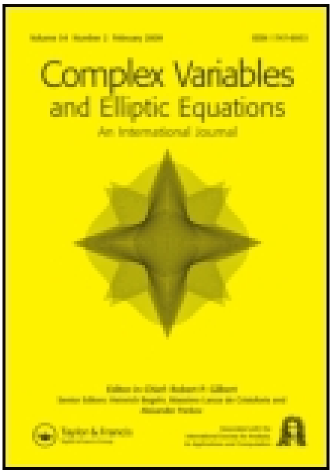

Complex Variables and Elliptic Equations: An International Journal
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcov20

Regularity up to the boundary for the gradient of solutions of linear elliptic systems with VMO-coefficients and $L^{1,\lambda}$ data

S. Leonardi [a] & J. Stará [b]

[a] Dipartimento di Matematica ed Informatica , Università di Catania , Viale A. Doria 6, 95125 Catania, Italy

[b] Department of Mathematical Analysis , Charles University , Sokolovská 83, 186 75 Praha 8, Czech Republic

Published online: 26 Jul 2011.

To cite this article: S. Leonardi & J. Stará (2011) Regularity up to the boundary for the gradient of solutions of linear elliptic systems with VMO-coefficients and $L^{1,\lambda}$ data, Complex Variables and Elliptic Equations: An International Journal, 56:12, 1085-1098, DOI: 10.1080/17476933.2010.534152

To link to this article: http://dx.doi.org/10.1080/17476933.2010.534152

Complex Variables and Elliptic Equations
Vol. 56, No. 12, December 2011, 1085–1098

Regularity up to the boundary for the gradient of solutions of linear elliptic systems with VMO-coefficients and $L^{1,\lambda}$ data†

S. Leonardi[a] and J. Stará[b]*

[a]*Dipartimento di Matematica ed Informatica, Università di Catania, Viale A. Doria 6, 95125 Catania, Italy;* [b]*Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic*

Communicated by R. Gilbert

(*Received 23 January 2010; final version received 25 June 2010*)

We prove that if a vector–function f belongs to the Morrey space $L^{1,\lambda}(\Omega, I\!R^N)$, with $\Omega \subset I\!R^n$, $n \geq 3$, $N \geq 2$, $\lambda \in [0, n-2]$, then there exists a unique very weak solution u of the system

$$\begin{cases} -D_i\,(A_{ij}(x)D_j\,u) = f & \text{in } \Omega, \\ u \in W_0^{1,1}(\Omega, I\!R^N), \end{cases}$$

such that Du belongs to the space $L^{q,n-q(n-\lambda-1)}(\Omega, I\!R^{nN})$ for any $q \in [1, \frac{n}{n-1}[$, provided the matrix of coefficients (A_{ij}) has $L^\infty \cap VMO$ entries.

Keywords: Elliptic systems; VMO-coefficients; L^1-data

AMS Subject Classifications: 35J25; 35D10

1. Introduction

This article is devoted to the study of existence and regularity of solutions to the Dirichlet problem associated to the system[1]

$$\begin{cases} A(u) \equiv -D_i\,(A_{ij}(x)D_j\,u) = f, \\ u \in W_0^{1,1}(\Omega, I\!R^N), \end{cases} \tag{1}$$

where Ω is an open bounded subset of $I\!R^n$ $(n \geq 3)$ with C^1-boundary, A an elliptic operator with VMO-coefficients and f belongs to the Morrey space $L^{1,\lambda}(\Omega,\ I\!R^N)$, $\lambda \in [0, n-2]$.

In a previous article [1] we established the local regularity of the very weak solution to the aforementioned problem in a suitable Morrey space; it is the aim of this article to extend that result up to the boundary of the domain Ω.

*Corresponding author. Email: stara@karlin.mff.cuni.cz
†Dedicated to Franco Nicolosi on his 70th birthday.

ISSN 1747–6933 print/ISSN 1747–6941 online

http://dx.doi.org/10.1080/17476933.2010.534152
http://www.tandfonline.com

We point out that the extension of the regularity property up to the boundary of the domain requires a new decomposition of the solution in a neighbourhood of a boundary point and for this purpose we will follow the idea of [2].

The study of linear elliptic equations ($N=1$) with L^1 (or measure) right-hand side and bounded coefficients was started by Stampacchia [3,4] and was later treated by many authors and by different approaches while, for elliptic systems ($N \geq 2$), several existence results were obtained under additional structural conditions: a short survey of known results was given in [1].

In this article we consider linear systems with VMO coefficients without further structural conditions and prove existence and regularity results for very weak solutions (briefly called Stampacchia solutions).

An important ingredient in our approach is the so called A-harmonic approximation Lemma of Duzaar and Steffen [5] (see also [6,7]), a new method allowing for a rapid and elegant implementation of certain comparison procedures.

A description of such a method has been given in the article [1].

We remark that recent regularity results in Morrey spaces of the type $L^{1,\lambda}$ are in the papers [8–12].

This article is organized as follows: we start with notations and a few auxiliary results in Section 2. In Section 3 we recall known [1,13] Morrey spaces regularity result saying that for $f \equiv D_i g_i$, with $g_i \in L^{2,\lambda}(\Omega, I\!R^N)$, the gradient Du of the solution u to the problem (1) belongs to the same space $L^{2,\lambda}(\Omega, I\!R^{nN})$.

This assertion allows us to state the existence of the Stampacchia solution to (1) for any $f \in L^1(\Omega, I\!R^N)$ in Section 4.

An analogue of Saint-Venant's principle for solutions of (1) with $f \equiv 0$ is given in Section 5. Finally, in Section 6 these results, combined with a Campanato-type approach, yield to the local regularity of Du in a suitable Morrey space.

2. Some notations and auxiliary results

In $I\!R^n$ ($n \geq 3$), with generic point $x = (x_1, x_2, \ldots, x_n)$, we shall denote by Ω a bounded open nonempty set with diameter d_Ω and C^1-boundary $\partial\Omega$.

