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# Regularity for minimizing sequences of some variational integrals

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**Abstract** This paper deals with regularity properties for minimizing sequences of some integral functionals related to the nonlinear elasticity theory. Under some structural conditions, we derive that the minimizing sequence and the derivatives of the sequences have some regularity properties by using the Ekeland variational principle.

**Keywords** regularity, minimizing sequences, variational integral, energy, Ekeland variational principle

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## 1 Introduction

This paper deals with regularity properties for minimizing sequences of integral functionals of the type

$$\mathcal{J}(u) = \int_\Omega f(x, Du(x))dx, \tag{1.1}$$

where $\Omega$ ia an open bounded subset of $\mathbb{R}^3$, $u = (u^1, u^2, u^3)^t : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is a vector-valued map, and $Du$ is the $3 \times 3$ Jacobian matrix of its partial derivatives, i.e.,

$$Du = \begin{pmatrix} Du^1 \\ Du^2 \\ Du^3 \end{pmatrix} = \begin{pmatrix} D_1u^1 & D_2u^1 & D_3u^1 \\ D_1u^2 & D_2u^2 & D_3u^2 \\ D_1u^3 & D_2u^3 & D_3u^3 \end{pmatrix}, \quad D_\beta u^\alpha = \frac{\partial u^\alpha}{\partial x_\beta}, \quad \alpha, \beta \in \{1, 2, 3\}.$$

We consider two special classes of (1.1). For the first class, we assume that there exist Carathéodory functions $F : \Omega \times \mathbb{R}^{3\times3} \to \mathbb{R}$, $G : \Omega \times \mathbb{R}^{3\times3} \to \mathbb{R}$ and $H : \Omega \times \mathbb{R} \to \mathbb{R}$, such that

$$f(x, \xi) = F(x, \xi) + G(x, \mathrm{adj}_2\xi) + H(x, \det \xi). \tag{1.2}$$

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For the second class, we assume that there exist Carathéodory functions $F^\alpha : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ ($\alpha = 1, 2, 3$), $G : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ and $H : \Omega \times \mathbb{R} \to \mathbb{R}$, such that

$$f(x,\xi) = \sum_{\alpha=1}^{3} F^\alpha(x, \xi^\alpha) + G(x, \mathrm{adj}_2\xi) + H(x, \det \xi). \tag{1.3}$$

In (1.2) and (1.3), $\det \xi$ is the determinant of the matrix

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} \xi^1_1 & \xi^1_2 & \xi^1_3 \\ \xi^2_1 & \xi^2_2 & \xi^2_3 \\ \xi^3_1 & \xi^3_2 & \xi^3_3 \end{pmatrix}, \quad \xi^\alpha = (\xi^\alpha_1, \xi^\alpha_2, \xi^\alpha_3) \in \mathbb{R}^3 \quad \text{for } \alpha \in \{1,2,3\}, \tag{1.4}$$

and $\mathrm{adj}_2\xi = ((\mathrm{adj}_2\xi)^\gamma_i) \in \mathbb{R}^{3\times 3}$ denotes the adjugate matrix of order 2 whose components are

$$(\mathrm{adj}_2\xi)^\gamma_i = (-1)^{\gamma+i} \det \begin{pmatrix} \xi^\alpha_k & \xi^\alpha_l \\ \xi^\beta_k & \xi^\beta_l \end{pmatrix}, \quad \gamma, i \in \{1,2,3\},$$

where $\alpha, \beta \in \{1,2,3\}\backslash\{\gamma\}$, $\alpha < \beta$ and $k, l \in \{1,2,3\}\backslash\{i\}$, $k < l$.

We remark that in the nonlinear elasticity theory, $\xi$, $\mathrm{adj}_2\xi$ and $\det \xi$ govern the deformations of line, surface and volume, respectively.

For some regularity results of the variational integral (1.1), we refer the reader to Ball [3], Acerbi and Fusco [1], Bauman et al. [4–7] and Dacorogna [17]. Partial regularity results, that is the regularity of solutions up to a set $\Omega_0$ and the study of the properties of the singular set are contained in [12, 19, 24–27, 37, 54, 56]. For the polyconvex case, only few everywhere regularity results are available; we mention [27], where the everywhere continuity was proved in the two-dimensional case, and [26], where Hölder continuity for extremals was dealt with, still in dimension two. Global pointwise bounds can be found in [20, 43, 46–48, 51].

Important contributions in this field follow from the work of Cupini et al. [16], where the authors gave a regularity result for local minimizers $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ of a special class of polyconvex functionals, i.e.,

$$\int_\Omega \left\{ \sum_{\alpha=1}^{3} [F_\alpha(x, Du^\alpha) + G_\alpha(x, (\mathrm{adj}_2 Du)^\alpha)] + H(x, \det Du) \right\} dx. \tag{1.5}$$

Under some structure assumptions on the energy density, the authors proved that local minimizers $u$ are locally bounded. This paper illustrated some ideas and methods which lead to local boundedness for local minimizers of some polyconvex functionals.

In the paper [13], Carozza et al. considered polyconvex functionals of the calculus of variations defined on maps from the three-dimensional Euclidean space into itself, i.e., they assumed that the integrand $f(x,\xi)$ in (1.1) is of the form

$$f(x,\xi) = F(x, |\xi|^2) + G(x, |\mathrm{adj}_2\xi|^2) + H(x, \det \xi). \tag{1.6}$$

Under some conditions on the structure of the functional, the authors proved local boundedness of minimizers.

In the paper [15], Cupini et al. proved local Hölder continuity of vectorial local minimizers of special classes of (1.1) with rank-one and polyconvex integrands

$$f(x,\xi) = \sum_{\alpha=1}^{N} F_\alpha(x, \xi^\alpha) + G(x, \xi). \tag{1.7}$$

The authors assumed that the energy densities in (1.7) satisfy suitable structure assumptions and may have neither the radial nor the quasi-diagonal structure and they obtained some regularity results.

Recently, Gao et al. [32] considered two special cases of the integrand $f(x,s,\xi)$. They assumed the splitting structure on the leading part of $f$:

$$f(x,u,Du)=\sum_{\alpha=1}^{3}F^{\alpha}(x,Du^{\alpha})+\text{lower order terms}. \tag{1.8}$$

Moreover, they assumed anisotropic behavior. Under two special cases of the "lower order terms" and some structural assumptions, they proved that minimizers either are bounded or have suitable integrability properties by using the classical Stampacchia lemma (see Lemma 2.9 below).

We also refer to [52] and the references therein for some aspects of the process/approach to interior regularity of weak solutions to a class of nonlinear elliptic equations in the divergence form, as well as of minimizers of integrals of the calculus of variations.

For the study of minimizing sequences of variational integrals, we mention [9], where some properties of the minimizing sequences for integral functionals are considered, and [46], where pointwise bounds are obtained for suitable minimizing sequences of functionals of the type

$$\int_{\Omega}(|Du|^{p}+h(\det Du))\,dx. \tag{1.9}$$

In both the above mentioned papers, the authors used the Ekeland's $\varepsilon$-variational principle (see Lemma 2.7 below). For some other applications of the Ekeland's $\varepsilon$-variational principle, we refer the reader to Leonetti et al. [45], in which the variational integral of the type (1.9) was considered with $h(t)$ is of logarithmic type. For some other results related to the variational integral (1.9), we refer the reader to [33,44].

In the present paper, we give some regularity properties for minimizing sequences of the variational integral (1.1) with the integrand $f(x,\xi)$ being as (1.2). In the mean time, we give uniform higher integrability for the gradients of minimizing sequences of the variational integral (1.1) with the integrand $f(x,\xi)$ being as (1.3). We mention that unlike [16], we do not make any convexity assumptions on the integrands.

For the first class, we assume that there exist constants

$$k_1,k_3>0,\quad k_2\geqslant 0, \tag{1.10}$$

$$1<p<3, \tag{1.11}$$

$$0<q<\frac{p}{2}, \tag{1.12}$$

$$0<r<\frac{p}{3} \tag{1.13}$$

$$\tag{1.14}$$

and nonnegative functions

$$a(x),b(x),c(x)\in L^{m}(\Omega),\quad m>1, \tag{1.15}$$

such that for almost all $x\in\Omega$, all $\xi\in\mathbb{R}^{3\times3}$ and all $t\in\mathbb{R}$,

$$k_1|\xi|^{p}-k_2\leqslant F(x,\xi)\leqslant k_1|\xi|^{p}+a(x), \tag{1.16}$$

$$-k_1|\xi|^{q}-k_2\leqslant G(x,\xi)\leqslant k_3|\xi|^{q}+b(x), \tag{1.17}$$

$$-k_1|t|^{r}-k_2\leqslant H(x,t)\leqslant k_3|t|^{r}+c(x). \tag{1.18}$$

In (1.16)–(1.18), the norm $|\xi|$ for a matrix $\xi=(\xi^i_j)\in\mathbb{R}^{3\times3}$ is defined by

$$|\xi|=\sum_{i=1}^{3}|\xi^{i}|=\sum_{i,j=1}^{3}|\xi^{i}_{j}|. \tag{1.19}$$

We remark that the norms for the matrix $\xi$ and the vectors $\xi^i$ defined above are different from the ones in [13,15,16,32].

Under the hypotheses (1.10)–(1.18) on $f(x,\xi)$ in (1.2), $\mathscr{J}$ is well defined. Our first theorem is the following theorem.

**Theorem 1.1.** *Consider the variational integral*

$$\mathcal{J}(u)=\int_\Omega\{F(x,Du(x))+G(x,\mathrm{adj}_2 Du(x))+H(x,\det Du(x))\}dx. \tag{1.20}$$

*We assume the conditions* (1.10)–(1.18)*. Let*

$$r_0=\min\left\{\frac{p}{2q},\frac{p-q}{q},\frac{p-r}{2r},m\right\}. \tag{1.21}$$

*Then there exist a minimizing sequence* $\{v_n\}$ *in* $W_0^{1,p}(\Omega)$ *and positive constants* $c_1,c_2$ *and* $c_3$ *depending only on* $p,q,r,m,k_1,k_2,k_3,|\Omega|$ *and* $\|a+b+c\|_{L^m(\Omega)}$*, such that*

(i) $1<r_0<\frac{3}{p}\Rightarrow\|v_n\|_{L^\sigma(\Omega)}\leqslant c_1$, $\sigma=(pr_0)^*$;

(ii) $r_0=\frac{3}{p}\Rightarrow\|\mathrm{e}^{c_2|v_n|}\|_{L^1(\Omega)}\leqslant|\Omega|(1+\frac{\mathrm{e}\pi^2}{6})$;

(iii) $r_0>\frac{3}{p}\Rightarrow\|v_n\|_{L^\infty(\Omega)}\leqslant c_3$.

