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Regularity results for solutions of nonlinear elliptic equations with $L^{1,\lambda}$ data

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Abstract

We consider the Dirichlet problem associated with equations whose prototype is

$$-\Delta_p u = f \quad \text{in } \Omega$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, $p \in [2, n[$, $-\Delta_p$ is the p-Laplacian operator and f belongs to the Morrey space $L^{1,\lambda}(\Omega)$ with $\lambda \in]0, n-p]$.

Firstly, we prove that the gradient of the truncation $T_j(u)$ belongs to $L^{p,\lambda}_{\text{loc}}(\Omega)$ for all $j > 0$ and, as a consequence, we establish regularity results in suitable weak Morrey spaces for u and its gradient.

MSC: 35J25; 35D10

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1. Introduction

In this paper we study the regularity of the solution of the following Dirichlet problem:

$$\begin{cases} -\text{div}(A(x, Du)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ ($n \geq 3$), $u \mapsto -\text{div}A(x, Du)$ is a strongly monotone operator mapping $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}$ ($p \in [2, n[$, $\frac{1}{p} + \frac{1}{p'} = 1$) and f belongs to the Morrey space $L^{1,\lambda}(\Omega)$.

The study of problem (1) with L^1-data (note that $L^1 = L^{1,0}$) was started in the linear case (i.e. $p = 2$ and $A(x, Du) = a(x) \cdot Du$, where $a(x)$ is a uniformly elliptic matrix with bounded coefficients) by G. Stampacchia who introduced the notion of "duality solution" (see [27,28]).

In the nonlinear framework the first attempt to solve problem (1) was made by L. Boccardo and T. Gallouët who proved in [6,5] the existence of a solution of (1) in the sense of distributions which belongs to the space $W_0^{1,q}(\Omega)$, for

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any $q \in [1, \frac{n(p-1)}{n-1}[$. Unfortunately, the distributional solutions of (1) are not unique (for a counterexample see [26]); however, in [3] it has been proved that the distributional solution u of (1) satisfying an additional condition, the so called entropy condition (see Section 2 for the definition), is unique and it is such that $u \in \mathcal{M}^{\frac{n(p-1)}{n-p}}(\Omega)$ and $Du \in \mathcal{M}^{\frac{n(p-1)}{n-1}}(\Omega)$, where $\mathcal{M}^{\alpha}(\Omega)$ is the weak Lebesgue space.

Other results concerning nonlinear higher order equations with L^1-right hand side can be found in [20].

Let us recall that a known result (see e.g. [10,9,15,21]) states that if $A(x, Du)$ is nonlinear, $p = 2$ and $f \in L^{2,\lambda}(\Omega)$ with $0 \leq \lambda < n$, then the weak solution of problem (1) has a gradient which belongs to the Morrey space $L^{2,\lambda}_{\mathrm{loc}}(\Omega)$.

In this paper assuming f in the Morrey space $L^{1,\lambda}(\Omega)$, $0 < \lambda \leq n - p$, for the entropy solution u of (1), we prove that the gradient of the truncation $T_j(u)$ belongs to $L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ for all $j > 0$.

As a consequence of this result we are able to establish the following regularity properties:

(i) If $\lambda \in]0, n - p[$ then, for all $H \subset\subset \Omega$, we have

$$u \in \mathcal{M}^{p_\lambda,\lambda}(H), \quad Du \in \mathcal{M}^{\bar{p}_\lambda,\lambda}(H)^{1}$$

where $p_\lambda = \frac{(p-1)(n-\lambda)}{n-\lambda-p}$ and $\bar{p}_\lambda = \frac{(p-1)(n-\lambda)}{n-\lambda-1}$;

(ii) if $\lambda = n - p$ then, for any cube $Q \subset\subset \Omega$, we have

$$u \in \mathrm{BMO}(Q), \quad Du \in \mathcal{M}^{\beta,\lambda}(Q), \quad \forall \beta < p.$$

Our results improve, at least locally, the regularity on u and Du obtained in previous quoted papers without increasing the integrability on the datum f.

If $p = 2$, if the operator is linear, and if Ω has C^1-boundary or it is a cube, the aforementioned results improve the analogous ones of paper [12] (see Remark 5.2).

We point out that our work completes the paper of T. Kilpeläinen [18] where the case $\lambda > n - p$ was considered (see Remark 2.2).

As regards the technique, as in the linear case, we use the approximation method used by L. Boccardo and T. Gallouët in the nonlinear L^1-setting (see e.g. [5,6,3]) mixed with the technique introduced by S. Campanato for handling equations and systems of equations with $L^{2,\lambda}$-data (see e.g. [8–10]).

2. Main notation, function spaces and statement of the results

In $\mathbb{R}^n$ $(n \geq 3)$, with generic point $x = (x_1, x_2, \ldots, x_n)$, we shall denote by Ω a bounded open nonempty set with diameter d_Ω.

For $\rho > 0$ and $x_o \in \mathbb{R}^n$ we define

$$B(x_o, \rho) = \{x \in \mathbb{R}^n : |x - x_o| < \rho\}$$
$$\Omega(x_o, \rho) = \Omega \cap B(x_o, \rho).$$

Moreover, if $u \in L^1(B)$ we define

$$u_B = \frac{1}{|B|} \int_B u(x) \mathrm{d}x$$

where $|B|$ is the n-dimensional Lebesgue measure of the set B.

Definition 2.1 (*Morrey Space*). Let $q \geq 1$ and $0 \leq \lambda < n$. By $L^{q,\lambda}(\Omega)$ we denote the space of all functions $u \in L^q(\Omega)$ for which

$$\|u\|_{L^{q,\lambda}(\Omega)} = \sup_{x_o \in \Omega, 0 < \rho \leq d_\Omega} \left\{ \rho^{-\lambda} \int_{\Omega(x_o,\rho)} |u(x)|^q \mathrm{d}x \right\}^{1/q}$$

is finite. $L^{q,\lambda}(\Omega)$ equipped with the above norm is a Banach space.

[1] We denote by $\mathcal{M}^{\alpha,\lambda}(H)$ the weak Morrey space. For the definition see Section 2.

Definition 2.2 (*Campanato Space*). Let $q \geq 1$ and $0 \leq \lambda < n + q$. By $\mathcal{L}^{q,\lambda}(\Omega)$ we denote the space of all functions $u \in L^q(\Omega)$ such that

$$[u]_{\mathcal{L}^{q,\lambda}(\Omega)} = \sup_{x_o \in \Omega, 0<\rho\leq d_\Omega} \left\{ \rho^{-\lambda} \int_{\Omega(x_o,\rho)} |u(x) - u_{\Omega(x_o,\rho)}|^q \mathrm{d}x \right\}^{1/q} < +\infty.$$

Moreover we introduce the notion of BMO class.

Definition 2.3 (*John–Nirenberg Space*). Let Q be a cube in $\mathbb{R}^n$. By BMO(Q) we denote the space of all functions $u \in L^1(Q)$ such that

$$[u]_{\mathrm{BMO}(Q)} = \sup_{\tilde{Q} \subset Q} \frac{1}{|\tilde{Q}|} \int_{|\tilde{Q}|} |u - u_{\tilde{Q}}| \mathrm{d}x < +\infty,$$

where the supremum is taken over all cubes with sides parallel to coordinate axes.

Let us recall that $\mathcal{L}^{q,n}(Q) \cong \mathrm{BMO}(Q)$, $\forall q \geq 1$.

