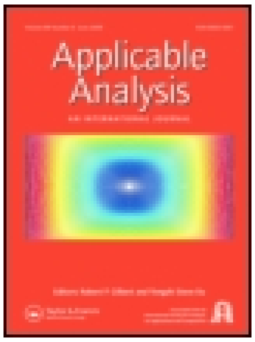

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Regularity for minimizers with positive Jacobian

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ABSTRACT

We deal with maps $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$ minimizing variational integrals $\int_\Omega[|Du(x)|^p+h(\det Du(x))]\,dx$, where $2\le p<n$, $h:(0,+\infty)\to[0,+\infty)$ is convex and blows when $\det Du\to 0^+$: $\lim_{t\to 0^+}h(t)=+\infty$. If such a blow up is a power of $|\ln(t)|$, then we derive regularity for the minimizer $u=(u^1,\dots,u^n)$. We are able also to deal with integrals containing all the minors: $\int_\Omega[|Du(x)|^p+\sum_{s=2}^{n-1}|\mathrm{adj}_s(Du(x))|^{q_s}+h(\det Du(x))]\,dx$ with $q_s\ge 1$.

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1. Introduction and main result

We consider variational integrals

$$\mathcal{E}(u)=\int_\Omega W(x,Du(x))\,dx \tag{1}$$

where $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$, $n\ge 3$ and Ω is a bounded open set. In the framework of nonlinear elasticity, a reasonable assumption is polyconvexity [1,2], that is, for each x, $W(x,F)$ can be written as a convex function of minors taken from the $n\times n$ matrix F. In this paper we are concerned with regularity properties for minimizers of (1) under polyconvexity: contributions to partial regularity can be found in [3–10]; on the other hand, maximum principle, pointwise bounds and L^∞ regularity appear in [11–17]; injectivity can be found in [18]. Let us mention that, in nonlinear elasticity, a natural assumption on the stored energy function $W(x,F)$ is given by

$$W(x,F)\to+\infty\quad \text{as}\det F\to 0^+ \tag{2}$$

see [1]. Such a condition says that an infinite amount of energy is needed to shrink a finite volume to zero. Assumption (2) leads us to the unilateral constraint $\det Du>0$. Let us mention that singular behaviour (2) brings us mathematical difficulties and the above-mentioned regularity results cannot be applied under (2).

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In [19], these difficulties are overcome and the author proved pointwise bounds for minimizers of some polyconvex integrals (1) under singularity (2). More precisely in [19] it is considered the simple model

$$W(x,F) = |F|^p + h(\det F) \tag{3}$$

where $2 \leq p$ and $h : (0,+\infty) \to [0,+\infty)$ is a convex function which blows up as $t \to 0^+$ sufficiently slow, that is

$$\lim_{t\to 0^+} h(t) = +\infty \tag{4}$$

and for some $\lambda \in (0,1)$ and $M \in [0,+\infty)$

$$h(\lambda t) \leq h(t) + M \quad \forall t \in (0,+\infty). \tag{5}$$

Note that $h(t) = -\ln(t)$ verifies (5) with $M = -\ln(\lambda)$ and any $\lambda \in (0,1)$. On the contrary, $h(t) = [\ln(t)]^2$ does not satisfy (5). In this paper we show how to deal with $[\ln(t)]^2$ and with any power of $\ln(t)$ by making the following weaker assumption: for some $\lambda, \gamma \in (0,1)$ and $c_1, M \in [0,+\infty)$ it results

$$h(\lambda t) \leq h(t) + c_1(h(t))^\gamma + M \quad \forall t \in (0,+\infty). \tag{6}$$

Let us note that the simple model (3) is useful when trying to understand the competition between $|F|^p$ and $h(\det F)$ when $|F|$ is small; on the other hand such a simple model (3) does not make $\mathcal{E}$ weakly lower semicontinuous, see Theorem 4.5, part (ii) in [20]. Weak lower semicontinuity is an important tool when proving existence of minimizers by direct methods. Following Remark 8.32 (iii) at page 405 in [2], we overcome such a trouble by taking all the minors in the density:

$$W(x,F) = |F|^p + \sum_{s=2}^{n-1} b_s |\mathrm{adj}_s(F)|^{q_s} + h(\det F). \tag{7}$$

When using Remark 8.32 we need $b_s > 0, q_s \geq p/(p-1)$ and $h(t) \geq b_n|t|^r - c$ with $r > 1$ and $b_n > 0$; in the present paper, devoted to regularity properties, we make the weaker assumptions

$$b_s \geq 0 \quad \text{and} \quad q_s \geq 1 \tag{8}$$

for all $s = 2, \dots, n-1$; moreover, we need no assumption about the behaviour of $h(t)$ for large t. Note that

$$|\mathrm{adj}_s(F)|^2 = \sum \left| \det \begin{pmatrix} F^{i_1}_{\alpha_1} \cdots\cdots F^{i_1}_{\alpha_s} \\ \cdots\cdots\cdots\cdots\cdots \\ F^{i_s}_{\alpha_1} \cdots\cdots F^{i_s}_{\alpha_s} \end{pmatrix} \right|^2 \tag{9}$$

where the sum is taken over all increasing s-tuples $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ and $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s \leq n$, see page 249 in [2]. In the sequel we will shortly write $W(Dv) \in L^1$ instead of $x \to W(x, Dv(x)) \in L^1$. We will prove the following

Theorem 1.1: *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$. Let $h : (0,+\infty) \to [0,+\infty)$ be a convex function verifying (6). Let us consider the variational integral (1) with the stored energy function (7) where $2 \leq p < n$ and (8). Let us assume that $u = (u^1, \dots, u^n) \in W^{1,p}(\Omega;\mathbb{R}^n)$ satisfies $\det Du > 0$ almost everywhere in Ω with $W(Du) \in L^1(\Omega)$ and*

$$\mathcal{E}(u) \leq \mathcal{E}(w) \tag{10}$$

for every $w \in u + W^{1,p}_0(\Omega;\mathbb{R}^n)$ with $\det Dw > 0$ almost everywhere in Ω and $W(Dw) \in L^1(\Omega)$. We let $|\Omega|$ be the Lebesgue measure of Ω.