We remark that in the assumption (1.16), we have the same constant $k_1$ on the left-hand side and the right-hand side: this is a structure condition that keeps away De Giorgi's counterexample [18] and that allows for $L^\infty$ estimates contained in Theorem 1.1(iii). We use the same constant $k_1$ from below and from above in the proof of Theorem 1.1 when writing the left-hand side of (3.8). We remark that solutions to vectorial problems might be irregular (see the surveys [53] and [42]).

We remark also that in (1.5), (1.7) and (1.8), the leading part of $f$ has splitting form. In [13], the integrand is of the form (1.6) with the functions $F(x,t)$, $G(x,t)$ and $H(x,s)$ satisfying some restrictive conditions (see [13, Theorem 2.2]). In Theorem 1.1 of the present paper, we assume that the integrand $f(x,\xi)$ is as in (1.2), and this kind of integrand seems to be more general. Meanwhile, our assumptions (1.10)–(1.18) seem to be weaker than the formers.

We have the following corollary.

**Corollary 1.2.** *Let*

$$\mathcal{J}(u)=\int_\Omega\{|Du(x)|^p+|\mathrm{adj}_2 Du(x)|^q+|\det Du(x)|^r\}dx \tag{1.22}$$

*with* $p$, $q$ *and* $r$ *satisfying* (1.11)–(1.13)*, respectively. Let*

$$r_0'=\min\left\{\frac{p}{2q},\frac{p-q}{q},\frac{p-r}{2r}\right\}.$$

*Then there exist a minimizing sequence* $\{v_n\}$ *in* $W_0^{1,p}(\Omega)$ *and positive constants* $c_1'$, $c_2'$ *and* $c_3'$*, depending only on* $p,q,r$ *and* $|\Omega|$*, such that*

(i) $1<r_0'<\frac{3}{p}\Rightarrow\|v_n\|_{L^{\sigma'}(\Omega)}\leqslant c_1'$, $\sigma'=(pr_0')^*$;

(ii) $r_0'=\frac{3}{p}\Rightarrow\|\mathrm{e}^{c_2'|v_n|}\|_{L^1(\Omega)}\leqslant|\Omega|(1+\frac{\mathrm{e}\pi^2}{6})$;

(iii) $r_0'>\frac{3}{p}\Rightarrow\|v_n\|_{L^\infty(\Omega)}\leqslant c_3'$.

For the second class, i.e., the integrand $f(x,\xi)$ is as in (1.3), we assume (1.10)–(1.15) and (1.17)–(1.18). Instead of (1.16), we assume that for almost all $x\in\Omega$ and all $\eta\in\mathbb{R}^3$,

$$k_1|\eta|^p-k_2\leqslant F^\alpha(x,\eta)\leqslant k_3|\eta|^p+a(x),\quad \alpha=1,2,3. \tag{1.15$'$}$$

Under the above-mentioned conditions, $\mathcal{J}$ is well defined.

Our second theorem is as follows.

**Theorem 1.3.** *Let* $\Omega$ *be a regular domain. Consider the variational integral*

$$\mathcal{J}(u)=\int_\Omega\left\{\sum_{\alpha=1}^3F^\alpha(x,Du^\alpha(x))+G(x,\mathrm{adj}_2 Du(x))+H(x,\det Du(x))\right\}dx. \tag{1.23}$$

*Under the assumptions* (1.10)–(1.15), (1.15)$'$ *and* (1.17)–(1.18)*, then there exist a minimizing sequence* $\{v_n\}$ *in* $W_0^{1,p}(\Omega)$ *and constants* $\delta,c_4>0$*, depending only on* $p,q,r,m,k_1,k_2,k_3,|\Omega|$ *and* $\|a+b+c\|_{L^m(\Omega)}$*, such that*

$$\|\nabla v_n\|_{L^{p+\delta}(\Omega)}\leqslant c_4.$$

Recall that a bounded open set $\Omega \subset \mathbb{R}^N$ is called *regular* if there exist $R_0 > 0$ and $0 < \theta_0 < 1$, such that for any $x_0 \in \partial\Omega$ and for any $0 < R < R_0$,

$$|B_R(x_0) \setminus \bar{\Omega}| \geqslant \theta_0 \omega_N R^N,$$

where $B_R(x_0)$ is the ball centered at $x_0$ with radius $R$ and $\omega_N = |B_1|$.

**Corollary 1.4.** *Let $\Omega$ be a regular domain and*

$$\mathcal{J}(u) = \int_\Omega \left\{ \sum_{\alpha=1}^{3} |Du^\alpha(x)|^p + |\mathrm{adj}_2 Du(x)|^q + |\det Du(x)|^r \right\} dx$$

*with $p$, $q$ and $r$ satisfying* (1.11)–(1.13)*, respectively. Then there exist a minimizing sequence $\{v_n\}$ in $W_0^{1,p}(\Omega)$ and constants $\delta', c_4' > 0$, depending only on $p, q, r$ and $|\Omega|$, such that*

$$\|\nabla v_n\|_{L^{p+\delta'}(\Omega)} \leqslant c_4'.$$

## 2 Preliminary results

This section is devoted to preliminary lemmas used in the proof of Theorems 1.1 and 1.3.

For the matrix (1.4) with the norm defined in (1.19), we have the following lemma.

**Lemma 2.1.** *Consider the matrix* (1.4)*. The following hold*:

(i) $|\xi| = |\xi^1| + |\xi^2| + |\xi^3|$;

(ii) $|(\mathrm{adj}_2\xi)^\alpha| \leqslant |\xi^\beta||\xi^\gamma|$, $\alpha \in \{1, 2, 3\}$, $\beta, \gamma = \{1, 2, 3\}\backslash\{\alpha\}$, $\beta < \gamma$;

(iii) $|\mathrm{adj}_2\xi| \leqslant |\xi|^2$;

(iv) $|\det \xi| \leqslant |\xi^1||\xi^2||\xi^3| \leqslant |\xi|^3$.

*Proof.* (i) The result is obvious by the definition (1.19).

(ii) For $\alpha \in \{1, 2, 3\}$, $\beta, \gamma = \{1, 2, 3\}\backslash\{\alpha\}$ and $\beta < \gamma$,

$$\begin{aligned}
|(\mathrm{adj}_2\xi)^\alpha| &= |(\mathrm{adj}\xi)_1^\alpha| + |(\mathrm{adj}\xi)_2^\alpha| + |(\mathrm{adj}\xi)_3^\alpha| \\
&= |\xi_2^\beta\xi_3^\gamma - \xi_2^\gamma\xi_3^\beta| + |\xi_1^\beta\xi_3^\gamma - \xi_1^\gamma\xi_3^\beta| + |\xi_1^\beta\xi_2^\gamma - \xi_1^\gamma\xi_2^\beta| \\
&\leqslant |\xi_2^\beta\xi_3^\gamma| + |\xi_2^\gamma\xi_3^\beta| + |\xi_1^\beta\xi_3^\gamma| + |\xi_1^\gamma\xi_3^\beta| + |\xi_1^\beta\xi_2^\gamma| + |\xi_1^\gamma\xi_2^\beta| \\
&\leqslant \sum_{i,j=1}^{3} |\xi_i^\beta||\xi_j^\gamma| \\
&= |\xi^\beta||\xi^\gamma|.
\end{aligned}$$

(iii) By (i) and (ii), one has

$$\begin{aligned}
|\mathrm{adj}_2\xi| &= |(\mathrm{adj}\xi)^1| + |(\mathrm{adj}_2\xi)^2| + |(\mathrm{adj}\xi)^3| \\
&\leqslant |\xi^2||\xi^3| + |\xi^1||\xi^3| + |\xi^1||\xi^2| \\
&\leqslant (|\xi^1| + |\xi^2| + |\xi^3|)(|\xi^1| + |\xi^2| + |\xi^3|) \\
&= |\xi|^2.
\end{aligned}$$

(iv) By the definition (1.19), we use (ii) to obtain

$$\begin{aligned}
|\det \xi| &= \left| \sum_{j=1}^{3} \xi_j^1 (\mathrm{adj}_2\xi)_j^1 \right| \leqslant \sum_{j=1}^{3} |\xi_j^1||(\mathrm{adj}_2\xi)_j^1| \\
&\leqslant |\xi^1| \sum_{j=1}^{3} |(\mathrm{adj}_2\xi)_j^1| = |\xi^1||(\mathrm{adj}_2\xi)^1| \leqslant |\xi^1||\xi^2||\xi^3| \leqslant |\xi|^3.
\end{aligned}$$

This completes the proof. $\square$

The next lemma illustrates that the integrand $f(x,\xi)$ defined in (1.2) is $p$-coercive and bounded from below.