Definition 2.4 (*Weak Lebesgue Space*). Let $q \geq 1$. By $\mathcal{M}^q(\Omega)$ we denote the space of all measurable functions u for which

$$\exists K > 0 \text{ such that } \quad |\{x \in \Omega : |u(x)| > \sigma\}| \leq \left(\frac{K}{\sigma}\right)^q, \quad \forall \sigma > 0.$$

Definition 2.5 (*Weak Morrey Space*). Let $q \geq 1$ and $0 \leq \lambda < n$. By $\mathcal{M}^{q,\lambda}(\Omega)$ we denote the space of all measurable functions u for which

$$\exists K > 0 \text{ such that } \quad \forall \sigma, \rho > 0, \quad \forall x_o \in \Omega,$$
$$\rho^{-\lambda} |\{x \in \Omega(x_o, \rho) : |u(x)| > \sigma\}| \leq \left(\frac{K}{\sigma}\right)^q.$$

If $u : \Omega \to \mathbb{R}$, we set

$$D_i \equiv \frac{\partial}{\partial x_i}, \qquad Du = (D_i u)_{i=1,\ldots,n}.$$

Let $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function (i.e. continuous in ξ for a.e. $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}^n$) satisfying the following conditions, for a.e. $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^n$, with $\xi \neq \eta$:

$$\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \Lambda_1 |\xi - \eta|^p, \tag{2}$$

$$|A(x, \xi)| \leq \Lambda_2 |\xi|^{p-1}, \tag{3}$$

where Λ_1, Λ_2 are two positive constants, $p \in [2, n[$ and $\langle \cdot, \cdot \rangle$ means the scalar product in $\mathbb{R}^n$.

We observe that, by virtue of assumptions (2) and (3), $u \mapsto -\mathrm{div}A(x, Du)$ is a Leray–Lions operator acting between $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$, $p' = \frac{p}{p-1}$.

Let us define the truncation operator. For a given constant $k > 0$ we define the cut-function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k \ \mathrm{sign}(s) & \text{if } |s| > k. \end{cases}$$

We introduce here the notion of entropy solution.

Definition 2.6. Let $f \in L^1(\Omega)$. By an entropy solution of the problem (1) we mean a function $u \in L^1(\Omega)$ such that

$$\begin{cases} T_k(u) \in W_0^{1,p}(\Omega), & \forall k > 0 \\ \displaystyle\int_\Omega A(x, Du) DT_k(u - v) \mathrm{d}x \leq \int_\Omega f T_k(u - v) \mathrm{d}x & \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{4}$$

Remark 2.1. The existence and uniqueness of the entropy solution of the problem (1) have been proved in [3] for any $p \in]1, n]$ and with (2) replaced by a strict monotonicity assumption (weaker than (2)).

The entropy solution belongs to $W_0^{1,q}(\Omega)$, for any $q \in [1, \frac{n(p-1)}{n-1}[$ if $p \in [2-\frac{1}{n}, n]$, while if $p \in]1, 2-\frac{1}{n}[$ another functional setting must be used, since $|Du|$ does not belong to $L^1(\Omega)$.

In order to obtain our regularity result the strong monotonicity assumption (2) must be required (see (18) in Theorem 3.2). For this reason, we restrict ourselves to the case $p \geq 2$ which in turn implies $\frac{n(p-1)}{n-1} > 1$.

Here we assume that

$$f \in L^{1,\lambda}(\Omega), \quad \lambda \in]0, n-p] \tag{5}$$

and will prove the following results:

Theorem 2.1. *Assume (2) and (3), and let $u \in W_0^{1,q}(\Omega)$, $q \in [1, \frac{n(p-1)}{n-1}[$, be the entropy solution of the problem (1). Then*

$$Du \in L^{q,\nu}_{\mathrm{loc}}(\Omega)$$

with $\nu = n - \frac{q}{p-1}(n-\lambda-1)$ and

$$DT_j(u) \in L^{p,\lambda}_{\mathrm{loc}}(\Omega), \quad \forall j > 0.$$

Moreover, for all $H \subset\subset \Omega$ there exist two positive constants c_1 and c_2 depending on the data such that

$$\|Du\|_{L^{q,\nu}(H)} \leq c_1 \tag{6}$$

and

$$\|DT_j(u)\|_{L^{p,\lambda}(H)} \leq c_2\, j^{1/p}. \tag{7}$$

As a consequence of the previous theorem we obtain the following regularity result for the solution u.

Theorem 2.2. *Assume (2) and (3), and let $u \in W_0^{1,q}(\Omega)$, $q \in [1, \frac{n(p-1)}{n-1}[$, be the entropy solution of the problem (1).*

(i) *If $\lambda \in]0, n-p[$ then, for all $H \subset\subset \Omega$, we have*

$$u \in \mathcal{M}^{p_\lambda,\lambda}(H)$$

where $p_\lambda = \frac{(p-1)(n-\lambda)}{n-\lambda-p}$;

(ii) *if $\lambda = n-p$ then $u \in \mathrm{BMO}(Q)$, where $Q \subset\subset \Omega$ is a cube in $\mathbb{R}^n$.*

Finally, we can derive the following

Theorem 2.3. *Assume (2) and (3), and let $u \in W_0^{1,q}(\Omega)$, $q \in [1, \frac{n(p-1)}{n-1}[$, be the entropy solution of the problem (1).*

(i) *If $\lambda \in]0, n-p[$ then, for all $H \subset\subset \Omega$, we have*

$$Du \in \mathcal{M}^{\bar{p}_\lambda,\lambda}(H)$$

where $\bar{p}_\lambda = \frac{(p-1)(n-\lambda)}{n-\lambda-1}$;

(ii) *if $\lambda = n-p$ then, for any cube $Q \subset\subset \Omega$, we have*

$$Du \in \mathcal{M}^{\beta,\lambda}(Q), \quad \forall \beta < p.$$

Remark 2.2. We point out that if $\lambda > n-p$ we are out of the L^1-framework since $L^{1,\lambda}(\Omega) \subset W^{-1,p'}(\Omega)$ (see the Corollary on p. 165 of [16] and Lemma 1 on p. 105 of [25]). For this case the Hölder continuity of u has been proved in [18].

Remark 2.3. In [3] it has been proved that there exists a unique entropy solution u of the problem (1) such that $u \in \mathcal{M}^{\frac{n(p-1)}{n-p}}(\Omega)$ and $Du \in \mathcal{M}^{\frac{n(p-1)}{n-1}}(\Omega)$.

In Theorems 2.2 and 2.3, our assumption (5), without increasing the summability of the datum f, improves the regularity of u and Du (at least locally).

Remark 2.4. In [5,6,19,13] the authors have proved that $Du \in L^{\frac{n(p-1)}{n-1}}(\Omega)$ and $u \in L^{\frac{n(p-1)}{n-p}}(\Omega)$ under the additional assumption $f \in L^1 \log L^1(\Omega)$ or $f \in L^1_{\frac{n}{n-1}}(\Omega)$ (see the Appendix for details about these spaces). We point out that $L^{1,\lambda}(\Omega)$ is not contained in nor contains $L^1 \log L^1(\Omega)$ as well as $L^1_{\frac{n}{n-1}}(\Omega)$; consequently our regularity results are independent of those studied in the above quoted papers and moreover improve them (at least locally).

Remark 2.5. Theorem 2.3 completes the result of paper [14] giving a regularity property also for Du.

Remark 2.6. We thank G. R. Mingione who informed us, after completion of this work, that he has proved, among others, Theorem 2.1 by a different method extending (ii) to the case $\beta = p$ (see [24]).