For $\rho > 0$ and $x_o \in I\!R^n$ we define

$$\begin{aligned} B(x_o, \rho) &= \{x \in I\!R^n : |x - x_o| < \rho\}, \\ \Omega(x_o, \rho) &= \Omega \cap B(x_o, \rho), \\ \mathrm{d}\,(x_o, \partial\Omega) &= \mathrm{dist}(x_o, \partial\Omega). \end{aligned}$$

If $y_o = (y_{o1}, \ldots, y_{on-1}, 0)$ we define

$$\begin{aligned} B^+(y_o, \rho) &= \{x \in B(y_o, \rho) : x_n > 0\}, \\ \Gamma(y_o, \rho) &= \{x \in B(y_o, \rho) : x_n = 0\}. \end{aligned}$$

Moreover, if $u \in L^1(\Omega(x_o, \rho), I\!R^N)$ we denote by

$$u_{\Omega(x_o,\rho)} = \frac{1}{|\Omega(x_o, \rho)|} \int_{\Omega(x_o,\rho)} u(x)\mathrm{d}x,$$

where $|\Omega(x_o, \rho)|$ is the n-dimensional Lebesgue measure of $\Omega(x_o, \rho)$.

Definition 2.1 (Morrey space) Let $q \geq 1$ and $0 \leq \lambda < n$. By $L^{q,\lambda}(\Omega, I\!R^N)$ we denote the linear space formed by the vector-functions $u \in L^q(\Omega, I\!R^N)$ for which

$$\|u\|_{L^{q,\lambda}(\Omega)} = \sup_{x_o \in \Omega,\, 0<\rho\leq d_\Omega} \left\{ \rho^{-\lambda} \int_{\Omega(x_o,\rho)} |u(x)|^q \, \mathrm{d}x \right\}^{1/q} < +\infty,$$

$L^{q,\lambda}(\Omega, I\!R^N)$ equipped with the above norm is a Banach space.

Definition 2.2 (Campanato space) Let $q \geq 1$ and $0 \leq \lambda < n+q$. By $\mathcal{L}^{q,\lambda}(\Omega, I\!R^N)$ we denote the space of all vector-functions $u \in L^q(\Omega, I\!R^N)$ such that

$$[u]_{\mathcal{L}^{q,\lambda}(\Omega)} = \sup_{x_o \in \Omega,\, 0<\rho\leq d_\Omega} \left\{ \rho^{-\lambda} \int_{\Omega(x_o,\rho)} |u(x) - u_{\Omega(x_o,\rho)}|^q \, \mathrm{d}x \right\}^{1/q} < +\infty.$$

Moreover, we introduce the notion of BMO and VMO classes.

Definition 2.3 (John–Nirenberg space) Let Q be a cube in $I\!R^n$. By BMO(Q) we denote the space of all functions $u \in L^1(Q, I\!R^{N^2})$ such that the semi-norm defined by

$$[u]_{BMO(Q)} = \sup_{\tilde{Q} \subset Q} \frac{1}{|\tilde{Q}|} \int_{|\tilde{Q}|} |u - u_{\tilde{Q}}| \, \mathrm{d}x$$

is finite, where the supremum is taken over all cubes with sides parallel to coordinate axes.

Let us recall that $\mathcal{L}^{q,n}(Q) \cong \mathrm{BMO}(Q)\ \forall q \geq 1$.

Definition 2.4 (Sarason space) For a matrix-function $w \in L^1(\Omega, I\!R^{N^2})$ and $r > 0$ we define

$$V(x,r) \equiv \sup_{0<\rho\leq r} \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |w(y) - w_{\Omega(x,\rho)}| \mathrm{d}y$$

and we introduce the VMO-continuity modulus for w

$$V(r) \equiv \sup_{x \in \Omega} V(x,r).$$

By VMO we denote the space of all matrix-functions $w \in L^1(\Omega, I\!R^{N^2})$ such that

$$V(r) < +\infty \quad \text{for all } 0 < r \leq d_\Omega$$

and

$$\lim_{r \to 0} V(r) = 0.$$

3. $L^{2,\lambda}$-regularity of Du on $\overline{\Omega}$

If $u : \Omega \to I\!R^N$, we set

$$D_i \equiv \frac{\partial}{\partial x_i}, \quad Du = (D_i u^r)_{\substack{i=1,\dots,n \\ r=1,\dots,N}}.$$

Let $A_{ij}(x)$, $i,\ j=1,2,\ldots,n$, be matrix-functions for which the following conditions be satisfied:

$$\begin{aligned} A_{ij}(x) &= \left(A_{ij}^{rs}(x)\right)_{r,s=1,\ldots,N} \in L^{\infty}(\Omega, I\!R^{N^2}) \cap VMO, \\ A_{ij}^{rs}(x) &= A_{ji}^{sr}(x) \quad \text{for a.a. } x \in \Omega. \end{aligned} \tag{2}$$

There exist two positive constants Λ_1 and Λ_2 such that

$$\begin{gathered} \Lambda_2\,|\xi|^2 \geq A_{ij}(x)\xi_i\xi_j \geq \Lambda_1|\xi|^2 \\ \text{for a.a. } x \in \Omega,\ \forall \xi = (\xi_i^r) \in I\!R^{nN}. \end{gathered} \tag{3}$$

For $x \in \overline{\Omega}$, $0 < r \leq d_\Omega$, we set

$$V(x,r) \equiv \sup_{\substack{0<\rho\leq r \\ i,j=1,2,\ldots,n}} \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |A_{ij}(y) - (A_{ij})_{\Omega(x,\rho)}|\,\mathrm{d}y. \tag{4}$$

In this section we are concerned with regularity of weak solution $u \in W_0^{1,2}(\Omega, I\!R^N)$ of the problem

$$D_i(A_{ij}(x)D_ju) = D_ig_i \quad \text{in } \Omega, \tag{5}$$

where Ω is an open bounded domain with C^1 boundary, $g_i \in L^{2,\lambda}(\Omega, I\!R^N)$ and the coefficients satisfy conditions (2) and (3).

