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Regularity for minimizers of integrals with nonstandard growth

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ABSTRACT

We deal with variational integrals

$$\int_\Omega f(x, Du(x))dx$$

and we consider a minimizer $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ among all functions that agree on the boundary $\partial\Omega$ with some fixed boundary value u_*. We assume that the boundary datum u_* makes the density $f(x, Du_*(x))$ more integrable and we prove that the minimizer u enjoy higher integrability.

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1. Introduction

We consider the variational integral

$$\int_\Omega f(x, Du(x))dx \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with $n \geq 2$, $u : \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Caratheodory function, that is, $x \to f(x, z)$ is measurable and $z \to f(x, z)$ is continuous. We assume p growth from below: there exist constants $p \in (1, n)$ and $\nu \in (0, +\infty)$, there exists a function $g_1 : \Omega \to [0, +\infty)$ such that

$$\nu|z|^p - g_1(x) \leq f(x, z) \tag{1.2}$$

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for almost every $x \in \Omega$ and for all $z \in \mathbb{R}^n$. We fix a boundary datum $u_* : \Omega \to \mathbb{R}$ such that $u_* \in W^{1,p}(\Omega)$ and

$$x \to f(x, Du_*(x)) \in L^1(\Omega). \tag{1.3}$$

The set of competing functions for the variational integral (1.1) is

$$\mathcal{C} = \{w \in u_* + W_0^{1,p}(\Omega) \text{ such that } x \to f(x, Dw(x)) \in L^1(\Omega)\}. \tag{1.4}$$

A function $u : \Omega \to \mathbb{R}$ is a minimizer for (1.1) if $u \in \mathcal{C}$ and it verifies

$$\int_\Omega f(x, Du(x))dx \le \int_\Omega f(x, Dw(x))dx \tag{1.5}$$

for all $w \in \mathcal{C}$. In this paper we deal with regularity of minimizers, [34,35]. Now we ask the following question: if the boundary datum u_* makes the density $f(x, Du_*(x))$ more integrable than (1.3) requires, does the minimizer u enjoy higher integrability? The answer is positive and in this paper we prove the following.

Theorem 1.1. *Assume that $g_1 \in L^\sigma(\Omega)$ and $x \to f(x, Du_*(x)) \in L^\sigma(\Omega)$ where $\sigma \in (1, +\infty)$. If $u \in \mathcal{C}$ minimizes the variational integral (1.1) under (1.2), then*

$$\sigma < \frac{n}{p} \Longrightarrow u - u_* \in L_{weak}^{\frac{np\sigma}{n-p\sigma}}(\Omega), \tag{1.6}$$

$$\sigma = \frac{n}{p} \Longrightarrow \exists \alpha > 0 : e^{\alpha|u-u_*|} \in L^1(\Omega), \tag{1.7}$$

$$\sigma > \frac{n}{p} \Longrightarrow u - u_* \in L^\infty(\Omega). \tag{1.8}$$

Note that $\frac{np\sigma}{n-p\sigma} > \frac{pn}{n-p}$ in (1.6).

Assume now that we have p growth from below and q growth from above: for some positive constants p, q, ν, K with $1 < p < n,\ p \le q$, for some functions $g_1, g_2 : \Omega \to [0, +\infty)$ we have

$$\nu|z|^p - g_1(x) \le f(x, z) \le K|z|^q + g_2(x) \tag{1.9}$$

for almost every $x \in \Omega$, for every $z \in \mathbb{R}^n$. Then

$$|f(x, z)| \le K|z|^q + g_2(x) + g_1(x) \tag{1.10}$$

and the σ-integrability of $|f(x, Du_*(x))|$ is guaranteed by $Du_* \in L^{q\sigma}$ and $g_1, g_2 \in L^\sigma$: this gives the following