3. Auxiliary results

In this section we assume that the structural conditions (2), (3) and (5) hold and we consider, at first, a weak solution v of the nonlinear equation

$$\operatorname{div}(A(x, Dv)) = 0 \quad \text{in } \Omega, \tag{8}$$

that is, a function $v \in W^{1,p}(\Omega)$ such that

$$\int_\Omega A(x, Dv) D\varphi \mathrm{d}x = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

Remark 3.1. Let us remark that if v is a solution of Eq. (8) then also $v - h$, $\forall h \in \mathbb{R}$, satisfies the same equation.

The key step in our paper will be to prove the following

Theorem 3.1 (*Saint-Venant Principle*). *Let $v \in W^{1,p}(\Omega)$ be a weak solution of Eq. (8).*
Then there exist two constants $\mu = \mu(n, \Lambda_1, \Lambda_2, p) \in]0, 1[$ and $c = c(n, q, p, \Lambda_1, \Lambda_2) > 0$ such that

$$\|Dv\|^q_{L^q(B(x_o,\rho_1))} \le c \left(\frac{\rho_1}{\rho_2}\right)^{n-q+\mu q} \|Dv\|^q_{L^q(B(x_o,\rho_2))} \tag{9}$$

$\forall x_o \in \Omega$, $\forall 0 < \rho_1 \le \rho_2 < \operatorname{dist}(x_o, \partial\Omega)$, $\forall q \in [1, p[$.

Before proving the previous theorem, let us state two useful lemmata which are interesting in themselves.

Lemma 3.1. *Let $v \in W^{1,p}(\Omega)$ be a weak solution of Eq. (8).*
Then there exist two constants $\mu = \mu(n, p, \Lambda_1, \Lambda_2) \in]0, 1[$ and $c = c(n, q, p, \Lambda_1, \Lambda_2) > 0$ such that

$$\operatorname{osc}_{B(x_o,\rho_1)} (v - h) \le c \left(\frac{\rho_1}{\rho_2}\right)^{\mu} \rho_2^{-n/q} \|v - h\|_{L^q(B(x_o,\rho_2))} \tag{10}$$

$\forall x_o \in \Omega$, $\forall 0 < \rho_1 \le \rho_2/2$, with $\rho_2 < \operatorname{dist}(x_o, \partial\Omega)$, $\forall q \in [1, p[$ and $\forall h \in \mathbb{R}$.

Proof. Fixing $x_o \in \Omega$, $\rho_2 \in]0, \operatorname{dist}(x_o, \partial\Omega)[$, $q \in [1, p[$ and $h \in \mathbb{R}$, by (6.7), p. 111 of [17], there exists μ as in the statement such that

$$\operatorname{osc}_{B(x_o,\rho_1)} v \le 2^{\mu+1} \left(\frac{\rho_1}{\rho_2}\right)^{\mu} \operatorname{osc}_{B(x_o,\rho_2/2)} v, \quad \forall 0 < \rho_1 \le \rho_2/2.$$

By (6.4), p. 110 of [17], and the above inequality we have

$$\sup_{B(x_o,\rho_2/2)} |v| \le c(n, p, q, \Lambda_1, \Lambda_2) \rho_2^{-n/q} \|v\|_{L^q(B(x_o,\rho_2))}. \tag{11}$$

The above two inequalities and Remark 3.1 yield

$$\operatorname{osc}_{B(x_o,\rho_1)} (v - h) \le c \, (\rho_1/\rho_2)^{\mu} \rho_2^{-n/q} \|v - h\|_{L^q(B(x_o,\rho_2))}. \quad \square \tag{12}$$

Lemma 3.2. *Let $v \in W^{1,p}(\Omega)$ be a weak solution of Eq. (8).*

Then there exists a positive constant $c = c(n, q, p, \Lambda_1, \Lambda_2)$ such that

$$\|Dv\|^q_{L^q(B(x_o,\rho/4))} \le c\rho^{-q}\|v - h\|^q_{L^q(B(x_o,\rho))} \tag{13}$$

$\forall x_o \in \Omega, \forall \rho \in]0, \mathrm{dist}(x_o, \partial\Omega)[, \forall q \in [1, p[, \forall h \in \mathbb{R}$.

Proof. Fixing $x_o \in \Omega$, $\rho \in]0, \mathrm{dist}(x_o, \partial\Omega)[$, $q \in [1, p[$ and $h \in \mathbb{R}$, by Caccioppoli's inequality (see Lemma 3.32, p. 65 of [17][2]) one has

$$\|Dv\|^p_{L^p(B(x_o,\rho/4))} \le c(\Lambda_1, \Lambda_2)\rho^{-p}\|v - h\|^p_{L^p(B(x_o,\rho/2))}.$$

On the other hand, Hölder's inequality and the above inequality yield

$$\begin{aligned}\|Dv\|^q_{L^q(B(x_o,\rho/4))} &\le c(n)\rho^{n(1-q/p)}\|Dv\|^q_{L^p(B(x_o,\rho/4))}\\ &\le c(n, q, \Lambda_1, \Lambda_2)\rho^{n(1-q/p)-q}\|v - h\|^q_{L^p(B(x_o,\rho/2))}.\end{aligned} \tag{14}$$

From (14) and (10) we deduce

$$\|Dv\|^q_{L^q(B(x_o,\rho/4))} \le c\,\rho^{n(1-q/p)-q+nq/p-n}\|v - h\|^q_{L^q(B(x_o,\rho))}. \quad \square$$

Proof of Theorem 3.1. The proof can be carried out like the proof of Theorem 3.1 of [12]. $\square$

Let us observe now that if $f \in L^{1,\lambda}(\Omega)$ then the sequence of functions $\{T_k(f)\}_{k\in\mathbb{N}}$ satisfies:

(a) $T_k(f) \in W^{-1,p}(\Omega) \cap L^{1,\lambda}(\Omega)$, $\forall k \in \mathbb{N}$,
(b) $T_k(f) \to f$ in $L^1(\Omega)$ as $k \to +\infty$,
(c) $\|T_k(f)\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}$, $\forall k \in \mathbb{N}$,
(d) $\|T_k(f)\|_{L^{1,\lambda}(\Omega)} \le \|f\|_{L^{1,\lambda}(\Omega)}$, $\forall k \in \mathbb{N}$.

Fixing $k \in \mathbb{N}$, let $u_k \in W^{1,p}_0(\Omega)$ be the weak solution[3] of the equation

$$-\mathrm{div}(A(x, Du_k)) = T_k(f) \quad \text{in } \Omega, \tag{15}$$

that is,

$$\begin{cases} u_k \in W^{1,p}_0(\Omega) \\ \displaystyle\int_\Omega A(x, Du_k)D\varphi \mathrm{d}x = \int_\Omega T_k(f)\varphi \mathrm{d}x, \quad \forall \varphi \in W^{1,p}_0(\Omega). \end{cases}$$

Fixing $k \in \mathbb{N}$, $x_o \in \Omega$ and $0 < \rho < \mathrm{dist}(x_o, \partial\Omega)$ let us now consider the weak solution w_k of the following Dirichlet problem:

$$\begin{cases} w_k \in W^{1,p}_0(B(x_o, \rho)) \\ -\mathrm{div}(A(x, D(u_k - w_k))) = 0 \quad \text{in } B(x_o, \rho), \end{cases} \tag{16}$$

that is a function $w_k \in W^{1,p}_0(B(x_o, \rho))$ such that

$$\int_{B(x_o,\rho)} A(x, D(u_k - w_k))D\varphi \mathrm{d}x = 0, \quad \forall \varphi \in W^{1,p}_0(B(x_o, \rho)).$$

Let us observe that problem (16) is equivalent to

$$\begin{cases} \tilde{w}_k \in W^{1,p}(B(x_o, \rho)) \\ \tilde{w}_k - u_k \in W^{1,p}_0(B(x_o, \rho)) \\ \mathrm{div}(A(x, D\tilde{w}_k)) = 0 \qquad \text{in } B(x_o, \rho) \end{cases}$$

which, for any $k \in \mathbb{N}$, admits a unique solution (see [22,23]).

