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Some Properties of Integral Operators on Generalized Morrey Spaces

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Abstract. In this note we study the behaviour of Hardy-Littlewood Maximal Operator and we obtain an estimate for commutators on Generalized Local Morrey Spaces. These results can be applied in the study of regularity of solutions of elliptic partial differential equations with discontinuous coefficients.

INTRODUCTION

In this paper we present some estimates for integral operators and commutators on certain Generalized Morrey Spaces that could be applied in the field of partial differential equations. First of all we introduce some definitions. Then, we present some new results dealing with commutators and elliptic partial differential equations.

We denote, throughout the paper, by

$$B(x,r) = \{y \in \mathbb{R}^n \, : \, |x-y| < r\}$$

a generic ball in $\mathbb{R}^n$ centered at x with radius r and by Ω an open subset of $\mathbb{R}^n$ such that $\partial\Omega \in C^{1,1}$.

We begin with the definition of the Morrey Space $M^{p,\lambda}(\mathbb{R}^n)$, introduced by Morrey in 1938 in his paper [5].

Definition 1 *Let $1 < p < \infty$, $0 < \lambda < n$. A measurable function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is in the Morrey Space $M^{p,\lambda}(\mathbb{R}^n)$ if the following norm is finite*

$$\|f\|_{M^{p,\lambda}} \equiv \|f\|_{M^{p,\lambda}(\mathbb{R}^n)} = \sup_{\substack{x\in\mathbb{R}^n \\ r>0}} \left(\frac{1}{r^{n-\lambda}} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p},$$

where $B(x,r)$ ranges in the class of the balls centered in x with radius r.

In 1991, Mizuhara in [4] defined the Generalized Morrey Spaces as follows.

Definition 2 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M^{p,\varphi} \equiv M^{p,\varphi}(\mathbb{R}^n)$ the Generalized Morrey Space, the space of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L^p(B(x,r))}.$$

Also by $WM^{p,\varphi} \equiv WM^{p,\varphi}(\mathbb{R}^n)$ we denote the Weak Generalized Morrey Space of all functions $f \in WL^p_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{WL^p(B(x,r))} < \infty,$$

where $WL^p(B(x,r))$ denotes the weak L^p-space consisting of all measurable functions f for which

$$\|f\|_{WL^p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL^p(\mathbb{R}^n)} < \infty.$$

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Useful in the sequel is to consider the local version of the previous concept. We assume that $\varphi(x, r)$ is a positive measurable function on $\mathbb{R}^n \times (0, \infty)$. Let $1 \leq p < \infty$. We can denote by $LM^{p,\varphi} \equiv LM^{p,\varphi}(\mathbb{R}^n)$ the Local Generalized Morrey Space, i.e. the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM^{p,\varphi}} = \sup_{r>0} \varphi(0, r)^{-1} |B(0, r)|^{-\frac{1}{p}} \|f\|_{L^p(B(0,r))}.$$

Also by $WLM^{p,\varphi} \equiv WLM^{p,\varphi}(\mathbb{R}^n)$ we denote the Weak Generalized Morrey Space of all functions $f \in WL^p_{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM^{p,\varphi}} = \sup_{r>0} \varphi(0, r)^{-1} |B(0, r)|^{-\frac{1}{p}} \|f\|_{WL^p(B(0,r))} < \infty.$$

In this case the Generalized Morrey Space is centered in $0 \in \mathbb{R}^n$. In place of the $0 \in \mathbb{R}^n$ we can take certain $x_0 \in \mathbb{R}^n$. So, for any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM^{p,\varphi}_{\{x_0\}} \equiv LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ the Local Generalized Morrey Space, i.e. the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM^{p,\varphi}_{\{x_0\}}} = \|f(x_0 + \cdot)\|_{LM^{p,\varphi}}.$$

Also by $WLM^{p,\varphi}_{\{x_0\}} \equiv WLM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ we denote the Weak Generalized Morrey Space of all functions $f \in WL^p_{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM^{p,\varphi}_{\{x_0\}}} = \|f(x_0 + \cdot)\|_{WLM^{p,\varphi}} < \infty.$$