Corollary 1.2. *Assume that $g_1, g_2 \in L^\sigma(\Omega)$ and $Du_* \in L^{q\sigma}(\Omega)$ where $\sigma \in (1, +\infty)$. If $u \in \mathcal{C}$ minimizes the variational integral (1.1) under (1.9), then (1.6)–(1.8) holds true.*

When $p < q$ in (1.9) we usually say that we have p, q-growth or nonstandard growth, [31,32]. The previous corollary deals with global regularity; local regularity has been studied in [31,20,36,10,6,8] under the additional assumption that p and q are close enough. When they are far apart, it is possible to build counterexamples in which bad boundary data force minimizers to be bad as well, see [24,30,31,26]. Let us recall that local regularity has been proved in [16] under the restriction

$$q < p\frac{n+2}{n}. \tag{1.11}$$

The additional assumption that the minimizer is locally *bounded* allows [6] to get local regularity under the less restrictive bound

$$q \le 2p + 2, \tag{1.12}$$

see also [10,17,5,4]. Our Theorem and Corollary guarantee higher integrability for minimizers; this results in the following question: is higher integrability enough to obtain local regularity under some bound

$$q < q_0 \tag{1.13}$$

for some q_0 between $p\frac{n+2}{n}$ and $2p+2$? Here we do not try to answer such a question. We now show some examples for which (1.9) applies.

Example 1.3. For $1 < p < q$, let us set

$$f_{p,q}(x,z) = |z|^p + a(x)|z|^q \tag{1.14}$$

where

$$0 \le a(x) \le A < +\infty, \tag{1.15}$$

see [41,42,18,19,12,11,14,1]. Then (1.9) holds true with $\nu = 1,\ g_1(x) = 0,\ K = A+1$ and $g_2(x) = 1$.

Example 1.4. For $1 < p$, let us set

$$f_{p,\log}(x,z) = |z|^p + a(x)|z|^p \ln(e + |z|) \tag{1.16}$$

where $a(x)$ satisfies (1.15), see [2,3,9]. Then (1.9) holds true with $q = p+\epsilon,\ \nu = 1,\ g_1(x) = 0,\ K = A+1+\frac{A}{\epsilon}$ and $g_2(x) = 0$, for every $\epsilon \in (0,1)$.

Example 1.5. Let us set

$$f_{deg}(x,z) = a(x)|z|^2 \tag{1.17}$$

where

$$a(x) > 0 \quad \text{for almost every } x \in \Omega, \tag{1.18}$$

$$a \in L^r(\Omega), \quad r \in (1,+\infty), \tag{1.19}$$

$$\frac{1}{a} \in L^t(\Omega), \quad t \in (1,+\infty), \tag{1.20}$$

see [41]. We use Young's inequality with exponents r and $\frac{r}{r-1}$ and we get

$$a(x)|z|^2 \le [a(x)]^r + |z|^{\frac{2r}{r-1}}. \tag{1.21}$$

Now we use Young's inequality with exponents $\frac{t+1}{t}$ and $t+1$ as follows:

$$|z|^{\frac{2t}{t+1}} = |z|^{\frac{2t}{t+1}}[a(x)]^{\frac{t}{t+1}}\frac{1}{[a(x)]^{\frac{t}{t+1}}} \le |z|^2 a(x) + \frac{1}{[a(x)]^t}, \tag{1.22}$$

then

$$|z|^{\frac{2t}{t+1}} - \frac{1}{[a(x)]^t} \le |z|^2 a(x). \tag{1.23}$$

This shows that (1.9) holds true with $p = \frac{2t}{t+1},\ q = \frac{2r}{r-1},\ \nu = 1,\ g_1(x) = \frac{1}{[a(x)]^t},\ K = 1$ and $g_2(x) = [a(x)]^r$. In order to apply our Corollary 1.2, we need $a \in L^{r\sigma}(\Omega)$ and $\frac{1}{a} \in L^{t\sigma}(\Omega)$, where $\sigma > 1$.