[2] In that lemma we take as η the cut-off function in $B(x_o, \rho/4)$.

[3] The existence of such a solution is ensured by the Leray–Lions theorem.

Theorem 3.2. *Let $w_k \in W_0^{1,p}(B(x_o,\rho))$ be the weak solution of the problem (16).*

Then there exist a constant $c > 0$ depending on $n, q, p, \Lambda_1, \Lambda_2$ such that

$$\|Dw_k\|^q_{L^q(B(x_o,\rho))} \leq c\,\rho^{n-\frac{q}{p-1}(n-\lambda-1)}\|f\|^{\frac{q}{p-1}}_{L^{1,\lambda}(\Omega)} \tag{17}$$

$\forall k \in \mathbb{N}, \forall x_o \in \Omega, \forall 0 < \rho < \mathrm{dist}(x_o, \partial\Omega), \forall q \in [1, \frac{n(p-1)}{n-1}[$.

Proof. We will use a standard procedure which has been used by many authors (see e.g. [29,30,2,4,13]).

For $h, t > 0$ let us set

$$\varphi_h(s) = \begin{cases} \mathrm{sign}\,(s) & \text{if } |s| > t+h \\ \dfrac{s - t\,\mathrm{sign}(s)}{h} & \text{if } t < |s| \leq t+h \\ 0 & \text{otherwise.} \end{cases}$$

Choosing $\varphi_h(w_k)$ as a test function in the weak formulations of problems (15) and (16) we obtain

$$\int_{B(x_o,\rho)} [A(x, Du_k) - A(x, D(u_k - w_k))]D\varphi_h(w_k)\mathrm{d}x = \int_{B(x_o,\rho)} T_k(f)\varphi_h(w_k)\mathrm{d}x.$$

Using the strong monotonicity assumption it follows that

$$\frac{1}{h}\int_{B(x_o,\rho)\cap\{t<|w_k|\leq t+h\}} |Dw_k|^p\mathrm{d}x \leq \frac{1}{\Lambda_1}\rho^\lambda\|f\|_{L^{1,\lambda}(\Omega)}. \tag{18}$$

By the previous inequality, letting $h \to 0$, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{B(x_0,\rho)\cap\{|w_k|>t\}} |Dw_k|^p\mathrm{d}x \leq \frac{1}{\Lambda_1}\rho^\lambda\|f\|_{L^{1,\lambda}(\Omega)} \tag{19}$$

Let $q < p$; by virtue of Holder's inequality, we have

$$\frac{1}{h}\int_{B(x_0,\rho)\cap\{t<|w_k|\leq t+h\}} |Dw_k|^q\mathrm{d}x \leq \left(\frac{1}{h}\int_{B(x_0,\rho)\cap\{t<|w_k|\leq t+h\}} |Dw_k|^p\mathrm{d}x\right)^{\frac{q}{p}} \times \left(\frac{|\{x \in B(x_0,\rho) : t < |w_k| \leq t+h\}|}{h}\right)^{1-\frac{q}{p}}. \tag{20}$$

Letting h go to zero, we have, for a.e. $t > 0$,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{B(x_0,\rho)\cap\{|w_k|>t\}} |Dw_k|^q\mathrm{d}x \leq \left(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{B(x_0,\rho)\cap\{|w_k|>t\}} |Dw_k|^p\mathrm{d}x\right)^{\frac{q}{p}} \left(-\mu'_\rho(t)\right)^{1-\frac{q}{p}} \tag{21}$$

where

$$\mu_\rho(t) = |\{x \in B(x_0,\rho) : |w_k(x)| > t\}|.$$

On the other hand it is well known that (see [30])

$$1 \leq C(n)\left(\mu_\rho(t)\right)^{\frac{1}{n}-1}\left(-\mu'_\rho(t)\right)^{1-\frac{1}{p}}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{B(x_0,\rho)\cap\{|w_k|>t\}} |Dw_k|^p\mathrm{d}x\right)^{\frac{1}{p}} \tag{22}$$

where $C(n)$ depends only on n. Using (22) raised to the power $\frac{q}{p-1}$ in the inequality (21) we get

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{B(x_0,\rho)\cap\{|w_k|>t\}} |Dw_k|^q\mathrm{d}x \leq C(n)^{\frac{q}{p-1}}\left(\mu_\rho(t)\right)^{\left(\frac{1}{n}-1\right)\frac{q}{p-1}}\left(-\mu'_\rho(t)\right) \times \left(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{B(x_0,\rho)\cap\{|w_k|>t\}} |Dw_k|^p\mathrm{d}x\right)^{\frac{q}{p-1}}.$$

By virtue of (19) we conclude that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{B(x_0,\rho)\cap\{|w_k|>t\}}|Dw_k|^q\mathrm{d}x \le c\left(\mu_\rho(t)\right)^{\left(\frac{1}{n}-1\right)\frac{q}{p-1}}\left(-\mu'_\rho(t)\right)\rho^{\lambda\frac{q}{p-1}}\|f\|^{\frac{q}{p-1}}_{L^{1,\lambda}(\Omega)}$$

with $c > 0$ constant depending only on n, Λ_1, q, p.

The previous inequality implies

$$\int_{B(x_0,\rho)}|Dw_k|^q\mathrm{d}x \le c\rho^{\lambda\frac{q}{p-1}}\|f\|^{\frac{q}{p-1}}_{L^{1,\lambda}(\Omega)}\int_0^{+\infty}\left(\mu_\rho(t)\right)^{\left(\frac{1}{n}-1\right)\frac{q}{p-1}}\left(-\mu'_\rho(t)\right)\mathrm{d}t. \tag{23}$$

Since the integral in the right hand side is finite for any $q < \frac{n(p-1)}{n-1}$ and it behaves as $\rho^{n-\frac{q(n-1)}{p-1}}$, from (23) we obtain (17).

4. Interior regularity

In this section we will connect the technique developed in [6] with the nowadays classical method of S. Campanato. We will prove, first, the following

Theorem 4.1. *Assume that assumptions (2), (3) and (5) hold and let u_k be the solution of problem (15). Then*

$$Du_k \in L^{q,\nu}_{\mathrm{loc}}(\Omega), \quad \forall q \in \left[1, \frac{n(p-1)}{n-1}\right[, \forall k \in \mathbb{N}, \tag{24}$$

with $\nu = n - \frac{q}{p-1}(n-\lambda-1)$ and

$$DT_j(u_k) \in L^{p,\lambda}_{\mathrm{loc}}(\Omega), \quad \forall j > 0\, \forall k \in \mathbb{N}. \tag{25}$$

Moreover, for all $H \subset\subset \Omega$ there exist two positive constants c_1, c_2 depending on $n, q, p, \lambda, \Lambda_1, \Lambda_2, d_\Omega$, $\mathrm{dist}(\overline{H}, \partial\Omega)$, $\|f\|_{L^{1,\lambda}(\Omega)}$ such that

$$\|Du_k\|_{L^{q,\nu}(H)} \le c_1, \quad \forall k \in \mathbb{N} \tag{26}$$

and

$$\|DT_j(u_k)\|_{L^{p,\lambda}(H)} \le c_2\, j^{1/p}, \quad \forall j > 0, \forall k \in \mathbb{N}. \tag{27}$$

Proof. Fix $k \in \mathbb{N}$, $x_o \in \Omega$ and $\rho \in]0, \mathrm{dist}(x_o, \partial\Omega)[$.