Theorem 3.1 [1] *Let $u \in W_0^{1,2}(\Omega, I\!R^N)$ be the weak solution to the problem* (5), *let conditions* (2) *and* (3) *be satisfied. Assume that $g_i \in L^{2,\lambda}(\Omega, I\!R^N)$ for $\lambda \in [0, n[$.*

Then $Du \in L^{2,\lambda}(\Omega, I\!R^{nN})$ and there exists a positive constant $c = c_V(n, \lambda, \Lambda_1, \Lambda_2, \Omega)^2$ such that the inequality

$$\|Du\|_{L^{2,\lambda}(\Omega)} \leq c\,\|g\|_{L^{2,\lambda}(\Omega)} \tag{6}$$

holds.

In particular, if

$$\lambda \in]n-2, n[\tag{7}$$

then $u \in C^{0,\gamma}(\overline{\Omega}, I\!R^N)$, with $\gamma = 1 - \frac{n-\lambda}{2}$, and the inequality

$$[u]_{C^{0,\gamma}(\overline{\Omega})} \leq c\,\|g\|_{L^{2,\lambda}(\Omega)} \tag{8}$$

holds.

Corollary 3.1 *Let $x_o \in I\!R^n$, $\Omega_R = \{x = x_o + Ry : y \in \Omega\}$, $0 < R \leq 1$ and assume the hypotheses of the Theorem. Then $u \in C^{0,\gamma}(\overline{\Omega_R}, I\!R^N)$ and there exists a positive constant $c = c_V(n,\ \lambda,\ \Lambda_1,\ \Lambda_2)$, which is independent of R, such that the inequality*

$$[u]_{C^{0,\gamma}(\overline{\Omega_R})} \leq c\,\|g\|_{L^{2,\lambda}(\Omega_R)} \tag{9}$$

holds.

Proof

The Corollary is true for $R = 1$ by the previous theorem.

The rest of the proof follows readily as in the Corollary 3.1 of [1].

The following Lemma is the analogue of Theorem 9.1 ([14] p. 339).

Lemma 3.1 [1] *Let* $u \in W^{1,2}(\Omega, I\!R^N)$ *be a solution to Equation* (5) *where* $g_i \in L^{2,\lambda}(\Omega, I\!R^N)$, *with* $\lambda \in [0, n[$, *and let conditions* (2) *and* (3) *be satisfied.*

Then there exist two positive constants $c = c(n, \lambda, \Lambda_1, \Lambda_2)$ *and* $\rho_o = \rho_V(n, \lambda, \Lambda_1, \Lambda_2)$ *such that*

$$\|Du\|_{L^{2,\lambda}(B(z_o, R/4))} \leq c \left(R^{-\lambda} \|Du\|_{L^2(B(z_o, R))} + \| g\|_{L^{2,\lambda}(\Omega)}\right) \tag{10}$$

for any $z_o \in \Omega$ *and* $0 < R < \min\{d(z_o, \partial\Omega), \rho_o\}$.

We will recall now the estimates on half ball obtained in [1].

For fixed $y_o = (y_{o1}, y_{o2}, \dots, 0)$ and $R > 0$, let us take into account the system

$$\begin{aligned} D_i(B_{ij}(x)D_ju') &= D_ig_i \quad \text{in } B^+(y_o, R), \\ u' &= 0 \quad \text{on } \Gamma(y_o, R) \end{aligned} \tag{11}$$

under the following structural assumptions:

$$\begin{aligned} &B_{ij}(x) = \left(B_{ij}^{rs}(x)\right)_{r,s=1,\dots,N} \in L^\infty(B^+(y_o, R), I\!R^{N^2}) \cap VMO, \\ &B_{ij}^{rs}(x) = B_{ji}^{sr}(x) \quad \text{for a.a. } x \in B^+(y_o, R). \end{aligned} \tag{12}$$

There exist two positive constants Λ_1' and Λ_2' such that

$$\Lambda_2' |\xi|^2 \geq B_{ij}(x)\xi_i\xi_j \geq \Lambda_1' |\xi|^2 \quad \text{for a.a. } x \in B^+(y_o, R)\ , \forall \xi = (\xi_i^r) \in I\!R^{nN}. \tag{13}$$

We denote by V' the VMO–continuity modulus for the matrix B_{ij}.

Definition 3.1 A vector-function $u' \in W^{1,2}(B^+(y_o, R), I\!R^N)$ is a weak solution of system (11) if

$$\begin{cases} \displaystyle\int_{B^+(y_o,R)} B_{ij}(x)D_iu'\, D_j\varphi, \mathrm{d}x = \int_{B^+(y_o,R)} D_ig\, D_j\varphi\, \mathrm{d}x & \forall \varphi \in W_0^{1,2}(B^+(y_o, R), I\!R^N) \\ u' = 0 & \text{on } \Gamma(y_o, R). \end{cases}$$

The following Lemma is the analogue of Theorem 13.1 ([14], p. 355).

Lemma 3.2 [1] *Let* $u' \in W^{1,2}(B^+(y_o, R), I\!R^N)$ *be a solution to problem* (11) *where* $g_i \in L^{2,\lambda}(B^+(y_o, R), I\!R^N)$, *with* $\lambda \in [0, n[$, *and let conditions* (12) *and* (13) *be satisfied.*

Then there exist two positive constants $c = c(n, \lambda, \Lambda_1', \Lambda_2')$ *and* $\overline{\rho} = \overline{\rho}_{V'}(n, \lambda, \Lambda_1', \Lambda_2')$ *such that*

$$\|Du'\|_{L^{2,\lambda}(B^+(y_o, R_o))} \leq c \left[(\min\{\overline{\rho}, R - R_o\})^{-\lambda} \|Du'\|_{L^2(B^+(y_o, R))} + \| g\|_{L^{2,\lambda}(B^+(y_o, R))}\right] \tag{14}$$

for any $0 < R_o < R$.