Example 1.6. Let us set

$$f_{var}(x,z) = |z|^{p(x)}, \tag{1.24}$$

where $1 < p \le p(x) \le q$, see [41,13], Chapter 2 in [37], [7]. Then (1.9) holds true with $\nu = 1,\ g_1(x) = 1,\ K = 1$ and $g_2(x) = 1$.

Example 1.7. For $a, b \in (0, +\infty)$ with $1 < a - b$, we set

$$f_{sin}(z) = |z|^{a+b\sin|z|}, \tag{1.25}$$

see [20,39,28,15,33]. Then (1.9) holds true with $p = a-b,\ q = a+b,\ \nu = 1,\ g_1(x) = 1,\ K = 1$ and $g_2(x) = 1$.

Example 1.8. When $2 < q$ and $z = (z_1, \dots, z_n)$, we set

$$f_{re-ma}(z) = |z|^2 + |\max\{z_n; 0\}|^q; \tag{1.26}$$

see [40]: this example looks useful when dealing with some reinforced materials. Note that

$$f_{re-ma}(z) \le 1 + |z_1|^2 + \cdots + |z_{n-1}|^2 + 2|z_n|^q \tag{1.27}$$

but it is false that

$$c_1(|z_1|^2 + \cdots + |z_{n-1}|^2 + |z_n|^q) - c_2 \le f_{re-ma}(z) \tag{1.28}$$

for some positive constants c_1 and c_2: take $z = (0, \dots, 0, t)$ and let t go to $-\infty$. Then we cannot use [21,25,29,22,23,27]. On the other hand, (1.9) holds true with $p = 2,\ \nu = 1,\ g_1(x) = 0,\ K = 2,\ g_2(x) = 1$.

2. Proof of Theorem 1.1

For $L \in (0, +\infty)$, we consider the set

$$\{|v| > L\} = \{x \in \Omega : |v(x)| > L\}.$$

Integrability of v can be obtained by estimates on the measure of $\{|v| > L\}$; these estimates are guaranteed by the inequality

$$\int_{\{|v|>L\}} |Dv|^p \le M|\{|v| > L\}|^s$$

as the following lemma states.

Lemma 2.1. *Assume that $v \in W_0^{1,p}(\Omega)$ with $1 < p < n$. Let M, γ, L_0 be positive constants such that*

$$\int_{\{|v|>L\}} |Dv|^p \le M|\{|v| > L\}|^{\frac{\gamma p}{p^*}} \tag{2.1}$$

for every $L > L_0$, where $p^* = \frac{np}{n-p}$. Then

$$\gamma < 1 \Longrightarrow v \in L^{\frac{p^*}{1-\gamma}}_{weak}(\Omega), \tag{2.2}$$

$$\gamma = 1 \Longrightarrow \exists \alpha > 0 : e^{\alpha|v|} \in L^1(\Omega), \tag{2.3}$$

$$\gamma > 1 \Longrightarrow v \in L^\infty(\Omega). \tag{2.4}$$

Such a lemma is Proposition 2.2 in [27] and its proof is based on Sobolev's inequality and Stampacchia's Lemma 4.1 in [38]. We want to use Lemma 2.1 with $v = u - u_*$. Then we apply the p growth from below in (1.2) and we get

$$\begin{aligned}\int_{\{|u-u_*|>L\}} |Du(x) - Du_*(x)|^p &\le 2^p \int_{\{|u-u_*|>L\}} |Du(x)|^p + 2^p \int_{\{|u-u_*|>L\}} |Du_*(x)|^p \\ &\le \frac{2^p}{\nu} \int_{\{|u-u_*|>L\}} f(x, Du(x)) + \frac{2^p}{\nu} \int_{\{|u-u_*|>L\}} g_1(x) \\ &\quad + 2^p \int_{\{|u-u_*|>L\}} |Du_*(x)|^p. \end{aligned} \tag{2.5}$$

