Regularity for entropy solutions to degenerate elliptic equations

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Abstract. We derive estimates for entropy solutions to degenerate elliptic equations of the form

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, u(x), \nabla u(x)) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where the Carathéodory function $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies degenerate coercivity condition

$$\mathcal{A}(x, s, \xi) \cdot \xi \geqslant \alpha \frac{|\xi|^p}{(1 + |s|)^\theta}$$

and controllable growth condition

$$|\mathcal{A}(x, s, \xi)| \leqslant \beta |\xi|^{p-1}$$

for $1 < p < n$, $0 \leqslant \theta < p - 1$ and $0 < \alpha \leqslant \beta < \infty$, and f lies in Marcinkiewicz spaces. We derive a generalized Stampacchia Lemma in order to prove the main theorem. Counterexample shows that some of the results in this paper are optimal. As an application of our results, we use a bootstrap argument to show that the entropy solutions of

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, u(x), \nabla u(x)) = |1 + u(x)|^{a-1}(1 + u(x)), & x \in \Omega, \\ u(x) = 0, & \in \partial\Omega, \\ |u(x)|^a \in L^1(\Omega) \end{cases}$$

are bounded provided that $0 \leqslant a < \frac{n(p-1-\theta)}{n-p}$.

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§1 Introduction

Throughout this paper we let Ω stand for a bounded domain in $\mathbb{R}^n, n \geqslant 2$, $1 < p < n$ and $0 \leqslant \theta < p - 1$. We shall consider degenerate operators

$$-\mathrm{div}\mathcal{A}(x, u, \nabla u),$$

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where $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory mapping with the following assumptions: for a.e. $x \in \mathbb{R}$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$,

$$\mathcal{A}(x, s, \xi) \cdot \xi \geqslant \alpha \frac{|\xi|^p}{(1 + |s|)^\theta} \tag{1.1}$$

and

$$|\mathcal{A}(x, s, \xi)| \leqslant \beta |\xi|^{p-1}, \tag{1.2}$$

where $0 < \alpha \leqslant \beta < \infty$ are constants.

Let $f \in L^1(\Omega)$. We consider the boundary value problem

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, u(x), \nabla u(x)) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

and our goal is to find estimates for u and ∇u in terms of the weak L^q norm of f. Since f is only an L^1 function, then we shall use the concept of an entropy solution that was introduced by Bénilan et. al. in [2], where the existence and uniqueness of such a solution was also established. A function u is called an entropy solution of the problem (1.3) if the truncations $T_k(u)$ belong to $W_0^{1,p}(\Omega)$ for each $k > 0$, and

$$\int_\Omega \mathcal{A}(x, u, \nabla u) \nabla T_k(u - v) dx = \int_\Omega f T_k(u - v) dx \tag{1.4}$$

for each $v \in C_0^\infty(\Omega)$. Here and in what follows T_k is the truncation operator at level k, $T_k(s) = \min(1, \frac{k}{|s|})s$. For some results related to entropy solutions, we refer the reader to [3, 9, 10, 25, 28, 40].

We note that, under (1.1), the coercivity can degenerate when $|s|$ is too big and $\theta > 0$. For some problems with degenerate coercivity, we refer the reader to Chapter 14 in the monograph [7], where some results are obtained: existence, uniqueness and regularity for the case $0 < \theta < 1$, and existence and nonexistence for the case $\theta > 1$. For some other results related to degenerate coercivity, we refer the reader to [1, 3, 5, 6, 8, 19, 21, 33].

We work in weak L^q spaces, known also as Marcinkiewicz spaces or Lorentz spaces $L^{(q,\infty)}$: if $q > 1$, then the space $L^q_{weak}(\Omega)$ consists of measurable functions g on Ω such that

$$\sup_{t>0} t |\{x \in \Omega : |g(x)| > t\}|^{\frac{1}{q}} < \infty. \tag{1.5}$$

This condition is equivalently stated as

$$|||g|||_q = \sup_{\substack{E \subset \Omega \\ |E| > 0}} \frac{1}{|E|^{\frac{1}{q'}}} \int_E |g| dx < \infty, \tag{1.6}$$

where q' is the conjugated exponent of q, $\frac{1}{q} + \frac{1}{q'} = 1$. It is well-known that $L^q_{weak}(\Omega)$ is a Banach space under $|||\cdot|||_q$ and, moreover, if the supremum in (1.5) is denoted by A, then

$$A \leqslant |||g|||_q \leqslant q' A.$$

A useful result is: if $g \in L^q_{weak}(\Omega)$ and $1 \leqslant \tau < q$, then $|g|^\tau \in L^{\frac{q}{\tau}}_{weak}(\Omega)$, and

$$\||g|^\tau\|_{\frac{q}{\tau}} = \|g\|_q^\tau. \tag{1.7}$$

For a detailed analysis of L^q_{weak} spaces we refer to [15]. Note that both (1.5) and (1.6) make sense also for $q = 1$; however then the latter is strictly stronger condition and coincides with the definition of L^1.

The following is our main theorem:

Theorem 1.1. *Let $f \in L^q_{weak}(\Omega)$, and that u is the entropy solution of (1.3).*
i) *If $1 \leqslant q < \frac{n}{p}$, then $u \in L^\gamma_{weak}(\Omega)$ with*

$$\gamma = \frac{nq(p-1-\theta)}{n-pq},$$

and

$$\|u\|_\gamma \leqslant c_1 \|f\|_q^{\frac{1}{p-1-\theta}},$$

where $c_1 > 0$ is a constant depending only on α, n, p, q and θ.
ii) *If $q = \frac{n}{p}$, then there exists a positive constant λ, such that*

$$e^{\lambda |u|^{1-\frac{\theta}{p-1}}} \in L^1(\Omega).$$

iii) *If $1 \leqslant q < \frac{np}{np-(n-p)(1+\theta)}$, then $\nabla u \in L^s_{weak}(\Omega)$ with*

$$s = \frac{nq(p-1-\theta)}{n-q(1+\theta)},$$

and

$$\|\nabla u\|_s \leqslant c_2 \|f\|_q^{\frac{1}{p-1-\theta}},$$

where $c_2 > 0$ is a constant depending only on α, n, p, q and θ.
iv) *If $q > \frac{np}{np-(n-p)(1+\theta)}$, then $u \in W^{1,p}_0(\Omega)$ and*

$$\|\nabla u\|_{L^p(\Omega)} \leqslant c_3 (1 + \|f\|_q)^{\frac{q(p-\theta)-1}{q(p-1-\theta)}},$$

where $c_3 > 0$ is a constant depending only on α, n, p, q, θ and $|\Omega|$.

As to related results, Kilpeläinen and Li proved in [26] some regularity results for non-degenerate case, that is, problem (1.3) for the operator $\mathcal{A}$ satisfies (1.1) with $\theta = 0$. We note that, for the case $\theta = 0$, our results i), iii) and iv) are just the same as Theorem 1.7 in [26]. The borderline case ii), $q = \frac{n}{p}$, was not appeared in [26]. Stampacchia Lemma (see [36]) is an important tool in dealing with regularity issues for solutions to elliptic partial differential equations and systems, see [16, 18, 19, 23, 24, 29, 30, 32]. We should mention that, in the degenerate case, it seems that one can not use the classical Stampacchia Lemma, because some power

of k naturally appears in the right hand side. To overcome this difficulty, we shall prove a generalized Stampacchia Lemma in the Appendix.

At the end of Section 2, we give an example showing that the results i) and iii) in Theorem 1.1 are optimal.