**Lemma 2.2.** *Under the assumptions* (1.10)–(1.18)*, there exists a positive constant* $M_1$*, depending only on* $p,q,r,k_1$ *and* $k_2$*, such that for any* $\xi\in\mathbb{R}^{3\times 3}$*,*

$$f(x,\xi)=F(x,\xi)+G(x,\mathrm{adj}_2\xi)+H(x,\det\xi)\geqslant\frac{k_1}{2}|\xi|^p-M_1\geqslant -M_1.$$

*Proof.* We use (1.16)–(1.18) and the results (iii) and (iv) in Lemma 2.1 in order to get

$$\begin{aligned}f(x,\xi)&=F(x,\xi)+G(x,\mathrm{adj}_2\xi)+H(x,\det\xi)\\&\geqslant k_1|\xi|^p-k_1|\mathrm{adj}_2\xi|^q-k_1|\det\xi|^r-3k_2\\&\geqslant k_1|\xi|^p-k_1|\xi|^{2q}-k_1|\xi|^{3r}-3k_2.\end{aligned}\tag{2.1}$$

The conditions $0<q<\frac{p}{2}$ and $0<r<\frac{p}{3}$ in (1.12) and (1.13) allow us to use Young's inequality

$$ab\leqslant\varepsilon\frac{a^t}{t}+\varepsilon^{-\frac{t'}{t}}\frac{b^{t'}}{t'},\quad a,b>0,\quad t>1,\quad \frac{1}{t}+\frac{1}{t'}=1,\quad \varepsilon>0\tag{2.2}$$

to derive (take $a=|\xi|^{2q}$, $b=k_1$, $\varepsilon=\frac{k_1}{4}$, $t=\frac{p}{2q}$ and $t'=\frac{p}{p-2q}$)

$$k_1|\xi|^{2q}\leqslant\frac{k_1}{4}\frac{2q}{p}|\xi|^p+\left(\frac{k_1}{4}\right)^{-\frac{2q}{p-2q}}\frac{p-2q}{p}k_1^{\frac{p}{p-2q}}\tag{2.3}$$

and (take $a=|\xi|^{3r}$, $b=k_1$, $\varepsilon=\frac{k_1}{4}$, $t=\frac{p}{3r}$ and $t'=\frac{p}{p-3r}$)

$$k_1|\xi|^{3r}\leqslant\frac{k_1}{4}\frac{3r}{p}|\xi|^p+\left(\frac{k_1}{4}\right)^{-\frac{3r}{p-3r}}\frac{p-3r}{p}k_1^{\frac{p}{p-3r}}.\tag{2.4}$$

Substituting (2.3) and (2.4) into (2.1) and noticing

$$\frac{k_1}{4}\left(\frac{2q}{p}+\frac{3r}{p}\right)<\frac{k_1}{2},$$

we have

$$\begin{aligned}f(x,\xi)&\geqslant\frac{k_1}{2}|\xi|^p-\left(\frac{k_1}{4}\right)^{-\frac{2q}{p-2q}}\frac{p-2q}{p}k_1^{\frac{p}{p-2q}}-\left(\frac{k_1}{4}\right)^{-\frac{3r}{p-3r}}\frac{p-3r}{p}k_1^{\frac{p}{p-3r}}-3k_2\\&=\frac{k_1}{2}|\xi|^p-M_1\geqslant -M_1,\end{aligned}$$

where

$$M_1=\left(\frac{p-2q}{p}4^{\frac{2q}{p-2q}}+\frac{p-3r}{p}4^{\frac{3r}{p-3r}}\right)k_1+3k_2.$$

This completes the proof. □

Analogously, under the assumptions (1.10)–(1.15), (1.15)$'$ and (1.17)–(1.18), the integrand $f(x,\xi)$ defined in (1.3) is $p$-coercive and bounded from below.

**Lemma 2.3.** *Under the assumptions* (1.10)–(1.15), (1.15)$'$ *and* (1.17)–(1.18)*, there exists a positive constant* $M_2$*, depending only on* $p,q,r,k_1$ *and* $k_2$*, such that for any* $\xi\in\mathbb{R}^{3\times 3}$*,*

$$f(x,\xi)=\sum_{\alpha=1}^{3}F^\alpha(x,\xi^\alpha)+G(x,\mathrm{adj}_2\xi)+H(x,\det\xi)\geqslant\frac{k_1}{2^{2p-1}}|\xi|^p-M_2\geqslant -M_2.$$

*Proof.* We use (1.15)′, (1.17), (1.18) and the results (iii) and (iv) in Lemma 2.1 in order to get

$$\begin{aligned}
f(x,\xi) &= \sum_{\alpha=1}^{3} F^\alpha(x,\xi^\alpha) + G(x,\mathrm{adj}_2\xi) + H(x,\det\xi) \\
&\geqslant k_1\sum_{\alpha=1}^{3}|\xi^\alpha|^p - k_1|\mathrm{adj}_2\xi|^q - k_1|\det\xi|^r - 5k_2 \\
&\geqslant k_1 4^{1-p}\left(\sum_{\alpha=1}^{3}|\xi^\alpha|\right)^p - k_1|\xi|^{2q} - k_1|\xi|^{3r} - 5k_2 \\
&= k_1 4^{1-p}|\xi|^p - k_1|\xi|^{2q} - k_1|\xi|^{3r} - 5k_2. \qquad (2.5)
\end{aligned}$$

The conditions $0 < q < \frac{p}{2}$ and $0 < r < \frac{p}{3}$ in (1.12) and (1.13) allow us to use Young's inequality (2.2) to derive (take $a = |\xi|^{2q}$, $b = k_1$, $\varepsilon = \frac{k_1}{4^p}$, $t = \frac{p}{2q}$ and $t' = \frac{p}{p-2q}$)

$$k_1|\xi|^{2q} \leqslant \frac{k_1}{4^p}\frac{2q}{p}|\xi|^p + \left(\frac{k_1}{4^p}\right)^{-\frac{2q}{p-2q}}\frac{p-2q}{p}k_1^{\frac{p}{p-2q}} \qquad (2.6)$$

and (take $a = |\xi|^{3r}$, $b = k_1$, $\varepsilon = \frac{k_1}{4^p}$, $t = \frac{p}{3r}$ and $t' = \frac{p}{p-3r}$)

$$k_1|\xi|^{3r} \leqslant \frac{k_1}{4^p}\frac{3r}{p}|\xi|^p + \left(\frac{k_1}{4^p}\right)^{-\frac{3r}{p-3r}}\frac{p-3r}{p}k_1^{\frac{p}{p-3r}}. \qquad (2.7)$$

Substituting (2.6) and (2.7) into (2.5) and noticing

$$\frac{k_1}{4^p}\left(\frac{2q}{p} + \frac{3r}{p}\right) < \frac{2k_1}{4^p},$$

we have

$$\begin{aligned}
f(x,\xi) &\geqslant \frac{2k_1}{4^p}|\xi|^p - \left(\frac{k_1}{4^p}\right)^{-\frac{2q}{p-2q}}\frac{p-2q}{p}k_1^{\frac{p}{p-2q}} - \left(\frac{k_1}{4^p}\right)^{-\frac{3r}{p-3r}}\frac{p-3r}{p}k_1^{\frac{p}{p-3r}} - 5k_2 \\
&= \frac{k_1}{2^{2p-1}}|\xi|^p - M_2 \geqslant -M_2,
\end{aligned}$$

where

$$M_2 = \left(\frac{p-2q}{p}4^{\frac{2pq}{p-2q}} + \frac{p-3r}{p}4^{\frac{3pr}{p-3r}}\right)k_1 + 5k_2.$$

This completes the proof. □

We recall Fatou Lemma, which can be found, for example, on page 23 in [8].

**Lemma 2.4.** *Let $f_n$ be a sequence of $L^1(E)$ functions such that*

(i) $f_n \geqslant 0$ *a.e. in* $E$;

(ii) $\int_E f_n(x)dx < +\infty$ *for every* $n \in \mathbb{N}$.

*Let*

$$f(x) = \liminf_{n\to\infty} f_n(x) \quad \textit{for a.e. } x \in E.$$

*Then*

$$\int_E f(x)dx \leqslant \liminf_{n\to\infty}\int_E f_n(x)dx.$$

We now prove the following lemma.

**Lemma 2.5.** *Consider the functional*

$$\mathcal{J}(u) = \int_\Omega \{F(x,Du(x)) + G(x,\mathrm{adj}_2 Du(x)) + H(x,\det Du(x))\}dx. \qquad (2.8)$$

*Let the functions* $F(x,\xi)$, $G(x,\xi)$ *and* $H(x,t)$ *satisfy the assumptions* (1.16)–(1.18), *respectively. Define*

$$V = \{w : \Omega \to R^3 \ \textit{such that}\ w \in W_0^{1,1}(\Omega;\mathbb{R}^3)\} \tag{2.9}$$

*and set*

$$d(w,v) = \|Dw - Dv\|_{L^1(\Omega)} \tag{2.10}$$

*for every* $w, v \in V$. *Then* $(V,d)$ *is a complete metric space and* $\mathcal{J}$ *is lower semicontinuous with respect to* $d$.

*Proof.* By the definition $d(w,v)$ in (2.10), one easily check that $d$ is a distance on $V$, and one should only note that $d(w,v) = 0$ implies $Dw = Dv$ a.e. $\Omega$, since $w = v = 0$ on $\partial\Omega$. Poincaré's inequality

$$\mathcal{S}\|u\|_{L^{p^*}(\Omega)} \leqslant \|Du\|_{L^p(\Omega)}, \quad \forall\, u \in W_0^{1,p}(\Omega;\mathbb{R}^N), \quad \mathcal{S} = \mathcal{S}(N,p) \tag{2.11}$$

with $p = 1$ gives us $w = v$ in $\Omega$. The completeness of $(V,d)$ is obvious.

Let $\{w_k\}_k \subset V$ be converging to $w_\infty \in V$ with respect to $d$ so that

$$Dw_k \to Dw_\infty \quad \text{in } L^1(\Omega).$$

Then there exists a subsequence $\{w_{s_k}\}_k$ such that

$$Dw_{s_k}(x) \to Dw_\infty(x)$$

for almost every $x \in \Omega$. Continuity of $\xi \mapsto f(x,\xi)$ implies

$$f(x, Dw_{s_k}(x)) \to f(x, Dw_\infty(x)) \quad \text{a.e. } x \in \Omega,$$

and then

$$f(x, Dw_{s_k}(x)) + M_1 \to f(x, Dw_\infty(x)) + M_1 \quad \text{a.e. } x \in \Omega,$$

where $M_1$ is the constant in Lemma 2.2. Such a pointwise convergence together with the facts that $f(x, Dw_{r_k}(x)) + M_1$ and $f(x, Dw_\infty(x)) + M_1$ are nonnegative, allows us to use the Fatou lemma (see Lemma 2.4), and we get

$$\mathcal{J}(w_\infty) \leqslant \liminf_{k\to\infty} \mathcal{J}(w_{s_k}) = \liminf_{k\to\infty} \mathcal{J}(w_k).$$

This means that $\mathcal{J}$ is lower semicontinuous with respect to $d$. □

Analogously, one has the following lemma.