To formulate the main hypotheses on the coefficients of elliptic operators under consideration, we need the John-Nirenberg space ([3]) of functions with bounded mean oscillation (BMO) and the Sarason class VMO of the functions with vanishing mean oscillation ([10]).
The concept of BMO class was defined by John-Nirenberg in 1961. Let $f(x)$ be a locally integrable function. We say that $f(x)$ belongs to BMO class if

$$\|f\|_* \equiv \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty,$$

where f_B stands for the integral average $\frac{1}{|B|} \int_B f(x)dx$ of the function $f(x)$ over the set B, and B ranges in the class of balls of $\mathbb{R}^n$.

Some years later, in 1975, Sarason presented the class of function with vanishing mean oscillation that is contained in the John-Nirenberg class and used by a lots of authors, see e.g. [6], [7], [8].

Let us consider $g(x) \in BMO$ and set

$$\eta(r) = \sup_{\substack{x \in \mathbb{R}^n \\ \rho \leq r}} \frac{1}{|B_\rho|} \int_{B_\rho} |g(x) - g_{B_\rho}| dx.$$

We say that $g(x) \in VMO$ if $\lim_{r \to 0^+} \eta(r) = 0$ and refer to $\eta(r)$ as the VMO-modulus of $g(x)$.

PRELIMINARY CONCEPTS AND RESULTS

An important tool for our results is the *Hardy-Littlewood Maximal Operator*. Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define the *Hardy-Littlewood Maximal Operator M* as

$$Mf(x) = \sup_{B(x,r)} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)| dy,$$

where $B(x, r)$ is the ball centered at x of radius r. A variant of Hardy-Littlewood Maximal function M is the *Sharp Maximal Function*

$$f^\#(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - f_B| dy,$$

where the supremum is taken over the balls B containing x.

Now we present the important notion of Calderón-Zygmund singular integral operator. We assume $K(x)$ be a real measurable function on $\mathbb{R}^n$ smooth out of the origin, homogeneous of degree zero such that the following conditions hold:

$$\int_{|x|=1} k(x)\, d\sigma = 0 \quad \text{and} \quad |K(x) - K(y)| \leq |x - y|, \ \text{ for } |x| = |y| = 1.$$

Let us consider the operator

$$(Kf)(x) = \int_{\mathbb{R}^n} \frac{K(x-y)}{|x-y|^n} f(y)\, dy \equiv \lim_{\substack{r\to+\infty \\ \varepsilon\to 0}} \int_{\varepsilon<|x-y|<r} \frac{K(x-y)}{|x-y|^n} f(y)\, dy$$

for every bounded function f having compact support. If K extends to a bounded operator we say it a Calderón-Zygmund singular integral operator.

Let K be a Calderón-Zygmund singular integral operator. We define the *commutator* as

$$[a, K](f) = a(x)(Kf)(x) - K(af)(x),$$

where $a \in L^1_{\text{loc}}(\mathbb{R}^n)$ and f is a suitable function.
We are ready to present two results contained in [2]. The first one is an estimate for the Hardy-Littlewood Operator:

Theorem 3 *Let $\varphi_1(x,r)$ and $\varphi_2(x,r)$ be positive and measurable functions on $\mathbb{R}^n \times (0, +\infty)$. Let $1 \leq q < \infty$ and the functions φ_1, φ_2 satisfy the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\, \tau^{\frac{n}{q}}}{t^{\frac{n}{q}}} \leq C\, \varphi_2(x,r),$$

where C does not depend on x and r. Then, for $1 \leq q < \infty$

$$\|Mf\|_{WM^{q,\varphi_2}} \leq c\, \|f\|_{M^{q,\varphi_1}} \leq c\, \|f^\sharp\|_{M^{q,\varphi_1}},$$

where c does not depend on f.