We apply Theorem 1.1 to investigate the regularity of solutions to the problem

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, u(x), \nabla u(x)) = |1 + u(x)|^{a-1}(1 + u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \\ |u(x)|^a \in L^1(\Omega). \end{cases} \tag{1.8}$$

We show:

Theorem 1.2. *If u is the entropy solution of (1.8) and $0 \leqslant a < \frac{n(p-1-\theta)}{n-p}$, then u is bounded in Ω.*

We give an example at the end of Section 3 to show that: if $a > a_c = \frac{n(p-1-\theta)}{n-p}$ in Theorem 1.2, then one can not expect that u is bounded.

The plan of the paper is as follows: in the next section we will prove the main theorem, we will also provide an example to show that some of the results in the main theorem are optimal. In Section 3, we will give an application of the main results by using a bootstrap argument. In the Appendix, we will provide a generalized Stampacchia lemma which will be used in the proof of the main theorem.

§2 Proof of Theorem 1.1

This section is dedicated to derive estimates that yield Theorem 1.1. Throughout the section we let u be an entropy solution of (1.3), $1 < p < n$ and $0 \leqslant \theta < p - 1$.

In order to prove Theorem 1.1 part i), we need a preliminary lemma, see Proposition 3 in [19].

Lemma 2.1. *Let $u : \Omega \to \mathbb{R}$ be a measurable function, let c_4, α, k_0 be positive constants and $0 < \beta < 1$. If, for every $k \geqslant k_0$, we have*

$$k^\alpha |\{|u| > 2k\}| \leqslant c_4 |\{|u| > k\}|^\beta,$$

then $u \in L^{\frac{\alpha}{1-\beta}}_{weak}(\Omega)$.

In fact, this lemma is a direct consequence of the classical Stampacchia Lemma (see Lemma 1 in the Appendix) and Remark 1 in [19].

Proof of Theorem 1.1.

i) We use $v = T_k(u)$, $k \geqslant k_0 = 1$, as a test function in (1.4). Note that

$$\nabla T_k(u - T_k(u)) = \begin{cases} 0, & |u| \leqslant k, \\ \nabla u, & k < |u| < 2k, \\ 0, & |u| \geqslant 2k, \end{cases}$$

then

$$
\begin{aligned}
& \frac{\alpha}{(3k)^\theta}\int_\Omega |\nabla T_k(u-T_k(u))|^p dx \\
= & \frac{\alpha}{(3k)^\theta}\int_{\{k<|u|<2k\}} |\nabla u|^p dx \\
\leqslant & \alpha\int_{\{k<|u|<2k\}} \frac{|\nabla u|^p}{(1+|u|)^\theta} dx \\
\leqslant & \int_{\{k<|u|<2k\}} \mathcal{A}(x,u,\nabla u)\nabla u dx \\
= & \int_\Omega \mathcal{A}(x,u,\nabla u)\nabla T_k(u-T_k(u)) dx \\
= & \int_\Omega f T_k(u-T_k(u)) dx \\
= & \int_{\{|u|>k\}} f T_k(u-T_k(u)) dx \\
\leqslant & k\int_{\{|u|>k\}} |f| dx \\
\leqslant & k|||f|||_q |\{|u|>k\}|^{\frac{1}{q'}},
\end{aligned}
\tag{2.1}
$$

here we have used (1.1), (1.6) and the fact

$$k \geqslant k_0 = 1 \Rightarrow 1+|u| \leqslant 1+2k \leqslant 3k \quad \text{on} \quad \{k<|u|<2k\}.$$

Hence by the Sobolev inequality

$$\left(\int_\Omega |w|^{p^*} dx\right)^{\frac{1}{p^*}} \leqslant c_* \left(\int_\Omega |\nabla w|^p dx\right)^{\frac{1}{p}}, \quad \forall w \in W_0^{1,p}(\Omega), \tag{2.2}$$

where $p^* = \frac{np}{n-p}$, c_* depends only on n and p, we have that

$$
\begin{aligned}
& k|\{|u|>2k\}|^{\frac{1}{p^*}} \\
= & \left(\int_{\{|u|>2k\}} |T_k(u-T_k(u))|^{p^*} dx\right)^{\frac{1}{p^*}} \\
\leqslant & \left(\int_\Omega |T_k(u-T_k(u))|^{p^*} dx\right)^{\frac{1}{p^*}} \\
\leqslant & c_* \left(\int_\Omega |\nabla(T_k(u-T_k(u)))|^p dx\right)^{\frac{1}{p}} \\
\leqslant & c_* \left(\frac{3^\theta k^{1+\theta}}{\alpha}\right)^{\frac{1}{p}} |||f|||_q^{\frac{1}{p}} |\{|u|>k\}|^{\frac{1}{pq'}},
\end{aligned}
$$

which easily implies

$$k^{\frac{p^*(p-1-\theta)}{p}} |\{|u|>2k\}| \leqslant c_*^{p^*} \left(\frac{3^\theta}{\alpha}\right)^{\frac{p^*}{p}} |||f|||_q^{\frac{p^*}{p}} |\{|u|>k\}|^{\frac{p^*}{pq'}}.$$

The assumption of Lemma 2.1 holds with

$$\alpha = \frac{p^*(p-1-\theta)}{p},$$

$$c_4 = c_*^{p^*} \left(\frac{3^\theta}{\alpha}\right)^{\frac{p^*}{p}} |||f|||_q^{\frac{p^*}{p}},$$

$$\beta = \frac{p^*}{pq'}$$

and

$$k_0 = 1,$$

here $\alpha > 0$ since $\theta < p-1$ and $\beta < 1$ since $q < \frac{n}{p}$. Lemma 2.1 yields the desired result

$$u \in L^\gamma_{weak}(\Omega),$$

where

$$\gamma = \frac{\alpha}{1-\beta} = \frac{\frac{p^*(p-1-\theta)}{p}}{1-\frac{p^*}{pq'}} = \frac{nq(p-1-\theta)}{n-pq}.$$

ii) Let $h > k > 0$. (1.4) can be rewritten as

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\nabla T_{h-k}(u-v)dx = \int_\Omega f T_{h-k}(u-v)dx, \quad \forall h > k > 0. \tag{2.3}$$

We use $v = T_k(u), k \geqslant k_0 = 1$, as a test function in (2.3) and we have

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\nabla T_{h-k}(u-T_k(u))dx = \int_\Omega f T_{h-k}(u-T_k(u))dx.$$

Let $\varphi = T_{h-k}(u-T_k(u))$. Note that $\varphi = 0$ on $\{|u| \leqslant k\}$ and

$$\nabla\varphi = \begin{cases} 0, & |u| \leqslant k, \\ \nabla u, & k < |u| < h, \\ 0, & |u| \geqslant h. \end{cases}$$

Condition (1.1) yields

$$\begin{aligned} &\alpha \int_{\{k<|u|<h\}} \frac{|\nabla u|^p}{(1+|u|)^\theta} dx \leqslant \int_{\{k<|u|<h\}} \mathcal{A}(x,u,\nabla u)\nabla u dx \\ &= \int_\Omega \mathcal{A}(x,u,\nabla u)\nabla\varphi dx = \int_\Omega f\varphi dx \leqslant \int_{\{|u|>k\}} |f||\varphi| dx. \end{aligned} \tag{2.4}$$

