with finite total variation $|\mu|(\Omega) < +\infty$ enjoying the following density condition

$$|\mu|(B_R) \leq M R^{\lambda}, \text{ for some } M > 0,\ \lambda \in [0, N]$$

for any ball $B_R \subset \Omega$.

This differentiability result has been extended to the very weak solutions of non-diagonal linear elliptic systems $(n \geq 2)$ without lower order terms in [26].

Here we present similar differentiability properties for the usual weak solutions to systems of nonlinear elliptic equations, under the Landes condition, with lower order terms having natural (or critical) growth with respect to the gradient and satisfying a sign condition.

We denote by $A(x, \xi)$ a matrix-valued function whose entries are the functions

$$A_i^{\nu} : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$$

for $i = 1, \ldots, N$ and $\nu = 1, \ldots, n$. Each entry is a Carathéodory function (i.e. continuous in $\xi \in \mathbb{R}^{nN}$ for a.e. $x \in \Omega$ and measurable in x for every ξ) satisfying the following conditions for a.e. $x \in \Omega$, for every nonnegative real number s and for every $\xi, \eta \in \mathbb{R}^{nN}$ such that $\xi \neq \eta$ ([1]):

$$\exists \Lambda_1 > 0 : (A_i^{\nu}(x,\xi) - A_i^{\nu}(x,\eta))(\xi_i^{\nu} - \eta_i^{\nu}) \geq \Lambda_1 \left(s^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2, \tag{3.2}$$

$$\exists\, \Lambda_2 > 0 : |A(x,\xi)| \leq \Lambda_2 \left(s^2 + |\xi|^2\right)^{\frac{p-2}{2}} |\xi|, \quad p \in [2, N[, \tag{3.3}$$

$$A_i^{\nu}(x, 0) = 0, \tag{3.4}$$

$$A_i^{\nu}(x,\xi) \left[\xi_i^{\nu} |\gamma|^2 - \gamma^{\nu} \gamma^{\mu} \xi_i^{\mu}\right] \geq 0 \quad \forall \gamma \in \mathbb{R}^N. \tag{3.5}$$

Remark 2. Since $p \geq 2$ the assumption (3.2) implies the strong monotonicity assumption

$$\left(A_i^{\nu}(x,\xi) - A_i^{\nu}(x,\eta)\right)\left(\xi_i^{\nu} - \eta_i^{\nu}\right) \geq c(\Lambda_1, p)\, |\xi - \eta|^p. \tag{3.6}$$

[1] We assume the use of Einstein's convention throughout the paper.

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For $s \geq 0$ we set

$$V(\xi) \equiv V_s(\xi) := (s^2 + |\xi|^2)^{\frac{p-2}{4}} \xi \quad \forall \xi \in \mathbb{R}^{nN}. \tag{3.7}$$

The assumptions (3.2) and (3.4) imply the ellipticity condition

$$A_i^\nu(x,\xi)\xi_i^\nu \geq \Lambda_1 |V(\xi)|^2, \text{ a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^{nN}. \tag{3.8}$$

Moreover, from Lemma 2.1 of [37] and (3.2) (see also [61]) we have the following properties

$$c(n,p)^{-1}(s^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \frac{|V(\xi) - V(\eta)|^2}{|\xi - \eta|^2} \leq c(n,p)(s^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}, \tag{3.9}$$

$$(A_i^\nu(x,\xi) - A_i^\nu(x,\eta))(\xi_i^\nu - \eta_i^\nu) \geq c(\Lambda_1, n, p) |V(\xi) - V(\eta)|^2. \tag{3.10}$$

The assumption (3.3) and Young's inequality yield

$$|A(x,\xi)| \leq c(\Lambda_2, p)\,(s^2 + |\xi|^2)^{\frac{p-1}{2}}. \tag{3.11}$$

Remark 3. The assumption (3.5) is the so- called "Landes condition". Note that it is automatically implied by (3.8), whenever $n = 1$.