(i) *If $\gamma < p/n$, then there exists a constant c_2 such that, for every component u^j, we have*

$$\inf_{\partial\Omega} u^j - c_2|\Omega|^{1/n-\gamma/p} \leq u^j(x) \leq \sup_{\partial\Omega} u^j + c_2|\Omega|^{1/n-\gamma/p} \tag{11}$$

for almost every $x \in \Omega$. The constant c_2 is given by

$$c_2 = C_S c_3^{1/p} 2^{(n-\gamma n)/(p-\gamma n)}$$

where $C_S = (n-1)p/(n-p)$ is the Sobolev constant,

$$c_3 = \frac{M|\Omega|^\gamma + c_1\|h(\det Du)\|^\gamma_{L^1(\Omega)}}{(1-\lambda^2)^{p/2}}.$$

(ii) *If $\gamma = p/n$, then for any $L \geq \sup_{\partial\Omega} u^j$*

$$|\{u^j > L\}| \leq |\Omega| e^{1-(e^{1/p^*} C_S c_3^{1/p})^{-1}(L-\sup_{\partial\Omega} u^j)} \tag{12}$$

and for any $L \geq -\inf_{\partial\Omega} u^j$

$$|\{u^j < -L\}| \leq |\Omega| e^{1-(e^{1/p^*} C_S c_3^{1/p})^{-1}(L+\inf\limits_{\partial\Omega} u^j)}. \tag{13}$$

(iii) *If $\gamma > p/n$, then for any $L > 0 \vee \sup_{\partial\Omega} u^j$*

$$|\{u^j > L\}| \leq 2^{np(n-p)/(n\gamma-p)^2}[(C_S^p c_3)^{n/(n\gamma-p)} + (2[0 \vee \sup_{\partial\Omega} u^j])^{np/(n\gamma-p)}|\Omega|]\left(\frac{1}{L}\right)^{np/(n\gamma-p)} \tag{14}$$

and for any $L > 0 \vee -\inf_{\partial\Omega} u^j$

$$|\{u^j < -L\}| \leq 2^{np(n-p)/(n\gamma-p)^2}[(C_S^p c_3)^{n/(n\gamma-p)} + (2[0 \vee -\inf_{\partial\Omega} u^j])^{np/(n\gamma-p)}|\Omega|]\left(\frac{1}{L}\right)^{np/(n\gamma-p)}. \tag{15}$$

In the previous Theorem 1.1 we assumed $p < n$: when $p \geq n$, Iwaniec et al. [21] show that every component of mappings with finite distorsion is weakly monotone: this means that every component enjoys maximum and minimum principle on every ball $B \subset \Omega$ [22]; note that mappings with positive Jacobian have finite distortion.

In order to give a corollary of Theorem 1.1, we need two function spaces. The first one is the exponential class $\mathrm{Exp}(\Omega, \mathbb{R}^n)$, which consists of all measurable vectors $f = (f^1, \dots, f^n)$ such that

$$\int_\Omega e^{\nu|f|} < \infty$$

for some $\nu > 0$. It is a Banach space under the norm

$$\|f\|_{\mathrm{Exp}(\Omega,\mathbb{R}^n)} = \inf\left\{\nu > 0 : \int_\Omega e^{|f|/\nu} \leq 2\right\}$$

The second one is the weak L^m space, known also as Marcinkiewicz space, which is defined as follows: if $m > 1$, then the Marcinkiewicz space $L^m_{\mathrm{weak}}(\Omega)$ consists of measurable functions f on Ω such that

$$A_m(f) = \sup_{t>0} t|\{x \in \Omega : |f(x)| > t\}|^{1/m} = \sup_{t>0} t f_*(t)^{1/m} < \infty,$$

where $f_*(t) = |\{x \in \Omega : |f(x)| > t\}|$ denotes the distribution function of f. We recall that $L^m(\Omega)$ is a proper subspace of $L^m_{\mathrm{weak}}(\Omega)$, and if $f \in L^m_{\mathrm{weak}}(\Omega)$ for some $m > 1$, then $f \in L^{m-\varepsilon}(\Omega)$ for every

$0 < \varepsilon \leq m-1$. We denote $L^m_{\text{weak}}(\Omega, \mathbb{R}^n)$ for vectors $f = (f^1, \ldots, f^n)$ such that $f^i \in L^m_{\text{weak}}(\Omega)$ for each i. For a detailed analysis of L^m_{weak} spaces we refer to [23].

