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Global integrability for weak solutions to some anisotropic elliptic equations

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ABSTRACT

We consider the boundary value problem

$$\begin{cases} \sum_{i=1}^{n} D_i(a_i(x, Du(x))) = 0, & x \in \Omega; \\ u(x) = u_*(x), & x \in \partial\Omega. \end{cases}$$

We show that higher integrability of the boundary datum u_* forces solutions u to have higher integrability as well. Assumptions on $a_i(x, z)$ are suggested by Euler equation of the anisotropic functional

$$\int_\Omega \sum_{i=1}^{n} \left(2|D_i u|^2 + |D_i u| \sin(|D_i u|)\right)^{\frac{p_i}{2}}.$$

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1. Introduction

We consider integral functionals

$$\mathcal{I}(u) = \int_\Omega f(x, Du(x))dx \tag{1.1}$$

where $u : \Omega \to \mathbb{R}$, Ω is a bounded open subset of $\mathbb{R}^n$ and $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$; about $f(x, z)$ we assume that $x \to f(x, z)$ is measurable and $z \to f(x, z)$ is continuous; u is taken from Sobolev space $W^{1,1}(\Omega)$. We are interested in functions u solving the Euler equation

$$\sum_{i=1}^{n} D_i\left(\frac{\partial f}{\partial z_i}(x, Du(x))\right) = 0 \tag{1.2}$$

in weak form, or more generally

$$\sum_{i=1}^{n} D_i(a_i(x, Du(x))) = 0, \tag{1.3}$$

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where $a_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ with $x \to a_i(x, z)$ measurable and $z \to a_i(x, z)$ continuous. In past years great attention has been paid to anisotropic functionals whose model is

$$\int_\Omega (|D_1 u|^{p_1} + |D_2 u|^{p_2} + \cdots + |D_n u|^{p_n}) dx \tag{1.4}$$

where the derivative $D_i u = \frac{\partial u}{\partial x_i}$ has the exponent p_i that might be different from the exponent p_j of the derivative $D_j u = \frac{\partial u}{\partial x_j}$, when $j \neq i$. Such a model suggests to consider energies $f(x, z)$ where

$$|f(x, z)| \leq c \left(1 + \sum_{i=1}^n |z_i|^{p_i}\right) \tag{1.5}$$

or Eq. (1.3) with coefficients $a_i(x, z)$ satisfying

$$|a_i(x, z)| \leq c(1 + |z_i|)^{p_i - 1}. \tag{1.6}$$

This anisotropic framework looks useful when dealing with some reinforced materials, see [15]; about theoretical viewpoint see [10], example 1.7.1, page 169. In the present paper we are interested in the integrability of solutions u to (1.3): does high integrability of boundary datum u_* improve the integrability of the solution u? A positive answer has been given in [8,4,2] when the operator is monotone:

$$\nu \sum_{i=1}^n |z - \tilde{z}|^{p_i} \leq \sum_{i=1}^n (a_i(x, z) - a_i(x, \tilde{z}))(z_i - \tilde{z}_i) \tag{1.7}$$

for some positive constant ν. Please, note that monotonicity forces f to be convex, when $a_i(z) = \frac{\partial f}{\partial z_i}(z)$. Recently, [9] shows that convexity of f is not necessary; only coercivity of f is required:

$$\nu_* \sum_{i=1}^n |z|^{p_i} \leq f(x, z), \tag{1.8}$$

for some positive constant ν_*. The result contained in [9] is valid for minimizers of (1.1). When f is no longer convex, stationary maps u need not to minimize $\mathcal{I}$, so we cannot use such a result. In the present paper we deal with stationary maps u and we show higher integrability, provided coercivity for $\frac{\partial f}{\partial z}$ is assumed:

$$\tilde{\nu} \sum_{i=1}^n |z|^{p_i} \leq \sum_{i=1}^n \frac{\partial f}{\partial z_i}(x, z) z_i, \tag{1.9}$$

for some positive constant $\tilde{\nu}$. More generally, higher integrability holds true for weak solutions u to (1.3) under coercivity for a:

$$\nu \sum_{i=1}^n |z|^{p_i} \leq \sum_{i=1}^n a_i(x, z) z_i, \tag{1.10}$$

for some positive constant ν. In order to state our theorem, let us assume that $p_1, \ldots, p_n \in (1, +\infty)$ with $\overline{p} < n$, where $\overline{p}$ is the harmonic mean, that is