For $\nu = 1, \ldots, n$ let $g^\nu : \Omega \times \mathbb{R}^n \times R^{nN} \to \mathbb{R}$ be Carathéodory functions and denote by $g(x,u,\xi)$ the vector-valued function whose νth component is g^ν. For $g(x,u,\xi)$ we will assume the following conditions for a.e. $x \in \Omega$, for every $u \in \mathbb{R}^N$ and for every $\xi \in \mathbb{R}^{nN}$:

$$|g(x,u,\xi)| \leq b(|u|)\,[d(x) + |\xi|^p], \tag{3.12}$$

and

$$|g(x,u,\xi)| \geq \sigma\,|V(\xi)|^2 \quad \forall u \in \mathbb{R}^N : |u| \geq 1, \tag{3.13}$$

where $b(\cdot)$ is a real valued, positive, increasing and continuous function, $d(x)$ is a nonnegative function in $L^{1,\lambda}(\Omega, \mathbb{R}^n)$, $\lambda \in]0, N-p]$, s is a nonnegative real number and σ is a positive real number.

Moreover, we assume the following angle condition

$$g^\nu(x,u,\xi)(u^\nu - \tau^\nu) \geq 0, \ \ \forall \tau, u \in \mathbb{R}^n : |\tau| \leq |u| \tag{3.14}$$

which amounts to a sign condition in the scalar case $n = 1$.

We consider the following system

$$\begin{cases} u \in W_0^{1,p}(\Omega, \mathbb{R}^n), \quad g(x,u,Du) \in L^1(\Omega, \mathbb{R}^n) \\ -D_i A_i^\nu(x,Du) + g^\nu(x,u,Du) = f^\nu \end{cases} \tag{3.15}$$

where, for any $\nu = 1, \ldots, n$, f^ν denotes the νth component of the vector

$$f \in L^{1,\lambda}(\Omega, \mathbb{R}^N), \quad \lambda \in]0, N-p]. \tag{3.16}$$

By a *weak solution* of the system of equations (3.15) we mean a vector- valued function $u \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ such that

$$\begin{cases} g(x,u(x),Du(x)) \in L^1(\Omega, \mathbb{R}^n) \\ \displaystyle\int_\Omega A_i^\nu(x,Du)\,D_i v^\nu\,dx + \int_\Omega g^\nu(x,u,Du)\,v^\nu\,dx = \int_\Omega f^\nu\,v^\nu\,dx \\ \forall\, v \in W_0^{1,p}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n). \end{cases} \tag{3.17}$$

The existence of a weak solution of the above problem has been proved in [8].

So that about the regularity we can prove the following

Theorem 1. *Let assumptions (3.2), (3.3) (3.4), (3.5), (3.12), (3.13), (3.14), (3.16) be satisfied and let $u \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ be a weak solution of the problem (3.15).*

Then

$$V(Du) \in L^{2,\lambda}_{loc}(\Omega, \mathbb{R}^{nN}), \quad Du \in L^{p,\lambda}_{loc}(\Omega, \mathbb{R}^{nN}) \tag{3.18}$$

and for any $\Omega' \subset\subset \Omega$ there exist two positive constants c_1 and c_2, depending only on data, such that

$$\|V(Du)\|_{L^{2,\lambda}(\Omega')} \leq c_1, \tag{3.19}$$

and

$$\|Du\|_{L^{p,\lambda}(\Omega')} \leq c_2. \tag{3.20}$$

Remark 4. The previous Morrey regularity result holds as well if $\lambda > N - p$ assuming also $u \in L^\infty(\Omega, \mathbb{R}^n)$.

To prove the differentiability of a weak solution u we shall require the following Hölder continuity assumption on the map $x \to A(x, \xi)$:

$$\begin{cases} \text{there exist } L > 0 \text{ and } \eta \in]0, 1] \text{ such that} \\ |A(x, \xi) - A(x_0, \xi)| \leq L|x - x_0|^\eta (s^2 + |\xi|^2)^{\frac{p-1}{2}}, \quad \forall x, x_0 \in \Omega, \xi \in \mathbb{R}^{nN}. \end{cases} \tag{3.21}$$

Theorem 2. *Let the assumptions (3.2), (3.3), (3.4), (3.5), (3.12), (3.13), (3.14), (3.16), (3.21) be satisfied and let $u \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ be a weak solution of the problem (3.15). Set*

$$\delta = \min\left\{1, \frac{\lambda}{2}\right\}. \tag{3.22}$$

Then

$$V(Du) \in W^{t,2}_{loc}(\Omega, \mathbb{R}^{nN}), \quad Du \in W^{2t/p,p}_{loc}(\Omega, \mathbb{R}^{nN}) \tag{3.23}$$

for every $t \in [0, \eta\delta[$.