In order to control $\int f(x, Du(x))$ we need the minimality of u. First of all, we define the variation v as follows:

$$v = \begin{cases} u - u_* + L & \text{if } u - u_* < -L, \\ 0 & \text{if } -L \le u - u_* \le L, \\ u - u_* - L & \text{if } u - u_* > L; \end{cases} \tag{2.6}$$

it turns out that

$$v \in W_0^{1,p}(\Omega) \tag{2.7}$$

and

$$Dv = (Du - Du_*)1_{\{|u-u_*|>L\}}, \tag{2.8}$$

where $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. Now we consider

$$w = u - v \tag{2.9}$$

and we have

$$w \in u_* + W_0^{1,p}(\Omega) \tag{2.10}$$

with

$$Dw = (Du)1_{\{|u-u_*|\le L\}} + (Du_*)1_{\{|u-u_*|>L\}}; \tag{2.11}$$

since u and u_* have finite energy, then (2.11) tells us that w has finite energy too:

$$x \to f(x, Dw(x)) \in L^1(\Omega). \tag{2.12}$$

Thus $w \in \mathcal{C}$ and we can use minimality (1.5):

$$\begin{aligned} &\int_{\{|u-u_*|\le L\}} f(x, Du(x)) + \int_{\{|u-u_*|>L\}} f(x, Du(x)) = \int_\Omega f(x, Du(x)) \le \int_\Omega f(x, Dw(x)) \\ &= \int_{\{|u-u_*|\le L\}} f(x, Du(x)) + \int_{\{|u-u_*|>L\}} f(x, Du_*(x)). \end{aligned} \tag{2.13}$$

Since u and u_* have finite energy, all the integrals are finite; then we can drop $\int_{\{|u-u_*|\le L\}} f(x, Du(x))$ from both sides and we get

$$\int_{\{|u-u_*|>L\}} f(x, Du(x)) \le \int_{\{|u-u_*|>L\}} f(x, Du_*(x)). \tag{2.14}$$

This inequality can be used in (2.5) and we get

$$\begin{aligned} &\int_{\{|u-u_*|>L\}} |Du(x) - Du_*(x)|^p \le \frac{2^p}{\nu} \int_{\{|u-u_*|>L\}} f(x, Du_*(x)) \\ &+ \frac{2^p}{\nu} \int_{\{|u-u_*|>L\}} g_1(x) + 2^p \int_{\{|u-u_*|>L\}} |Du_*(x)|^p = \int_{\{|u-u_*|>L\}} h(x) \end{aligned} \tag{2.15}$$

where

$$h(x) = \frac{2^p}{\nu} f(x, Du_*(x)) + \frac{2^p}{\nu} g_1(x) + 2^p |Du_*(x)|^p. \tag{2.16}$$

The assumptions on Du_*, g_1 and f guarantee that

$$h \in L^\sigma(\Omega); \tag{2.17}$$

then, using Hölder inequality, we get

$$\int_{\{|u-u_*|>L\}} h \le \left(\int_\Omega h^\sigma \right)^{\frac{1}{\sigma}} |\{|u - u_*| > L\}|^{\frac{\sigma-1}{\sigma}}; \tag{2.18}$$

we insert this inequality into (2.15) and we obtain

$$\int_{\{|u-u_*|>L\}} |Du - Du_*|^p \le \|h\|_{L^\sigma(\Omega)} |\{|u-u_*| > L\}|^{\frac{\sigma-1}{\sigma}}. \tag{2.19}$$

Now

$$\frac{\sigma-1}{\sigma} = \frac{1-\frac{1}{\sigma}}{1-\frac{p}{n}} \frac{p}{p^*} \tag{2.20}$$

and we can apply Lemma 2.1 with $\gamma = \frac{1-\frac{1}{\sigma}}{1-\frac{p}{n}}$. This ends the proof of Theorem 1.1. □

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