We recall here the proof of the Theorem 3.1 for the reader's convenience, since the procedure will be used later on.

Proof of Theorem 3.1 Since Ω is of class C^1 and bounded (see e.g. [15], p. 305), there exist a positive R and an open finite covering $\{\mathcal{B}(\overline{y}^j, R)\}_{j=1,\dots,\nu}$ of $\partial\Omega$ such that for all

$\overline{y}^j$ there exists a C^1-function ζ^j, defined on a domain $D \subset I\!R^{n-1}$ such that with respect to a suitable system of coordinates $\{y_1, \ldots, y_n\}$, with the origin at $\overline{y}^j$:

(a) the set $\partial\Omega \cap \mathcal{B}(\overline{y}^j, R)$ can be represented by an equation of the type

$$y_n = \zeta^j(y_1, \ldots, y_{n-1}),$$

(b) each $y \in \Omega \cap \mathcal{B}(\overline{y}^j, R)$ satisfies

$$y_n > \zeta^j(y_1, \ldots, y_{n-1}).$$

Without loss of generality we can suppose that the system of coordinates is such that the hyperplane tangent to $\partial\Omega$ at $\overline{y}^j$ has equation $y_n = 0$, that

$$\zeta^j(\overline{y}^j) = D\zeta^j(\overline{y}^j) = 0 \tag{15}$$

and that R is such that $\max_{j=1,\ldots,\nu} \max_{\overline{\mathcal{B}(\overline{y}^j, R)\cap\Omega}} |D\zeta^j| < 1/2$.

For such domains the portion of boundary within the ball $\mathcal{B}(\overline{y}^j, R)$ can be straightened by means of the smooth transformation[3].

$$\begin{cases} \psi_i(y) = y_i - (\overline{y}^j)_i & \text{for } i = 1, 2, \ldots, n-1, \\ \psi_n(y) = y_n - \zeta^j(y_1, \ldots, y_{n-1}). \end{cases} \tag{16}$$

It turns out that $\psi(y) = (\psi_1(y), \ldots, \psi_n(y))$ is a $C^1(\mathcal{B}(\overline{y}^j, R))$-diffeomorphism verifying the following properties (see e.g. [15], p. 305 or [14], Theorem V, p. 375):

(i) $\psi(\mathcal{B}(\overline{y}^j, R) \cap \partial\Omega) = \{x \in I\!R^n : x_n = 0, |x_i| < R, \text{ for } i = 1, \ldots, n-1\}$,
(ii) $\frac{1}{2}|y - \overline{y}^j| \le |\psi(y)| \le \frac{3}{2}|y - \overline{y}^j| \ \forall y \in \mathcal{B}(\overline{y}^j, R) \cap \Omega$,
(iii) $B^+(0, R/2) \subset \psi(\mathcal{B}(\overline{y}^j, R) \cap \Omega) \subset B^+(0, \frac{3}{2}R)$, $\quad \mathcal{B}(\overline{y}^j, \frac{2}{3}R) \cap \Omega \subset \psi^{-1}(B^+(0, R)) \subset \mathcal{B}(\overline{y}^j, 2R) \cap \Omega$.

If $z \in B^+(0, R)$ we set

$$\begin{aligned} B_{ik}(z) &= A_{rs}(\psi^{-1}(z)) \frac{\partial\psi_i}{\partial y_r}(\psi^{-1}(z)) \frac{\partial\psi_k}{\partial y_s}(\psi^{-1}(z)), \\ g_i(z) &= f_r(\psi^{-1}(z)) \frac{\partial\psi_i}{\partial y_r}(\psi^{-1}(z)), \\ u'(z) &= u(\psi^{-1}(z)), \end{aligned} \tag{17}$$

where we have used the fact that the absolute value of the Jacobian determinant of $\psi^{-1}(z)$ is equal to 1.

Let us observe that Lemma 2.1 of [1] guarantees that the coefficients $B_{\mathrm{ik}}(z)$ still satisfy hypothesis (12).

Moreover, from definition (16) and the fact that $\max_{\overline{\mathcal{B}(\overline{y}^j, R)\cap\Omega}} |D\zeta^j| < 1/2$, it follows that

$$(1/2)^2 \Lambda_1 |\eta|^2 \le B_{ik}\eta_i\eta_k \le (3/2)^2 \Lambda_2 |\eta|^2 \quad \forall \eta = (\eta_i) \in I\!R^{nN}. \tag{18}$$

Thus a change of variables in the system (5) yields

$$\begin{cases} u' \in W^{1,2}(B^+(0,R)), \\ u' = 0 \quad \text{on } \Gamma(0,R), \\ \displaystyle\int_{B^+(0,R)} B_{ik}\, D_k u'\, D_i\varphi \,\mathrm{d}z = \int_{B^+(0,R)} D_i g\, D_i \varphi\,\mathrm{d}z \quad \forall \varphi \in W_0^{1,2}(B^+(0,R)). \end{cases} \tag{19}$$

To the problem (19) we apply Lemma 3.2 and so we conclude that Du' lies in $L^{2,\lambda}(B^+(0,R_o))$, $R_o \in]0, R[$, with norm estimate (14).

As a consequence, the matrix-function $Du'(\psi(y))$, $y \in \mathcal{B}(\bar{y}^j, r) \cap \Omega$, $r \in]0, \frac{2}{3}R_o[$, belongs to $L^{2,\lambda}(\mathcal{B}(\bar{y}^j, r) \cap \Omega)$ that is, by the chain rule, $Du \in L^{2,\lambda}(\mathcal{B}(\bar{y}^j, r) \cap \Omega)$.

Thus by changing back coordinates in (14) we deduce that

$$\|Du\|_{L^{2,\lambda}(\mathcal{B}(\bar{y}^j, r)\cap\Omega)} \le c\big[\|Du\|_{L^2(\Omega)} + \|f\|_{L^{2,\lambda}(\Omega)}\big], \tag{20}$$

where $c = c_V(n, \lambda, \Lambda_1, \Lambda_2, R, R - R_o)$.