Moreover, for every couple of open subset $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exist two positive constants c_1 and c_2, independent on u, such that

$$[V(Du)]^2_{W^{t,2}(\Omega')} \leq c_1 \left[\int_{\Omega''} (s^p + |Du|^p)\, dx + \|V(Du)\|^2_{L^{2,\theta}(\Omega'')}\right] \tag{3.24}$$

and

$$[Du]^p_{W^{2t/p,p}(\Omega')} \leq c_2 \left[\int_{\Omega''} (s^p + |Du|^p)\, dx + \|Du\|^p_{L^{p,\theta}(\Omega'')}\right]. \tag{3.25}$$

Remark 5. In the case $\lambda > N - p$ the differentiability result stated above holds for the bounded weak solutions of the problem (3.15).

Remark 6. As a consequence of the fractional Sobolev embedding Theorem 2.5 we gain a better integrability on Du. Namely,

$$Du \in L^{\frac{pn}{n-2t}}_{loc}(\Omega, \mathbb{R}^{nN}) \text{ for every } t \in [0, \eta\delta[$$

where δ is the number defined in (3.22).

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3.1. Optimality of the result

The above result turns out to be optimal for this class of systems. As a matter of fact, as shown below, the differentiability of a solution fails whether $\lambda = 0$, that is under the sole requirement that f is just in $L^1(\Omega, \mathbb{R}^n)$, while in the case of the operator without lower order term a small amount of differentiability still holds (see [61]).

Given a vector $u \in \mathbb{R}^n$ and a real number $k > 0$ let us denote by $T_k(u)$ the vector-valued function whose components are defined by

$$[T_k(u)]^\nu = \begin{cases} u^\nu & \text{if } |u| \leq k \\ k\dfrac{u^\nu}{|u|} & \text{if } |u| > k \end{cases} \tag{3.26}$$

for $\nu = 1, \dots, n$. Moreover, if $v \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ then $T_k(v) \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ and for any $i = 1, \dots, N$ and $\nu = 1, \dots, n$ and it holds

$$D_i[T_k(v)]^\nu = \begin{cases} D_i v^\nu & \text{if } |v| \leq k \\ \dfrac{k}{|v|}\left[D_i v^\nu - \dfrac{1}{|v|^2} v^\nu v^\mu D_i v^\mu\right] & \text{if } |v| > k \end{cases}$$

see [49].

Let $n = 2$ (or $n = 1$), $\Omega = B(0, 1/2)$ and, for a. e. $x \in B(0, 1/2)$, define

$$u(x) = \left(\int_{|x|}^{1/2} \frac{1}{\rho^{n/2}|log\rho|} d\rho, \int_{|x|}^{1/2} \frac{1}{\rho^{n/2}|log\rho|} d\rho\right).$$

We can readily prove that $u \in W_0^{1,2}(B(0, 1/2), \mathbb{R}^2)$ is a solution of the Dirichlet problem associated to the system

$$-\Delta u + T_1(u)|Du|^2 = f(x)$$

where

$$f(x) = \frac{1 - (n/2 - 1)log|x|}{|x|^{n/2+1} log^2|x|} + T_1(u(x))|Du(x)|^2.$$

Easy calculations show that

$$Du \notin L^{2,\sigma}_{loc}(\Omega, \mathbb{R}^{2N}) \text{ for any } \sigma \in]0, N[,$$

this implies that the vector-valued function f belongs to $L^1(\Omega, \mathbb{R}^2)$ but does not belong to $L^{1,\sigma}(\Omega, \mathbb{R}^2)$ for any $\sigma \in]0, N[$.

Moreover

$$Du \notin W^{t,2}_{loc}(\Omega, \mathbb{R}^{2n}) \text{ for any } t \in]0, 1[$$

since otherwise, being $W^{t,2}_{loc}(\Omega, \mathbb{R}^{2n}) \subset L^{\frac{2n}{n-2t}}_{loc}(\Omega, \mathbb{R}^{2n})$, it would be $Du \in L^{2,n-2t}_{loc}(\Omega, \mathbb{R}^{2n})$.

4. Operators with degenerate lower order term

Here we present some results about the regularity of a solution of the Dirichlet problem associated to the singular equation

$$-\operatorname{div}(a(x)\mathrm{D}u) + M\,\frac{|\mathrm{D}u|^2}{u^\theta} = f(x) \ \text{ in } \Omega \tag{4.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary, $a(x)$ is a L^∞-matrix satisfying the standard ellipticity condition, $\theta \in]0,1[$, M is a positive constant and f is sufficiently regular i.e. it belongs to a suitable Morrey space $L^{q,\lambda}(\Omega)$, with $q \geq 1$, to be specified later on.

