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On the global well-posedness for the 3D axisymmetric incompressible Keller–Segel–Navier–Stokes equations

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Abstract. In this paper, we study the Cauchy problem for the three-dimensional incompressible Keller–Segel–Navier–Stokes equations. By taking advantage of the geometry of axisymmetric flow without swirl, we obtain the global well-posedness for the system.

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1. Introduction

Broadcast spawning is a fertilization strategy that males and females release sperm and egg gametes into the surrounding flow. This phenomenon is studied by authors in [7,8]. Some experiments also indicate that chemotaxis plays a role in coral spawning problem: Sperm will be attracted to what eggs release, see [2,3,13,14]. The following model [7] is introduced to analyze the above phenomenon.

$$\rho_t + u\cdot\nabla\rho = \Delta\rho + \chi\nabla\cdot(\rho\nabla(\Delta)^{-1}\rho) - \varepsilon\rho^q, \quad \rho(x,0), \quad x\in\mathbb{R}^d. \tag{1.1}$$

Here ρ denotes the unknown population density, which relates to the supposition that the densities of sperm and egg gametes are identical. The given vector field u represents the ambient ocean flow, which is divergence-free and independent of ρ. The term $\chi\nabla\cdot(\rho\nabla(\Delta)^{-1}\rho)$ means the standard chemotactic phenomenon. The last term on the right-hand side of (1.1) models the reaction (fertilization). The parameters χ and ε denote the positive chemotactic sensitivity constant and the strength of the fertilization process, respectively. Under the conditions that q is an integer larger than two and initial data are sufficiently smooth with the decaying condition in $\mathbb{R}^d$ ($d = 2, 3$), authors prove the global result in [7]. The critical case $q = 2$ is shown in [8] by giving logarithmic improvements of the decay of the total mass $\int_{\mathbb{R}^d}\rho(x,t)\mathrm{d}x$.

Recently, the following system is considered in [1]:

$$\begin{cases} \rho_t + u\cdot\nabla\rho = \Delta\rho - \chi\nabla\cdot(\rho\nabla c) - \varepsilon\rho^q,\\ c_t + u\cdot\nabla c = \Delta c - c + \rho,\\ \nabla\cdot u = 0,\\ \rho(0,x) = \rho_0(x), c(0,x) = c_0(x). \end{cases} \tag{1.2}$$

The above model also describes the coral broadcast spawning phenomenon in $\mathbb{R}^d$ ($d = 2, 3$). From mathematical point, for $q = 2$ and $q > 2$, they prove the local and global well-posedness of regular solutions. Besides the total mass of the egg (sperm) density asymptotically approaches a strictly positive constant is obtained.

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There are other papers [4,9,10,15,18–23] devoted to describing different biological process of chemoattraction in the past years. Our goal here is to investigate the model (1.2) for the case $q = 3$, corresponding to the subcritical case. Suppose that the chemical is also transported by the fluid and the fluid velocity is modeled through a Navier–Stokes equation. Then we have the following model,

$$\begin{cases} \rho_t + u\cdot\nabla\rho = \Delta\rho - \chi\nabla\cdot(\rho\nabla c) - \varepsilon\rho^3, \\ c_t + u\cdot\nabla c = \Delta c - a_1 c + \rho, \\ u_t + \kappa(u\cdot\nabla u) + \nabla P = \Delta u - \rho\nabla\phi, \\ \nabla\cdot u = 0, \\ \rho(0,x) = \rho_0(x), c(0,x) = c_0(x), u(0,x) = u_0(x). \end{cases} \tag{1.3}$$

The unknowns ρ, c, u and P are the cell density, the chemical concentration, the fluid velocity and the pressure of the fluid separately. The parameters a_1, κ, χ and ε are nonnegative constants. The external force $-\rho\nabla\phi$ is exerted on the fluid by cells. The potential function $\phi = \phi(x,t)$ is considered as centrifugal force or gravity force. The form of gravity is $\phi = kx_1$ for a constant $k \in \mathbb{R}$ which depends on fluid mass density, cell mass density and gravity acceleration. Generally, ϕ is supposed to be a sufficiently smooth function. In $\mathbb{R}^2$, as far as we know, the first global existence of weak solution for (1.3) with $\kappa = 0$ is shown in [5]. Then the global results are extended to $\kappa = 1$ [17] in smoothly bounded domain in $\mathbb{R}^2$. Jin [6] further proves the existence of large time periodic strong solutions for (1.3) with $\kappa = 1$ in bounded domain in $\mathbb{R}^2$. For the three-dimensional case, similar results are obtained in [6,16] for (1.3) with $\kappa = 0$ in a bounded domain.