The following result is an estimate for the commutator $[a, K](f)$:

Theorem 4 *Let $\varphi(x,r)$ be a positive and measurable function on $\mathbb{R}^n \times (0, +\infty)$. Let $x_0 \in \mathbb{R}^n$, $1 < q < s < p < +\infty$, K be a Calderón-Zygmund singular integral operator and the function φ satisfy the following conditions*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\, \tau^{\frac{nq}{p}}}{t^{\frac{nq}{p}}} \leq C\, \varphi(x_0,r), \quad \sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\, \tau^{\frac{ns}{p}}}{t^{\frac{ns}{p}}} \leq C\, \varphi(x_0,r)$$

and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\, \tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \leq C\, \varphi(x_0,r),$$

where C does not depend on r. If $a \in BMO(\mathbb{R}^n)$, $\forall f \in LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$, we have

$$\|[a,K](f)\|_{LM^{p,\varphi}_{\{x_0\}}} \leq c\, \|a\|_*\, \|f\|_{LM^{p,\varphi}_{\{x_0\}}} \leq c\, \|a\|_*\, \|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

for some constant $c \geq 0$ independent of a and f.

Let us consider the following elliptic equation of second order in nondivergence form:

$$\sum_{i,j=1}^n a_{ij} u_{x_i x_j} = f. \tag{1}$$

where f is assumed to be in some Generalized Morrey space $LM^{p,\varphi}_{\{x_0\}}(\Omega)$ and $a_{ij} \in VMO(\Omega)$. Are known some regularity results on Morrey spaces $L^{p,\lambda}$ (see [9]) of the second derivatives of a solution of the previous equation. In order to obtain local regularity results, we use the boundedness of some integral operators on Generalized Morrey spaces.

We assume the following regularity and ellipticity assumptions on the coefficients of the partial differential equation under consideration:

$$\begin{cases} a_{ij}(x) \in L^\infty(\Omega) \cap VMO, \quad \forall i, j = 1, ..., n \\ a_{ij}(x) = a_{ji}(x), \quad \forall i, j = 1, ..., n \\ \exists\, \lambda > 0 : \ \lambda^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.a. } x \in \Omega. \end{cases} \tag{2}$$

Set η_{ij} for the VMO-modulus of the function $a_{ij}(x)$ and let $\eta(r) = \left(\sum_{i,j=1}^n \eta_{ij}^2\right)^{1/2}$. We denote by $\Gamma(x,t)$ the normalized fundamental solution of the differential operator $\mathcal{L}$ associated to (1) and we set also

$$\Gamma_i(x,\xi) = \frac{\partial}{\partial \xi_i}\Gamma(x,\xi), \qquad \Gamma_{ij}(x,\xi) = \frac{\partial}{\partial \xi_i \partial \xi_j}\Gamma(x,\xi),$$

It is well known that $\Gamma_{ij}(x,\xi)$ are Calderón–Zygmund kernels in the ξ variable.

The following theorem is a local Morrey type regularity result for solutions of the differential equation under consideration.

Theorem 5 *Let the ellipticity assumptions* (2) *be true,* $1 < q < s < p < +\infty$*,* K *be a Calderón-Zygmund singular integral operator and we assume that the function* $\varphi(x,r)$*, defined on* $\mathbb{R}^n \times (0,\infty)$*, is positive and measurable and such that the following conditions are fulfilled:*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\,\tau^{\frac{nq}{p}}}{t^{\frac{nq}{p}}} \le C\,\varphi(x_0,r), \quad \sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\,\tau^{\frac{ns}{p}}}{t^{\frac{ns}{p}}} \le C\,\varphi(x_0,r)$$

and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\,\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\,dt \le C\,\varphi(x_0,r),$$

where C *does not depend on* r*. Then, there exists a constant* γ *independent of* u *and* f *and there exists a number* σ*, also independent of* u *and* f*, such that for every ball* $B_R \Subset \Omega$ *having radius* $R < \sigma$ *and every* $u \in W^{2,p}(B_R)$ *such that* $\partial_{ij}u \in LM^{p,\varphi}(B_R)$*, we have*

$$\|\partial_{ij}u\|_{LM^{p,\varphi}(B_R)} \le \gamma\|\mathcal{L}u\|_{LM^{p,\varphi}(B_R)}, \qquad \forall i,j = 1,\ldots,n.$$

Proof. Using the theorems obtained in [2] and inspired by the paper [1], we obtain the result. □

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