There is a huge literature about the problems with quadratic term in the gradient (see [9,13,43]) also for high-order equations whose coefficients satisfy a strengthened ellipticity condition (see [21] and see also [30–32] for more details concerning such kind of high-order equation), but they do not consider a singularity in the lower order term.

The problem (4.1) has been studied in the paper [4] by D. Arcoya, J. Carmona, T. Leonori, P. J. Martínez-Aparicio, L. Orsina, F. Petitta and in the paper [10] by L. Boccardo where the source term f belonged to $\mathrm{L}^q(\Omega)$ with $q \geq 1$.

Here we will extend to the gradient of a solution the Morrey property of the right-hand side f and we will show that, in some cases, we can improve some results contained in [4,10] without increasing the summability of f.

It remains an open problem how to extend Mingione's theory to the case when the right-hand side f is a Radon measure or it is slightly more regular, but not more than L^1, for instance when it stays in $\mathrm{L}^{1,\lambda}$ with $0 \leq \lambda \leq N-2$ (see e.g. [3]). In this case we cannot perform the decomposition of a solution in the "good part" and the "bad part" as it was done e.g. in [26] (see also [28,38,39,50,51,53–59]).

We now denote by $a(x)$ a symmetric matrix whose entries $a_{ij}(x)$ for $i,j = 1,\dots,N$ are bounded functions satisfying the following standard structural conditions for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$:

$$\exists\, \Lambda_1,\, \Lambda_2 > 0 : \Lambda_1 |\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda_2 |\xi|^2 . \tag{4.2}$$

Let us consider the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(a(x)\mathrm{D}u) + M\,\dfrac{|\mathrm{D}u|^2}{u^\theta} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.3}$$

where M denotes a positive constant, $\theta > 0$ and f belongs to a suitable Morrey space to be specified later on.

By weak solution of the above problem we mean a function u such that

$$\begin{cases} u \in \mathrm{H}^1_0(\Omega), \qquad u > 0, \qquad \dfrac{|\mathrm{D}u|^2}{u^\theta} \in \mathrm{L}^1(\Omega) \\ \displaystyle\int_\Omega a(x)\,\mathrm{D}u\mathrm{D}\varphi\,\mathrm{d}x + M\int_\Omega \frac{|Du|^2}{u^\theta}\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x\,, \quad \forall\,\varphi \in \mathrm{H}^1_0(\Omega)\cap\mathrm{L}^\infty(\Omega). \end{cases} \tag{4.4}$$

Existence (and nonexistence) of positive solutions of the problem (4.3) has been studied in [4,10,29]. Under hypothesis (4.2), if $0 < \theta < 2$,

$$f \in \mathrm{L}^m(\Omega) \quad \text{with } m = \min\Big\{\Big(\frac{2^*}{\theta}\Big)', (2^*)'\Big\} \tag{4.5}$$

and it satisfies moreover the assumption

$$\operatorname{ess\,inf}\big\{f(x), x \in \omega\big\} > 0 \qquad \text{for all } \omega \subset\subset \Omega. \tag{4.6}$$

Remark 4.1. If $0 < \theta < 1$, then $m = \big(\frac{2^*}{\theta}\big)'$ and the assumption (4.6) has been weakened by Boccardo in [10]. Namely, it can be assumed that $f \geq 0$ and $f \not\equiv 0$ in Ω. In the case $\theta = 1$, one has to assume moreover that $\Lambda_1 > 2M$. In [5], Corollary 2.12, it has been established the uniqueness of the solution.

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4.1. Gradient integrability in the case of regular right-hand side

The following theorem states the regularity of $\mathrm{D}u$ in a suitable Morrey space which, in turn, will be a fundamental tool to obtain the fractional differentiability of the gradient of a solution u (see Theorem 4.4 below). It could be also used other kinds of Morrey spaces, similar to weighted Lebesgue spaces, as it was done in [52].

We will focus on local estimates just in the case

$$0 < \theta < 1 \tag{4.7}$$

and we will prove the following

Theorem 4.2. *Let the assumptions (4.2), (4.6) and (4.7) be satisfied and assume that the function f is such that*

$$f \in \mathrm{L}^{m,\lambda}(\Omega), \tag{4.8}$$

with $m = \left(\frac{2^*}{\theta}\right)' = \frac{2N}{2N-\theta(N-2)}$ and $0 < \lambda < N$.