In this paper, we choose $\chi = \kappa = \varepsilon = a_1 = 1$, then (1.3) can be simplified as the following one

$$\begin{cases} \rho_t + u\cdot\nabla\rho = \Delta\rho - \nabla\cdot(\rho\nabla c) - \rho^3, \\ c_t + u\cdot\nabla c = \Delta c - c + \rho, \\ u_t + u\cdot\nabla u + \nabla P = \Delta u - \rho\nabla\phi, \\ \nabla\cdot u = 0, \\ \rho(0,x) = \rho_0(x), c(0,x) = c_0(x), u(0,x) = u_0(x). \end{cases} \tag{1.4}$$

In this paper, we study the global existence for (1.4) with special geometry, namely axisymmetric without swirl. An axisymmetric vector field u is called without swirl if it has the form:

$$u(t,x) = u^r(t,r,z)e_r + u^z(t,r,z)e_z,\ x = (x_1,x_2,z),\ r = (x_1^2 + x_2^2)^{\frac{1}{2}},$$

where (e_r, e_z, e_θ) is the cylindrical basis of $\mathbb{R}^3$. Direct computations yield that the vorticity $\omega = \mathrm{curl} u$ of the vector field takes the form:

$$\omega = (\partial_z u^r - \partial_r u^z)e_\theta := \omega_\theta e_\theta.$$

On the other hand, we know

$$u\cdot\nabla = u^r\partial_r + u^z\partial_z, \quad \omega\cdot\nabla u = \frac{u^r}{r}\omega$$

in the cylindrical coordinates. Therefore, the vorticity ω satisfies

$$\partial_t\omega + u\cdot\nabla\omega - \Delta\omega = \frac{u^r}{r}\omega - \nabla\times(\rho\nabla\phi).$$

Let $g = \rho\nabla\phi = g^r e_r + g^\theta e_\theta + g^z e_z$, where

$$\begin{aligned} g^r &= \frac{x_1\rho\partial_1\phi + x_2\rho\partial_2\phi}{r} = \frac{\rho(x_1\partial_1\phi + x_2\partial_2\phi)}{r}, \\ g^\theta &= \frac{x_1\rho\partial_2\phi - x_2\rho\partial_1\phi}{r} = \frac{\rho(x_1\partial_2\phi - x_2\partial_1\phi)}{r}, \\ g^z &= \rho\partial_z\phi = \rho\partial_z\phi. \end{aligned}$$

If $x_1\partial_2\phi - x_2\partial_1\phi = 0$ and $\rho = \rho(r,z)$, $\phi = \phi(r,z)$, then we have $g = \rho\nabla\phi$ is a axisymmetric vector field without swirl. Hence,

$$\nabla \times (\rho\nabla\phi) = \nabla \times g = (\partial_z g^r - \partial_r g^z)e_\theta.$$

Set $x^h = (x_1, x_2)$ and $\nabla_h = (\partial_1, \partial_2)$, we simply compute that

$$\begin{aligned}\partial_z g^r &= \frac{1}{r}\partial_z\rho x^h \cdot \nabla_h\phi + \frac{\rho}{r}x^h \cdot \nabla_h(\partial_z\phi),\\ \partial_r g^z &= \partial_r\rho\partial_z\phi + \rho\partial^2_{r,z}\phi.\end{aligned}$$

Thus,

$$\nabla \times (\rho\nabla\phi) = \left(\frac{1}{r}\partial_z\rho x^h \cdot \nabla_h\phi + \frac{\rho}{r}x^h \cdot \nabla_h(\partial_z\phi) - \partial_r\rho\partial_z\phi - \rho\partial^2_{r,z}\phi\right) e_\theta.$$