Let $t = \frac{n(p-\theta)}{n-\theta}$, then

$$t < p \Leftrightarrow p < n,$$

$$t^* = \frac{nt}{n-t} = \frac{t\theta}{p-t} = \frac{n(p-\theta)}{n-p}$$

and

$$\theta < p-1 \Rightarrow (t^*)' < \frac{n}{p} = q.$$

Hölder's inequality with exponents $\frac{p}{t}$ and $\frac{p}{p-t}$ implies

$$\begin{aligned}
& \int_{\{k<|u|<h\}} |\nabla u|^t dx \\
= & \int_{\{k<|u|<h\}} \frac{|\nabla u|^t}{(1+|u|)^{\frac{t\theta}{p}}} (1+|u|)^{\frac{t\theta}{p}} dx \\
\leqslant & \left(\int_{\{k<|u|<h\}} \frac{|\nabla u|^p}{(1+|u|)^{\theta}} dx \right)^{\frac{t}{p}} \left(\int_{\{k<|u|<h\}} (1+|u|)^{\frac{t\theta}{p-t}} dx \right)^{1-\frac{t}{p}} \\
\leqslant & \; 2^{\frac{t\theta}{p}} \left(\int_{\{k<|u|<h\}} \frac{|\nabla u|^p}{(1+|u|)^{\theta}} dx \right)^{\frac{t}{p}} \left(\int_{\{k<|u|<h\}} (k+|u-T_k(u)|)^{\frac{t\theta}{p-t}} dx \right)^{1-\frac{t}{p}} \\
\leqslant & \; 2^{\frac{t\theta}{p}} \left(\frac{1}{\alpha} \int_{\{|u|>k\}} |f||\varphi| dx \right)^{\frac{t}{p}} \left(\int_{\{k<|u|<h\}} (k+|u-T_k(u)|)^{\frac{t\theta}{p-t}} dx \right)^{1-\frac{t}{p}},
\end{aligned} \tag{2.5}$$

here we have used the facts $k \geqslant k_0 = 1$ and

$$1+|u| \leqslant 2|u| \leqslant 2(k+|u-T_k(u)|) \quad \text{on} \quad \{|u|>k\}.$$

We use Hölder's inequality with exponents t^* and $(t^*)'$ and Sobolev inequality (2.2) again in order to get

$$\begin{aligned}
& \int_{\{|u|>k\}} |f||\varphi| dx \\
\leqslant & \left(\int_{\{|u|>k\}} |f|^{(t^*)'} dx \right)^{\frac{1}{(t^*)'}} \left(\int_{\{|u|>k\}} |\varphi|^{t^*} dx \right)^{\frac{1}{t^*}} \\
\leqslant & \left(\int_{\{|u|>k\}} |f|^{(t^*)'} dx \right)^{\frac{1}{(t^*)'}} \left(\int_{\Omega} |\varphi|^{t^*} dx \right)^{\frac{1}{t^*}} \\
\leqslant & \; c_* \left(\int_{\{|u|>k\}} |f|^{(t^*)'} dx \right)^{\frac{1}{(t^*)'}} \left(\int_{\Omega} |\nabla \varphi|^t dx \right)^{\frac{1}{t}} \\
= & \; c_* \left(\int_{\{|u|>k\}} |f|^{(t^*)'} dx \right)^{\frac{1}{(t^*)'}} \left(\int_{\{k<|u|<h\}} |\nabla u|^t dx \right)^{\frac{1}{t}}.
\end{aligned} \tag{2.6}$$

Since $(t^*)' < \frac{n}{p} = q$ and $f \in L^q_{weak}(\Omega)$, then $|f|^{(t^*)'} \in L^{\frac{q}{(t^*)'}}_{weak}(\Omega)$ and

$$\begin{aligned}
& \int_{\{|u|>k\}} |f|^{(t^*)'} dx \leqslant \||f|^{(t^*)'}\|_{\frac{q}{(t^*)'}} |\{|u|>k\}|^{\frac{1}{(q/(t^*)')'}} \\
& = \|f\|_q^{(t^*)'} |\{|u|>k\}|^{\frac{(n-p)(p-1-\theta)}{n(p-\theta)-n+p}}.
\end{aligned} \tag{2.7}$$

Combining (2.6) with (2.7) we have

$$\begin{aligned} & \int_{\{|u|>k\}} |f||\varphi| dx \\ \leqslant \quad & c_* |||f|||_q |\{|u|>k\}|^{\frac{(n-p)(p-1-\theta)}{n(p-\theta)}} \left(\int_{\{k<|u|<h\}} |\nabla u|^t dx \right)^{\frac{1}{t}} . \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.5) one has

$$\begin{aligned} & \int_{\{k<|u|<h\}} |\nabla u|^t dx \\ \leqslant \quad & 2^{\frac{t\theta}{p-1}} \left(\frac{c_* |||f|||_q}{\alpha} \right)^{\frac{t}{p-1}} |\{|u|>k\}|^{\frac{(n-p)(p-1-\theta)}{(p-1)(n-\theta)}} \\ \times \quad & \left(\int_{\{k<|u|<h\}} (k+|u-T_k(u)|)^{\frac{t\theta}{p-t}} dx \right)^{\frac{p-t}{p-1}} . \end{aligned} \tag{2.9}$$

We use the following inequalities, satisfied by every $a, b \geqslant 0$:

$$(a+b)^m \leqslant 2^{m-1}(a^m + b^m) \quad \text{if} \;\; m \geqslant 1 \tag{2.10}$$

and

$$(a+b)^m \leqslant a^m + b^m \quad \text{if} \;\; 0 < m \leqslant 1, \tag{2.11}$$

in order to derive that

$$\begin{aligned} & \int_{\{k<|u|<h\}} (k+|u-T_k(u)|)^{\frac{t\theta}{p-t}} dx \\ \leqslant \quad & 2^{\frac{t\theta}{p-t}-1} \left(\int_{\{|u|>k\}} k^{\frac{t\theta}{p-t}} dx + \int_{\{k<|u|<h\}} |u-T_k(u)|^{\frac{t\theta}{p-t}} dx \right) \\ \leqslant \quad & 2^{\frac{t\theta}{p-t}-1} \left(k^{\frac{t\theta}{p-t}} |\{|u|>k\}| + \int_\Omega |\varphi|^{\frac{t\theta}{p-t}} dx \right), \end{aligned} \tag{2.12}$$

where we recall that $\varphi = T_{h-k}(u - T_k(u))$. By Sobolev inequality (2.2) again

$$\begin{aligned} & \int_\Omega |\varphi|^{\frac{t\theta}{p-t}} dx = \int_\Omega |\varphi|^{t^*} dx \\ \leqslant \quad & c_*^{t^*} \left(\int_\Omega |\nabla \varphi|^t dx \right)^{\frac{t^*}{t}} = c_*^{t^*} \left(\int_{\{k<|u|<h\}} |\nabla u|^t dx \right)^{\frac{t^*}{t}} . \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.12) we arrive at

$$\begin{aligned} & \int_{\{k<|u|<h\}} (k+|u-T_k(u)|)^{\frac{t\theta}{p-t}} dx \\ \leqslant \quad & 2^{\frac{t\theta}{p-t}-1} \left[k^{\frac{t\theta}{p-t}} |\{|u|>k\}| + c_*^{t^*} \left(\int_{\{k<|u|<h\}} |\nabla u|^t dx \right)^{\frac{t^*}{t}} \right] . \end{aligned} \tag{2.14}$$

A direct calculation shows

$$\frac{t^*}{t}\frac{p-t}{p-1}=\frac{\theta}{p-1},\quad \frac{(n-p)(p-1-\theta)}{(p-1)(n-\theta)}+\frac{p-t}{p-1}=\frac{n-p}{n-\theta}. \tag{2.15}$$