**Lemma 2.6.** *Consider the functional*

$$\mathcal{J}(u) = \int_\Omega \left\{ \sum_{\alpha=1}^{3} F^\alpha(x, Du^\alpha(x)) + G(x, \mathrm{adj}_2 Du(x)) + H(x, \det Du(x)) \right\} dx. \tag{2.12}$$

*We assume* (1.10)–(1.15), (1.15)′ *and* (1.17)–(1.18). *Define*

$$V = \{w : \Omega \to R^3 \ \textit{such that}\ w \in W_0^{1,1}(\Omega;\mathbb{R}^3)\}, \tag{2.13}$$

*and set*

$$d(w,v) = \|Dw - Dv\|_{L^1(\Omega)} \tag{2.14}$$

*for every* $w, v \in V$. *Then* $(V,d)$ *is a complete metric space and* $\mathcal{J}$ *is lower semicontinuous with respect to* $d$.

Let us recall the Ekeland $\varepsilon$-variational principle (see [21–23, 28, 36]).

**Lemma 2.7.** *Let $(V, d)$ be a complete metric space and let $\mathcal{F} : V \to (-\infty, +\infty]$ be a lower semicontinous function such that $\inf_V \mathcal{F}$ is finite. Let $\varepsilon > 0$ and $u \in V$ be such that*

$$\mathcal{F}(u) \leqslant \inf_{v \in V} \mathcal{F}(v) + \varepsilon. \tag{2.15}$$

*Then there exists a $v \in V$ such that*

(i) $d(u, v) \leqslant 1$;

(ii) $\mathcal{F}(v) \leqslant \mathcal{F}(u)$;

(iii) *$v$ minimizes the functional $\mathcal{G}(w) = \mathcal{F}(w) + \varepsilon d(v, w)$.*

For some applications of the Ekeland $\varepsilon$-variational principle, we refer the reader to [8, 9, 17, 45, 46].

In the proof of Theorem 1.1, we need the following lemma (see [9, Lemma 2.3]).

**Lemma 2.8.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set; let $v$ be a function in $W_0^{1,p}(\Omega)$ with $1 < p < N$; let $\psi_0$, $\psi_1$ and $\psi_2$ be nonnegative measurable functions, and $\gamma_1$ and $\gamma_2$ be real numbers such that*

$$\begin{cases} \psi_0 \in L^{r_0}(\Omega), \quad 1 < r_0 < \dfrac{N}{p}, \\ \psi_1 \in L^{r_1}(\Omega), \quad r_1 > \dfrac{N}{p}, \quad 0 \leqslant \gamma_1 < p^* \dfrac{r_1 - 1}{r_1}, \\ \psi_2 \in L^{r_2}(\Omega), \quad r_2 > N, \quad 0 \leqslant \gamma_2 < \dfrac{N}{N-1} \dfrac{r_2 - 1}{r_2}. \end{cases} \tag{2.16}$$

*Suppose that for every $k > 0$,*

$$\int_{A_k} |\nabla v|^p dx \leqslant \int_{A_k} [\psi_0 + \psi_1 |v|^{\gamma_1} + \psi_2 |v|^{\gamma_2}] dx, \tag{2.17}$$

*where $A_k = \{x \in \Omega : |v(x)| > k\}$. Then there exists a positive constant $c_5$, depending on the various parameters and on the $W_0^{1,p}(\Omega)$ norm of $v$, such that*

$$\|v\|_{L^\sigma(\Omega)} \leqslant c_5, \quad \sigma = (pr_0)^*.$$

The following well-known Stampacchia lemma is from [55, Lemma 4.1].

**Lemma 2.9.** *Let $c_6$, $\alpha$ and $\beta$ be positive constants and $k_0$ a real number. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be decreasing and such that*

$$\varphi(h) \leqslant \frac{c_6}{(h-k)^\alpha} [\varphi(k)]^\beta \tag{2.18}$$

*for every $h$ and $k$ with $h > k \geqslant k_0$. It results that*

(i) *if $\beta > 1$, then*

$$\varphi(k_0 + d) = 0,$$

*where*

$$d^\alpha = c_6 [\varphi(k_0)]^{\beta - 1} 2^{\frac{\alpha\beta}{\beta - 1}};$$

(ii) *if $\beta = 1$, then for any $k \geqslant k_0$,*

$$\varphi(k) \leqslant \varphi(k_0) \mathrm{e}^{1 - (c_6 \mathrm{e})^{-\frac{1}{\alpha}} (k - k_0)};$$

(iii) *if $\beta < 1$ and $k_0 > 0$, then for any $k \geqslant k_0$,*

$$\varphi(k) \leqslant 2^{\frac{\alpha}{(1-\beta)^2}} \{c_6^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} \varphi(k_0)\} \left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}}.$$

Stampacchia lemma is an efficient tool in dealing with regularity issues of minima of variational integrals as well as solutions to elliptic equations and systems. This lemma has also been generalized in many respects, and we refer to [29, 31, 39, 41] for more details. For some other results related to Lemma 2.9, we refer to [2, 8, 10, 11, 14, 29, 30, 35] and [31, 34, 38, 40, 49, 50].

In the proof of Theorem 1.3, we need the following lemma (see [9, Lemma 2.6]).

**Lemma 2.10.** *Let $\Omega \subset \mathbb{R}^N$ be a regular bounded, open set; let $v$ be a function in $W_0^{1,p}(\Omega)$ with $1 < p < N$, $\psi_0$, $\psi_1$ and $\psi_2$ be nonnegative measurable functions, and $\gamma_1$ and $\gamma_2$ be real numbers such that (2.16) holds. Let us assume that there exists a $Q \geqslant 1$ such that for every $\varphi \in W_0^{1,p}(\Omega)$,*

$$\int_{\mathrm{supp}\varphi} |\nabla v|^p dx \leqslant Q \int_{\mathrm{supp}\varphi} \{|\nabla v + \nabla\varphi|^p + \psi_0 + \psi_1(|v| + |\varphi|)^{\gamma_1} + \psi_2(|v| + |\varphi|)^{\gamma_2}\}dx.$$

*Then there exist $\delta > 0$ and $c_7 > 0$, depending on the various parameters and on $\|v\|_{W_0^{1,p}}$, such that $\nabla v \in L^{p+\delta}(\Omega)$, and*

$$\|\nabla v\|_{L^{p+\delta}(\Omega)} \leqslant c_7(\|\nabla v\|_{L^p(\Omega)} + \|\psi_1\|_{L^{r_1}(\Omega)} + \|\psi_2\|_{L^{r_2}(\Omega)} + \|\psi_0\|_{L^{r_0}(\Omega)} + 1).$$

## 3 Proofs of Theorems 1.1 and 1.3

This section is devoted to the proof of the main theorems.

For convenience of the reader, we outline the scheme of the proofs of Theorems 1.1 and 1.3. For $\mathcal{F} = \mathcal{J}$ and $\{u_n\} \subset W_0^{1,p}(\Omega)$ any minimizing sequence of variational functionals (1.20) or (1.23), the previous Lemmas 2.5 and 2.2 allow us to use the Ekeland $\varepsilon$-variational principle (see Lemma 2.7) to derive that there exists another minimizing sequence $\{v_n\} \subset V$ satisfying some sorts of variational inequalities. For such a sequence, we use an appropriate test function, Lemma 2.8 or Lemma 2.10, and the Stampacchia lemma (see Lemma 2.9), and we derive the desired results.

*Proof of Theorem* 1.1. We let $V$ be as in (2.9) and the distance $d$ be given by (2.10). Let $\mathcal{F} = \mathcal{J}$ and $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a minimizing sequence of $\mathcal{J}$, i.e.,

$$\mathcal{J}(u_n) \to \inf_V \mathcal{J}(u) \quad \text{as } n \to +\infty. \tag{3.1}$$

Assume that $\mathcal{J}(u_n) > \inf_V \mathcal{J}(u)$ for every $n \in \mathbb{N}$. We set

$$\varepsilon_n = \mathcal{J}(u_n) - \inf_V \mathcal{J}(u),$$

where $\{\varepsilon_n\}$ is a sequence of positive real numbers, converging to zero. Then

$$\mathcal{J}(u_n) \leqslant \inf_V \mathcal{J}(u) + \varepsilon_n. \tag{3.2}$$

It is no loss of generality to assume $\varepsilon_n \leqslant 1$. Lemmas 2.5 and 2.2 tell us that $(V, d)$ is a complete metric space, $\mathcal{J}$ is sequentially lower semicontinuous with respect to $d$, and $\inf_V \mathcal{J}$ is finite: $\inf_V \mathcal{J} \geqslant -M_1|\Omega|$. We use the Ekeland $\varepsilon$-variational principle (see Lemma 2.7) and we derive that there exists a sequence $\{v_n\} \subset V$ such that

$$\int_\Omega |Dv_n - Du_n| dx \leqslant 1, \tag{3.3}$$

$$\mathcal{J}(v_n) \leqslant \mathcal{J}(u_n) \tag{3.4}$$

and

$$\mathcal{J}(v_n) \leqslant \mathcal{J}(w) + \varepsilon_n \int_\Omega |Dv_n - Dw| dx \quad \text{for every } w \in V. \tag{3.5}$$

(3.4) together with (3.1) implies

$$\mathcal{J}(v_n) \to \inf_V \mathcal{J}(u) \quad \text{as } n \to +\infty,$$

which means that $\{v_n\}$ is a minimizing sequence.