In $B(x_o,\rho)$ we can write $u_k = v_k + w_k$ where $v_k \in W^{1,p}(B(x_o,\rho))$ is a weak solution of the problem

$$\mathrm{div}(A(x, Dv_k)) = 0 \quad \text{in } B(x_o,\rho)$$

and $w_k \in W^{1,p}_0(B(x_o,\rho))$ is the weak solution of the Dirichlet problem

$$\begin{cases} -\mathrm{div}(A(x, D(u_k - w_k))) = 0 & \text{in } B(x_o,\rho) \\ w_k = 0 & \text{on } \partial B(x_o,\rho). \end{cases} \tag{28}$$

Gathering together (9) and (17) we deduce, for any $\sigma < \rho$,

$$\begin{aligned}\|Du_k\|^q_{L^q(B(x_o,\sigma))} &\le c\left[\left(\frac{\sigma}{\rho}\right)^{n-q+\mu q}\|Dv_k\|^q_{L^q(B(x_o,\rho))} + \rho^{\lambda\frac{q}{p-1}+n-\frac{(n-1)q}{p-1}}\|f\|^{\frac{q}{p-1}}_{L^{1,\lambda}(\Omega)}\right] \\ &\le c\left[\left(\frac{\sigma}{\rho}\right)^{n-q+\mu q}\|Du_k\|^q_{L^q(B(x_o,\rho))} + \rho^{\lambda\frac{q}{p-1}+n-\frac{(n-1)q}{p-1}}\|f\|^{\frac{q}{p-1}}_{L^{1,\lambda}(\Omega)}\right].\end{aligned}$$

Set now,

$$\varphi(\sigma) = \|Du_k\|^q_{L^q(B(x_o,\sigma))}, \qquad A = c,$$
$$\alpha = n - q + \mu q, \qquad \beta = \lambda \frac{q}{p-1} + n - \frac{(n-1)q}{p-1},$$
$$\Phi(\sigma) = c\|f\|^{\frac{q}{p-1}}_{L^{1,\lambda}(\Omega)}, \qquad \varepsilon = \alpha - \beta,$$

and observe that $\lambda \leq n - p$ implies $\epsilon > 0$; thus we can apply Lemma 1.1, p. 7 of [9], to the above inequality so that we obtain, $\forall \sigma < \rho$,

$$\|Du_k\|^q_{L^q(B(x_o,\sigma))} \leq c \left[\left(\frac{\sigma}{\rho}\right)^{\lambda \frac{q}{p-1} + n - \frac{(n-1)q}{p-1}} \|Du_k\|^q_{L^q(B(x_o,\rho))} + \sigma^{\lambda \frac{q}{p-1} + n - \frac{(n-1)q}{p-1}} \|f\|^{\frac{q}{p-1}}_{L^{1,\lambda}(\Omega)} \right]. \tag{29}$$

Observe that

$$\lambda \frac{q}{p-1} + n - \frac{(n-1)q}{p-1} > \lambda.$$

Now (24) and (26) can be proved as in Theorem 4.1 of [12] taking into account the uniform norm estimate in $W^{1,q}(\Omega)$ of u_k of [5].

Let us prove (25) and the uniform norm estimate (27).

Let now $\eta(x)$ be the standard cut-off function in $B(x_0, \sigma)$ and choose as test function in the weak formulation of problem (15) the function $\eta^p T_j(u_k)$ with $j > 0$ fixed.

Using (2) and (3) and the Hölder inequality we obtain

$$\begin{aligned}\int_{B(x_0,\sigma/2)} |DT_j(u_k)|^p \mathrm{d}x &\leq cj \left[\sigma^{-1} \int_{B(x_0,\sigma)} |Du_k|^{p-1} \mathrm{d}x + \sigma^{\lambda} \|f\|_{L^{1,\lambda}(\Omega)} \right] \\ &\leq cj \left[\sigma^{-1+n(1-\frac{p-1}{q})} \left(\int_{B(x_0,\sigma)} |Du_k|^q \mathrm{d}x \right)^{\frac{p-1}{q}} + \sigma^{\lambda} \|f\|_{L^{1,\lambda}(\Omega)} \right].\end{aligned} \tag{30}$$

Joining together inequalities (30) and (29) we get, $\forall \sigma < \rho$,

$$\begin{aligned}\int_{B(x_0,\sigma/2)} |DT_j(u_k)|^p \,\mathrm{d}x &\leq cj \left\{ \sigma^{-1+n(1-\frac{p-1}{q})} \left[\left(\frac{\sigma}{\rho}\right)^{\lambda + n\frac{p-1}{q} - n + 1} \|Du_k\|^{p-1}_{L^q(B(x_o,\rho))} \right.\right. \\ &\quad \left.\left. + \sigma^{\lambda + n\frac{p-1}{q} - n + 1} \|f\|_{L^{1,\lambda}(\Omega)} \right] + \sigma^{\lambda} \|f\|_{L^{1,\lambda}(\Omega)} \right\} \\ &\leq cj \left[\sigma^{-1+n(1-\frac{p-1}{q})} \left(\frac{\sigma}{\rho}\right)^{\lambda + n\frac{p-1}{q} - n + 1} \|Du_k\|^{p-1}_{L^q(B(x_o,\rho))} + \sigma^{\lambda} \|f\|_{L^{1,\lambda}(\Omega)} \right].\end{aligned}$$

The required assertions now follow again arguing as in Theorem 4.1 of [12]. □

5. Proof of the main results and final remarks

Proof of Theorem 2.1. We recall (see [5,3]) that

$$\begin{aligned}&\|Du_k\|_{L^q(\Omega)} \leq c_3, \quad \forall k \in \mathbb{N}, \forall q \in \left[1, \frac{n(p-1)}{n-1}\right[, \\ &\|DT_j(u_k)\|_{L^p(\Omega)} \leq c_4 j^{1/p}, \quad \forall j > 0, \forall k \in \mathbb{N},\end{aligned} \tag{31}$$

with c_3 and c_4 positive constants independent of k.

This information allows us to deduce the following facts:

(a) $u_k \rightharpoonup u$ in $W^{1,q}(\Omega)$ as $k \to +\infty$,
(b) $u_k \to u$ in $L^q(\Omega)$ and a.e. in Ω as $k \to +\infty$,
(c) $Du_k \to Du$ a.e. in Ω as $k \to +\infty$ (see [6]),
(d) the function u is the entropy solution of the Dirichlet problem (1) (see [3]),
(e) $DT_j(u_k) \rightharpoonup DT_j(u)$ in $L^p(\Omega)$ (for fixed j) as $k \to +\infty$ (see [3]).