Since R_o is arbitrary, it can be chosen sufficiently close to R so that the family $\{\mathcal{B}(\bar{y}^j, r)\}_{j=1,\dots,\nu}$ still cover $\partial\Omega$.

On the other hand, set

$$\delta := \min_{\partial\Omega} d\left(x, \mathbb{R}^n \Big\backslash \bigcup_{j=1}^{\nu} \mathcal{B}(\bar{y}^j, r)\right) > 0,$$

the open set

$$H = \{x \in \Omega : d(x, \partial\Omega) > \delta/2\} \subset\subset \Omega$$

is such that H, $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_\nu$ cover $\overline{\Omega}$.

The aforementioned remarks, the use of Corollary 3.2 of [1] and Lax–Milgram Theorem prove the Theorem (see e.g. [14, p. 365, 366] or [16, pp. 252–255]).

4. Existence and uniqueness of the Stampacchia solution

Stampacchia proved in [3,4], by duality method, the existence and uniqueness of the weak solution to Dirichlet boundary problem for elliptic equations with non smooth coefficients and right-hand side measure.

Definition 4.1 Let $f \in L^1(\Omega, \mathbb{R}^N)$. We say that a vector-function $u \in W_0^{1,1}(\Omega, \mathbb{R}^N)$ is a very weak solution (briefly Stampacchia solution) of the system (1) if it satisfies

$$\int_\Omega u\, A(\varphi)\mathrm{d}x = \int_\Omega f\,\varphi\,\mathrm{d}x$$
$$\forall \varphi \in \Phi = \left\{\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N) \cap C^0(\bar{\Omega}, \mathbb{R}^N) : A(\varphi) \in C^0(\bar{\Omega}, \mathbb{R}^N)\right\}. \tag{21}$$

The proof of the existence and uniqueness of the Stampacchia solution to (1) follows the same steps as in papers [1,10,17].

THEOREM 4.1 [1] *Let Ω be a bounded domain with C^1-boundary and $f \in L^1(\Omega, \mathbb{R}^N)$. Let conditions* (2) *and* (3) *be satisfied.*

Then there exists a unique Stampacchia solution u of problem (1) *such that* $u \in W_0^{1,q}(\Omega, I\!R^N)$ *for any* $q < \frac{n}{n-1}$.

Moreover, there exists a positive constant $c = c_V(n, q, \Lambda_1, \Lambda_2, \Omega)$ *such that*

$$\|u\|_{W_0^{1,q}(\Omega)} \leq c\, \|f\|_{L^1(\Omega)}. \tag{22}$$

Corollary 4.1 *Let* $x_o \in I\!R^n$, $\Omega_R = \{x = x_o + Ry : y \in \Omega\}$, $0 < R \leq 1$ *and assume the hypotheses of the Theorem. Then there exists a unique Stampacchia solution u of problem* (1) *such that* $u \in W_0^{1,q}(\Omega_R, I\!R^N)$ *for any* $q < \frac{n}{n-1}$.

Moreover, there exists a positive constant $c = c_V(n, q, \Lambda_1, \Lambda_2)$, *which is independent of R, such that*

$$\|u\|_{W_0^{1,q}(\Omega_R)} \leq cR^{1-n\left(1-\frac{1}{q}\right)}\, \|f\|_{L^1(\Omega_R)}. \tag{23}$$

Proof The proof follows readily from Theorem 4.1 of [1] using a standard homotopy argument on Ω and taking into account (9) from Corollary 3.1.

5. Saint-Venant's principle

In this section we consider weak solutions of the homogeneous systems

$$-D_i(A_{ij}(x)D_j v) = 0 \quad \text{in } \Omega. \tag{24}$$

As the right-hand side $g \equiv 0$, Lemmas 3.3 and 3.4 of [1] hold with any $\lambda \geq 0$. For $\lambda > n-2$ any weak solution u', $v \in W^{1,2}$ to problems (11) and (24), respectively, is locally Hölder continuous on Ω and $B^+(y_o, R) \cup \Gamma(y_o, R)$, respectively.

Exploiting Lemma 3.2 and arguing as in Lemma 5.1 and Theorem 5.1 of [1] we get the following 'half-ball version' of Hölder semi-norm estimate and Saint-Venant principle.

Lemma 5.1 *Let* u' *be a solution to problem* (11), *let conditions* (12) *and* (13) *be satisfied and assume that* $\lambda \in]n-2, n[$, $\gamma = 1 - \frac{n-\lambda}{2}$, $q \in [1, 2[$.

Then there exists a positive constant $c = c(\overline{\rho}, n, \lambda, \Lambda'_1, \Lambda'_2)^4$ *such that it holds*

$$[u']_{C^{0,\gamma}(\overline{B^+(y_o, \rho/4)})} \leq c\, \rho^{n(1/2-1/q)-\lambda/2} \|Du'\|_{L^q(B^+(y_o,\rho))} \tag{25}$$

$\forall\, 0 < \rho < \frac{1}{16} R$.

Theorem 5.1 (Saint-Venant Principle on B^+) *Let* u' *be a solution to problem* (11), *let conditions* (12) *and* (13) *be satisfied and assume that* $\lambda \in]n-2, n[$, $\gamma = 1 - \frac{n-\lambda}{2}$, $q \in [1, 2[$.

Then, there exist a positive constant c independent of y_o *such that, for any weak solution v to system* (11), *it holds*

$$\|Du'\|^q_{L^q(B^+(y_o,\rho_1))} \leq c\left(\frac{\rho_1}{\rho_2}\right)^{n-q+\gamma q} \|Du'\|^q_{L^q(B^+(y_o,\rho_2))} \tag{26}$$

$\forall 0 \leq \rho_1 \leq \rho_2 < R$.