We now prove that such a minimizing sequence $\{v_n\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$, i.e., there exists a constant $c_8$, depending only on $p, q, r, k_1, k_2$, $|\Omega|$ and $\|a + b + c\|_{L^m(\Omega)}$, such that

$$\|v_n\|_{W_0^{1,p}(\Omega)} \leqslant c_8. \tag{3.6}$$

In fact, Lemma 2.2 tells us that $f(x,\xi)$ is $p$-coercive:

$$f(x, Dv_n) + M_1 \geqslant \frac{k_1}{2}|Dv_n|^p,$$

from which, and considering the facts (3.4), (3.2) and $\varepsilon_n \leqslant 1$, we derive

$$\begin{aligned}\int_\Omega |Dv_n|^p dx &\leqslant \frac{2}{k_1}(\mathcal{J}(v_n) + M_1|\Omega|) \leqslant \frac{2}{k_1}(\mathcal{J}(u_n) + M_1|\Omega|)\\ &\leqslant \frac{2}{k_1}\Big(\inf_V \mathcal{J}(u) + 1 + M_1|\Omega|\Big) < +\infty.\end{aligned}$$

(3.6) is proved.

Now, for any fixed $n \in \mathbb{N}$ and any $k > 0$ we take

$$w = \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix} = \begin{pmatrix} T_k(v_n^1) \\ v_n^2 \\ v_n^3 \end{pmatrix} \in V,$$

where $T_k(v_n^1)$ is the truncation function of $v_n^1$ at level $k$, i.e.,

$$T_k(v_n^1) = \max\{-k, \min\{v_n^1, k\}\}.$$

We use the above $w$ in (3.5) and we have

$$\begin{aligned}&\int_\Omega \{F(x, Dv_n) + G(x, \mathrm{adj}_2 Dv_n) + H(x, \det Dv_n)\}dx\\ &\quad\leqslant \int_\Omega \{F(x, Dw) + G(x, \mathrm{adj}_2 Dw) + H(x, \det Dw)\}dx + \varepsilon_n \int_\Omega |Dv_n - Dw|dx. \end{aligned}\tag{3.7}$$

Let us define the superlevel set

$$A^1_{n,k} = \{x \in \Omega : |v_n^1(x)| > k\}.$$

Since $Dw = Dv_n$ in $\Omega\backslash A^1_{n,k}$, (3.7) yields

$$\begin{aligned}&\int_{A^1_{n,k}} \{F(x, Dv_n) + G(x, \mathrm{adj}_2 Dv_n) + H(x, \det Dv_n)\}dx\\ &\quad\leqslant \int_{A^1_{n,k}} \{F(x, Dw) + G(x, \mathrm{adj}_2 Dw) + H(x, \det Dw)\}dx + \varepsilon_n \int_{A^1_{n,k}} |Dv_n - Dw|dx.\end{aligned}$$

(1.16)–(1.18) merge into

$$\begin{aligned}&k_1 \int_{A^1_{n,k}} |Dv_n|^p dx - k_1 \int_{A^1_{n,k}} |\mathrm{adj}_2 Dv_n|^q dx - k_1 \int_{A^1_{n,k}} |\det Dv_n|^r dx - 3\int_{A^1_{n,k}} k_2 dx\\ &\quad\leqslant k_1 \int_{A^1_{n,k}} |Dw|^p dx + k_3 \int_{A^1_{n,k}} |\mathrm{adj}_2 Dw|^q dx + k_3 \int_{A^1_{n,k}} |\det Dw|^r dx\\ &\qquad + \int_{A^1_{n,k}} (a(x) + b(x) + c(x))dx + \varepsilon_n \int_{A^1_{n,k}} |Dv_n - Dw|dx,\end{aligned}$$

from which we derive

$$\begin{aligned}&k_1 \int_{A^1_{n,k}} (|Dv_n|^p - |Dw|^p)dx\\ &\quad\leqslant k_1 \int_{A^1_{n,k}} |\mathrm{adj}_2 Dv_n|^q dx + k_3 \int_{A^1_{n,k}} |\mathrm{adj}_2 Dw|^q dx\end{aligned}$$

$$
\begin{aligned}
&+k_1\int_{A^1_{n,k}}|\det Dv_n|^r dx+k_3\int_{A^1_{n,k}}|\det Dw|^r dx\\
&+\int_{A^1_{n,k}}|Dv_n-Dw|dx+\int_{A^1_{n,k}}(a(x)+b(x)+c(x)+3k_2)dx, \qquad (3.8)
\end{aligned}
$$

where we have used again $\varepsilon_n \leqslant 1$. Since

$$
Dw=\begin{pmatrix} Dv_n^1\cdot 1_{\Omega\setminus A^1_{n,k}}\\ Dv_n^2\\ Dv_n^3\end{pmatrix},
$$

where $1_E(x)$ is the characteristic function of the set $E$, i.e., $1_E(x)=1$ if $x\in E$ and $1_E(x)=0$ if $x\notin E$, we obtain that for $x\in A^1_{n,k}$,

$$
Dw=\begin{pmatrix}0\\ Dv_n^2\\ Dv_n^3\end{pmatrix}=\begin{pmatrix}0&0&0\\ D_1v_n^2&D_2v_n^2&D_3v_n^2\\ D_1v_n^3&D_2v_n^3&D_3v_n^3\end{pmatrix}, \qquad (3.9)
$$

$$
\mathrm{adj}_2 Dw=\begin{pmatrix}(\mathrm{adj}_2 Dv_n)^1\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}(\mathrm{adj}_2 Dv_n)^1_1&(\mathrm{adj}_2 Dv_n)^1_2&(\mathrm{adj}_2 Dv_n)^1_3\\ 0&0&0\\ 0&0&0\end{pmatrix} \qquad (3.10)
$$

and

$$
\det Dw=0. \qquad (3.11)
$$

From (3.9), we obtain

$$
|Dw|=|Dv_n^2|+|Dv_n^3|.
$$

We apply a basic inequality

$$
a^p+b^p\leqslant (a+b)^p,\quad \forall\, a,b\geqslant 0,\quad \forall\, p\geqslant 1
$$

to obtain

$$
|Dv_n^1|^p\leqslant |Dv_n|^p-|Dw|^p. \qquad (3.12)
$$

From (3.10), we obtain

$$
|\mathrm{adj}_2 Dw|=|(\mathrm{adj}_2 Dv_n)^1|. \qquad (3.13)
$$

Substituting (3.12), (3.13), (3.11) and (3.9) into (3.8), we arrive at

$$
\begin{aligned}
&k_1\int_{A^1_{n,k}}|Dv_n^1|^p dx\\
&\leqslant k_1\int_{A^1_{n,k}}|\mathrm{adj}_2 Dv_n|^q dx+k_3\int_{A^1_{n,k}}|(\mathrm{adj}_2 Dv_n)^1|^q dx\\
&\quad +k_1\int_{A^1_{n,k}}|\det Dv_n|^r dx+\int_{A^1_{n,k}}|Dv_n^1|dx+\int_{A^1_{n,k}}(a(x)+b(x)+c(x)+3k_2)dx\\
&=: k_1I_1+k_3I_2+k_1I_3+I_4+I_5. \qquad (3.14)
\end{aligned}
$$

Our nearest goal is to estimate $I_i$, $i=1,2,3,4,5$.

We use (i) and (ii) of Lemma 2.1 and Young's inequality in order to get

$$
I_1=\int_{A^1_{n,k}}|\mathrm{adj}_2 Dv_n|^q dx
$$

$$
\begin{aligned}
&= \int_{A^1_{n,k}} (|(\mathrm{adj}_2 Dv_n)^1| + |(\mathrm{adj}_2 Dv_n)^2| + |(\mathrm{adj}_2 Dv_n)^3|)^q dx \\
&\leqslant \int_{A^1_{n,k}} (|Dv_n^2||Dv_n^3| + |Dv_n^1||Dv_n^3| + |Dv_n^1||Dv_n^2|)^q dx \\
&\leqslant 2^q \int_{A^1_{n,k}} (|Dv_n^2||Dv_n^3|)^q dx + 2^q \int_{A^1_{n,k}} |Dv_n^1|^q (|Dv_n^2| + |Dv_n^3|)^q dx \\
&\leqslant 2^q \int_{A^1_{n,k}} (|Dv_n^2||Dv_n^3|)^q dx + 2^q \varepsilon \int_{A^1_{n,k}} |Dv_n^1|^p dx \\
&\quad + 2^q C_\varepsilon \int_{A^1_{n,k}} (|Dv_n^2| + |Dv_n^3|)^{\frac{pq}{p-q}} dx.
\end{aligned}
$$

We use Lemma 2.1(ii) again and we have

$$
I_2 = \int_{A^1_{n,k}} |(\mathrm{adj}_2 Dv_n)^1|^q dx \leqslant \int_{A^1_{n,k}} (|Dv_n^2||Dv_n^3|)^q dx.
$$

$I_3$ can be estimated by using Lemma 2.1(iv) and Young's inequality, i.e.,

$$
\begin{aligned}
I_3 &= \int_{A^1_{n,k}} |\det Dv_n|^r dx \\
&\leqslant \int_{A^1_{n,k}} |Dv_n^1|^r (|Dv_n^2||Dv_n^3|)^r dx \\
&\leqslant \varepsilon \int_{A^1_{n,k}} |Dv_n^1|^p dx + C_\varepsilon \int_{A^1_{n,k}} (|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}} dx.
\end{aligned}
$$

$I_4$ can be estimated by using Young inequality, i.e.,

$$
I_4 = \int_{A^1_{n,k}} |Dv_n^1| dx \leqslant \varepsilon \int_{A^1_{n,k}} |Dv_n^1|^p dx + \int_{A^1_{n,k}} C_\varepsilon dx.
$$

To sum up, we substitute the above estimates for $I_i$ $(i = 1, 2, 3, 4)$ into (3.14) and we have

$$
\begin{aligned}
&k_1 \int_{A^1_{n,k}} |Dv_n^1|^p dx \\
&\quad \leqslant (k_1 2^q + k_3) \int_{A^1_{n,k}} (|Dv_n^2||Dv_n^3|)^q dx + (k_1 2^q + k_1 + 1)\varepsilon \int_{A^1_{n,k}} |Dv_n^1|^p dx \\
&\qquad + k_1 2^q C_\varepsilon \int_{A^1_{n,k}} (|Dv_n^2| + |Dv_n^3|)^{\frac{pq}{p-q}} dx + k_1 C_\varepsilon \int_{A^1_{n,k}} (|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}} dx \\
&\qquad + \int_{A^1_{n,k}} C_\varepsilon dx + \int_{A^1_{n,k}} (a(x) + b(x) + c(x) + 3k_2) dx.
\end{aligned}
$$

We take $\varepsilon$ small enough such that $(k_1 2^q + k_1 + 1)\varepsilon < k_1$, and then the second term on the right-hand side of the above inequality is absorbed by the left-hand side. Thus,

$$
\begin{aligned}
\int_{A^1_{n,k}} |Dv_n^1|^p dx &\leqslant c_9 \int_{A^1_{n,k}} [(|Dv_n^2||Dv_n^3|)^q + (|Dv_n^2| + |Dv_n^3|)^{\frac{pq}{p-q}} \\
&\qquad + (|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}} + (a(x) + b(x) + c(x) + 3k_2 + 1)] dx \\
&=: c_9 \int_{A^1_{n,k}} \psi_0 dx. \qquad (3.15)
\end{aligned}
$$

where $c_9$ is a constant depending only on $p, q, r, k_1, k_2$ and $k_3$.