Let $u \in \mathrm{H}^1_0(\Omega)$ be the solution of the problem (4.3) in the sense of (4.4).

Then $\mathrm{D}u \in \mathrm{L}^{2,\mu}_{\mathrm{loc}}(\Omega)$ for every $\mu < \bar{\mu}$, where

$$\bar{\mu} = \begin{cases} \dfrac{2}{2-\theta}\dfrac{\lambda}{m} & \text{if } 0 < \lambda < N - 2m \\ N-2 & \text{if } \lambda \geq N - 2m. \end{cases} \tag{4.9}$$

Moreover, there exists a positive constant c, independent of u, such that

$$\|\mathrm{D}u\|_{\mathrm{L}^{2,\mu}_{\mathrm{loc}}(\Omega)} \leq c \tag{4.10}$$

for every $\mu < \bar{\mu}$.

Remark 4.3. Theorem 4.2 naturally extends Theorem 1.1 in [4], Theorems 3.1 and 4.1 in [10] and Theorem 1.1 in [29] to the framework of Morrey spaces.

Note that if $0 < \lambda < N - 2$ then $\lambda < \bar{\mu} \leq N - 2$.

Furthermore, observe that if $f \in \mathrm{L}^{\bar{m}}(\Omega)$, with $\bar{m} > m = \left(\frac{2^*}{\theta}\right)'$, then $f \in \mathrm{L}^{m,\lambda}(\Omega)$ with $\lambda = N\left(1 - \frac{m}{\bar{m}}\right)$, where $\lambda < N - 2m$ if $\bar{m} < \frac{N}{2}$, otherwise $\lambda \geq N - 2m$.

Once the Morrey estimate (4.10) is established, following the proof of Theorem 4 in [27] exploiting the method introduced in [61] (see also [1,26,40]). We are able to prove the fractional differentiability of $\mathrm{D}u$. Namely, the following Theorem holds.

Theorem 4.4. *Let the assumptions (4.2), (4.7) and (4.8) be satisfied. Assume that $a_{ij} \in \mathrm{C}^{0,\eta}(\Omega)$, $0 < \eta < 1$, and let $u \in \mathrm{H}^1_0(\Omega)$ be a positive weak solution of the problem (4.3).*

Then

$$\mathrm{D}u \in \mathrm{W}^{t,2}_{\mathrm{loc}}(\Omega) \tag{4.11}$$

for every $t \in [0, \eta\delta[$ and for every $\delta < \min\left\{1, \frac{\bar{\mu}}{2}\right\}$, with $\bar{\mu}$ defined in (4.9).

Moreover, for every couple of open subset $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists a positive constant c, independent of u, such that

$$[\mathrm{D}u]^2_{\mathrm{W}^{t,2}(\Omega')} \leq c \left[\int_{\Omega''} |\mathrm{D}u|^2 \, dx + \|\mathrm{D}u\|^2_{L^{2,\mu}(\Omega'')} \right], \tag{4.12}$$

for every $\mu < \bar{\mu}$.

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4.2. Counterexamples to the regularity of the gradient

In this section we construct two counterexamples to show that our results are somehow optimal.

4.2.1. Counterexample to Morrey regularity

This example shows that the Morrey regularity of the right-hand side is transmitted to the gradient of a solution through the lower order term.

Namely, we produce a solution $v \in \mathrm{H}^1_0(\Omega)$ of an equation of the type (4.3) without lower order term (i.e. $M = 0$) which has no gradient in $\mathrm{L}^{2,\lambda}(\Omega)$ for any $\lambda > 0$, although the right-hand side belongs to a suitable Morrey space.

Let $v : B_{1/2}(0) \to \mathbb{R}$ be the function defined as

$$v(x) = \int_{|x|}^{1/2} \frac{1}{t^{N/2}|\log t|}\,\mathrm{d}\rho\,. \tag{4.13}$$

Observe that, being

$$\frac{\partial v(x)}{\partial x_i} = -\frac{x_i}{|x|}\frac{1}{|x|^{\frac{N}{2}}\log|x|} \quad \text{and} \quad |\mathrm{D}v(x)|^2 = \frac{1}{|x|^N\log^2|x|},$$

it can readily seen that $v \in W^{1,1}_0$ and that

$$\mathrm{D}v \in \mathrm{L}^2\big(B_{1/2}(0)\big), \quad \mathrm{D}v \notin \mathrm{L}^{2,\mu}\big(B_{1/2}(0)\big) \quad \text{for any } \mu > 0. \tag{4.14}$$