Substituting (2.14) into (2.9), using (2.15), (2.11) and Young inequality, we obtain

$$\begin{aligned}
&\int_{\{k<|u|<h\}}|\nabla u|^t dx\\
\leqslant\ & 2^{\frac{t\theta}{p-1}}\left(\frac{c_*|||f|||_q}{\alpha}\right)^{\frac{t}{p-1}}|\{|u|>k\}|^{\frac{(n-p)(p-1-\theta)}{(p-1)(n-\theta)}}\\
&\times 2^{\frac{t(1+\theta)-p}{p-1}}\left[k^{\frac{t\theta}{p-1}}|\{|u|>k\}|^{\frac{p-t}{p-1}}+c_*^{\frac{t^*(p-t)}{p-1}}\left(\int_{\{k<|u|<h\}}|\nabla u|^t dx\right)^{\frac{t^*}{t}\frac{p-t}{p-1}}\right]\\
=\ & 2^{\frac{t(1+2\theta)-p}{p-1}}\left(\frac{c_*|||f|||_q}{\alpha}\right)^{\frac{t}{p-1}}\left[k^{\frac{t\theta}{p-1}}|\{|u|>k\}|^{\frac{n-p}{n-\theta}}\right.\\
&\left.+c_*^{\frac{t^*(p-t)}{p-1}}|\{|u|>k\}|^{\frac{(n-p)(p-1-\theta)}{(p-1)(n-\theta)}}\left(\int_{\{k<|u|<h\}}|\nabla u|^t dx\right)^{\frac{\theta}{p-1}}\right]\\
\leqslant\ & 2^{\frac{t(1+2\theta)-p}{p-1}}\left(\frac{c_*|||f|||_q}{\alpha}\right)^{\frac{t}{p-1}}\left[k^{\frac{t\theta}{p-1}}|\{|u|>k\}|^{\frac{n-p}{n-\theta}}\right.\\
&\left.+c(\varepsilon)\ c_*^{\frac{t^*(p-t)}{p-1-\theta}}|\{|u|>k\}|^{\frac{n-p}{n-\theta}}+\varepsilon\int_{\{k<|u|<h\}}|\nabla u|^t dx\right].
\end{aligned}$$

Take ε small enough such that

$$2^{\frac{t(1+2\theta)-p}{p-1}}\left(\frac{c_*|||f|||_q}{\alpha}\right)^{\frac{t}{p-1}}\varepsilon<1,$$

then the last term in the right-hand side of the above inequality is absorbed by the left-hand side. Thus

$$\begin{aligned}
\int_{\{k<|u|<h\}}|\nabla u|^t dx &\leqslant c_5\left[k^{\frac{t\theta}{p-1}}|\{|u|>k\}|^{\frac{n-p}{n-\theta}}+|\{|u|>k\}|^{\frac{n-p}{n-\theta}}\right]\\
&\leqslant 2c_5 k^{\frac{t\theta}{p-1}}|\{|u|>k\}|^{\frac{n-p}{n-\theta}},
\end{aligned} \tag{2.16}$$

where c_5 is a constant depending only on n,p,θ,α and $|||f|||_q$, and we have used again the fact $k\geqslant k_0=1$.

We use Sobolev inequality (2.2) again in order to derive

$$\begin{aligned}
&\int_{\{k<|u|<h\}}|\nabla u|^t dx=\int_\Omega|\nabla\varphi|^t dx\geqslant c_*^{-t}\left(\int_\Omega|\varphi|^{t^*}dx\right)^{\frac{t}{t^*}}\\
&\geqslant c_*^{-t}\left(\int_{\{|u|>h\}}|\varphi|^{t^*}dx\right)^{\frac{t}{t^*}}\geqslant c_*^{-t}(h-k)^t|\{|u|>h\}|^{\frac{t}{t^*}}\\
&=c_*^{-t}(h-k)^t|\{|u|>h\}|^{\frac{n-p}{n-\theta}}.
\end{aligned} \tag{2.17}$$

Estimates (2.16) and (2.17) merge into

$$|\{|u|>h\}| \leqslant \frac{(2c_5c_*)^{\frac{n-\theta}{n-p}}}{(h-k)^{\frac{n(p-\theta)}{n-p}}} k^{\frac{n\theta(p-\theta)}{(n-p)(p-1)}} |\{|u|>k\}|. \tag{2.18}$$

Thus condition (3) in the Appendix holds with

$$\tilde{c} = (2c_5c_*)^{\frac{n-\theta}{n-p}},$$

$$\alpha = \frac{n(p-\theta)}{n-p},$$

$$\beta = 1$$

and

$$k_0 = 1,$$

here we note that

$$\theta < p-1 \Rightarrow \frac{n\theta(p-\theta)}{(n-p)(p-1)} = \tilde{\theta}\frac{n(p-\theta)}{n-p}, \quad \text{with} \quad \tilde{\theta} = \frac{\theta}{p-1} < 1,$$

thus one can use the result of Lemma 3 part (ii) in the Appendix to derive that, for every $k \geqslant k_0 = 1$,

$$|\{|u|>k\}| \leqslant |\{|u|>1\}| e^{1-\left(\frac{k-1}{\tau}\right)^{1-\tilde{\theta}}} \leqslant |\Omega| e^{1+2\lambda} e^{-2\lambda k^{1-\tilde{\theta}}}, \tag{2.19}$$

with τ be as in (4) and $2\lambda = \tau^{\tilde{\theta}-1}$. (2.19) implies

$$|\{|u|>k\}| \leqslant c_6 e^{-2\lambda k^{1-\tilde{\theta}}}, \tag{2.20}$$

where c_6 is a constant depending only on n, p, α, θ and $|\Omega|$. Hence

$$|\{e^{\lambda|u|^{1-\tilde{\theta}}} > e^{\lambda k^{1-\tilde{\theta}}}\}| = |\{|u|>k\}| \leqslant c_6 e^{-2\lambda k^{1-\tilde{\theta}}}.$$

We let $\tilde{k} = e^{\lambda k^{1-\tilde{\theta}}}$ and we have

$$|\{e^{\lambda|u|^{1-\tilde{\theta}}} > \tilde{k}\}| = \frac{c_6}{\tilde{k}^2}.$$

We now use Lemma 3.11 in [7] which states that the sufficient and necessary condition for $f \in L^r(\Omega)$, $r \geqslant 1$, is

$$\sum_{k=0}^{\infty} k^{r-1} |\{|f|>k\}| < +\infty.$$

We use the above lemma for $f = e^{\lambda|u|^{1-\tilde{\theta}}}$ and $r = 1$, since

$$\sum_{k=0}^{\infty} |\{e^{\lambda|u|^{1-\tilde{\theta}}} > \tilde{k}\}| \leqslant c_6 \sum_{k=0}^{\infty} \frac{1}{\tilde{k}^2} < +\infty,$$

then $e^{\lambda|u|^{1-\tilde{\theta}}} = e^{\lambda|u|^{1-\frac{\theta}{p-1}}} \in L^1(\Omega)$, as desired.

iii) We first prove

$$\frac{np}{np-(n-p)(1+\theta)} \leqslant \frac{n}{p}, \quad \text{for } 1+\theta < p < n, \tag{2.21}$$

which is equivalent to prove that

$$p^2 \leqslant np-(n-p)(1+\theta), \quad \text{for } 1+\theta < p < n. \tag{2.22}$$

To this end, we need only to prove

$$p^2-(n+1+\theta)p+n(1+\theta) \leqslant 0.$$

Since the two roots for the equation

$$p^2-(n+1+\theta)p+n(1+\theta) = 0$$

are $p_1 = 1+\theta$ and $p_2 = n$, then the condition $1+\theta < p < n$ implies (2.22), and (2.21) is proved. Thus, in the case $1 \leqslant q < \frac{np}{np-(n-p)(1+\theta)}$, the result of i) holds true, i.e.,

$$u \in L^{\gamma}_{weak}(\Omega), \quad \gamma = \frac{nq(p-1-\theta)}{n-pq}. \tag{2.23}$$