We recall the definition for $r_0$ in (1.21) and we distinguish the following proof into three cases:

**Case 1.** $1 < r_0 < \frac{3}{p}$.

It is obvious that

$$(|Dv_n^2||Dv_n^3|)^q \in L^{\frac{p}{2q}}(\Omega),$$
$$(|Dv_n^2|+|Dv_n^3|)^{\frac{pq}{p-q}} \in L^{\frac{p-q}{q}}(\Omega),$$
$$(|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}} \in L^{\frac{p-r}{2r}}(\Omega)$$

and

$$(a(x)+b(x)+c(x)+3k_2+1) \in L^m(\Omega),$$

which imply $\psi_0 \in L^{r_0}(\Omega)$. We are now in a position to use Lemma 2.8 to derive that there exists a constant $c_5$, depending on $p,q,r,m,k_1,k_2,k_3,|\Omega|$ and $\|a+b+c\|_{L^m(\Omega)}$ and on the $W_0^{1,p}(\Omega)$ norm of $v_n^1$, such that

$$\|v_n^1\|_{L^\sigma(\Omega)} \leqslant c_5, \quad \sigma = (pr_0)^*.$$

Notice from (3.6) that $v_n$ is uniformly bounded in $W_0^{1,p}(\Omega)$ by a constant $c_8$ depending only on $p,q,r,k_1,k_2,|\Omega|$ and $\|a+b+c\|_{L^m(\Omega)}$, and then the constant $c_5$ above can be independent of the $W_0^{1,p}(\Omega)$ norm of $v_n^1$, i.e., there exists a constant $c_1^1$, depending only on $p,q,r,m,k_1,k_2,k_3,|\Omega|$ and $\|a+b+c\|_{L^m(\Omega)}$, such that

$$\|v_n^1\|_{L^\sigma(\Omega)} \leqslant c_1^1, \quad \sigma = (pr_0)^*.$$

Similarly, there exist constant $c_1^2$ and $c_1^3$, depending only on $p,q,r,m,k_1,k_2,k_3,|\Omega|$ and $\|a+b+c\|_{L^m(\Omega)}$, such that

$$\|v_n^2\|_{L^\sigma(\Omega)} \leqslant c_1^2, \quad \sigma = (pr_0)^*,$$
$$\|v_n^3\|_{L^\sigma(\Omega)} \leqslant c_1^3, \quad \sigma = (pr_0)^*.$$

Therefore,

$$\|v_n\|_{L^\sigma(\Omega)} \leqslant \|v_n^1\|_{L^\sigma(\Omega)} + \|v_n^2\|_{L^\sigma(\Omega)} + \|v_n^3\|_{L^\sigma(\Omega)} \leqslant c_1^1 + c_1^2 + c_1^3 =: c_1, \quad \sigma = (pr_0)^*,$$

as desired.

**Case 2.** $r_0 = \frac{3}{p}$.

We start by estimating the right-hand side terms in (3.15):

$$\int_{A_{n,k}^1} (|Dv_n^2||Dv_n^3|)^q dx \leqslant \left(\int_{A_{n,k}^1} (|Dv_n^2||Dv_n^3|)^{\frac{p}{2}} dx\right)^{\frac{2q}{p}} |A_{n,k}^1|^{1-\frac{2q}{p}}, \tag{3.16}$$

$$\int_{A_{n,k}^1} (|Dv_n^2|+|Dv_n^3|)^{\frac{pq}{p-q}} dx \leqslant \left(\int_{A_{n,k}^1} (|Dv_n^2|+|Dv_n^3|)^p dx\right)^{\frac{q}{p-q}} |A_{n,k}^1|^{1-\frac{q}{p-q}}, \tag{3.17}$$

$$\int_{A_{n,k}^1} (|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}} dx \leqslant \left(\int_{A_{n,k}^1} (|Dv_n^2||Dv_n^3|)^{\frac{p}{2}} dx\right)^{\frac{2r}{p-r}} |A_{n,k}^1|^{1-\frac{2r}{p-r}}, \tag{3.18}$$

$$\int_{A_{n,k}^1} (a(x)+b(x)+c(x)+3k_2+1) dx \leqslant \|a+b+c+3k_2+1\|_{L^m(\Omega)} |A_{n,k}^1|^{1-\frac{1}{m}}. \tag{3.19}$$

Combining (3.15) with (3.16)–(3.19), and recalling the definition $r_0$ in (1.21), one has

$$\begin{aligned}\int_{A_{n,k}^1} |Dv_n^1|^p dx &\leqslant c_{10}(|A_{n,k}^1|^{1-\frac{2q}{p}} + |A_{n,k}^1|^{1-\frac{q}{p-q}} + |A_{n,k}^1|^{1-\frac{2r}{p-r}} + |A_{n,k}^1|^{1-\frac{1}{m}}) \\ &\leqslant c_{10}|A_{n,k}^1|^{1-\frac{1}{r_0}},\end{aligned} \tag{3.20}$$

where $c_{10}$ is a constant depending only on $p, q, r, m, k_1, k_2, k_3, |\Omega|$ and $\|a+b+c\|_{L^m(\Omega)}$ and the $W_0^{1,p}(\Omega)$ norm of $v_n^2$ and $v_n^3$. Due to (3.6) again, the constant $c_{10}$ can be depending only on $p, q, r, m, k_1, k_2, k_3, |\Omega|$ and $\|a+b+c\|_{L^m(\Omega)}$. For any $h > k \geqslant 0$, the Poincaré inequality (2.11) gives us

$$\begin{aligned}\int_{A^1_{n,k}} |Dv_n^1|^p dx &= \int_\Omega |D(v_n^1 - T_k(v_n^1))|^p dx \\ &\geqslant \mathcal{S}^p \bigg( \int_\Omega |v_n^1 - T_k(v_n^1)|^{p^*} dx \bigg)^{\frac{p}{p^*}} \\ &= \mathcal{S}^p \bigg( \int_{A^1_{n,k}} |v_n^1 - T_k(v_n^1)|^{p^*} dx \bigg)^{\frac{p}{p^*}} \\ &\geqslant \mathcal{S}^p \bigg( \int_{A^1_{n,h}} |v_n^1 - T_k(v_n^1)|^{p^*} dx \bigg)^{\frac{p}{p^*}} \\ &\geqslant \mathcal{S}^p (h-k)^p |A^1_{n,h}|^{\frac{p}{p^*}}.\end{aligned}$$

We substitute the above inequality into (3.20) and we have that for any $h > k \geqslant 0$,

$$\mathcal{S}^p (h-k)^p |A^1_{n,h}|^{\frac{p}{p^*}} \leqslant c_{10} |A^1_{n,k}|^{1-\frac{1}{r_0}},$$

from which we derive

$$|A^1_{n,h}| \leqslant \frac{c_{11}}{(h-k)^{p^*}} |A^1_{n,k}|^{(1-\frac{1}{r_0})\frac{p^*}{p}}, \quad \forall\, h > k \geqslant 0, \tag{3.21}$$

where $c_{11}$ is a constant depending only on $p, q, r, m, k_1, k_2, k_3, |\Omega|$ and $\|a+b+c\|_{L^m(\Omega)}$. We use the facts that

$$r_0 = \frac{3}{p} \quad \text{and} \quad \bigg(1 - \frac{1}{r_0}\bigg)\frac{p^*}{p} = 1,$$

and we have

$$|A^1_{n,h}| \leqslant \frac{c_{11}}{(h-k)^{p^*}} |A^1_{n,k}|, \quad \forall\, h > k \geqslant 0.$$

Thus (2.18) holds with

$$k_0 = 0, \quad \varphi(k) = |A^1_{n,k}|, \quad c_6 = c_{11}, \quad \alpha = p^* \quad \text{and} \quad \beta = 1.$$

We use the Stampacchia lemma (see Lemma 2.9(ii)) to derive that for any $k \geqslant 0$,

$$|A^1_{n,k}| = |\{|v_n^1| > k\}| \leqslant |\{x \in \Omega : |u^1_{n,k}| > 0\}| \mathrm{e}^{1-(c_{11}\mathrm{e})^{-\frac{1}{p^*}} k} \leqslant c_{12} \mathrm{e}^{-6c_2 k}, \tag{3.22}$$

where

$$c_{12} = |\Omega| \mathrm{e}, \quad 6c_2 = (c_{11}\mathrm{e})^{-\frac{1}{p^*}}. \tag{3.23}$$

From (3.22), we have that for any $k \geqslant 0$,

$$|\{\mathrm{e}^{3c_2|v_n^1|} > \mathrm{e}^{3c_2 k}\}| = |\{|v_n^1| > k\}| \leqslant \frac{c_{12}}{(\mathrm{e}^{3c_2 k})^2}.$$

Let $\tilde{k} = \mathrm{e}^{3c_2 k}$. Then for any $\tilde{k} \geqslant 1$,

$$|\{\mathrm{e}^{3c_2|v_n^1|} > \tilde{k}\}| \leqslant \frac{c_{12}}{\tilde{k}^2}.$$