$$\frac{1}{\overline{p}} \sum_{i=1}^n \frac{1}{p_i}; \tag{1.11}$$

condition $\overline{p} < n$ allows us to consider the Sobolev exponent $\overline{p}^* = \frac{n\overline{p}}{n - \overline{p}}$. As far as the boundary datum u_* is concerned, we assume that

$$u_* \in W^{1,1}(\Omega) \quad \text{with } D_i u_* \in L^{q_i}(\Omega), \quad q_i \in (p_i, +\infty) \tag{1.12}$$

for every $i = 1, \ldots, n$. Let us introduce the Sobolev space

$$W_0^{1,(p_i)}(\Omega) = \left\{ v \in W_0^{1,1}(\Omega) : \ D_i v \in L^{p_i}(\Omega) \ \forall i = 1, \ldots, n \right\}. \tag{1.13}$$

In this paper we will prove the following

Theorem 1.1. *Let $u \in u_* + W_0^{1,(p_i)}(\Omega)$ verify*

$$\int_\Omega \sum_{i=1}^n a_i(x, Du(x))D_i\varphi(x)dx = 0 \quad \forall \varphi \in W_0^{1,(p_i)}(\Omega) \tag{1.14}$$

under growth condition (1.6) and coercivity (1.10). Then

$$u \in u_* + L^t_{weak}(\Omega) \tag{1.15}$$

where

$$t = \frac{\overline{p}\ \overline{p}^*}{\overline{p} - b\overline{p}^*} \tag{1.16}$$

with

$$0 < b \le \min_{i=1,\dots,n}\left(\frac{1}{p_i} - \frac{1}{q_i}\right) \quad \text{and} \quad b < \frac{\overline{p}}{\overline{p}^*}. \tag{1.17}$$

Remark 1.2. When $p_i \ge 2$ for every $i = 1, \dots, n$ we consider

$$f(z) = \sum_{i=1}^n \left(2|z_i|^2 + |z_i|\sin(|z_i|)\right)^{\frac{p_i}{2}}. \tag{1.18}$$

Then $f \in C^2(\mathbb{R}^n)$. Set $a_i(z) = \frac{\partial f}{\partial z_i}(z)$ and $\tilde{p} = \max_{i=1,\dots,n} p_i$. Since f is not convex, then (1.7) cannot hold true. On the other hand, (1.6) is satisfied with $c = 3^{\tilde{p}}\tilde{p}$, (1.10) holds true provided $\nu = 2$. Thus we can apply our Theorem 1.1 to this example. Please, note also that (1.5) is satisfied with $c = n6^{\tilde{p}}$ and (1.8) holds true provided $\nu_* = 1$.

The previous Theorem 1.1 deals with global integrability; in the anisotropic setting, local integrability has been studied in [5] and [3] under the additional assumption $p_i < \overline{p}^*$ for every i. Such an inequality forces exponents $p_1, \dots, p_n$ to be close enough; when they are spread out, it is possible to get counterexamples in which bad boundary data forces solutions to be bad as well, see [6,11,12,7] and sections 5,6 in [13]. We end this introduction by remarking that this paper is concerned with higher integrability of u; as far as higher integrability of Du is concerned, a delicate interplay between the regularity of $x \to f(x, z)$ and the growth of $z \to f(x, z)$ appears: see [1].

2. Proof of Theorem 1.1

Let us consider $L \in (0, +\infty)$ and

$$\varphi = \begin{cases} u - u_* + L & \text{if } u - u_* < -L, \\ 0 & \text{if } -L \le u - u_* \le L, \\ u - u_* - L & \text{if } u - u_* > L; \end{cases} \tag{2.1}$$

so that

$$\varphi \in W_0^{1,(p_i)}(\Omega) \tag{2.2}$$

and

$$D\varphi = (Du - Du_*)1_{\{|u-u_*|>L\}}, \tag{2.3}$$

where $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. Now

$$\sum_{i=1}^n \int_{\{|u-u_*|>L\}} |D_iu - D_iu_*|^{p_i} \le \sum_{i=1}^n c_1\left(\int_{\{|u-u_*|>L\}} |D_iu|^{p_i} + \int_{\{|u-u_*|>L\}} |D_iu_*|^{p_i}\right) = (I) + (II);$$