We next prove that, for all $k \geqslant k_0 = 1$,

$$\int_{\{|u|<k\}} |\nabla u|^p dx \leqslant c_7 |||u|||_{\gamma}^{\frac{\gamma}{q'}} k^{1+\theta-\frac{\gamma}{q'}} |||f|||_q, \tag{2.24}$$

where $c_7 = \frac{3^{\theta}}{\alpha\left(2^{1+\theta-\frac{\gamma}{q'}}-1\right)}$. We use (2.1) in order to derive that: for all $k \geqslant 1$,

$$\alpha \int_{\{k<|u|<2k\}} |\nabla u|^p dx \leqslant 3^{\theta} k^{1+\theta} |\{|u|>k\}|^{\frac{1}{q'}} |||f|||_q, \tag{2.25}$$

whence

$$\begin{aligned}
& \alpha \int_{\{|u|<k\}} |\nabla u|^p dx \\
= \; & \alpha \sum_{j=0}^{\infty} \int_{\{2^{-j-1}k<|u|<2^{-j}k\}} |\nabla u|^p dx \\
\leqslant \; & \sum_{j=0}^{\infty} 3^{\theta} (2^{-j-1}k)^{1+\theta} |\{|u|>2^{-j-1}k\}|^{\frac{1}{q'}} |||f|||_q \\
\leqslant \; & \sum_{j=0}^{\infty} 3^{\theta} |||u|||_{\gamma}^{\frac{\gamma}{q'}} (2^{-j-1}k)^{1+\theta-\frac{\gamma}{q'}} |||f|||_q \\
\leqslant \; & \frac{3^{\theta} |||u|||_{\gamma}^{\frac{\gamma}{q'}}}{2^{1+\theta-\frac{\gamma}{q'}}-1} k^{1+\theta-\frac{\gamma}{q'}} |||f|||_q,
\end{aligned}$$

here we notice that $q < \frac{np}{np-(n-p)(1+\theta)} \Leftrightarrow 1+\theta-\frac{\gamma}{q'} > 0$. (2.24) is proved.

Let t_0 be such that

$$\frac{\gamma t_0^p |||u|||_\gamma^{\frac{\gamma}{q}}}{c_7 |||f|||_q (1+\theta-\frac{\gamma}{q})} = 1. \tag{2.26}$$

We obtain from (2.24) that, for $t > t_0$,

$$\begin{aligned} |\{|\nabla u| > t\}| &= |\{|\nabla u| > t\} \cap \{|u| > k\}| + |\{|\nabla u| > t\} \cap \{|u| < k\}| \\ &\leqslant |\{|u| > k\}| + t^{-p} \int_{\{|u|<k\}} |\nabla u|^p dx \\ &\leqslant |\{|u| > k\}| + c_7 t^{-p} |||u|||_\gamma^{\frac{\gamma}{q'}} k^{1+\theta-\frac{\gamma}{q'}} |||f|||_q \\ &\leqslant |||u|||_\gamma^\gamma \left(k^{-\gamma} + c_7 t^{-p} |||u|||_\gamma^{-\frac{\gamma}{q}} |||f|||_q k^{1+\theta-\frac{\gamma}{q'}} \right). \end{aligned} \tag{2.27}$$

Next we minimize this in k, i.e. choose

$$k = \left(\frac{\gamma t^p |||u|||_\gamma^{\frac{\gamma}{q}}}{c_7 |||f|||_q (1+\theta-\frac{\gamma}{q'})} \right)^{\frac{q}{q+\gamma+\theta q}}.$$

Condition (2.26) and $t > t_0$ ensure $k > 1$. (2.27) with k as above yields

$$|\{|\nabla u| > t\}| \leqslant c_8 |||u|||_\gamma^{\gamma\left(1-\frac{\gamma}{q+\gamma+q\theta}\right)} t^{-\frac{pq\gamma}{q+\gamma+q\theta}} |||f|||_q^{\frac{q\gamma}{q+\gamma+q\theta}},$$

where c_8 is a constant depending on n, p, α and θ. Now we observe

$$\frac{pq\gamma}{q+\gamma+q\theta} = \frac{nq(p-1-\theta)}{n-q(1+\theta)} = s,$$

and by **i)**, $|||u|||_\gamma \leqslant c_1 |||f|||_q^{\frac{1}{p-1-\theta}}$, hence for $t > t_0$,

$$|\{|\nabla u| > t\}| \leqslant c_8 t^{-s} |||u|||_\gamma^{\frac{s(1+\theta)}{p}} |||f|||_q^{\frac{s}{p}} \leqslant c_9 t^{-s} |||f|||_q^{\frac{s}{p-1-\theta}}, \tag{2.28}$$

where $c_9 = c_8 c_1^{\frac{s(1+\theta)}{p}}$. We let c_{10} be such that

$$|\Omega| \leqslant c_{10} t_0^{-s} |||f|||_q^{\frac{s}{p-1-\theta}},$$

then for $0 < t \leqslant t_0$,

$$|\{|\nabla u| > t\}| \leqslant |\Omega| \leqslant c_{10} t_0^{-s} |||f|||_q^{\frac{s}{p-1-\theta}} \leqslant c_{10} t^{-s} |||f|||_q^{\frac{s}{p-1-\theta}}. \tag{2.29}$$

(2.28) and (2.29) merge into

$$|\{|\nabla u| > t\}| \leqslant \max\{c_9, c_{10}\} t^{-s} |||f|||_q^{\frac{s}{p-1-\theta}}, \quad \forall t > 0,$$

from which the result follows.

iv) We take $k=1$ and $v=0$ in (1.4). Then

$$\begin{aligned} &\frac{\alpha}{2^\theta}\int_{\{|u|\leqslant 1\}}|\nabla u|^p dx \leqslant \alpha\int_{\{|u|\leqslant 1\}}\frac{|\nabla u|^p}{(1+|u|)^\theta}dx \\ \leqslant\ &\int_\Omega \mathcal{A}(x,u,\nabla u)\nabla T_1(u)dx = \int_\Omega fT_1(u)dx \leqslant |\Omega|^{\frac{1}{q'}}|||f|||_q. \end{aligned} \tag{2.30}$$

We use (2.25) again and we have

$$\begin{aligned} &\alpha\int_{\{|u|>1\}}|\nabla u|^p dx \\ =\ &\alpha\sum_{j=0}^{\infty}\int_{\{2^j<|u|<2^{j+1}\}}|\nabla u|^p dx \\ \leqslant\ &3^\theta\sum_{j=0}^{\infty}2^{j(1+\theta)}|\{|u|>2^j\}|^{\frac{1}{q'}}|||f|||_q \\ \leqslant\ &3^\theta\sum_{j=0}^{\infty}2^{j\left(1+\theta-\frac{\gamma}{q'}\right)}|||u|||_\gamma^{\frac{\gamma}{q'}}|||f|||_q \\ \leqslant\ &3^\theta\sum_{j=0}^{\infty}2^{j(1+\theta-\frac{\gamma}{q'})}|||f|||_q^{1+\frac{1}{q'(p-1-\theta)}}, \end{aligned}$$

In the case $q>\frac{np}{np-(n-p)(1+\theta)}$, one has $1+\theta-\frac{\gamma}{q'}<0$. Thus

$$\int_{\{|u|>1\}}|\nabla u|^p dx \leqslant c_{11}|||f|||_q^{1+\frac{1}{q'(p-1-\theta)}} = c_{11}|||f|||_q^{\frac{q(p-\theta)-1}{q(p-1-\theta)}}, \tag{2.31}$$

here c_{11} is a constant depending only on n,p,α,θ and $|\Omega|$. (2.30) and (2.31) yield the desired result.

The proof of Theorem 1.1 has been completed.