Moreover, set

$$g(x) \equiv \frac{1-\left(\frac{N}{2}-1\right)\log|x|}{|x|^{\frac{N}{2}+1}\log^2|x|} = \Delta v(x),$$

we have that

$$\lim_{\rho\to 0^+} \frac{\int_{B_\rho}\left|\frac{1-\left(\frac{N}{2}-1\right)\log|x|}{|x|^{\frac{N}{2}+1}\log^2|x|}\right|^m \mathrm{d}x}{\rho^\lambda} < +\infty$$

where $m = \left(\frac{2^*}{\theta}\right)'$, θ is a fixed number in $]0,1[$, $\lambda = \frac{N(1-\theta)(N-2)}{2N-\theta(N-2)} < N-2m$.

So that

$$g(x) \in L^{m,\lambda}(B_{1/2}(0))\,.$$

According to Theorem 4.2, for any fixed positive constant M, the function $u(x)$, solution of the problem

$$\begin{cases} u \in \mathrm{H}^1_0\big(B_{1/2}(0)\big) \\ -\Delta u + M\dfrac{|Du|^2}{u^\theta} = g & \text{in } B_{1/2}(0)\,, \end{cases}$$

is such that

$$Du \in \mathrm{L}^{2,\mu}_{loc}\big(B_{1/2}(0)\big)$$

with $\mu > \lambda$.

On the other hand, since as well

$$m \equiv \left(\frac{2^*}{\theta}\right)' < \frac{2(N-\lambda)}{N-\lambda+2}\,,$$

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then, according to Theorem 3.1 by Adams [2] (see also [62]), a solution u of the problem (with $M = 0$)

$$\begin{cases} u \in \mathrm{W}_0^{1,1}\big(B_{1/2}(0)\big) \\ -\ \Delta u = g & \text{in } B_{1/2}(0) \end{cases} \tag{4.15}$$

is such that

$$Du \in L^{\frac{m(N-\lambda)}{N-\lambda-m},\lambda}(B_{1/2}(0))\,.$$

It is worthwhile to observe that, with our choice of m and λ, it is

$$\frac{m(N-\lambda)}{N-\lambda-m} = 2\left(1-\frac{\lambda}{N}\right)\ ([2]).$$

Moreover we stress that even if the function $u(x) = -v(x) \in H_0^1(B_{1/2}(0))$ ([3]) is the weak solution of the problem (4.15), as previously seen, it is

$$\mathrm{D}u = -\mathrm{D}v \notin \mathrm{L}^{2,\mu}_{loc}\big(B_{1/2}(0)\big)$$

for any $\mu > 0$.

4.2.2. Counterexample to differentiability

By this second example we show that the presence of the lower-order term does not ensure the differentiability of a solution if the right-hand side does not belong to any Morrey space, i.e. if it is just in a Lebesgue space.

The first counterexample is flexible enough to be modified for our purposes and thus, for the sake of simplicity, we keep the notation of the previous subsection for $v(x), g(x), \theta, m, \lambda$.

We preliminarily observe that, being

$$t^{N/2}|\ln|t|| = o\,(t) \qquad \text{as } t \to 0,$$

then

$$\lim_{|x|\to 0} v(x) = +\infty.$$

Moreover it can be readily proved that

$$v^{\theta}(x) = O\left(\frac{1}{|x|^{\theta\left(\frac{N}{2}-1\right)}\big|\log|x|\big|^{\theta}}\right) \qquad \text{as } |x| \to 0$$

so that the function

$$h(x) \equiv \frac{|Dv(x)|^2}{(v(x))^{\theta}}$$

is such that

$$h(x) \in \mathrm{L}^{m}\big(B_{1/2}(0)\big), \quad h(x) \notin \mathrm{L}^{m,\mu}\big(B_{1/2}(0)\big) \ \text{ for any } \mu > 0.$$

[2] Note that, in general, it is

$$L^2(\cdot) \subset L^{2\left(1-\frac{\nu}{N}\right),\nu}(\cdot) \quad 0 < \nu < N\,.$$

[3] $v(x)$ from (4.13).

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As a consequence, the function $h(x)-g(x)\notin \mathrm{L}^{m,\mu}\big(B_{1/2}(0)\big)$ for any $\mu>0$.

In this way, the function $v(x)\in \mathrm{H}^1_0(B_{1/2}(0))$ is solution of the equation

$$-\Delta v+\frac{|\mathrm{D}v|^2}{v^\theta}=h(x)-g(x)$$

but, as we will see, its gradient is not fractionally differentiable.