Corollary 1.1: *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$. Let $h : (0, +\infty) \to [0, +\infty)$ be a convex function verifying* (6). *Let us consider the variational integral* (1) *with the stored energy function* (7) *where $2 \leq p < n$ and* (8). *Fix $u_* = (u^1_*, \ldots, u^n_*) \in \mathrm{Lip}(\bar{\Omega}, \mathbb{R}^n)$. Let us assume that $u = (u^1, \ldots, u^n) \in u_* + W^{1,p}_0(\Omega; \mathbb{R}^n)$ satisfies* $\det Du > 0$ *almost everywhere in Ω with $W(Du) \in L^1(\Omega)$ and*

$$\mathcal{E}(u) \leq \mathcal{E}(w)$$

for every $w \in u_* + W^{1,p}_0(\Omega; \mathbb{R}^n)$ with $\det Dw > 0$ *almost everywhere in Ω and $W(Dw) \in L^1(\Omega)$.*

(1) *If $\gamma < p/n$, then $u \in L^\infty(\Omega, \mathbb{R}^n)$;*
(2) *If $\gamma = p/n$, then $u \in Exp(\Omega, \mathbb{R}^n)$;*
(3) *If $\gamma > p/n$, then $u \in L^{np/(n\gamma - p)}_{\text{weak}}(\Omega, \mathbb{R}^n)$.*

Remark 1.1: For $\gamma = 0$ or $c_1 = 0$ in (6) we obtain (5), already treated in [19] and satisfied by any convex function h with $h(t) = -\ln(t)$ for every $t \in (0, t_0)$, where $0 < t_0 \leq 1$.

In Section 3 we give examples of functions satisfying the condition (6).

Let us mention [24–27] concerning some two-dimensional problems. It would be interesting to understand whether an exponent $\gamma \in [p/n, 1)$ could produce an unbounded minimizer or not. We note that [28–30] deals with a 'singular' example in dimension $n = 2$ but such an example shows a C^1 map that is neither $C^{1,\beta}$ nor $W^{2,2}$. In the present paper we are concerned with dimension $n \geq 3$ and we would need an unbounded map.

2. Proof of Theorem 1.1 and Corollary 1.1

Proof of Theorem 1.1: For $\sup_{\partial\Omega} u^j < L_0 \leq L < +\infty$ and λ as in (6), we define $w = (w^1, \ldots, w^n)$ as follows: $w^j = u^j - (1-\lambda)[(u^j - L) \vee 0]$ and $w^i = u^i$ if $i \neq j$. Since $\sup_{\partial\Omega} u^j < L$, we have $(u^j - L) \vee 0 \in W^{1,p}_0(\Omega)$ and $w \in u + W^{1,p}_0(\Omega; \mathbb{R}^n)$. Note that $Dw = Du$ on $\{u^j \leq L\}$. Moreover

$$Dw^i = \begin{Bmatrix} Du^i & \text{if } i \neq j \\ \lambda Du^i & \text{if } i = j, \end{Bmatrix} \quad \text{on} \{u^j > L\} \tag{16}$$

thus

$$\det Dw = \lambda \det Du > 0 \quad \text{on} \{u^j > L\} \tag{17}$$

and, using (6),

$$0 \leq h(\det Dw) \leq h(\det Du) + c_1 (h(\det Du))^\gamma + M \quad \text{on} \{u^j > L\} \tag{18}$$

Note that

$$|\mathrm{adj}_s(Dw)|^2 = \sum \left| \det \begin{pmatrix} \dfrac{\partial w^{i_1}}{\partial x_{\alpha_1}} \cdots \dfrac{\partial w^{i_1}}{\partial x_{\alpha_s}} \\ \cdots\cdots\cdots\cdots \\ \dfrac{\partial w^{i_s}}{\partial x_{\alpha_1}} \cdots \dfrac{\partial w^{i_s}}{\partial x_{\alpha_s}} \end{pmatrix} \right|^2 = \sum \left| \det \frac{\partial (w^{i_1}, \ldots, w^{i_s})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_s})} \right|^2$$

so that, if $j \notin \{i_1, \ldots, i_s\}$, then

$$\det \frac{\partial (w^{i_1}, \ldots, w^{i_s})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_s})} = \det \frac{\partial (u^{i_1}, \ldots, u^{i_s})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_s})}$$

otherwise, if $j \in \{i_1, \dots, i_s\}$, then

$$\det \frac{\partial(w^{i_1}, \dots, w^{i_s})}{\partial(x_{\alpha_1}, \dots, x_{\alpha_s})} = \lambda \det \frac{\partial(u^{i_1}, \dots, u^{i_s})}{\partial(x_{\alpha_1}, \dots, x_{\alpha_s})}$$

These properties guarantee that

$$|\mathrm{adj}_s(Dw)| \le |\mathrm{adj}_s(Du)| \quad \text{on}\{u^j > L\}. \tag{19}$$

Then $W(Dw) \in L^1(\Omega)$ so we can use such a w in (10) and we get

$$\begin{aligned}
&\int_{\{u^j > L\}} \left[|Du|^p + \sum_{s=2}^{n-1} b_s |\mathrm{adj}_s(Du)|^{q_s} + h(\det Du) \right] \mathrm{d}x \\
&\quad \le \int_{\{u^j > L\}} \left[|Dw|^p + \sum_{s=2}^{n-1} b_s |\mathrm{adj}_s(Dw)|^{q_s} + h(\det Dw) \right] \mathrm{d}x \\
&\quad \le \int_{\{u^j > L\}} \left[|Dw|^p + \sum_{s=2}^{n-1} b_s |\mathrm{adj}_s(Du)|^{q_s} + h(\det Du) + c_1 (h(\det Du))^\gamma + M \right] \mathrm{d}x
\end{aligned}$$

so that

$$\int_{\{u^j > L\}} |Du|^p \,\mathrm{d}x \le \int_{\{u^j > L\}} [|Dw|^p + c_1 (h(\det Du))^\gamma + M] \,\mathrm{d}x \tag{20}$$