coercivity (1.10) allows us to handle (I) so that

$$\begin{aligned}(I) + (II) &\le \frac{c_1}{\nu}\int_{\{|u-u_*|>L\}} \sum_{i=1}^n a_i(Du)D_iu + c_1\sum_{i=1}^n \int_{\{|u-u_*|>L\}} |D_iu_*|^{p_i} \\ &= \frac{c_1}{\nu}\int_{\{|u-u_*|>L\}} \sum_{i=1}^n a_i(Du)(D_iu - D_iu_*) + \frac{c_1}{\nu}\int_{\{|u-u_*|>L\}} \sum_{i=1}^n a_i(Du)D_iu_* + c_1\sum_{i=1}^n \int_{\{|u-u_*|>L\}} |D_iu_*|^{p_i} \\ &= (IV) + (V) + (VI).\end{aligned}$$

We keep in mind that u verifies Eq. (1.14); when φ is taken as in (2.1), $D\varphi$ is given by (2.3), then we have $(IV) = 0$. Growth condition (1.6) allows us to control (V), thus we have

$$(IV) + (V) + (VI) \le \frac{c_2}{\nu} \sum_{i=1}^{n} \int_{\{|u-u_*|>L\}} (1 + |D_i u|)^{p_i - 1} |D_i u_*| + c_1 \sum_{i=1}^{n} \int_{\{|u-u_*|>L\}} |D_i u_*|^{p_i} = (VII) + (VIII).$$

In (VII) we use Hölder inequality with exponents $\frac{p_i}{p_i-1}$, p_i and we get

$$(VII) + (VIII) \le c_3 \sum_{i=1}^{n} \left(\int_{\Omega} (1 + |D_i u|)^{p_i} \right)^{\frac{p_i-1}{p_i}} \left(\int_{\{|u-u_*|>L\}} |D_i u_*|^{p_i} \right)^{\frac{1}{p_i}} + c_1 \sum_{i=1}^{n} \int_{\{|u-u_*|>L\}} |D_i u_*|^{p_i} = (IX) + (X).$$

Let us set

$$c_4 = \max_{i=1,\dots,n} \left(\int_{\Omega} (1 + |D_i u|)^{p_i} \right)^{\frac{p_i-1}{p_i}};$$

then

$$(IX) \le c_5 \sum_{i=1}^{n} \left(\int_{\{|u-u_*|>L\}} |D_i u_*|^{p_i} \right)^{\frac{1}{p_i}}.$$

Moreover, there exists $L_0 = L_0(u, u_*) > 0$ such that, for every $L \ge L_0$ the measure of $\{|u - u_*| > L\}$ is small enough to ensure that

$$\int_{\{|u-u_*|>L\}} |D_i u_*|^{p_i} \le 1;$$

thus

$$\int_{\{|u-u_*|>L\}} |D_i u_*|^{p_i} \le \left(\int_{\{|u-u_*|>L\}} |D_i u_*|^{p_i} \right)^{\frac{1}{p_i}}$$

and

$$(X) \le c_1 \sum_{i=1}^{n} \left(\int_{\{|u-u_*|>L\}} |D_i u_*|^{p_i} \right)^{\frac{1}{p_i}}.$$

We collect all the previous inequalities and we get

$$\sum_{i=1}^{n} \int_{\{|u-u_*|>L\}} |D_i u - D_i u_*|^{p_i} \le c_6 \sum_{i=1}^{n} \left(\int_{\{|u-u_*|>L\}} |D_i u_*|^{p_i} \right)^{\frac{1}{p_i}}.$$

Let us consider t_i, to be chosen later, such that $p_i < t_i \le q_i$ for $i = 1, \dots, n$. We use Hölder inequality and we get

$$\left(\int_{\{|u-u_*|>L\}} |D_i u_*|^{p_i} \right)^{\frac{1}{p_i}} \le \left(\int_{\{|u-u_*|>L\}} |D_i u_*|^{t_i} \right)^{\frac{1}{t_i}} |\{|u - u_*| > L\}|^{\frac{t_i - p_i}{t_i p_i}}.$$