Since the Laplacian operator has the from $\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz}$ in the cylindrical coordinates, then ω_θ satisfies

$$\partial_t\omega_\theta + u \cdot \nabla\omega_\theta - \Delta\omega_\theta + \frac{\omega_\theta}{r^2} = \frac{u^r}{r}\omega_\theta - \frac{1}{r}\partial_z\rho x^h \cdot \nabla_h\phi - \frac{\rho}{r}x^h \cdot \nabla_h(\partial_z\phi) + \partial_r\rho\partial_z\phi + \rho\partial^2_{r,z}\phi.$$

Then, the evolution of the quantity $\frac{\omega_\theta}{r}$ satisfies

$$\begin{aligned}(\partial_t + u \cdot \nabla)\frac{\omega_\theta}{r} - \Delta\frac{\omega_\theta}{r} - \frac{2}{r}\partial_r\frac{\omega_\theta}{r} = &-\frac{1}{r^2}\partial_z\rho x^h \cdot \nabla_h\phi - \frac{\rho}{r^2}x^h \cdot \nabla_h(\partial_z\phi)\\ &+\frac{\partial_r\rho\partial_z\phi}{r} + \frac{\rho\partial^2_{r,z}\phi}{r}.\end{aligned} \tag{1.5}$$

The aim of this paper is to consider the global well-posedness of problem (1.4) with axisymmetric initial data. Suppose the initial data satisfy

$$X = \begin{cases} (1)\, \rho_0 \in L^1 \cap H^1,\ \rho_0 > 0,\\ (2)\, c_0 \in H^2, c_0 > 0,\\ (3)\, u_0 \in H^2, \nabla\phi \in L^\infty,\ \nabla^2\phi \in L^\infty,\ \nabla^3\phi \in L^\infty,\ \nabla\phi(0) = 0,\\ (4)\, u_0, \phi \text{ are all axisymmetric without swirl and } x_1\partial_2\phi - x_2\partial_1\phi = 0,\\ (5)\, \text{supp } \phi \text{ dose not intersect the axis}(Oz)\text{and } \prod_z (\text{supp } \phi) \text{ is a compact set,}\\ \quad \text{where } \prod_z \text{ denotes the orthogonal projector over}(Oz).\end{cases}$$

Now we present our result as follows:

Theorem 1.1. *Let the triple* $(\rho_0, c_0, u_0) \in X$ *and* $\frac{\omega_0}{r} \in L^2$, $\omega_0 = \nabla \times u_0$. *Then system* (1.4) *has a unique global solution* (ρ, c, u) *such that*

$$\begin{aligned}&\rho \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^3)) \cap L^\infty_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^3)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^3)),\\ &c \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^3)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^3(\mathbb{R}^3)),\\ &u \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^3)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^3(\mathbb{R}^3)).\end{aligned}$$

Remark 1.1. For the case $q \geq 3$ of system (1.4), the above theorem also holds.

Notation Throughout the paper, C stands for a generic constant and changes from line to line; $\|\cdot\|_p$ denotes the norm of the Lebesgue space L^p. Finally, $\mathcal{D}(\mathbb{R}^3)$ is a space of smooth compactly supported functions on $\mathbb{R}^3$, and space $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz class of smooth and rapidly decreasing functions.

2. Preliminaries

In this section, we give some notations and recall basic properties which will be used throughout the paper.

We first introduce the dynamic partition of the unity to define Besov spaces. One may check [12] for more details. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be supported in $\mathcal{C} = \{\xi \in \mathbb{R}^3,\ \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \text{for} \quad \xi \neq 0.$$

Defining $\chi(\xi) = 1 - \sum\limits_{q\in\mathbb{N}} \varphi(2^{-q}\xi)$. For $f \in \mathcal{S}'$, we set Littlewood–Paley operators as follows

$$\Delta_{-1} f = \chi(D) f; \quad \forall q \in \mathbb{N},\ \Delta_q f = \varphi(2^{-q}D) f.$$

The following low-frequency cutoff will be also used:

$$S_q f = \sum_{-1\le j\le q-1} \Delta_j f.$$

Now, let us recall the definition of the Besov space. For $s \in \mathbb{R}$, $1 \le p, r \le \infty$, the inhomogeneous space $B^s_{p,r}$ is the set of tempered distribution f such that

$$\|f\|_{B^s_{p,r}} := \left(\sum_{q\ge -1} 2^{qsr} \|\Delta_q f\|_p^r \right)^{\frac{1}{r}} < \infty.$$

It is worthwhile to remark that $B^s_{2,2}$ and $B^s_{\infty,\infty}$ coincide with the usual Sobolev spaces H^s and the usual Hölder space C^s for $s \in \mathbb{R}^+ \backslash \mathbb{Z}$, respectively.