Note that, on $\{u^j > L\}$,

$$|Du|^2 = |Dw|^2 + (1 - \lambda^2)|Du^j|^2 \tag{21}$$

thus,

$$|Du|^p \ge |Dw|^p + (1 - \lambda^2)^{p/2} |Du^j|^p \tag{22}$$

since $p \ge 2$. Equations (20) and (22) merge into

$$(1 - \lambda^2)^{p/2} \int_{\{u^j > L\}} |Du^j|^p \,\mathrm{d}x \le M|\{u^j > L\}| + c_1 \int_{\{u^j > L\}} (h(\det Du))^\gamma \,\mathrm{d}x. \tag{23}$$

From Hölder inequality with exponents $1/\gamma$ and $1/(1 - \gamma)$ it follows that

$$\int_{\{u^j > L\}} (h(\det Du))^\gamma \,\mathrm{d}x \le \|h(\det Du)\|^\gamma_{L^1(\{u^j > L\})} |\{u^j > L\}|^{1-\gamma};$$

using last inequality in (23), we have

$$\begin{aligned}
(1 - \lambda^2)^{p/2} \int_{\{u^j > L\}} |Du^j|^p \,\mathrm{d}x &\le M|\{u^j > L\}|^{1-\gamma} |\{u^j > L\}|^\gamma \\
&\quad + c_1 \|h(\det Du)\|^\gamma_{L^1(\{u^j > L\})} |\{u^j > L\}|^{1-\gamma} \\
&\le (M|\Omega|^\gamma + c_1 \|h(\det Du)\|^\gamma_{L^1(\Omega)}) |\{u^j > L\}|^{1-\gamma},
\end{aligned}$$

that is

$$\int_{\{u^j > L\}} |Du^j|^p \,\mathrm{d}x \le c_3 |\{u^j > L\}|^{1-\gamma} \tag{24}$$

where c_3 denotes the constant

$$c_3 = \frac{M|\Omega|^\gamma + c_1 \|h(\det Du)\|^\gamma_{L^1(\Omega)}}{(1-\lambda^2)^{p/2}}.$$

Note that $1_{\{u^j>L\}} Du^j = D[(u^j - L) \vee 0]$ and $(u^j - L) \vee 0 \in W^{1,p}_0(\Omega)$; therefore, since $p < n$, the following Sobolev inequality holds

$$\left(\int_{\{u^j>L\}} |u^j - L|^{p^*} \, dx \right)^{1/p^*} \leq C_S \left(\int_{\{u^j>L\}} |Du^j|^p \, dx \right)^{1/p}$$

where $1/p^* = 1/p - 1/n$ and $C_S = (n-1)p/(n-p)$ is the Sobolev constant; using this inequality in (24) we get

$$\int_{\{u^j>L\}} |u^j - L|^{p^*} \, dx \leq C_S^{p^*} c_3^{p^*/p} |\{u^j > L\}|^{p^*(1/p - \gamma/p)}. \tag{25}$$

For any $V > L$ we have

$$\begin{aligned}(V-L)^{p^*} |\{u^j > V\}| &= \int_{\{u^j>V\}} |V - L|^{p^*} \, dx \leq \int_{\{u^j>V\}} |u^j - L|^{p^*} \, dx \\ &\leq \int_{\{u^j>L\}} |u^j - L|^{p^*} \, dx;\end{aligned} \tag{26}$$

then, putting together (25) and (26), we have

$$|\{u^j > V\}| \leq \frac{C_S^{p^*} c_3^{p^*/p}}{(V-L)^{p^*}} |\{u^j > L\}|^{p^*(1/p - \gamma/p)} \tag{27}$$

for every V,L with $V > L \geq L_0$. Then we can use Lemma 4.1 at page 93 of [31] that we write for the convenience of the reader. See also [32,33]. ■

Lemma 2.1: *Let c_*, α, β be positive constants. Let $\varphi : [L_0, +\infty) \to [0, +\infty)$ be decreasing and such that*

$$\varphi(V) \leq \frac{c_*}{(V-L)^\alpha} [\varphi(L)]^\beta \tag{28}$$

for every V,L with $V > L \geq L_0$. It results that:

(i) *if $\beta > 1$ then we have*

$$\varphi(L_0 + d) = 0,$$

where

$$d = \{c_* [\varphi(L_0)]^{\beta-1} 2^{\alpha\beta/(\beta-1)}\}^{1/\alpha}; \tag{29}$$

(ii) *if $\beta = 1$ then for any $L \geq L_0$ we have*

$$\varphi(L) \leq \varphi(L_0) e^{1-(ec_*)^{-1/\alpha}(L-L_0)};$$

(iii) *if $\beta < 1$ and $L_0 > 0$ then for any $L \geq L_0$ we have*

$$\varphi(L) \leq 2^{\alpha/(1-\beta)^2} [c_*^{1/(1-\beta)} + (2L_0)^{\alpha/(1-\beta)} \varphi(L_0)] \left(\frac{1}{L}\right)^{\alpha/(1-\beta)}.$$

We use the previous lemma with $\varphi(L) = |\{u^j > L\}|$, $c_* = C_S^{p^*} c_3^{p^*/p}$, $\alpha = p^*$ and $\beta = p^*(1/p - \gamma/p)$; from (27) we get:

(i) If $\gamma < p/n$ then $\beta > 1$, thus

$$|\{u^j > L_0 + d\}| = 0$$

this implies

$$u^j \le L_0 + d = L_0 + C_S c_3^{1/p} |\{u^j > L_0\}|^{1/n-\gamma/p}\, 2^{(n-\gamma n)/(p-\gamma n)}$$

almost everywhere in Ω. In order to get the right-hand side of (11), we control $|\{u^j > L_0\}|$ by means of $|\Omega|$ and we take a sequence $\{(L_0)_k\}_k$ with $(L_0)_k \to \sup_{\partial\Omega} u^j$. Let us show how we obtain the left-hand side of (11). We change sign to u^j; since $n \ge 3$, it turns out that $\{1,\dots,n\} \setminus \{j\} \ne \emptyset$ thus we take $s = \min\{1,\dots,n\} \setminus \{j\}$ and we change sign to u^s too; we get $v = (v^1,\dots,v^n)$ with $v^i = -u^i$ if $i \in \{j,s\}$, $v^i = u^i$ if $i \in \{1,\dots,n\} \setminus \{j,s\}$. Since we have only two changes of sign, $\det Dv = \det Du > 0$; thus we can apply the right-hand side of (11) to v and we are done.

(ii) If $\gamma = p/n$ then $\beta = 1$, and for any $L \ge L_0$ we have

$$|\{u^j > L\}| \le |\{u^j > L_0\}| e^{1-(e^{1/p^*} C_S c_3^{1/p})^{-1}(L-L_0)}$$

In order to get (12) we control $|\{u^j > L_0\}|$ by means of $|\Omega|$ and we take a sequence $\{(L_0)_k\}_k$ with $(L_0)_k \to \sup_{\partial\Omega} u^j$. Let us show how we obtain (13). As in the proof of part (i), we need only to change signs to u^j and u^s where $s = \min\{1,\dots,n\} \setminus \{j\}$.

(iii) If $\gamma > p/n$ then $\beta < 1$. For any $L \ge L_0 > 0 \vee \sup_{\partial\Omega} u^j$ we have

$$|\{u^j > L\}| \le 2^{np(n-p)/(n\gamma-p)^2} [(C_S^p c_3)^{n/(n\gamma-p)} + (2L_0)^{np/(n\gamma-p)} |\{u^j > L_0\}|] \left(\frac{1}{L}\right)^{np/(n\gamma-p)}$$

In order to get (14) we control $|\{u^j > L_0\}|$ by means of $|\Omega|$ and we take a sequence $\{(L_0)_k\}_k$ with $(L_0)_k \to 0 \vee \sup_{\partial\Omega} u^j$. Let us show how we obtain (15). As in the proof of part (i), we need only to change signs to u^j and u^s where $s = \min\{1,\dots,n\} \setminus \{j\}$. This ends the proof of Theorem 1.1.

Proof of Corollary 1.1: By Theorem 1.1 and the assumption $u_* \in Lip(\bar{\Omega}, \mathbb{R}^n)$, we have

(1) If $\gamma < p/n$, then (11) implies for any $j \in \{1,\dots,n\}$,

$$|u^j(x)| \le \|u_*\|_{L^\infty(\Omega,\mathbb{R}^n)} + c_2 |\Omega|^{1/n-\gamma/p}$$

thus

$$\|u\|_{L^\infty(\Omega,\mathbb{R}^n)} \le \sum_{j=1}^n \|u^j\|_{L^\infty(\Omega)} \le n(\|u_*\|_{L^\infty(\Omega,\mathbb{R}^n)} + c_2|\Omega|^{1/n-\gamma/p}) < \infty$$

(2) If $\gamma = p/n$, then (12) and (13) imply for any $j \in \{1,\dots,n\}$, for any $L \ge \|u_*\|_{L^\infty(\Omega)}$

$$\left.\begin{array}{l} |\{u^j > L\}| \\ |\{u^j < -L\}| \end{array}\right\} \le |\Omega| e^{1-(e^{1/p^*} C_S c_3^{1/p})^{-1}(L-\|u_*^j\|_{L^\infty(\Omega)})} \le c_4 e^{-\mu L} \tag{30}$$

where

$$c_4 = |\Omega| e^{1+\mu\|u_*^j\|_{L^\infty(\Omega)}} \quad \text{and} \quad \mu = (e^{1/p^*} C_S c_3^{1/p})^{-1}$$

Equation (30) yields

$$|\{|u^j| > L\}| = |\{u^j > L\}| + |\{u^j < -L\}| \le 2c_4 e^{-\mu L} \quad \text{for } L \ge \|u_*\|_{L^\infty(\Omega)}$$

Since

$$\{|u| > nL\} \subset \bigcup_{j=1}^{n} \{|u^j| > L\}$$

then

$$|\{|u| > nL\}| \le \sum_{j=1}^{n} |\{|u^j| > L\}| \le 2nc_4 e^{-\mu L} \quad \text{for } L \ge \|u_*\|_{L^\infty(\Omega)} \tag{31}$$

Since $|\{|u| > nL\}| \le |\Omega|$, then for $0 \le L < \|u_*\|_{L^\infty(\Omega)}$ one has

$$|\{|u| > nL\}| \le |\Omega| \le |\Omega| e^{\mu \|u_*\|_{L^\infty(\Omega)}} e^{-\mu L} \tag{32}$$