6. Global regularity of the Stampacchia solution

First of all let us introduce the truncation operator. For a given constant $k>0$ we define the cut off function $T_k : I\!R \to I\!R$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k\,\mathrm{sign}(s) & \text{if } |s| > k. \end{cases}$$

For a vector-function $f=(f^r(x))_{r=1,\dots,N}$, $x\in\Omega$, we define the truncated vector-function $f_k=(T_k(f^r))_{r=1,\dots,N}$ point-wise: for every $x\in\Omega$ the value of f_k at x is just $(T_k(f^r(x)))_{r=1,\dots,N}$.

Throughout this section we shall assume that the right-hand side of (1)

$$f\in L^{1,\lambda}(\Omega, R^N), \quad \lambda\in[0, n-2].$$

For such a vector-function let us consider a sequence of functions $\{f_k\}_{k\in I\!N}$ such that

(i) $f_k\in W^{-1,2}(\Omega, I\!R^N)\cap L^{1,\lambda}(\Omega, I\!R^N)$ $\forall k\in I\!N$,
(ii) $f_k\to f$ in $L^1(\Omega, I\!R^N)$ as $k\to+\infty$,
(iii) $\|f_k\|_{L^1(\Omega)}\le\|f\|_{L^1(\Omega)}$ $\forall k\in I\!N$,
(iv) $\|f_k\|_{L^{1,\lambda}(\Omega)}\le\|f\|_{L^{1,\lambda}(\Omega)}$ $\forall k\in I\!N$.

An example of a sequence satisfying the above requirements is the sequence $\{T_k(f)\}_{k\in IN}$.

For fixed $k\in I\!N$, let u_k be the weak solution of the system

$$-D_i(A_{ij}(x)D_ju_k)=f_k \quad \text{in } \Omega \tag{27}$$

that is,

$$\begin{cases} u_k\in W_0^{1,2}(\Omega, I\!R^N), \\ \displaystyle\int_\Omega A_{ij}(x)D_ju_kD_i\varphi\,\mathrm{d}x=\int_\Omega f_k\,\varphi\,\mathrm{d}x \quad \forall\varphi\in W_0^{1,2}(\Omega, I\!R^N). \end{cases}$$

As in Section 3 we get the global estimate by combining the local interior estimate, the estimate on half ball and the local flattening of the boundary of Ω.

Let us recall the following local estimate proved in [1].

Theorem 6.1 [1] *Assume that hypotheses* (2), (3) *hold and let u_k be the weak solution of problem* (27).

Then

$$Du_k\in L^{q,\nu}_{loc}(\Omega, I\!R^{nN}) \quad \forall q\in\left[1, \frac{n}{n-1}\right[\;\; \forall k\in IN,$$

with $\nu=n-q(n-\lambda-1)$, and for all $H\subset\subset\Omega$ there exists a positive constant $c=c_V(n,\lambda,q,\Lambda_1,\Lambda_2,d(\overline{H},\partial\Omega))$ such that

$$\|Du_k\|_{L^{q,\nu}(H)}\le c\left[\|Du_k\|_{L^q(\Omega)}+\|f\|_{L^{1,\lambda}(\Omega)}\right] \quad \forall k\in IN. \tag{28}$$

We now consider the case of a half ball, i.e.

$$\begin{aligned} D_i(B_{ij}(x)D_ju_k) &= f_k \quad \text{in } B^+(y_o, R), \\ u_k &= 0 \quad \text{on } \Gamma(y_o, R) \end{aligned} \tag{29}$$

and prove the following theorem.

THEOREM 6.2 *Assume that hypotheses* (12), (13) *hold and let u_k be the weak solution of problem* (29).

Then

$$Du_k \in L^{q,\nu}(B^+(y_o, R_o), I\!R^{nN}) \quad \forall q \in \left[1, \frac{n}{n-1}\right[, \quad \forall k \in IN,$$

with $\nu = n - q(n - \lambda - 1)$, and for all $R_o < R$. Moreover, there exists a positive constant $c = c_{V'}(n, \lambda, q, \Lambda_1', \Lambda_2', R_o)$ such that

$$\begin{aligned} \|Du_k\|_{L^{q,\nu}(B^+(y_o, R_o))} \leq c\,\big[&\|Du_k\|_{L^q(B^+(y_o, R))} \\ &+ \|f\|_{L^{1,\lambda}(B^+(y_o, R))}\big] \quad \forall k \in I\!N. \end{aligned} \tag{30}$$

Proof We will follow the idea of the proof of Theorem 6.1 of [1].

Fix $k \in I\!N$, $x_o \in \Gamma(y_o, R)$, $\rho \in]0, \min\{1, \overline{\rho}, R - |x_o - y_o|\}$[5] and extend f_k by zero to $I\!R^n$.

Let G be a bounded C^1-domain in $I\!R^n{}_+ = \{x = (x', x_n) \in I\!R^n : x_n > 0\}$ containing $B^+(0, 1)$[6] and, for any positive ρ, denote by

$$G_\rho = \{x = x_o + \rho y : y \in G\}.$$

Then $B^+(x_o, \rho) \subset G_\rho \subset I\!R^n{}_+$.

For a cut off function $\eta \in C^\infty(I\!R^n)$ with $\operatorname{supp}\eta \in B(0, 1)$; $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B(0, 1/4)$, put $\varphi(x) = \eta(\frac{x - x_o}{\rho})$ for $x \in G_\rho$ and $g_k(x) = f_k(x)\varphi(x)$ on G_ρ.