Indeed, if it happened to be

$$\mathrm{D}v\in \mathrm{W}^{t,2}_{\mathrm{loc}}(\Omega)\quad \text{for some } \bar t\in]0,1[$$

then, in force of the embedding $\mathrm{W}^{\bar t,2}_{\mathrm{loc}}(\Omega)\subset \mathrm{L}^{\frac{2N}{N-2\bar t}}_{\mathrm{loc}}(\Omega)$ (see Theorem 2.5), it should also be $\mathrm{D}v\in \mathrm{L}^{2,\frac{2N}{N-2\bar t}}_{\mathrm{loc}}(\Omega)$. But this is false as we have shown in (4.14).

4.3. Gradient regularity in the case of measure right-hand side

In this section we will consider the problem (4.3), under the assumptions (4.2) and (4.6), when the right-hand side satisfies

$$f\in \mathrm{L}^{1,\lambda}(\Omega),\quad \lambda\in]0,N-2[. \tag{4.16}$$

By a solution of the problem (4.3) we now mean a function u such that

$$\begin{cases} u\in \mathrm{W}^{1,1}_0(\Omega), \qquad u>0, \qquad \dfrac{|\mathrm{D}u|^2}{u^\theta}\in \mathrm{L}^1(\Omega)\\ \displaystyle\int_\Omega a(x)\,\mathrm{D}u\mathrm{D}\varphi\,\mathrm{d}x+M\int_\Omega \frac{|\mathrm{D}u|^2}{u^\theta}\varphi\,\mathrm{d}x=\int_\Omega f\varphi\,\mathrm{d}x\,,\quad \forall\,\varphi\in C^\infty_0(\Omega). \end{cases} \tag{4.17}$$

Here we prove the following

Theorem 4.5. *Let the hypotheses (4.2), (4.6), (4.7) and (4.16) be satisfied. Then there exists a solution u of the problem (4.17)) such that*

- $\mathrm{D}u\in \mathrm{L}^{\bar\sigma,\lambda}_{\mathrm{loc}}(\Omega)$ *where* $\bar\sigma=\frac{(N-\lambda)(2-\theta)}{N-\lambda-\theta}$,
- $\mathrm{D}T_k(u)\in \mathrm{L}^{2,\lambda}_{\mathrm{loc}}(\Omega)$ *for every* $k\in\mathbb{N}$.

with corresponding norm estimates.

Remark 4.6. Observe that the presence of the lower order term improves the regularity of the solution in comparison with analogous results for an operator only in principal part (see [24,25,61]).

Remark 4.7. If the right hand side $f\in L^{1,\lambda}(\Omega)$, with $N-2<\lambda<N$ and $\theta\in]0,2[$, then there exists a solution u of the problem (4.3) which belongs to $\mathrm{L}^\infty(\Omega)$, thereby improving Theorem 1.1 in [4] (see Remark 2.6 therein contained) and Theorem 1.5 part (1) in [29].

4.4. Gradient regularity in the case of right-hand side "below the duality exponent"

Here we deal with an intermediate case, in the sense that the right hand side is not merely a measure and it does not belong to the right dual space (see [54,62]).

For further details on topics which are closed related to the present ones, also dealing with higher order operators and weighted Morrey spaces the reader can refer to [40–42,61,63] and also to [24,26–28,38,39,43,50–53,55–60].

Theorem 4.8. *Let the assumptions (4.2), (4.6) and (4.7) be satisfied and assume that the function f is such that*

$$f \in \mathrm{L}^{m,\lambda}(\Omega), \tag{4.18}$$

with $1 < m < \left(\frac{2^*}{\theta}\right)'$ ([4]) *and* $\lambda \in]0, N[$.

Then, there exists a weak solution of the problem (4.3), in the sense of definition (4.17), such that

$$\mathrm{D}u \in \mathrm{L}^{\sigma(\mu),\mu}_{\mathrm{loc}}(\Omega), \quad \textit{for every } \mu < \bar{\mu}$$

where

$$\sigma(\mu) = \frac{m(N-\mu)(N-2)(2-\theta)}{(N-\mu)(N-2m) + m[2N - \theta(N-2)] - 2N} \quad \textit{and} \tag{4.19}$$
$$\bar{\mu} = \min\left\{\frac{N-2}{N-2m}\,\lambda, N-2\right\}.$$