Puting together (31) and (32) we get

$$|\{|u| > nL\}| \le c_5 e^{-\mu L}, \quad \text{for all } L \ge 0 \tag{33}$$

where

$$c_5 = \max\{2nc_4, |\Omega| e^{\mu \|u_*\|_{L^\infty(\Omega)}}\}$$

We now prove ■

Lemma 2.2: *Let $u = (u^1, \dots, u^n)$ be a measurable vector; let $C > 0$, $\mu > 0$ be constants such that*

$$|\{|u| > t\}| \le Ce^{-\mu t}, \quad \forall t > 0,$$

then

$$\int_\Omega e^{\nu |u|} < \infty \quad \forall \nu : \quad 0 < \nu < \mu.$$

Proof: Let $k \in \mathbb{N}$, then

$$\begin{aligned}
\int_\Omega |u|^k &= k \int_0^\infty t^{k-1} |\{|u| > t\}| \le Ck \int_0^\infty t^{k-1} e^{-\mu t} \, \mathrm{d}t \\
&= -\frac{Ck}{\mu} \int_0^\infty t^{k-1} \, \mathrm{d}e^{-\mu t} = -\frac{Ck}{\mu} \left[t^{k-1} e^{-\mu t} |_0^\infty - \int_0^\infty e^{-\mu t} \, \mathrm{d}t^{k-1} \right] \\
&= \frac{Ck(k-1)}{\mu} \int_0^\infty e^{-\mu t} t^{k-2} \, \mathrm{d}t = \frac{Ck(k-1)(k-2)}{\mu^2} \int_0^\infty e^{-\mu t} t^{k-3} \, \mathrm{d}t \\
&= \cdots = \frac{Ck(k-1) \cdots 2 \cdot 1}{\mu^{k-1}} \int_0^\infty e^{-\mu t} \, \mathrm{d}t = \frac{Ck!}{\mu^k}
\end{aligned}$$

This implies

$$\int_\Omega \frac{(\nu |u|)^k}{k!} \le C \left(\frac{\nu}{\mu} \right)^k$$

thus for $0 < \nu < \mu$ we use Taylor's expansion to derive that

$$\begin{aligned}
\int_\Omega e^{\nu |u|} &= \int_\Omega \left[1 + \nu |u| + \frac{(\nu |u|)^2}{2!} + \frac{(\nu |u|)^3}{3!} + \cdots \right] \\
&\le C \left[|\Omega| + \frac{\nu}{\mu} + \left(\frac{\nu}{\mu} \right)^2 + \left(\frac{\nu}{\mu} \right)^3 + \cdots \right] = C \left[|\Omega| + \sum_{j=1}^\infty \left(\frac{\nu}{\mu} \right)^j \right] < \infty
\end{aligned}$$

This ends the proof of Lemma 2.2.

We use the previous lemma and we have for any $0 < \nu < \mu/n$

$$\int_\Omega e^{\nu|u|} < \infty$$

this shows that $u \in \mathrm{Exp}(\Omega, \mathbb{R}^n)$.

(3) If $\gamma > p/n$, then (14) and (15) imply that, for any $j \in \{1, \dots, n\}$, for any $L > \|u_*\|_{L^\infty(\Omega)}$,

$$|\{u^j > L\}| \leq c_6 \left(\frac{1}{L}\right)^{np/(n\gamma-p)} \quad \text{and} \quad |\{u^j < -L\}| \leq c_6 \left(\frac{1}{L}\right)^{np/n\gamma-p}$$

where

$$c_6 = 2^{np(n-p)/(n\gamma-p)^2}[(C_S^p c_3)^{n/(n\gamma-p)} + (2\|u_*\|_{L^\infty(\Omega)})^{np/(n\gamma-p)}|\Omega|]$$

thus

$$|\{|u^j| > L\}| = |\{u^j > L\}| + |\{u^j < -L\}| \leq 2c_6 \left(\frac{1}{L}\right)^{np/(n\gamma-p)}$$

this implies

$$\begin{aligned} A^{np/(n\gamma-p)}_{np/(n\gamma-p)}(u^j) &= \sup_{L>0} L^{np/(n\gamma-p)}|\{|u^j| > L\}| \\ &\leq \sup_{0<L\leq\|u_*\|_{L^\infty(\Omega)}} L^{np/(n\gamma-p)}|\{|u^j| > L\}| + \sup_{L>\|u_*\|_{L^\infty(\Omega)}} L^{np/(n\gamma-p)}|\{|u^j| > L\}| \\ &\leq \|u_*\|^{np/(n\gamma-p)}_{L^\infty(\Omega)}|\Omega| + 2c_6 < \infty. \end{aligned}$$

This ends the proof of Corollary 1.1. ■

3. Final remarks

Remark 3.1: Let us note that the proof of theorem 1.1 does not use convexity of h: we only need measurability of $x \to h(\det Dw(x))$; this is guaranteed, for example, if h is continuous.

Remark 3.2: Let us mention that a basic step in the proof of theorem 1.1 is the choice of the test function w; such a test function is taken from [19] and here we emphasize the feature of w. Because of the constraint $\det Dw > 0$, test functions with a flat part, such as those arising from a truncation argument, are not allowed. Our function w^j can also be written as $w^j = \psi(u^j)$ where

$$\psi(s) = \begin{cases} s & \text{if } s \leq L \\ L + \lambda(s - L) & \text{if } s > L. \end{cases} \tag{34}$$

This means that, on $\{s > L\}$, we keep increasing, since $\lambda > 0$, but with less 'speed', since $\lambda < 1$. Positivity of λ keeps under control the constraint $\det Dw > 0$; smallness of λ allows us to lower $|Dw|^p$ with respect to $|Du|^p$.

Remark 3.3: As we have seen in the proof, the additional piece $(h(t))^\gamma$ in (6) shows up as $(h(\det Du))^\gamma$; since $h(\det Du) \in L^1$, then Hölder inequality allows us to estimate the integral of $(h(\det Du))^\gamma$ over $\{u^j > L\}$ by means of $|\{u^j > L\}|^{1-\gamma}$: this is enough to make the proof going to the end. A posteriori, the role of $(h(t))^\gamma$ is similar to the one that a function $M(x) \in L^r$ would have if it would be put in place of the constant M in (5), see also [34].