Let $w_k \in W_0^{1,2}(G_\rho, I\!R^N)$ be the weak solution of the Dirichlet problem

$$\begin{cases} -D_i(B_{ij}(x)D_jw_k) = g_k & \text{in } G_\rho \\ w_k = 0 & \text{on } \partial G_\rho \end{cases} \tag{31}$$

and observe that for $v_k = u_k - w_k$ we have

$$-D_i(B_{ij}(x)D_jv_k) = 0 \quad \text{in } B^+(y_o, \rho/4).$$

Since any weak solution of the problem (31) is also a very weak solution of the same problem then, by (23) and by item (iv) it follows that, for any $q \in \left[1, \frac{n}{n-1}\right[$,

$$\begin{aligned} \|w_k\|^q_{W^{1,q}(B^+(x_o, \rho))} \leq \|w_k\|^q_{W^{1,q}(G_\rho)} &\leq c\rho^{n+q-nq}\|g_k\|^q_{L^1(G_\rho))} \\ &\leq c\rho^{n+q-nq}\|f_k\|^q_{L^1(B^+(x_o,\rho))}. \end{aligned} \tag{32}$$

Gathering together (26) and (32) we deduce that, for any $\sigma < \rho/4$,

$$\begin{aligned}\|D\,u_k\|^q_{L^q(B^+(x_o,\sigma))} &\le c\left[\left(\frac{\sigma}{\rho}\right)^{n-q+\gamma q}\|Dv_k\|^q_{L^q(B^+(x_o,\rho))} + \rho^{n+q+\lambda q-nq}\|f\|^q_{L^{1,\lambda}(B^+(x_o,\rho))}\right]\\ &\le c\left[\left(\frac{\sigma}{\rho}\right)^{n-q+\gamma q}\|Du_k\|^q_{L^q(B^+(x_o,\rho))} + \rho^{n+q+\lambda q-nq}\|f\|^q_{L^{1,\lambda}(B^+(x_o,R))}\right].\end{aligned}$$

An application of Lemma 1.1 of [18, p. 7] to the above inequality gives

$$\begin{aligned}\|D\,u_k\|^q_{L^q(B^+(x_o,\sigma))} &\le c\left[\left(\frac{\sigma}{\rho}\right)^{n+q+\lambda q-nq}\|Du_k\|^q_{L^q(B^+(x_o,\rho))} + \sigma^{n+q+\lambda q-nq}\|f\|^q_{L^{1,\lambda}(B^+(x_o,R))}\right].\end{aligned} \tag{33}$$

The proof can now be completed as in Lemma 3.4 of [1].

Now we are in the position to prove the following theorem.

Theorem 6.3 *Assume Ω to be a bounded domain with C^1-boundary and hypotheses* (2), (3) *to be satisfied.*

Let, moreover, u be the Stampacchia solution of problem (1).

Then

$$Du \in L^{q,\nu}(\Omega, \mathbb{R}^{nN}), \quad \forall q \in \left[1, \frac{n}{n-1}\right[,$$

with $\nu = n - q(n - \lambda - 1)$, and there exists a positive constant $c = c_V(n, \lambda, q, \Lambda_1, \Lambda_2, \Omega)$ such that

$$\|Du\|_{L^{q,\nu}(\Omega)} \le c\left[\|Du\|_{L^q(\Omega)} + \|f\|_{L^{1,\lambda}(\Omega)}\right]. \tag{34}$$

Proof We have already remarked (see Theorem 2.1, formula (22)) that

$$\|Du_k\|_{L^q(\Omega)} \le c_V(n, q, \Lambda_1, \Lambda_2, \Omega)\,\|f\|_{L^1(\Omega)} \quad \forall k \in \mathbb{N},\ \forall q \in \left[1, \frac{n}{n-1}\right[.$$

This information allows us to deduce that there exists a subsequence $\{u_{n_k}\} \subset \{u_k\}$ such that

(a) $u_{n_k} \rightharpoonup v$ in $W^{1,q}(\Omega, \mathbb{R}^N)$ as $k \to +\infty$ $\forall q \in [1, \frac{n}{n-1}[$,
(b) $u_{n_k} \to v$ in $L^q(\Omega, \mathbb{R}^N)$ and a.e. in Ω as $k \to +\infty$ $\forall q \in [1, \frac{n}{n-1}[$,
(c) the function v is a Stampacchia solution of the Dirichlet problem (1).

By the uniqueness of the Stampacchia solution we can conclude that $v = u$.
To achieve the thesis we need only to show that $\mathrm{Du} \in L^{q,\nu}(\Omega)$.
For this purpose let us fix $H \subset\subset \Omega$, $x_o \in H$ and $\rho \in]0, d_H]$.
Since, by (a), we have

$$Du_{n_k} \rightharpoonup Du \ \text{ in } L^q(H(x_o, \rho), \mathbb{R}^N).$$

By virtue of (28) we obtain

$$\begin{aligned}\|D\,u\|^q_{L^q(H(x_o,\rho))} &\le \liminf_{k\to+\infty} \|D\,u_{n_k}\|^q_{L^q(H(x_o,\rho))}\\ &\le \rho^\nu \liminf_{k\to+\infty} \|D\,u_{n_k}\|^q_{L^{q,\nu}(H)}\\ &\le c\,[\|Du\|_{L^q(\Omega)} + \|f\|_{L^{1,\lambda}(\Omega)}]\,\rho^\nu,\end{aligned}$$

where $c = c_V(n, \lambda, q, \Lambda_1, \Lambda_2, d(\overline{H}, \partial\Omega)) > 0$.

The above inequality proves the estimate in the interior of Ω.

In an analogous manner, as in the proof of the Theorem 3.1 from [1], formula (30) yields the estimate, for any $\rho \in]0, \frac{2}{3}R_o[$ and $R_o \in]0, R[$,

$$\|Du_k\|_{L^q(\mathcal{B}(\overline{y}^j,\rho)\cap\Omega)} \le c\,\rho^\nu\big[\|Du_k\|_{L^q(\Omega)} + \|f\|_{L^{1,\lambda}(\Omega)}\big], \tag{35}$$

where $c = c_V(n, \lambda, \Lambda_1, \Lambda_2, R - R_o)$ and $\mathcal{B}(\overline{y}^j, \rho)$ is the generic element of an open finite covering $\{\mathcal{B}(\overline{y}^j, R)\}_{j=1,\dots,\nu}$ of $\partial\Omega$.