To conclude the proof we need only to show that $Du \in L^{q,\nu}_{\text{loc}}(\Omega)$ and that $DT_j(u) \in L^{p,\lambda}_{\text{loc}}(\Omega)$, $\forall j > 0$.
To this end let us fix $H \subset\subset \Omega$, $x_o \in H$, $\rho \in]0, d_H]$ and $j > 0$.
By (a), (e), (26) and (27) we have

$$\begin{aligned}\|Du\|^q_{L^q(H(x_o,\rho))} &\le \liminf_{k\to+\infty} \|Du_k\|^q_{L^q(H(x_o,\rho))}\\ &\le \rho^\nu \liminf_{k\to+\infty} \|Du_k\|^q_{L^{q,\nu}(H)} \le c_1\rho^\nu\end{aligned}$$

and

$$\begin{aligned}\|DT_j(u)\|^p_{L^p(H(x_o,\rho))} &\le \liminf_{k\to+\infty} \|DT_j(u_k)\|^p_{L^p(H(x_o,\rho))}\\ &\le \rho^\lambda \liminf_{k\to+\infty} \|DT_j(u_k)\|^p_{L^{p,\lambda}(H)} \le c_2 j\rho^\lambda.\end{aligned}$$

The above inequalities conclude the proof. □

Before proving Theorem 2.2 let us state the following useful

Lemma 5.1. *Let $v \in W^{1,p}_0(\Omega)$ such that $Dv \in L^{p,\lambda}_{\text{loc}}(\Omega)$, with $\lambda \in]0, n-p[$. Then*

$$v \in L^{p_\lambda,\lambda}_{\text{loc}}(\Omega)$$

where $\frac{1}{p_\lambda} = \frac{1}{p} - \frac{1}{n-\lambda}$ and for all $H \subset\subset \Omega$ there exists a positive constant $c = c(n, p, \lambda, H)$ such that

$$\|v\|_{L^{p_\lambda,\lambda}(H)} \le c\,[\|Dv\|_{L^p(\Omega)} + \|Dv\|_{L^{p,\lambda}(H)}]. \tag{32}$$

Proof. Observe that by Lemma 4.22 of [1], for fixed $H \subset\subset \Omega$, there exists $H' \subset\subset \Omega$ with the cone property such that $H \subset H'$.

Moreover, since $v \in W^{1,p}_0(\Omega)$, following the proof of Corollary 2.1 in [12], it can be proved that $v \in L^{p,\lambda}(H')$ with norm estimate

$$\|v\|_{L^{p,\lambda}(H')} \le c\,[\|Dv\|_{L^p(\Omega)} + \|Dv\|_{L^{p,\lambda}(H')}]. \tag{33}$$

Indeed, by the Sobolev embedding theorem and the standard properties of Morrey spaces it turns out that

$$v \in L^{p^*,0}(\Omega) \subset L^{p,\mu_0}(\Omega) \subset L^{p,\mu_0}_{\text{loc}}(\Omega), \quad \forall \mu_0 \in]0, p],$$

with norm estimate

$$\|v\|_{L^{p,\mu_0}_{\text{loc}}(\Omega)} \le c\|Dv\|_{L^p(\Omega)}. \tag{34}$$

(i) If $\lambda \in]0, p]$ then $v \in L^{p,\lambda}_{\text{loc}}(\Omega)$ and (34) holds with λ instead of μ_0. Thus (33) is proved.
(ii) If $\lambda > p$ then $v, Dv \in L^{p,\mu_0}_{\text{loc}}(\Omega)$ and, by virtue of Theorem 1.2, p. 72 of [7], we have

$$v \in L^{p,\mu}_{\text{loc}}(\Omega), \quad \forall \mu < p + \mu_0,$$

with norm estimate

$$\begin{aligned}\|v\|_{L^{p,\mu}_{\text{loc}}(\Omega)} &\le c\,[\|v\|_{L^{p,\mu_0}_{\text{loc}}(\Omega)} + \|Dv\|_{L^{p,\mu_0}_{\text{loc}}(\Omega)}]\\ &\le c\,[\|D\,v\|_{L^{p,\mu_0}_{\text{loc}}(\Omega)} + \|Dv\|_{L^p(\Omega)}]\end{aligned} \tag{35}$$

by virtue of (34).

If now $\lambda < p + \mu_0$ then $v, Dv \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ and we have the estimate (35) with λ instead of μ_0 and μ. Thus (33) holds again.

If instead $\lambda \geq p + \mu_0$ then $v, Dv \in L^{p,\mu_1}_{\mathrm{loc}}(\Omega)$, with $\mu_1 = \frac{p}{2} + \mu_0$, and a new application of the same theorem yields

$$v \in L^{p,\mu}_{\mathrm{loc}}(\Omega), \quad \forall \mu < p + \mu_1$$

with norm estimate

$$\begin{aligned}\|v\|_{L^{p,\mu}_{\mathrm{loc}}(\Omega)} &\leq c\,[\|v\|_{L^{p,\mu_1}_{\mathrm{loc}}(\Omega)} + \|Dv\|_{L^{p,\mu_1}_{\mathrm{loc}}(\Omega)}]\\ &\leq c\,[\|Dv\|_{L^{p,\mu_1}_{\mathrm{loc}}(\Omega)} + \|Dv\|_{L^{p}(\Omega)}]\end{aligned} \tag{36}$$

by virtue of (35) and the embedding properties of Morrey spaces.

Iterating the above procedure we can prove that

$$v, Dv \in L^{p,\mu_m}(\Omega), \quad \forall m \in \mathbb{N}$$

with

$$\mu_m = \begin{cases} \lambda & \text{if } \lambda < p + \mu_{m-1} \\ \dfrac{p}{2} + \mu_{m-1} & \text{if } \lambda \geq p + \mu_{m-1} \end{cases}$$

and norm estimate

$$\|v\|_{L^{p,\mu_m}_{\mathrm{loc}}(\Omega)} \leq c\,[\|Dv\|_{L^{p,\mu_m}_{\mathrm{loc}}(\Omega)} + \|Dv\|_{L^{p}(\Omega)}], \quad \forall m \in \mathbb{N}.$$

Since, for all $m \in \mathbb{N}$,

$$\mu_m = \begin{cases} \lambda & \text{if } \lambda < p + \mu_{m-1} \\ m\dfrac{q}{2} + \mu_0 & \text{if } \lambda \geq p + \mu_{m-1}, \end{cases}$$

after a finite number of steps we obtain (33).

On the other hand, by a well known representation formula, we have

$$v(x) \leq c(n) \int_{\mathbb{R}^n} \frac{V_1(y) + V_2(y)}{|x-y|^{n-1}} \mathrm{d}y, \quad \text{for a.e. } x \in H \tag{37}$$

where

$$V_1(y) = \begin{cases} |v(y)| & y \in H \\ 0 & y \in \mathbb{R}^n \setminus H \end{cases}$$

and

$$V_2(y) = \begin{cases} |Dv(y)| & y \in H \\ 0 & y \in \mathbb{R}^n \setminus H. \end{cases}$$

The result can be proved by applying to formula (37) a slight modification of the proof of Theorem 2 of [11] and from (33).

Remark 5.1. The result of the previous lemma can be refined if Ω has the cone property and $Dv \in L^{p,\lambda}(\Omega)$. Namely, it can be proved that

$$\|v\|_{L^{p_\lambda,\lambda}(\Omega)} \leq c\|Dv\|_{L^{p,\lambda}(\Omega)}. \tag{38}$$

Proof of Theorem 2.2. (i) Let us fix $H \subset\subset \Omega$, $x_o \in H$, $\rho \in]0, d_H]$ and $j > 0$. Thus we have

$$\begin{aligned}|\{x \in H(x_o,\rho) : |u(x)| > j\}| &\leq \left(\frac{1}{j}\right)^{\frac{p(n-\lambda)}{n-\lambda-p}} \int_{H(x_o,\rho)} |T_j u|^{\frac{p(n-\lambda)}{n-\lambda-p}} \mathrm{d}x\\ &\leq \rho^{\lambda} \left(\frac{1}{j}\right)^{\frac{p(n-\lambda)}{n-\lambda-p}} \|T_j u\|^{\frac{p(n-\lambda)}{n-\lambda-p}}_{L^{\frac{p(n-\lambda)}{n-\lambda-p},\lambda}(H)}.\end{aligned} \tag{39}$$

Gathering together (39), (7) and (31), by virtue of Lemma 5.1, we obtain

$$|\{x \in H(x_o,\rho) : |u(x)| > j\}| \le c\,\rho^{\lambda}\left(\frac{1}{j}\right)^{\frac{p(n-\lambda)}{n-\lambda-p}} j^{\frac{n-\lambda}{n-\lambda-p}}$$

which concludes the proof of the first case.