Lemma 3.1: *Assume that $h : (0, +\infty) \to [0, +\infty)$ is convex and verifies* (4); *assume that for some $\lambda \in (0, 1)$, $\gamma \in [0, 1)$, $M, c \in [0, +\infty)$ and for some $0 < t_0 \le 1$, we have*

$$h(\lambda t) \le h(t) + ch(t)^{\gamma} + M \quad \forall t \in (0, t_0]. \tag{35}$$

Then (6) holds for every $t \in (0, +\infty)$ with a new constant $M^* = M + h(t_0) + ch(t_0)^{\gamma}$, that is

$$h(\lambda t) \le h(t) + ch(t)^{\gamma} + M + h(t_0) + ch(t_0)^{\gamma} \quad \forall t \in (0, +\infty).$$

Proof of Lemma 3.1: Being h convex, either h decreases in $(0, +\infty)$ or h blows up as $t \to +\infty$.
If h decreases in $(0, +\infty)$, then for all $t > t_0$ we have

$$\begin{aligned} h(\lambda t) &\le h(\lambda t_0) \le h(t_0) + ch(t_0)^{\gamma} + M \\ &\le h(t) + ch(t)^{\gamma} + M + h(t_0) + ch(t_0)^{\gamma} \end{aligned}$$

and the lemma is proved.

Let us assume now that h blows up as $t \to +\infty$ and let us denote with t^* the point where h assumes its minimum value. Let $t \in (t_0, +\infty)$. If $t \ge t^*/\lambda$, then $t^* \le \lambda t \le t$; being h increasing for $s > t^*$, we have

$$h(\lambda t) \le h(t) \le h(t) + ch(t)^{\gamma} + M^*.$$

If $t_0 < t < t^*/\lambda$, then $\lambda t_0 < \lambda t < t^*$. Since h decreases for $s < t^*$ and in view of the assumption, we have

$$\begin{aligned} h(\lambda t) &\le h(\lambda t_0) \le h(t_0) + ch(t_0)^{\gamma} + M \\ &\le h(t) + ch(t)^{\gamma} + M + h(t_0) + ch(t_0)^{\gamma} \end{aligned}$$

and the lemma is proved. ■

Remark 3.4: The following class of functions

$$h(t) = \begin{cases} |\ln t|^k & \forall t \in (0, 1] \\ 0 & \forall t \in [1, +\infty). \end{cases} \qquad k \in \{2, 3, 4, \ldots\} \tag{36}$$

satisfies the assumptions of Lemma 3.1 and therefore Theorem 1.1 can be applied to it, but these functions do not satisfy the assumption (5).

Indeed it is easy to verify that $h \in C^1(0, +\infty)$ and h is convex. Moreover, in order to prove (35), let us consider $t_0 = e^{-1}$. Then for any $\lambda \in (0, 1)$ and for any $t \in (0, e^{-1}]$ we have

$$\begin{aligned} |\ln(\lambda t)|^k &= |\ln \lambda + \ln t|^k = (|\ln \lambda| + |\ln t|)^k \\ &= \sum_{j=0}^{k} \binom{k}{j} |\ln \lambda|^j |\ln t|^{k-j} \\ &\le |\ln t|^k + |\ln \lambda|^k + \sum_{j=1}^{k-1} \binom{k}{j} \max\{1, |\ln \lambda|^{k-1}\} |\ln t|^{k-1} \\ &= |\ln t|^k + |\ln \lambda|^k + (2^k - 2) \max\{1, |\ln \lambda|^{k-1}\} |\ln t|^{k-1}. \end{aligned}$$

Then, setting $M = |\ln \lambda|^k$, $c = (2^k - 2) \max\{1, |\ln \lambda|^{k-1}\}$ and $\gamma = \frac{k-1}{k}$, we have

$$h(\lambda t) \le h(t) + M + c[h(t)]^{\gamma}.$$

We now prove that (36) does not satisfy (5). That is, we need to prove that, $\forall\lambda \in (0,1)$, $\forall M \in [0,+\infty)$, there exists $t = t(M,\lambda) \in (0,+\infty)$ such that

$$h(\lambda t) > h(t) + M.$$

We will take $t \in (0,1)$ so that we need to prove

$$|\ln(\lambda t)|^k > |\ln t|^k + M. \tag{37}$$

We take $0 < t < e^{-(M/k|\ln\lambda|)^{1/(k-1)}} \le 1$, so that

$$|\ln t| > \left(\frac{M}{k|\ln\lambda|}\right)^{1/(k-1)} \ge 0$$

Then, for such a t one has

$$\begin{aligned}
|\ln(\lambda t)|^k &= |\ln\lambda + \ln t|^k = (|\ln\lambda| + |\ln t|)^k \\
&= \sum_{j=0}^{k} \binom{k}{j} |\ln\lambda|^j |\ln t|^{k-j} \\
&\ge \binom{k}{0} |\ln\lambda|^0 |\ln t|^k + \binom{k}{1} |\ln\lambda| |\ln t|^{k-1} \\
&= |\ln t|^k + k|\ln\lambda| |\ln t|^{k-1} \\
&> |\ln t|^k + k|\ln\lambda| \frac{M}{k|\ln\lambda|} = |\ln t|^k + M
\end{aligned}$$

and (37) is proved.