We will use two kinds of mixed space–time Besov spaces. The first mixed space is $L^q_T B^s_{p,r}$. For $T > 0$ and $q \ge 1$, we define $L^q_T B^s_{p,r}$ as the set of all tempered distribution f satisfying

$$\|f\|_{L^q_T B^s_{p,r}} := \left\| \left(\sum_{j\ge -1} 2^{jsr} \|\Delta_j f\|_{L^p}^r \right)^{\frac{1}{r}} \right\|_{L^q_T} < \infty.$$

The second mixed space is $\widetilde{L}^q_T B^s_{p,r}$ which is the set of tempered distribution f satisfying

$$\|f\|_{\widetilde{L}^q_T B^s_{p,r}} := \left(\sum_{j\ge -1} 2^{jsr} \|\Delta_j f\|_{L^q_T L^p}^r \right)^{\frac{1}{r}} < \infty.$$

And the norm $\|\cdot\|_{L^q_T}$ is defined as following

$$\|f\|_{L^q_T} = \left(\int_0^T |f(\tau)|^q \mathrm{d}\tau \right)^{\frac{1}{q}}.$$

We have the following embeddings, which are the immediate results of Minkowski's inequality.

Let $s \in \mathbb{R}$, $q \ge 1$ and $(p, r) \in [1, \infty]^2$, then we have

$$L^q_T B^s_{p,r} \hookrightarrow \widetilde{L}^q_T B^s_{p,r}, \quad \text{if } r \ge q \quad \text{and} \quad \widetilde{L}^q_T B^s_{p,r} \hookrightarrow L^q_T B^s_{p,r} \quad \text{if } r \le q.$$

Lemma 2.1. [12] *Let $1 \le p \le q \le \infty$. Assume that $f \in L^p$, then there exists a constant C independent of f, j such that*

$$\begin{aligned} &\mathrm{supp}\hat{f} \subset \{|\xi| \le C2^j\} \Longrightarrow \|\partial^\alpha f\|_{L^q} \le C 2^{j|\alpha| + 3j(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p}, \\ &\mathrm{supp}\hat{f} \subset \left\{ \frac{1}{C} 2^j \le |\xi| \le C2^j \right\} \Longrightarrow \|f\|_{L^p} \le C 2^{-j|\alpha|} \sup_{|\beta|=|\alpha|} \|\partial^\beta f\|_{L^p}. \end{aligned}$$

Lemma 2.2. [12] *Let the divergence-free vector field u be axisymmetric without swirl, and denote $\omega = \omega_\theta e_\theta$ its curl. Then*

$$\|u\|_\infty \le C\|\omega_\theta\|_2^{\frac{1}{2}}\|\omega_\theta\|_{\dot H^1}^{\frac{1}{2}}.$$
$$\|\frac{u^r}{r}\|_\infty \le C\|\frac{\omega_\theta}{r}\|_2^{\frac{1}{2}}\|\frac{\omega_\theta}{r}\|_{\dot H^1}^{\frac{1}{2}}.$$