We next show that parts i) and iii) in Theorem 1.1 are sharp.

Example 3.1. We denote by $B=B_1(0)$ the unit ball of $\mathbb{R}^n$ and we consider the function $u(x)=|x|^{-a}-1$, $a>0$. The truncations $T_k(u)$ of u belong to $W_0^{1,p}(B)$ for each $k>0$. A direct computation shows that

$$\nabla u = -\frac{ax}{|x|^{a+2}} \tag{2.32}$$

and

$$-\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^\theta}\right) = a^{p-1}(n-p-a(p-1-\theta))|x|^{-p-a(p-1-\theta)}$$

if $x\neq 0$. Now we observe that

$$f = a^{p-1}(n-p-a(p-1-\theta))|x|^{-p-a(p-1-\theta)} \in L^1(B)$$

if and only if

$$a(1-p+\theta)-p>-n,$$

that is,

$$a<\frac{n-p}{p-1-\theta}.$$

Then u is an entropy solution of

$$\begin{cases} -\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta}}\right)=f, & x\in B,\\ u=0, & x\in\partial B.\end{cases}$$

Note that the operator $\mathcal{A}(x,u,\nabla u)=\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta}}$ satisfies (1.1) and (1.2) with $\alpha=\beta=1$. Moreover, it follows that if

$$a=\frac{n-pq}{q(p-1-\theta)},$$

then $f\in L^q_{weak}(B)$ and $u\in L^{\gamma}_{weak}(B)$ if and only if

$$\gamma\leqslant\frac{nq(p-1-\theta)}{n-pq}.$$

Furthermore, by (2.32), $\nabla u\in L^s(B)$ if and only if

$$s\leqslant\frac{nq(p-1-\theta)}{n-q(1+\theta)}.$$

This shows that parts i) and iii) of Theorem 1.1 are sharp.

§3 Proof of Theorem 1.2.

In this section, we apply Theorem 1.1 to investigate the regularity of solutions to the problem (1.8) and prove Theorem 1.2.

We divide the proof into two steps: Step 1, $f=|1+u|^{a-1}(1+u)\in L^q_{weak}(\Omega)$, for some $q>\frac{n}{p}$; Step 2, any weak solution $u\in W^{1,p}_0(\Omega)$ to the boundary value problem

$$\begin{cases} -\mathrm{div}\mathcal{A}(x,u(x),\nabla u(x))=f(x), & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega,\end{cases}\tag{3.1}$$

where $f\in L^q_{weak}(\Omega)$, $q>\frac{n}{p}$, is bounded.

Step 1. We use a bootstrap argument. Since $f\in L^1(\Omega)$ we have by Theorem 1.1 part i) that

$$u\in L^{\gamma_1}_{weak}(\Omega),\quad \gamma_1=\frac{n(p-1-\theta)}{n-p}.$$

Therefore

$$f\in L^{q_1}_{weak}(\Omega),\quad q_1=\frac{\gamma_1}{a}.$$

Now we repeatedly use Theorem 1.1 part i). At the jth step we obtain

$$u \in L^{\gamma_j}_{weak}(\Omega), \text{ where } \quad \gamma_j = \frac{nq_{j-1}(p-1-\theta)}{n-pq_{j-1}}$$

and

$$f \in L^{q_j}_{weak}(\Omega), \quad q_j = \frac{\gamma_j}{a},$$

here we put $q_0 = 1$. By recursion

$$q_j = \frac{n}{n\left(\frac{a}{p-1-\theta}\right)^j - p\sum_{k=1}^{j}\left(\frac{a}{p-1-\theta}\right)^k},$$

provided that $q_{j-1} < \frac{n}{p}$. Since $0 \leqslant a < \frac{n(p-1-\theta)}{n-p}$, it is immediate that q_j is an increasing sequence and moreover, there is an $\delta > 0$ such that

$$\begin{aligned}
& \sum_{k=0}^{j}\left(\frac{a}{p-1-\theta}\right)^{k-j} \\
= & \sum_{k=0}^{j}\left(\frac{a}{p-1-\theta}\right)^{-k} \\
\geqslant & \ \delta + \sum_{k=0}^{j}\left(1-\frac{p}{n}\right)^k \\
= & \ \delta + \frac{n}{p} - \frac{n}{p}\left(1-\frac{n}{p}\right)^{j+1} \\
> & \ \frac{n}{p},
\end{aligned}$$

if j is large enough. One easily checks that for such a j it holds that $q_j > \frac{n}{p}$. Therefore we conclude that $f \in L^q_{weak}(\Omega)$ for some $q > \frac{n}{p}$. Since $q > \frac{n}{p} > \frac{np}{np-(n-p)(1+\theta)}$, then Theorem 1.1 part iv) tells us that an entropy solution is an ordinary $W^{1,p}_0(\Omega)$ solution.

Step 2. To prove any weak solution $u \in W^{1,p}_0(\Omega)$ of (3.1) is bounded, we define, for $s \in \mathbb{R}$,

$$H(s) = \int_0^s \frac{1}{(1+|t|)^{\frac{\theta}{p-1}}}dt.$$

For $k > 0$, we take

$$G_k(H(u)) = H(u) - T_k(H(u))$$

as a test function in the weak formulation of (3.1) and use assumption (1.1) we obtain

$$\alpha\int_\Omega |\nabla G_k(H(u))|^p dx \leqslant \int_{\tilde{A}_k} fG_k(H(u))dx, \tag{3.2}$$

where

$$\tilde{A}_k = \{x \in \Omega : |H(u)(x)| > k\}.$$

We use Sobolev inequality (2.2) and we have

$$\int_\Omega |\nabla G_k(H(u))|^p dx \geqslant c_*^p \left(\int_\Omega |G_k(H(u))|^{p^*} dx \right)^{\frac{p}{p^*}}. \tag{3.3}$$

We note that

$$1 \leqslant p < n \Rightarrow q > \frac{n}{p} > (p^*)'.$$

Hölder's inequality with exponents p^* and $(p^*)'$ yields

$$\begin{aligned} \int_{\tilde{A}_k} f G_k(H(u)) dx &\leqslant \left(\int_{\tilde{A}_k} |f|^{(p^*)'} \right)^{\frac{1}{(p^*)'}} \left(\int_\Omega |G_k(H(u))|^{p^*} \right)^{\frac{1}{p^*}} \\ &\leqslant |||f|||_q |\tilde{A}_k|^{\frac{1}{\left(\frac{q}{(p^*)'}\right)'} \frac{1}{(p^*)'}} \left(\int_\Omega |G_k(H(u))|^{p^*} \right)^{\frac{1}{p^*}}. \end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) we arrive at

$$\alpha c_*^p \left(\int_\Omega |G_k(H(u))|^{p^*} dx \right)^{\frac{p-1}{p^*}} \leqslant |||f|||_q |\tilde{A}_k|^{\frac{1}{\left(\frac{q}{(p^*)'}\right)'} \frac{1}{(p^*)'}}. \tag{3.5}$$

For $h > k$, we have

$$\int_\Omega |G_k(H(u))|^{p^*} dx \geqslant \int_{\tilde{A}_h} |G_k(H(u))|^{p^*} dx \geqslant (h-k)^{p^*} |\tilde{A}_h|,$$

which together with (3.5) yields

$$|\tilde{A}_h| \leqslant \left(\frac{|||f|||_q}{\alpha c_*^p} \right)^{\frac{p^*}{p-1}} \frac{1}{(h-k)^{p^*}} |\tilde{A}_k|^{\frac{1}{\left(\frac{q}{(p^*)'}\right)'} \frac{1}{(p^*)'} \frac{p^*}{p-1}}.$$

The assumption of Lemma 1 in the Appendix holds with

$$\varphi(k) = |\tilde{A}_k|,$$

$$\tilde{c}_1 = \left(\frac{|||f|||_q}{\alpha c_*^p} \right)^{\frac{p^*}{p-1}},$$

$$\alpha = p^*,$$

$$k_0 = 0$$

and

$$\beta = \frac{1}{\left(\frac{q}{(p^*)'}\right)'} \frac{1}{(p^*)'} \frac{p^*}{p-1},$$

here $\beta > 1$ since $q > \frac{n}{p}$. Lemma 1 in the Appendix implies that there exists $d > 0$ such that $|\tilde{A}_d| = |\{x \in \Omega : |H(u)(x)| > d\}| = \varphi(d) = 0$, which yields $H(u)(x) \leqslant d$, a.e. Ω. The desired result follows.