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References

[1] Ball JM. Convexity conditions and existence theorems in nonlinear elasticity. Arch Rational Mech Anal. 1976;63:337–403.
[2] Dacorogna B. Direct methods in the calculus of variations. New York (NY): Springer; 2007. (Applied mathematical sciences; vol. 78).
[3] Evans LC. Quasiconvexity and partial regularity in the calculus of variations. Arch Rational Mech Anal. 1986;95:227–252.
[4] Acerbi E, Fusco N. A regularity theorem for minimizers of quasiconvex integrals. Arch Rational Mech Anal. 1987;99:261–281.
[5] Giaquinta M, Modica G. Partial regularity of minimizers of quasiconvex integrals. Ann Inst Henri Poincaré, Analyse non lineaire. 1986;3:185–208.
[6] Fusco N, Hutchinson J. Partial regularity in problems motivated by nonlinear elasticity. SIAM J Math Anal. 1991;22:1516–1551.
[7] Fuchs M, Seregin G. Partial regularity of the deformation gradient for some model problems in nonlinear twodimensional elasticity. Algebra i Analiz. 1994;6:128–153.
[8] Fuchs M, Reuling J. Partial regularity for certain classes of polyconvex functionals related to nonlinear elasticity. Manuscripta Math. 1995;87:13–26.
[9] Passarelli di Napoli A. A regularity result for a class of polyconvex functionals. Ricerche Mat. 1999;48:379–393.

[10] Esposito L, Mingione G. Partial regularity for minimizers of degenerate polyconvex energies. J Convex Anal. 2001;8:1–38.
[11] Leonetti F. Maximum principle for vector-valued minimizers of some integral functionals. Boll Un Mat Ital. 1991;5-A:51–56.
[12] Fusco N, Hutchinson JE. Partial regularity and everywhere continuity for a model problem from nonlinear elasticity. J Austral Math Soc (Ser A). 1994;57:158–169.
[13] D'Ottavio A, Leonetti F, Musciano C. Maximum principle for vector valued mappings minimizing variational integrals. Atti Sem Math Univ Modena. 1998;46(Suppl.):677–683.
[14] Leonetti F, Siepe F. Maximum principle for vector valued minimizers. J Convex Anal. 2005;12:267–278.
[15] Leonetti F, Siepe F. Bounds for vector valued minimizers of some integral functionals. Ricerche Mat. 2005;54:303–312.
[16] Cupini G, Leonetti F, Mascolo E. Local boundedness for minimizers of some polyconvex integrals. Arch Rational Mech Anal. 2017;224:269–289.
[17] Carozza M, Gao H, Giova R, et al., A boundedness result for minimizers of some polyconvex integrals. J Optim Theory Appl. 2018;178:699–725. DOI:10.1007/s10957-018-1335-0.
[18] Bauman P, Phillips D. Univalent minimizers of polyconvex functionals in two dimensions. Arch Rational Mech Anal. 1994;126:161–181.
[19] Leonetti F. Pointwise estimates for a model problem in nonlinear elasticity. Forum Math. 2006;18:529–534.
[20] Ball JM, Murat F. $W^{1,p}$ quasiconvexity and variational problems for multiple integrals. J Funct Anal. 1984;58:225–253.
[21] Iwaniec T, Koskela P, Onninen J. Mapping of finite distortion: monotonicity and continuity. Invent Math. 2001;144:507–531.
[22] Manfredi JJ. Weakly monotone functions. J Geom Anal. 1994;3:393–402.
[23] Stein E, Weiss G. Introduction to Fourier analysis on Euclidean spaces. Princeton, NJ: Princeton University Press; 1971.
[24] Bauman P, Owen NC, Phillips D. Maximum principles and a priori estimates for a class of problems from nonlinear elasticity. Ann Inst Henri Poincare Analyse Nonlineaire. 1991;8:119–157.
[25] Fuchs M, Seregin G. Hölder continuity for weak estremals of two-dimensional variational problems related to nonlinear elasticity. Adv Math Sci Appl. 1997;7:413–425.
[26] Bevan JJ. Explicit examples of Lipschitz, one-homogeneous solutions of log-singular planar elliptic systems. Nonlinear Anal. 2015;125:659–680.
[27] Bevan JJ. A condition for the Hölder regularity of local minimizers of a nonlinear elastic energy in two dimensions. Arch Rational Mech Anal. 2017;225:249–285.
[28] Bauman P, Phillips D, Owen NC. Maximal smoothness of solutions to certain Euler-Lagrange equations from nonlinear elasticity. Proc Roy Soc Edinb Sect A. 1991;119:241–263.
[29] Bevan J, Yan X. Minimizers with topological singularities in two dimensional elasticity. ESAIM Control Optim Calc Var. 2008;14:192–209.
[30] Yan X. Maximal smoothness for solutions to equilibrium equations in 2D nonlinear elasticity. Proc Amer Math Soc. 2007;135:1717–1724.
[31] Stampacchia G. Equations elliptiques du second ordre a coefficientes discontinus. Semin. de Math. Superieures, Univ. de Montreal, Vol. 16, 1966.
[32] Kovalevsky AA, Voitovich MV. On the improvement of summability of generalized solutions of the Dirichlet problem for nonlinear equations of the fourth order with strengthened ellipticity. Ukrainian Math J. 2006;58:1717–1733.
[33] Gao H, Leonetti F, Wang L. Remarks on Stampacchia Lemma. J Math Anal Appl. 2018;458:112–122.
[34] Leonetti F, Petricca PV. Regularity for vector valued minimizers of some anisotropic integral functionals. JIPAM J Inequal Pure Appl Math. 2006;7(3). Article 88, 7 p.