We would like that the exponent $\frac{t_i - p_i}{t_i p_i}$ would be independent of i. To this aim, according to assumption (1.17), we take b such that $0 < b \le \min_{i=1,\dots,n} \left(\frac{1}{p_i} - \frac{1}{q_i} \right)$ and we define

$$t_i = \frac{p_i}{1 - b p_i};$$

then $p_i < t_i \le q_i$ and

$$\frac{t_i - p_i}{t_i p_i} = b.$$

Let us set

$$c_7 = \max_{i=1,\dots,n} \left(\int_{\Omega} |D_i u_*|^{t_i} \right)^{\frac{1}{t_i}};$$

then

$$\sum_{i=1}^{n} \int_{\{|u-u_*|>L\}} |D_i u - D_i u_*|^{p_i} \le c_8 |\{|u - u_*| > L\}|^{b}.$$

Now we lower the left hand side by considering only the ith piece of the sum; then we take the $\frac{1}{p_i}$ power; eventually we take the product over all i and we get

$$\prod_{i=1}^{n}\left(\int_{\{|u-u_*|>L\}}|D_iu-D_iu_*|^{p_i}\right)^{\frac{1}{p_i}} \le \prod_{i=1}^{n}\left(c_8|\{|u-u_*|>L\}|^b\right)^{\frac{1}{p_i}}$$
$$= \left(c_8|\{|u-u_*|>L\}|^b\right)^{\sum_{i=1}^{n}\frac{1}{p_i}} = c_8^{\frac{n}{\bar{p}}}|\{|u-u_*|>L\}|^{\frac{nb}{\bar{p}}}.$$

Let us come back to the test function (2.1): according to (2.2), φ is zero on the boundary of Ω, then we can use the anisotropic Sobolev embedding (see [16] and Theorem 3.1 in [9]), in order to get

$$\left(\int_\Omega|\varphi|^{\bar{p}^*}\right)^{\frac{1}{\bar{p}^*}} \le c_9\left[\prod_{i=1}^{n}\left(\int_\Omega|D_i\varphi|^{p_i}\right)^{\frac{1}{p_i}}\right]^{\frac{1}{n}}$$
$$= c_9\left[\prod_{i=1}^{n}\left(\int_{\{|u-u_*|>L\}}|D_iu-D_iu_*|^{p_i}\right)^{\frac{1}{p_i}}\right]^{\frac{1}{n}}$$
$$\le c_9c_8^{\frac{1}{\bar{p}}}|\{|u-u_*|>L\}|^{\frac{b}{\bar{p}}}.$$

Since

$$|\varphi| = (|u-u_*|-L)1_{\{|u-u_*|>L\}},$$

for $\tilde{L}>L$ we get

$$(\tilde{L}-L)^{\bar{p}^*}|\{|u-u_*|>\tilde{L}\}| = \int_{\{|u-u_*|>\tilde{L}\}}(\tilde{L}-L)^{\bar{p}^*}$$
$$\le \int_{\{|u-u_*|>\tilde{L}\}}(|u-u_*|-L)^{\bar{p}^*} \le \int_{\{|u-u_*|>L\}}(|u-u_*|-L)^{\bar{p}^*} = \int_\Omega|\varphi|^{\bar{p}^*}.$$

The previous inequalities merge into

$$|\{|u-u_*|>\tilde{L}\}| \le \frac{c_{10}}{(\tilde{L}-L)^{\bar{p}^*}}|\{|u-u_*|>L\}|^{\frac{b\bar{p}^*}{\bar{p}}},$$

for every $\tilde{L}, L$ such that $\tilde{L}>L\ge L_0>0$. Now we use Lemma 4.1 at page 93 of [14]; for the convenience of the reader, such a Lemma has been explicitly written in [9] as Lemma 3.2; application of such a Lemma requires that $b<\frac{\bar{p}}{\bar{p}^*}$: such an inequality has been assumed in (1.17). We use the aforementioned lemma and we get, for $s\ge L_0$,

$$|\{|u-u_*|>s\}| \le c_{11}\left(\frac{1}{s}\right)^{\frac{\bar{p}\,\bar{p}^*}{\bar{p}-b\bar{p}^*}};$$

such an inequality guarantees that $u-u_*\in L^t_{weak}(\Omega)$ with $t=\frac{\bar{p}\,\bar{p}^*}{\bar{p}-b\bar{p}^*}$. This ends the proof of Theorem 1.1. □

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