We use the above inequality and (3.23) in order to get

$$\begin{aligned}\int_\Omega \mathrm{e}^{3c_2|v_n^1|} dx &= \int_0^{+\infty} |\{\mathrm{e}^{3c_2|v_n^1|} > t\}| dt \\ &= \sum_{\tilde{k}=0}^{+\infty} \int_{\tilde{k}}^{\tilde{k}+1} |\{\mathrm{e}^{3c_2|v_n^1|} > t\}| dt\end{aligned}$$

$$\begin{aligned}
&\leqslant |\Omega| + \sum_{\tilde{k}=1}^{+\infty} \int_{\tilde{k}}^{\tilde{k}+1} |\{e^{3c_2|v_n^1|} > \tilde{k}\}| dt \\
&= |\Omega| + \sum_{\tilde{k}=1}^{+\infty} |\{e^{3c_2|v_n^1|} > \tilde{k}\}| \\
&\leqslant |\Omega| + \sum_{\tilde{k}=1}^{+\infty} \frac{c_{12}}{\tilde{k}^2} \\
&= |\Omega| + c_{12} \frac{\pi^2}{6} \\
&= |\Omega| \biggl(1 + \frac{e\pi^2}{6}\biggr).
\end{aligned}$$

Similarly, one can use the same method to derive

$$\int_\Omega e^{3c_2|v_n^2|} dx \leqslant |\Omega| \biggl(1 + \frac{e\pi^2}{6}\biggr)$$

and

$$\int_\Omega e^{3c_2|v_n^3|} dx \leqslant |\Omega| \biggl(1 + \frac{e\pi^2}{6}\biggr).$$

Hölder's inequality gives

$$\int_\Omega e^{c_2|v_n|} dx = \int_\Omega e^{c_2(|v_n^1| + |v_n^2| + |v_n^3|)} dx \leqslant \prod_{i=1}^{3} \biggl(\int_\Omega e^{3c_2|v_n^i|} dx\biggr)^{\frac{1}{3}} \leqslant |\Omega| \biggl(1 + \frac{e\pi^2}{6}\biggr).$$

**Case 3.** $r_0 > \frac{3}{p}$.

We start from (3.21). In this case,

$$\biggl(1 - \frac{1}{r_0}\biggr) \frac{p^*}{p} > 1,$$

and (2.18) holds with

$$k_0 = 0, \quad \varphi(k) = |A_{n,k}^1|, \quad c_6 = c_{11}, \quad v\alpha = p^* \quad \text{and} \quad \beta = \biggl(1 - \frac{1}{r_0}\biggr) \frac{p^*}{p} > 1.$$

We use the Stampacchia Lemma (see Lemma 2.9(i)) to derive that there exists a constant $d$, depending only on $p, q, r, m, k_1, k_2, k_3$, $|\Omega|$ and $\|a + b + c\|_{L^m(\Omega)}$, such that

$$|\{x \in \Omega : |v_n^1| \geqslant d\}| = 0,$$

i.e.,

$$|v_n^1| \leqslant d \quad \text{a.e. } \Omega.$$

Similarly, one can use the same method to derive

$$|v_n^2|, |v_n^3| \leqslant d \quad \text{a.e. } \Omega.$$

Hence,

$$|v_n| = |v_n^1| + |v_n^2| + |v_n^3| \leqslant 3d = c_3 \quad \text{a.e. } \Omega.$$

The proof of Theorem 1.1 is completed. □

Now we are going to prove Theorem 1.3.

*Proof of Theorem* 1.3. As in the proof of Theorem 1.1, we let $V$ be as in (2.9) and the distance $d$ be given by (2.10). Let $\mathcal{F} = \mathcal{J}$. We use Lemmas 2.6 and 2.3 and by Lemma 2.7 we derive that there exists a minimizing sequence $\{v_n\} \subset V$ such that (3.3)–(3.5) hold true. Lemma 2.3 implies that such a minimizing sequence is uniformly bounded, i.e., (3.6) holds true.

Let $\varphi \in W_0^{1,p}(\Omega)$ and we take

$$w = \begin{pmatrix} v_n^1 + \varphi \\ v_n^2 \\ v_n^3 \end{pmatrix}.$$

It is easy to see $w \in W_0^{1,p}(\Omega;\mathbb{R}^3) \subset W_0^{1,1}(\Omega;\mathbb{R}^3)$, which allows us to take the above $w$ in (3.5) and we have

$$\begin{aligned}
&\int_\Omega \Big\{ \sum_{\alpha=1}^3 F^\alpha(x, Dv_n^\alpha) + G(x, \mathrm{adj}_2 Dv_n) + H(x, \det Dv_n)\Big\} dx \\
&\quad \leqslant \int_\Omega \Big\{ \sum_{\alpha=1}^3 F^\alpha(x, Dw^\alpha) + G(x, \mathrm{adj}_2 Dw) + H(x, \det Dw)\Big\} dx + \varepsilon_n \int_\Omega |D\varphi| dx.
\end{aligned}$$

Since for $x \in \Omega \setminus \mathrm{supp}\varphi$, one has $Dw = Dv_n$ and $D\varphi = 0$, from the above inequality we derive

$$\begin{aligned}
&\int_{\mathrm{supp}\varphi} \Big\{ \sum_{\alpha=1}^3 F^\alpha(x, Dv_n^\alpha) + G(x, \mathrm{adj}_2 Dv_n) + H(x, \det Dv_n)\Big\} dx \\
&\quad \leqslant \int_{\mathrm{supp}\varphi} \Big\{ \sum_{\alpha=1}^3 F^\alpha(x, Dw^\alpha) + G(x, \mathrm{adj}_2 Dw) + H(x, \det Dw)\Big\} dx \\
&\qquad + \varepsilon_n \int_{\mathrm{supp}\varphi} |D\varphi| dx.
\end{aligned}$$

We use the assumptions (1.15)$'$, (1.17) and (1.18), and noticing that $Dw^2 = Dv_n^2$ and $Dw^3 = Dv_n^3$, we have

$$\begin{aligned}
&k_1 \int_{\mathrm{supp}\varphi} |Dv_n^1|^p dx - k_1 \int_{\mathrm{supp}\varphi} |\mathrm{adj}_2 Dv_n|^q dx - k_1 \int_{\mathrm{supp}\varphi} |\det Dv_n|^r dx - 3 \int_{\mathrm{supp}\varphi} k_2 dx \\
&\quad \leqslant k_3 \int_{\mathrm{supp}\varphi} |Dw^1|^p dx + k_3 \int_{\mathrm{supp}\varphi} |\mathrm{adj}_2 Dw|^q dx + k_3 \int_{\mathrm{supp}\varphi} |\det Dw|^r dx \\
&\qquad + \int_{\mathrm{supp}\varphi} (a(x) + b(x) + c(x)) dx + \varepsilon_n \int_{\mathrm{supp}\varphi} |D\varphi| dx,
\end{aligned}$$

from which, and noticing $Dw^1 = Dv_n^1 + D\varphi$, we derive

$$\begin{aligned}
k_1 \int_{\mathrm{supp}\varphi} |Dv_n^1|^p dx &\leqslant k_3 \int_{\mathrm{supp}\varphi} |Dv_n^1 + D\varphi|^p dx + k_1 \int_{\mathrm{supp}\varphi} |\mathrm{adj}_2 Dv_n|^q dx \\
&\quad + k_3 \int_{\mathrm{supp}\varphi} |\mathrm{adj}_2 Dw|^q dx + k_1 \int_{\mathrm{supp}\varphi} |\det Dv_n|^r dx \\
&\quad + k_3 \int_{\mathrm{supp}\varphi} |\det Dw|^r dx + \int_{\mathrm{supp}\varphi} |D\varphi| dx \\
&\quad + \int_{\mathrm{supp}\varphi} [a(x) + b(x) + c(x) + 3k_2] dx \\
&= k_3 I_6 + k_1 I_7 + k_3 I_8 + k_1 I_9 + k_3 I_{10} + I_{11} + I_{12},
\end{aligned} \tag{3.24}$$

where we have used again $\varepsilon_n \leqslant 1$.

$I_7$ can be estimated by using (i) and (ii) of Lemma 2.1 and Young's inequality, i.e.,

$$I_7 = \int_{\mathrm{supp}\varphi} |\mathrm{adj}_2 Dv_n|^q dx$$

$$
\begin{aligned}
&= \int_{\mathrm{supp}\varphi} (|(\mathrm{adj}_2 Dv_n)^1| + |(\mathrm{adj}_2 Dv_n)^2| + |(\mathrm{adj}_2 Dv_n)^3|)^q dx \\
&\leqslant 2^q \int_{\mathrm{supp}\varphi} (|Dv_n^2||Dv_n^3|)^q dx + 2^q \int_{\mathrm{supp}\varphi} |Dv_n^1|^q (|Dv_n^2| + |Dv_n^3|)^q dx \\
&\leqslant 2^q \int_{\mathrm{supp}\varphi} (|Dv_n^2||Dv_n^3|)^q dx + 2^q \varepsilon \int_{\mathrm{supp}\varphi} |Dv_n^1|^p dx \\
&\quad + 2^q C_\varepsilon \int_{\mathrm{supp}\varphi} (|Dv_n^2| + |Dv_n^3|)^{\frac{pq}{p-q}} dx. \qquad (3.25)
\end{aligned}
$$

We use (i) and (ii) of Lemma 2.1 and Young's inequality again to estimate $I_8$ as follows:

$$
\begin{aligned}
I_8 &= \int_{\mathrm{supp}\varphi} |\mathrm{adj}_2 Dw|^q dx \\
&= \int_{\mathrm{supp}\varphi} (|(\mathrm{adj}_2 Dw)^1| + |(\mathrm{adj}_2 Dw)^2| + |(\mathrm{adj}_2 Dw)^3|)^q dx \\
&\leqslant \int_{\mathrm{supp}\varphi} (|Dv_n^2||Dv_n^3| + |Dv_n^1 + D\varphi||Dv_n^3| + |Dv_n^1 + D\varphi||Dv_n^2|)^q dx \\
&\leqslant 2^q \int_{\mathrm{supp}\varphi} (|Dv_n^2||Dv_n^3|)^q dx \\
&\quad + 2^q \int_{\mathrm{supp}\varphi} |Dv_n^1 + D\varphi|^q (|Dv_n^2| + |Dv_n^3|)^q dx \\
&\leqslant 2^q \int_{\mathrm{supp}\varphi} (|Dv_n^2||Dv_n^3|)^q dx + \frac{2^q q}{p} \int_{\mathrm{supp}\varphi} |Dv_n^1 + D\varphi|^p dx \\
&\quad + \frac{2^q (p-q)}{p} \int_{\mathrm{supp}\varphi} (|Dv_n^2| + |Dv_n^3|)^{\frac{pq}{p-q}} dx. \qquad (3.26)
\end{aligned}
$$