Moreover, there exists a positive constant c, independent of u, such that

$$\|\mathrm{D}u\|_{\mathrm{L}^{\sigma(\mu),\mu}_{\mathrm{loc}}(\Omega)} \le c. \tag{4.20}$$

Remark 4.9. Let us consider the equation

$$\Delta u = f \quad \text{in } \mathbb{R}^N.$$

A well known result due to *D*. Adams [2] states that if $f \in \mathrm{L}^{m,\lambda}$ with $1 < m < \frac{2(N-\lambda)}{N-\lambda+2}$, then $\mathrm{D}u \in \mathrm{L}^{\frac{m(N-\lambda)}{N-\lambda-m},\lambda}$ and the exponent $\frac{m(N-\lambda)}{N-\lambda-m}$ is sharp. We point out that the presence of a lower order term having quadratic growth with respect to $|\mathrm{D}u|$ and singular with respect to u, increases, at least locally, the summability of the gradient of the solution.

As a matter of the fact, let $f \in \mathrm{L}^{m,\lambda}$ with $1 < m < \frac{2(N-\lambda)}{N-\lambda+2}$ and $0 < \lambda < N-2$, and let us fix $\bar{\theta} = \frac{N(N-\lambda-2)}{(N-2)(N-\lambda)}$. Note that this choice implies $\frac{2(N-\lambda)}{N-\lambda+2} = \frac{2N}{2N-\theta(N-2)}$.

Thus, by Theorem 4.8 there exists a weak solution of the homogeneous Dirichlet problem related to the equation

$$-\Delta u + M\frac{|\mathrm{D}u|^2}{u^{\bar{\theta}}} = f \quad \text{in } \Omega$$

such that $\mathrm{D}u \in \mathrm{L}^{s,\lambda}_{\mathrm{loc}}(\Omega)$, where

$$s = \frac{m(N-\lambda)(N-2)(2-\bar{\theta})}{(N-\lambda)(N-2m) + m[2N - \bar{\theta}(N-2)] - 2N} > \frac{m(N-\lambda)}{N-\lambda-m}.$$

On the other hand, for every $\theta \in]0, 1[$, let us fix $\bar{\lambda} = N\frac{(N-2)(1-\theta)}{N-\theta(N-2)}$. This choice implies $\frac{2(N-\bar{\lambda})}{N-\bar{\lambda}+2} = \frac{2N}{2N-\theta(N-2)}$.

Assume that $f \in \mathrm{L}^{m,\bar{\lambda}}$ with $1 < m < \frac{2(N-\bar{\lambda})}{N-\bar{\lambda}+2}$, by Theorem 4.8 there exists a solution of the homogeneous Dirichlet problem related to the equation

$$-\Delta u + M\frac{|\mathrm{D}u|^2}{u^{\theta}} = f \quad \text{in } \Omega$$

such that $\mathrm{D}u \in \mathrm{L}^{s,\bar{\lambda}}_{\mathrm{loc}}(\Omega)$, where $s = \frac{m(N-\bar{\lambda})(N-2)(2-\theta)}{(N-\bar{\lambda})(N-2m)+m[2N-\theta(N-2)]-2N} > \frac{m(N-\bar{\lambda})}{N-\bar{\lambda}-m}$.

[4] Note that if $\theta \in]0, 1[$ then $\min\left\{\left(\frac{2^*}{\theta}\right)', \left(2^*\right)'\right\} = \frac{2N}{2N-\theta(N-2)}$.

Remark 4.10. The result of Theorem 4.8 links up continuously with the results already proved in [20].

In fact, when $m \to \bar{m} = \left(\frac{2^*}{\theta}\right)'$ then

$$\sigma(\mu) \to 2 \quad \text{and} \quad \bar{\mu} \to \min\left\{\frac{2}{2-\theta}\frac{\lambda}{\bar{m}},\ N-2\right\}$$

while, when $m \to 1$ then

$$\sigma(\mu) \to 1 \quad \text{and} \quad \bar{\mu} \to \lambda.$$

Moreover, if $\lambda \to 0$ then $\sigma(\mu) \to \frac{mN(2-\theta)}{N-m\theta}$ and we obtain, at least formally, the results already proved in [10,29].

Remark 4.11. If $\lambda \geq N-2m$ Theorem 4.8 gives us a solution whose gradient belongs in $\mathrm{L}^{s,\mu_s}_{\mathrm{loc}}(\Omega)$, for every $s < 2$, for some $\mu_s \in]0, N-2[$.

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