(ii) Let $Q \subset\subset \Omega$ be a cube, $x_o \in Q$ and $\rho \in]0, d_Q[$. Then, the Poincaré inequality and (6) imply

$$\begin{aligned}\int_{Q(x_o,\rho)} |u - u_{Q(x_o,\rho)}|^q \mathrm{d}x &\le c\rho^q \int_{Q(x_o,\rho)} |Du|^q \mathrm{d}x\\ &\le c\,\rho^{q+(n-q)}\,\|D\,u\|^q_{L^{q,n-q}_{\mathrm{loc}}(\Omega)}\\ &\le cc_1\rho^n.\end{aligned}$$

Thus, $u \in \mathcal{L}^{q,n}(Q) \cong \mathrm{BMO}(Q)$. $\square$

Remark 5.2. If $p = 2$, if the operator is linear and Ω has C^1-boundary (and thus the cone property) or is a cube the result of Theorem 2.2 can be extended to the whole Ω. Observe that this result improves the analogous ones of the paper [12].

To this end, in the proof of case (i) of Theorem 2.2 (the proof of case (ii) remains identical) it will be enough to replace (7) by the following:

$$\|DT_j(u)\|_{L^{2,\lambda}(\Omega)} \le cj^{1/2}, \quad \forall j > 0 \tag{40}$$

and then use Remark 5.1.

To prove (40) we will argue as in the proof of Theorems 4.1 and 2.1.

Let us fix $x_o \in \Omega$, $\rho \in]0, d_\Omega]$ and let $\eta(x)$ be the standard cut-off function in $\Omega(x_0,\rho)$.

Then, choosing $\eta^2\,T_j(u_k)$, with $j > 0$ fixed, as test function in the weak formulation of problem (15) we obtain

$$\int_{\Omega(x_0,\rho/2)} |DT_j(u_k)|^2 \mathrm{d}x \le cj\left[\rho^{-1+n(1-\frac{1}{q})}\left(\int_{\Omega(x_0,\rho)} |Du_k|^q\,\mathrm{d}x\right)^{\frac{1}{q}} + \rho^{\lambda}\|f\|_{L^{1,\lambda}(\Omega)}\right].$$

So that, if $\sigma \in]0, d_\Omega/2]$, the above inequality and formula (18) from Theorem 4.1 of [12] together with Lemma 4.2 of [12] yield

$$\sigma^{-\lambda}\int_{\Omega(x_o,\sigma)} |DT_j(u_k)|^2 \mathrm{d}x \le cj[1 + \|f\|_{L^{1,\lambda}(\Omega)}]. \tag{41}$$

If $\sigma \in]d_\Omega/2, d_\Omega]$ inequality (41) is obvious and in both cases (40) follows by approximation as in the proof of Theorem 2.1.

Proof of Theorem 2.3. The proof follows the lines of Lemma 4.2 in [3]. We will prove here only the case $\lambda = n - p$ for the reader's convenience.

Fix a cube $Q \subset\subset \Omega$, $x_o \in Q$, $\rho > 0$ and recall that since $u \in \mathrm{BMO}(Q)$ then, by embedding, $u \in \mathcal{M}^{q,\lambda}(Q)$, for any $q > 1$, that is

$$\forall q > 1, \exists K > 0 \text{ such that } \quad \rho^{-\lambda}|\{x \in Q(x_o,\rho) : |u(x)| > \sigma\}| \le \left(\frac{K}{\sigma}\right)^q, \quad \forall\sigma > 0.$$

Setting, for $\sigma, \gamma > 0$,

$$\Phi(\sigma,\gamma) = |\left\{x \in Q(x_o,\rho) : |u(x)| > \sigma, |Du(x)|^p > \gamma\right\}|$$

we get

$$\Phi(0,\gamma) \le \Phi(\sigma,0) + \frac{1}{\gamma}\int_0^{\gamma} [\Phi(0,s) - \Phi(\sigma,s)]\mathrm{d}s \le c\rho^{\lambda}\left[\sigma^{-q} + \frac{\sigma}{\gamma}\right].$$

Minimization of the above formula in σ and setting $\gamma = t^p$ give the result.

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Appendix

In this section we repeat some examples, taken from [12], which indicate the relationships among the spaces $L^{1,\lambda}$, $L^1 \log L^1$ and $L^1_{\frac{n}{n-1}}$.

Let us start with the definitions of the last two spaces.

Definition A.1. We denote by $L^1 \log L^1(\Omega)$ the space of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f| \log(1 + |f|) \mathrm{d}x < +\infty.$$

Definition A.2. We denote by $L^1_{\frac{n}{n-1}}(\Omega)$ the space of the functions $f \in L^1(\Omega)$ such that

$$\int_0^{|\Omega|} [\sigma f^{**}(\sigma)]^{\frac{n}{n-1}} \frac{\mathrm{d}\sigma}{\sigma} < +\infty$$

where

$$f^{**}(\sigma) = \frac{1}{\sigma} \sup_{|F|=\sigma} \int_F |f| \mathrm{d}x.$$

Remark A.1. Let $f(x) \in L^1(B(0,\epsilon))$; then the function $g : \Omega = B(0,\epsilon) \times]0,1[\to \mathbb{R}$ defined by the law

$$g(x,t) = f(x)$$

belongs to the space $L^{1,1}(\Omega)$.

Indeed, fixing $y_o = (x_o, t_o) \in \Omega$ and $\rho \in]0, d_\Omega]$, we have

$$\begin{aligned}
\int_{\Omega(y_o,\rho)} |g(x,t)| \mathrm{d}x\mathrm{d}t &= \int_{\Omega(y_o,\rho)} |f(x)| \mathrm{d}x\mathrm{d}t \\
&\le \int_{(B(0,\epsilon)\cap B(x_o,\rho))\times(]0,1[\cap]t_0-\rho,t_0+\rho[)} |f(x)| \mathrm{d}x\mathrm{d}t \\
&\le \rho \int_{B(0,\epsilon)} |f(x)| \mathrm{d}x.
\end{aligned}$$

Finally we are ready to review the relationships among the above spaces.

(a) $L^1 \log L^1 \not\subset L^{1,\lambda}$.

It is well known that the function $f(x) = |x|^{-\alpha}$ belongs to $L^1 \log L^1(B(0,R))$ for $\alpha \in]0,n[$.

On the other hand, for any $\rho \in]0,R]$, it turns out that

$$\rho^{-\lambda} \int_{B(0,\rho)\cap B(0,R)} |x|^{-\alpha} \mathrm{d}x = \rho^{-\lambda} \int_{B(0,\rho)} |x|^{-\alpha} \mathrm{d}x$$

and the right hand side blows up as $\rho \to 0^+$ if $\alpha \in]n-\lambda, n[$.

(b) $L^{1,\lambda} \not\subset L^1 \log L^1$.