We use Lemma 2.1(iv) and Young's inequality to estimate $I_9$ and $I_{10}$ as follows:

$$
\begin{aligned}
I_9 &= \int_{\mathrm{supp}\varphi} |\det Dv_n|^r dx \\
&\leqslant \int_{\mathrm{supp}\varphi} |Dv_n^1|^r (|Dv_n^2||Dv_n^3|)^r dx \\
&\leqslant \varepsilon \int_{\mathrm{supp}\varphi} |Dv_n^1|^p dx + C_\varepsilon \int_{\mathrm{supp}\varphi} (|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}} dx, \qquad (3.27) \\
I_{10} &= \int_{\mathrm{supp}\varphi} |\det Dw|^r dx \\
&\leqslant \int_{\mathrm{supp}\varphi} |Dv_n^1 + D\varphi|^r (|Dv_n^2||Dv_n^3|)^r dx \\
&\leqslant \frac{r}{p} \int_{\mathrm{supp}\varphi} |Dv_n^1 + D\varphi|^p dx + \frac{p-r}{p} \int_{\mathrm{supp}\varphi} (|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}} dx. \qquad (3.28)
\end{aligned}
$$

We estimate $I_{11}$ as

$$
\begin{aligned}
I_{11} &= \int_{\mathrm{supp}\varphi} |D\varphi| dx \\
&= \int_{\mathrm{supp}\varphi} |Dv_n^1 + D\varphi - Dv_n^1| dx \\
&\leqslant \int_{\mathrm{supp}\varphi} |Dv_n^1 + D\varphi| dx + \int_{\mathrm{supp}\varphi} |Dv_n^1| dx \\
&\leqslant \frac{1}{p} \int_{\mathrm{supp}\varphi} |Dv_n^1 + D\varphi|^p dx + \int_{\mathrm{supp}\varphi} \frac{1}{p'} dx
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon\int_{\mathrm{supp}\varphi}|Dv_n^1|^p dx+C_\varepsilon\int_{\mathrm{supp}\varphi}dx\\
=&\ \frac{1}{p}\int_{\mathrm{supp}\varphi}|Dv_n^1+D\varphi|^p dx+\varepsilon\int_{\mathrm{supp}\varphi}|Dv_n^1|^p dx\\
&+\int_{\mathrm{supp}\varphi}\left(\frac{1}{p'}+C_\varepsilon\right)dx.
\end{aligned}
\tag{3.29}
$$

Substituting (3.25)–(3.29) into (3.24), we arrive at

$$
\begin{aligned}
k_1\int_{\mathrm{supp}\varphi}|Dv_n^1|^p dx \leqslant&\ k_3\int_{\mathrm{supp}\varphi}|Dv_n^1+D\varphi|^p dx+k_1 2^q\int_{\mathrm{supp}\varphi}(|Dv_n^2||Dv_n^3|)^q dx\\
&+k_1 2^q\varepsilon\int_{\mathrm{supp}\varphi}|Dv_n^1|^p dx+k_1 2^q C_\varepsilon\int_{\mathrm{supp}\varphi}(|Dv_n^2|+|Dv_n^3|)^{\frac{pq}{p-q}}dx\\
&+k_3 2^q\int_{\mathrm{supp}\varphi}(|Dv_n^2||Dv_n^3|)^q dx+k_3\frac{2^q q}{p}\int_{\mathrm{supp}\varphi}|Dv_n^1+D\varphi|^p dx\\
&+k_3\frac{2^q(p-q)}{p}\int_{\mathrm{supp}\varphi}(|Dv_n^2|+|Dv_n^3|)^{\frac{pq}{p-q}}dx\\
&+k_1\varepsilon\int_{\mathrm{supp}\varphi}|Dv_n^1|^p dx+k_1 C_\varepsilon\int_{\mathrm{supp}\varphi}(|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}}dx\\
&+k_3\frac{r}{p}\int_{\mathrm{supp}\varphi}|Dv_n^1+D\varphi|^p dx+k_3\frac{p-r}{p}\int_{\mathrm{supp}\varphi}(|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}}dx\\
&+\frac{1}{p}\int_{\mathrm{supp}\varphi}|Dv_n^1+D\varphi|^p dx+\varepsilon\int_{\mathrm{supp}\varphi}|Dv_n^1|^p dx\\
&+\int_{\mathrm{supp}\varphi}\left(\frac{1}{p'}+C_\varepsilon\right)dx+\int_{\mathrm{supp}\varphi}[a(x)+b(x)+c(x)+3k_2]dx\\
=&\ (k_1 2^q+k_1+1)\varepsilon\int_{\mathrm{supp}\varphi}|Dv_n^1|^p dx\\
&+\left(k_3+\frac{k_3 2^q q}{p}+\frac{k_3 r}{p}+\frac{1}{p}\right)\int_{\mathrm{supp}\varphi}|Dv_n^1+D\varphi|^p dx+\int_{\mathrm{supp}\varphi}\psi_0 dx,
\end{aligned}
\tag{3.30}
$$

where

$$
\begin{aligned}
\psi_0=&\ (k_1+k_3)2^q(|Dv_n^2||Dv_n^3|)^q\\
&+\left[k_1 2^q C_\varepsilon+k_3\frac{2^q(p-q)}{p}\right](|Dv_n^2|+|Dv_n^3|)^{\frac{pq}{p-q}}\\
&+\left[k_1 C_\varepsilon+k_3\frac{p-r}{p}\right](|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}}\\
&+\left(\frac{1}{p'}+C_\varepsilon\right)+[a(x)+b(x)+c(x)+3k_2].
\end{aligned}
$$

We take $\varepsilon$ small enough such that

$$(k_1 2^q+k_1+1)\varepsilon<k_1,$$

and then the first term on the right-hand side of (3.30) is absorbed by the left-hand side. Thus, (3.30) yields

$$\int_{\mathrm{supp}\varphi}|Dv_n^1|^p dx\leqslant c_{13}\int_{\mathrm{supp}\varphi}\{|Dv_n^1+D\varphi|^p+\psi_0\}dx,$$

where $c_{13}$ is a constant depending only on $p,q,r,k_1,k_2$ and $k_3$.

What we need to show next is

$$\psi_0\in L^{r_0}(\Omega),\quad 1<r_0<\frac{3}{p}. \tag{3.31}$$

In fact, by the conditions (1.11)–(1.15) and $v_n \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
&(|Dv_n^2||Dv_n^3|)^q \in L^{\frac{p}{2q}}(\Omega),\\
&(|Dv_n^2|+|Dv_n^3|)^{\frac{pq}{p-q}} \in L^{\frac{p-q}{q}}(\Omega),\\
&(|Dv_n^2||Dv_n^3|)^{\frac{pr}{p-r}} \in L^{\frac{p-r}{2r}}(\Omega),\\
&a(x)+b(x)+c(x) \in L^m(\Omega).
\end{aligned}$$

The above inclusions together with the definition of $r_0$ in (1.21) imply (3.31) (we remark that the condition $r_0 < \frac{3}{p}$ is irrelevant because if $\psi_0 \in L^r(\Omega)$ for some $r \geqslant \frac{3}{p}$, then we also have $\psi_0 \in L^{r_0}(\Omega)$ and $1 < r_0 < \frac{3}{p}$). We are now in a position to use Lemma 2.10 to derive that there exist constants $\delta, c_7 > 0$, depending on $p, q, r, m, k_1, k_2, k_3$ and $|\Omega|$ and on the $W_0^{1,p}(\Omega)$ norm of $v_n^1$, such that

$$\|\nabla v_n^1\|_{L^{p+\delta}(\Omega)} \leqslant c_7(\|\nabla v_n^1\|_{L^p(\Omega)} + \|\psi_0\|_{L^{r_0}(\Omega)} + 1) \leqslant c_4^1.$$

We use (3.6) again and we know that the constants $\delta$, $c_7$ and $c_4^1$ above are depending only on $p, q, r, m, k_1, k_2, k_3, |\Omega|$ and $\|a+b+c\|_{L^m(\Omega)}$.

Similarly, there exist constants $c_4^2, c_4^3 > 0$, depending on $p, q, r, m, k_1, k_2, k_3, |\Omega|$ and $\|a+b+c\|_{L^m(\Omega)}$, such that

$$\|\nabla v_n^2\|_{L^{p+\delta}(\Omega)} \leqslant c_4^2 \quad \text{and} \quad \|\nabla v_n^3\|_{L^{p+\delta}(\Omega)} \leqslant c_4^3.$$

Thus,

$$\|\nabla v_n\|_{L^{p+\delta}(\Omega)} \leqslant \sum_{i=1}^3 \|\nabla v_n^i\|_{L^{p+\delta}(\Omega)} \leqslant \sum_{i=1}^3 c_4^i =: c_4.$$

This completes the proof of Theorem 1.3. □

## 4 Conclusion

In this paper, we obtain some regularity properties for minimizing sequences of the variational integral (1.1) with the integrand $f(x,\xi)$ being as (1.2). Meanwhile, we prove uniform higher integrability for the gradients of minimizing sequences of the variational integral (1.1) with the integrand $f(x,\xi)$ being as (1.3). It is worthwhile to note that we do not need any convexity assumptions on the integrands via the Ekeland variational principle. So we weaken the assumptions compared with the ones of preceding results. Moreover, it is also interesting to study Hölder continuity for minimizing sequences as well as other further regularity properties for gradients of minimizing sequences of the variational integral (1.1) in the future.

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