The above two inequalities together with the standard argument of covering and flattening $\partial\Omega$ ends the proof (see Theorem 3.1).

Corollary 6.1 *Assume the same hypotheses of Theorem* 6.3 *and that* $\lambda \in]0, n-2[$. *Then*

$$u \in L^{\beta,\nu}(\Omega, I\!R^N)$$

for all $\beta \in [1, \frac{q(n-\lambda-1)}{n-\lambda-2}[$.

The proof of the Corollary 6.1 is an easy consequence of the following useful Lemma (proved in a slightly different form in [1, Lemma 6.1] and in [9, Lemma 5.1]) applied to each component of u.

Lemma 6.1 *Let* $v \in W^{1,p}_0(\Omega, I\!R)$ *such that* $Dv \in L^{p,\kappa}(\Omega, I\!R^n)$, *with* $\kappa \in]0, n-p[$. *Then*

$$v \in L^{p_\kappa,\kappa}(\Omega, I\!R),$$

where $\frac{1}{p_\kappa} = \frac{1}{p} - \frac{1}{n-\kappa}$.

Moreover, there exists a positive constant $c = c(n, p, \Omega)$ *such that*

$$\|v\|_{L^{p_\kappa,\kappa}(\Omega)} \le c\,[\|D\,v\|_{L^p(\Omega)} + \|D\,v\|_{L^{p,\kappa}(\Omega)}]. \tag{36}$$

Corollary 6.2 [1] *Assume that the hypotheses of Theorem* 6.3 *is satisfied and suppose that* $\lambda = n-2$. *Then the solution u of problem* (1) *belongs to* $BMO(Q)$.

Acknowledgments

Stará was supported by grants GACR 201/09/0917 and MSM 0021620839.

Notes

1. Einstein's convention will be used throughout this article.
2. As a permanent convention we will denote by $c_V(\dots,\Omega)$ a constant which depends on various parameters, on the coefficients of the system through the smallness of their *VMO*-continuity modulus and on the geometrical properties of the involved domain Ω.
3. For the sake of simplicity we will drop the index j relative to the diffeomorphism ψ^j.
4. The same $\overline{\rho}$ of Lemma 3.2.
5. The same $\overline{\rho}$ of Lemma 3.4 of [1].
6. For example, set $G = G_1 \cup G_2$ where $G_1 = \{y = [y', y_n] \in I\!R^n : |y'| < 1, y_n \in]0, 1[\}$ and $G_2 = \{y \in I\!R^n$: there exists $z = [z', 1/2]$ with $|z'| = 1$ so that $|z - y| < 1/2\}$.

References

[1] S. Leonardi and J. Stará, *Regularity results for the gradient of solutions of linear elliptic systems with VMO-coefficients and $L^{1,\lambda}$ data*, Forum Math. 22(5) (2010), pp. 913–940.

[2] J. Wolf, On the boundary regularity of suitable weak solution to the Navier–Stokes equations, Preprint No. 09–03, University of Magdeburg (www.math.uni-magdeburg.de/Forschung/index/).

[3] W. Littman, G. Stampacchia, and H.F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Sc. Norm. Super. Pisa 17 (1963), pp. 43–77.

[4] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) 15(1) (1965), pp. 189–258.

[5] F. Duzaar and K. Steffen, *Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals*, J. Reine Angew. Math. 546 (2002), pp. 73–138.

[6] F. Duzaar and J.F. Grotowski, *Optimal interior partial regularity for nonlinear elliptic systems: The method of A-harmonic approximation*, Manuscripta Math. 103 (2000), pp. 267–298.

[7] J.F. Grotowski, *Boundary regularity results for nonlinear elliptic systems in divergence form*, Calc. Var. 15 (2002), pp. 353–388.

[8] G.R. Cirmi and S. Leonardi, *Regularity results for the gradient of solutions of linear elliptic equations with $L^{1,\lambda}$ data*, Ann. Mat. Pura Appl. 185(4) (2006), pp. 537–553.

[9] G.R. Cirmi and S. Leonardi, *Regularity results for the gradient of nonlinear elliptic equations with $L^{1,\lambda}$ data*, Nonlinear Anal. T.M.A. 69(1) (2008), pp. 230–244.

[10] G.R. Cirmi, S. Leonardi, and J. Stará, *Regularity results for the gradient of solutions of a class of linear elliptic systems with $L^{1,\lambda}$ data*, Nonlinear Anal. T.M.A. 68(12) (2008), pp. 3609–3624.

[11] Q. Huang, *Estimates on the generalized Morrey spaces $L^{p,\lambda}_{\Phi}$ and BMO for elliptic systems*, Indiana Univ. Math. J. 45 (1996), pp. 397–439.

[12] G. Mingione, *The Calderón–Zygmund theory for elliptic problems with measure data*, Ann. Sc. Norm. Super. Pisa Cl. Sci. VI(5) (2007), pp. 195–261.

[13] M. Steinhauer, *Funktionenräume vom Campanato-Typ und Regularitätseigenschaften elliptischer Systeme zweiter Ordnung*, Diploma Thesis, University of Bonn.

[14] S. Campanato, *Equazioni ellittiche del II ordine e spazi $\mathcal{L}^{2,\lambda}$*, Ann. Mat. Pura Appl. 69 (1965), pp. 321–382.

[15] A. Kufner, O. John, and S. Fučik, *Function Spaces*, Noordhoff International Publishing Leyden, Academia, Prague, 1977.

[16] P. Acquistapace, *On BMO regularity for linear elliptic systems*, Ann. Mat. Pura Appl. (IV) CLXI (1992), pp. 231–269.

[17] S. Leonardi, J. Kottas, and J. Stará, *Hölder regularity of the solutions of some classes of elliptic systems in convex non smooth domains*, Nonlinear Anal. T.M.A. 60(5) (2005), pp. 925–944.

[18] S. Campanato, *Sistemi Ellittici in Forma Divergenza, Regolarità All'interno*, Quaderni Scuola Normale Superiore Pisa, Pisa, 1980.