Let $0 < \epsilon < 1$ and

$$f(x) = \frac{1}{|x|^n \log^2 |x|^n}, \quad x \in B(0,\epsilon).$$

By Remark A.1 the function $g : \Omega = B(0,\epsilon) \times]0,1[\to \mathbb{R}$ defined by the law

$$g(x,t) = f(x)$$

belongs to the space $L^{1,1}(\Omega)$.

On the other hand, the maximal function of g does not belong to $L^1(\Omega)$ and thus g does not belong to the space $L^1 \log L^1$.

(c) $L^1_{\frac{n}{n-1}} \not\subset L^{1,\lambda}$.

The function

$$f(x) = \frac{1}{|x|^n \log^2 |x|^n}, \quad x \in B(0,\epsilon), 0 < \epsilon < 1,$$

belongs to the space $L^1_{\frac{n}{n-1}}(B(0,\epsilon))$ (see [13]) but direct calculations show that f does not belong to $L^{1,\lambda}(B(0,\epsilon))$ for any $\lambda \in]0,n[$.

(d) $L^{1,\lambda} \not\subset L^1_{\frac{n}{n-1}}$.

Let $n > 2$, $\Omega \subset \mathbb{R}^{n-1}$ and choose a function $f \in L^1(\Omega)$ such that $f \notin L^1_{\frac{n-1}{n-2}}(\Omega)$.

By virtue of Remark A.1, the function $g : \Omega \times]0,1[\to \mathbb{R}$ defined by the law

$$g(x,t) = f(x)$$

belongs to the space $L^{1,1}(\Omega \times]0,1[)$.

We will prove now that $g \notin L^1_{\frac{n}{n-1}}(\Omega \times]0,1[)$.

In fact, if g belongs to $L^1_{\frac{n}{n-1}}(\Omega \times]0,1[)$ then, by definition, we would have

$$\int_0^{|\Omega \times]0,1[|} [\sigma g^{**}(\sigma)]^{\frac{n}{n-1}} \frac{\mathrm{d}\sigma}{\sigma} < +\infty$$

where

$$g^{**}(\sigma) = \frac{1}{\sigma} \sup_{|F|=\sigma} \int_F |g(x,t)| \mathrm{d}x \mathrm{d}t.$$

Observing now that

$$f^{**}(\sigma) \le g^{**}(\sigma), \quad \sigma \in]0, |\Omega|[$$

and that $\sigma f^{**}(\sigma)$ is an increasing function, we deduce

$$\begin{aligned}\int_0^{|\Omega|} [\sigma f^{**}(\sigma)]^{\frac{n-1}{n-2}} \frac{\mathrm{d}\sigma}{\sigma} &= \int_0^{|\Omega|} [\sigma f^{**}(\sigma)]^{\frac{n}{n-1}} [\sigma f^{**}(\sigma)]^{\frac{n-1}{n-2}-\frac{n}{n-1}} \frac{\mathrm{d}\sigma}{\sigma} \\ &\le [|\Omega| f^{**}(|\Omega|)]^{\frac{n-1}{n-2}-\frac{n}{n-1}} \int_0^{|\Omega|} [\sigma g^{**}(\sigma)]^{\frac{n}{n-1}} \frac{\mathrm{d}\sigma}{\sigma} < +\infty\end{aligned}$$

whence $f \in L^1_{\frac{n-1}{n-2}}(\Omega)$ which contradicts the assumption.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press Inc., 1975.

[2] A. Alvino, P.L. Lions, G. Trombetti, Comparison results for elliptic and parabolic equations via Schwarz symmetrization, Ann. Inst. H. Poincaré 7 (1990).

[3] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Sc. Norm. Super. Pisa IV XXII (2) (1995).

[4] M.F. Betta, V. Ferone, A. Mercaldo, Regularity for solutions of nonlinear elliptic equations, Bull. Sci. Math. 118 (1994).

[5] L. Boccardo, T. Gallouët, Non linear elliptic and parabolic equations involving measure data, J. Funct. Anal. 87 (1) (1989) 149–169.

[6] L. Boccardo, T. Gallouët, Nonlinear elliptic equations with right-hand side measures, Comm. Partial Differential Equations 17 (1992) 641–655.

[7] S. Campanato, Proprietá di inclusione per spazi di Morrey, Ric. Mat. 12 (1963).

[8] S. Campanato, Equazioni ellittiche del II^o ordine e spazi $\mathcal{L}^{2,\lambda}$, Ann. Mat. Pura Appl. 69 (1965).

[9] S. Campanato, Sistemi Ellittici in forma divergenza, Regolarità all'interno, Quaderni della Scuola Normale Superiore di Pisa (1980).

[10] S. Campanato, Sistemi differenziali del 2^o ordine di tipo ellittico, Quaderno n.1 del dottorato di Ricerca in Matematica (1991).

[11] F. Chiarenza, M. Frasca, Morrey spaces and Hardy–Littlewood maximal function, Rend. Mat. VII (7) (1987).
[12] G.R. Cirmi, S. Leonardi, Regularity results for the gradient of the solutions of linear elliptic equations with $L^{1,\lambda}$ data, Ann. Mat. Pura Appl. 185 (4) (2006).
[13] T. Del Vecchio, Nonlinear elliptic equations with measure data, Potential Anal. 4 (2) (1995).
[14] G. Di Fazio, Poisson equations and Morrey spaces, J. Math Anal. Appl. 163 (1992).
[15] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems, in: Annals of Mathematics Studies, vol. 105, Princeton University Press, Princeton, NJ, 1983.
[16] L.I. Hedberg, Th.H. Wolff, Thin sets in nonlinear potential theory, Ann. Inst. Fourier 33 (4) (1983).
[17] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford Science Publ., 1993.
[18] T. Kilpeläinen, Hölder continuity of solutions to quasilinear elliptic equations involving measures, Potential Anal. 3 (1994).
[19] A.A. Kovalesvskii, Integrability of solutions of nonlinear elliptic equations with right-hand sides from classes close to L^1, Math. Notes 70 (3) (2001) 337–346.
[20] A.A. Kovalesvskii, F. Nicolosi, Solvability of Dirichlet problem for a class of degenerate nonlinear high-order equations with L^1-data, in: Proceedings of the Third World Congress of Nonlinear Analysts, Part 1 (Catania, 2000), Nonlinear Anal. 47 (1) (2001) 435–446.
[21] O.A. Lady'zhenskaya, N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, 1968.
[22] J. Leray, J.L. Lions, Quelques résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.
[23] J.L. Lions, Quelques méthodes de résultats de Visik sur les problèmes elliptiques non linéaires par les de Minty–Browder, Bull. Soc. Math. France 93 (1965) 97–107.
[24] G.R. Mingione, The Calderón–Zygmund theory for elliptic problems with measure data, Ann. Sc. Norm. Super. Pisa, Cl. Sci. Ser. V 6 (2007).
[25] J.M. Rakotoson, Equivalence between the growth of $\int_{B(x,r)} |\nabla y| \mathrm{d}y$ and T in the equation $P[u] = T$, J. Differential Equations 86 (1990).
[26] J. Serrin, Pathological solutions of elliptic differential equations, Ann. Sc. Norm. Super. Pisa 18 (1964).
[27] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble) 15 (1) (1965).
[28] G. Stampacchia, Équations elliptiques du second ordre à coefficients discontinus, Université de Montréal 1966.
[29] G. Talenti, Elliptic equations and rearrangements, Ann. Sc. Norm. Super. Pisa 4 (3) (1976).
[30] G. Talenti, Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces, Ann. Mat. Pura Appl. 120 (4) (1979).