Example 3.2. Suppose that

$$a > a_c = \frac{n(p-1-\theta)}{n-p}.$$

Let

$$b = \frac{p}{a-(p-1-\theta)}.$$

It is easy to see from $a > a_c$ that

$$b = \frac{p}{a-(p-1-\theta)} < \frac{p}{\frac{n(p-1-\theta)}{n-p} - (p-1-\theta)} = \frac{n-p}{p-1-\theta}. \tag{3.6}$$

Let us consider the function

$$u = \lambda |x|^{-b} - 1 \tag{3.7}$$

in the ball $B_{\lambda^b}(0)$, where λ is a positive constant to be determined later. Direct calculations show that

$$-\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^\theta}\right) = f,$$

where

$$\begin{aligned} f &= \lambda^{p-1-\theta} b^{p-1}(n-p-b(p-1-\theta))|x|^{-p-b(p-1-\theta)} \\ &= \lambda^{p-1-\theta} b^{p-1}(n-p-b(p-1-\theta))|x|^{-ab}, \end{aligned}$$

and

$$|1+u|^{a-1}(1+u) = \lambda^a |x|^{-ab}.$$

If we let λ to be a number satisfying

$$\lambda^{p-1-\theta} b^{p-1}(n-p-b(p-1-\theta)) = \lambda^a,$$

then the function u in (3.7) satisfies the equation in (1.8) in the ball $B_{\lambda^b}(0)$. It is easy to see that $u = 0$ on $\partial B_{\lambda^b}(0)$ and $u^a \in L^1(B_{\lambda^b}(0))$ since by (3.6) that

$$p + b(p-1-\theta) < p + \frac{n-p}{p-1-\theta}(p-1-\theta) = n,$$

but u is not bounded.

Appendix

The well-known Stampacchia Lemma (see Lemma 4.1 in [36]) reads as follows:

Lemma 1. *Let $\tilde{c}_1, \alpha, \beta$ be positive constants. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be decreasing and such that*

$$\varphi(h) \leqslant \frac{\tilde{c}_1}{(h-k)^\alpha}[\varphi(k)]^\beta \tag{1}$$

for every h, k with $h > k \geqslant k_0$. It results that:

(i) *if $\beta > 1$ then we have*

$$\varphi(k_0 + d) = 0,$$

where

$$d^\alpha = \tilde{c}_1[\varphi(k_0)]^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}.$$

(ii) *if $\beta = 1$ then for any $k \geqslant k_0$ we have*

$$\varphi(k) \leqslant \varphi(k_0) e^{1-(\tilde{c}_1 e)^{-\frac{1}{\alpha}}(k-k_0)}.$$

(iii) *if $\beta < 1$ and $k_0 > 0$ then for any $k \geqslant k_0$ we have*

$$\varphi(k) \leqslant 2^{\frac{\alpha}{(1-\beta)^2}} \left\{ \tilde{c}_1^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} \varphi(k_0) \right\} \left(\frac{1}{k} \right)^{\frac{\alpha}{1-\beta}}.$$

Mammoliti proved a lemma in [6] in order to deal with regularity for solutions to some elliptic equations with degenerate coercivity, see Lemma A.1 in the Appendix in [6]:

Lemma 2. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function such that

$$\varphi(h) \leqslant \frac{\tilde{c}_2}{(h-k)^\alpha} k^{\tilde{\theta}\alpha}[\varphi(k)]^\beta, \qquad \forall h > k > 0, \tag{2}$$

for some positive constant $\tilde{c}_2$, with $\alpha > 0$, $0 \leqslant \tilde{\theta} < 1$ and $\beta > 1$. Then there exists $k^* > 0$ such that $\varphi(k^*) = 0$.

After carefully reading the proof of Lemma A.1 in the Appendix in [6], we find that the above lemma can be stated as follows:

Lemma 2′. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function such that

$$\varphi(h) \leqslant \frac{\tilde{c}_2}{(h-k)^\alpha} k^{\tilde{\theta}\alpha}[\varphi(k)]^\beta, \qquad \forall h > k \geqslant k_0 > 0, \tag{2′}$$

for some positive constant $\tilde{c}_2$, with $\alpha > 0$, $0 \leqslant \tilde{\theta} < 1$ and $\beta > 1$. Then there exists $k^* > 0$ such that $\varphi(k^*) = 0$.

We remark that the difference between Lemma 2 and Lemma $2'$ is that, in the condition $(2')$ we need only to assume the inequality holds true for all $h > k \geqslant k_0 > 0$.

We now compare Lemma $2'$ with Lemma 1 part (i). The difference between the assumptions (1) and $(2')$ is that a factor $k^{\tilde{\theta}\alpha}$ appears in the right-hand side of $(2')$. When $\tilde{\theta} = 0$, they are the same. Thus Lemma $2'$ has slightly more general hypotheses.

We now prove a generalized Stampacchia Lemma in order to deal with regularity for entropy solutions to elliptic equations with degenerate coercivity.

Lemma 3. Let $\tilde{c}, \alpha, \beta, k_0$ be positive constants and $0 \leqslant \tilde{\theta} < 1$. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be decreasing and such that

$$\varphi(h) \leqslant \frac{\tilde{c}}{(h-k)^{\alpha}} k^{\tilde{\theta}\alpha} [\varphi(k)]^{\beta} \tag{3}$$

for every h, k with $h > k \geqslant k_0 > 0$. It results that:

(i) if $\beta > 1$ then there exists $k^* > 0$ such that $\varphi(k^*) = 0$.

(ii) if $\beta = 1$ then for any $k \geqslant k_0$ we have

$$\varphi(k) \leqslant \varphi(k_0) e^{1-\left(\frac{k-k_0}{\tau}\right)^{1-\tilde{\theta}}},$$

where

$$\tau = \max\left\{k_0, (\tilde{c} e 2^{\tilde{\theta}\alpha}(1-\tilde{\theta})^{\alpha})^{\frac{1}{(1-\tilde{\theta})\alpha}}\right\}. \tag{4}$$

(iii) if $\beta < 1$ then for any $k \geqslant k_0$ we have

$$\varphi(k) \leqslant 2^{\frac{\alpha(1-\tilde{\theta})}{(1-\beta)^2}} \left\{\tilde{c}^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha(1-\tilde{\theta})}{1-\beta}} \varphi(k_0)\right\} \left(\frac{1}{k}\right)^{\frac{\alpha(1-\tilde{\theta})}{1-\beta}}.$$

Proof. **(i)** Let $\beta > 1$. We note that the result is nothing but Lemma $2'$.

(ii) Let $\beta = 1$ and τ be as in (4). For any $s \in \mathbb{N}$ we let

$$k_s = k_0 + \tau s^{\frac{1}{1-\tilde{\theta}}},$$

then

$$k_{s+1} - k_s = \tau\left[(s+1)^{\frac{1}{1-\tilde{\theta}}} - s^{\frac{1}{1-\tilde{\theta}}}\right].$$

We use Taylor's formula in order to get

$$k_{s+1} - k_s = \tau\left[\frac{1}{1-\tilde{\theta}} s^{\frac{\tilde{\theta}}{1-\tilde{\theta}}} + \frac{\tilde{\theta}}{2(1-\tilde{\theta})^2} \xi^{\frac{2\tilde{\theta}-1}{1-\tilde{\theta}}}\right] \geqslant \frac{\tau}{1-\tilde{\theta}} s^{\frac{\tilde{\theta}}{1-\tilde{\theta}}}, \tag{5}$$

where ξ lies in the interval between s and $s+1$. (3) with $\beta = 1$ and (5) merge into

$$\varphi(k_{s+1}) \leqslant \frac{c\left[k_0 + \tau s^{\frac{1}{1-\tilde{\theta}}}\right]^{\tilde{\theta}\alpha}}{\left(\frac{\tau}{1-\tilde{\theta}}\right)^{\alpha} s^{\frac{\tilde{\theta}\alpha}{1-\tilde{\theta}}}} \varphi(k_s). \tag{6}$$

(4) implies $k_0 \leqslant \tau \leqslant \tau s^{\frac{1}{1-\tilde{\theta}}}$. From (6) and the definition of τ,

$$\varphi(k_{s+1}) \leqslant \frac{c\left(2\tau s^{\frac{1}{1-\tilde{\theta}}}\right)^{\tilde{\theta}\alpha}}{\left(\frac{\tau}{1-\tilde{\theta}}\right)^{\alpha} s^{\frac{\tilde{\theta}\alpha}{1-\tilde{\theta}}}}\varphi(k_s) = \frac{c(2\tau)^{\tilde{\theta}\alpha}}{\left(\frac{\tau}{1-\tilde{\theta}}\right)^{\alpha}}\varphi(k_s) \leqslant \frac{1}{e}\varphi(k_s).$$

By recursion,

$$\varphi(k_s) \leqslant \frac{1}{e^s}\varphi(k_0).$$

For any $k \geqslant k_0$, there exists $s \in \mathbb{N}$ such that

$$k_0 + \tau(s-1)^{\frac{1}{1-\tilde{\theta}}} \leqslant k < k_0 + \tau s^{\frac{1}{1-\tilde{\theta}}},$$

therefore,

$$\varphi(k) \leqslant \varphi\left(k_0 + \tau(s-1)^{\frac{1}{1-\tilde{\theta}}}\right) = \varphi(k_{s-1}) \leqslant e^{1-s}\varphi(k_0) \leqslant \varphi(k_0)e^{1-\left(\frac{k-k_0}{\tau}\right)^{1-\tilde{\theta}}}.$$

(iii) Let $\beta < 1$. In (3) we let $h = 2k$ and we have

$$\varphi(2k) \leqslant \frac{\tilde{c}}{k^{(1-\tilde{\theta})\alpha}}[\varphi(k)]^{\beta}. \tag{7}$$

We use Remark 1 in [19], which states that (1) (with $\alpha(1-\tilde{\theta})$ in place of α) and (7) are equivalent in the case $0 < \beta < 1$. So that one can use Lemma 1 part (iii) and the desired result follows. □

Let us take a special case $h = 2k$ in (1), it becomes

$$\varphi(2k) \leqslant \frac{\tilde{c}_1}{k^{\alpha}}[\varphi(k)]^{\beta}, \quad \forall h > k \geqslant k_0. \tag{8}$$

In [19], the authors compared the two conditions (1) and (8), and obtained:
(i) for the case $0 < \beta < 1$, (1) and (8) are equivalent;
(ii) for the case $\beta = 1$, (1) is stronger than (8);
(iii) for the case $\beta > 1$, (1) is stronger than (8).
Motivated by [19], we now compare the two conditions (3) and

$$\varphi(2k) \leqslant \frac{\tilde{c}}{k^{(1-\tilde{\theta})\alpha}}[\varphi(k)]^{\beta}, \quad \forall k \geqslant k_0 > 0. \tag{9}$$

Surprisingly, for $0 < \tilde{\theta} < 1$ and $0 < \beta < 1$, (3) and (9) are not equivalent! In fact, we have the following

Lemma 4. Let $\tilde{c}, \alpha, k_0$ be positive constants, $0 < \beta < 1$ and $0 < \tilde{\theta} < 1$. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be decreasing. Then

$$(3) \not\Leftarrow (9). \tag{10}$$

More precisely, the function

$$\varphi(k) = k^{-1}$$

verifies (9) but it does not satisfy (3).

Proof. Let $\alpha, \beta, \tilde{\theta}$ and k_0 be such that

$$\alpha > 1, \tag{11}$$

$$1 - \beta > (1 - \tilde{\theta})\alpha, \tag{12}$$

and

$$k_0 = (2\tilde{c})^{-\frac{1}{1-\beta-(1-\tilde{\theta})\alpha}}. \tag{13}$$

(12) and (13) guarantee that

$$\frac{1}{k^{1-\beta-(1-\tilde{\theta})\alpha}} \leqslant 2\tilde{c}, \quad \forall k \geqslant k_0. \tag{14}$$

We consider the function $\varphi(k) = k^{-1}$. (14) implies

$$\varphi(2k) = \frac{1}{2k} \leqslant \frac{\tilde{c}}{k^{(1-\tilde{\theta})\alpha}} \frac{1}{k^{\beta}} = \frac{\tilde{c}}{k^{(1-\tilde{\theta})\alpha}} [\varphi(k)]^{\beta}, \quad \forall k \geqslant k_0 > 0.$$

This shows that, under the conditions (12)-(13), the function $\varphi(k) = k^{-1}$ satisfies (9). Now we are going to show that (3) does not hold true for $\varphi(k) = k^{-1}$: by contradiction, if (3) would hold true, then

$$\frac{1}{h} \leqslant \frac{\tilde{c}}{(h-k)^{\alpha}} k^{\tilde{\theta}\alpha} \frac{1}{k^{\beta}}, \quad \forall h > k \geqslant k_0 > 0.$$

We take $k = k_0$ and we have

$$\frac{1}{h} \leqslant \frac{\tilde{c} k_0^{\tilde{\theta}\alpha - \beta}}{(h-k_0)^{\alpha}}, \quad \forall h > k_0 > 0,$$

that is

$$\frac{(h-k_0)^{\alpha}}{h} \leqslant \tilde{c} k_0^{\tilde{\theta}\alpha - \beta}, \quad \forall h > k_0 > 0,$$

but this is false, since by (11), $\frac{(h-k_0)^{\alpha}}{h} \to +\infty$ as $h \to +\infty$. □

For the other two cases $\beta = 1$ and $\beta > 1$, we also have

$$(3) \not\Leftarrow (9).$$

In fact, the function

$$\varphi(k) = e^{-(\ln k)^2}, \quad k \in [1, +\infty)$$

satisfies (9) with $\beta = 1$, $\tilde{\theta} = \frac{1}{2}$, $\alpha = 4\ln 2$ and $\tilde{c} = 2^{-\ln 2}$, but is does not satisfy (3) by Lemma 3 part (ii). The function

$$\varphi(k) = e^{-k^p}, \quad p = \log_2(2\beta), \quad k \in [1, +\infty)$$

verifies (9) with some $\beta > 1$, with $\tilde{c} = 1$, with any $0 < \tilde{\theta} < 1$, with any $\alpha > 0$, and with a suitable $k_0 = k_0(\alpha, \beta) \geqslant 1$, but is does not satisfy (3) by Lemma 3 part (iii). The detailed analysis can be found in [19].

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