3. Solutions to the regularized problem

In this section,we will prove the global well-posedness for the following regularized system:

$$\begin{cases} \partial_t\rho^\varepsilon + (u^\varepsilon * \sigma^\varepsilon)\cdot\nabla\rho^\varepsilon - \Delta\rho^\varepsilon = -\nabla\cdot(\rho^\varepsilon\nabla(c^\varepsilon * \sigma^\varepsilon)) - (\rho^\varepsilon)^3, \\ \partial_t c^\varepsilon + (u^\varepsilon * \sigma^\varepsilon)\cdot\nabla c^\varepsilon - \Delta c^\varepsilon = -c^\varepsilon + \rho^\varepsilon, \\ \partial_t u^\varepsilon + (u^\varepsilon * \sigma^\varepsilon)\cdot\nabla u^\varepsilon - \Delta u^\varepsilon + \nabla P^\varepsilon = -(\rho^\varepsilon\nabla\phi) * \sigma^\varepsilon, \\ \operatorname{div} u^\varepsilon = 0, \\ (\rho^\varepsilon, c^\varepsilon, u^\varepsilon)|_{t=0} = (\rho_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon) = (\rho_0 * \sigma^\varepsilon, c_0 * \sigma^\varepsilon, u_0 * \sigma^\varepsilon), \end{cases} \tag{3.1}$$

where σ^ε is a standard mollifier, namely

$$\sigma^\varepsilon = \frac{1}{\varepsilon^2}\sigma\left(\frac{x}{\varepsilon}\right),$$

and satisfies the following conditions

$$\sigma(|x|) \in \mathcal{C}_0^\infty(\mathbb{R}^3), \quad \sigma \ge 0, \quad \int_{\mathbb{R}^3} \sigma \mathrm{d}x = 1.$$

Proposition 3.1. *Let $\nabla\phi \in L^\infty(\mathbb{R}^3)$ and the initial data $(\rho_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon) \in (H^s(\mathbb{R}^3))^3$ with $s > \frac{3}{2}$. Assume that $\rho_0^\varepsilon > 0, c_0^\varepsilon > 0$, and $\rho_0^\varepsilon \in L^1(\mathbb{R}^3)$. Then there exists a unique global solution $(\rho^\varepsilon, c^\varepsilon, u^\varepsilon) \in (\mathcal{C}([0,\infty); H^s(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}([0,\infty); H^{s+1}(\mathbb{R}^3)))^3$ for regularized system (3.1). Moreover, $\rho^\varepsilon(x,t) > 0$ and $c^\varepsilon(x,t) > 0$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$.*

Proof. Here we employ the standard method as used in [21]. For $k \ge 1$, let J_k be the spectral cutoff defined by

$$\widehat{J_k f}(\xi) = 1_{[0,k]}(|\xi|)\widehat{f}(\xi) \quad \text{for} \quad \xi \in \mathbb{R}^3.$$

We construct the following system in the space $H_k^s(\mathbb{R}^3) \triangleq \{f \in H^s(\mathbb{R}^3) | \operatorname{supp}\widehat{f} \subset B(0,k)\}$:

$$\begin{cases} \partial_t\rho^{k,\varepsilon} + J_k(J_k(u^{k,\varepsilon} * \sigma^\varepsilon)\cdot\nabla(J_k\rho^{k,\varepsilon})) = \Delta J_k^2\rho^{k,\varepsilon} - \nabla\cdot J_k(J_k\rho^{k,\varepsilon}\nabla(c^{k,\varepsilon} * \sigma^\varepsilon)) - (\rho^{k,\varepsilon})^3, \\ \partial_t c^{k,\varepsilon} + J_k(J_k(u^{k,\varepsilon} * \sigma^\varepsilon)\cdot\nabla(J_k c^{k,\varepsilon})) = \Delta J_k^2 c^{k,\varepsilon} - J_k^2 c^{k,\varepsilon} + J_k^2\rho^{k,\varepsilon}, \\ \partial_t u^{k,\varepsilon} + J_k(J_k(u^{k,\varepsilon} * \sigma^\varepsilon)\cdot\nabla(J_k u^{k,\varepsilon})) + \nabla J_k P^{k,\varepsilon} = \Delta J_k^2 u^{k,\varepsilon} - J_k((\rho^{k,\varepsilon}\nabla\phi) * \sigma^\varepsilon), \\ \operatorname{div} u^{k,\varepsilon} = 0, \\ (\rho^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})|_{t=0} = J_k(\rho_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon). \end{cases} \tag{3.2}$$

With the Leray method and the incompressibility condition $\operatorname{div} u^{k,\varepsilon} = 0$, we remove the pressure $P^{k,\varepsilon}$ by projecting the third equation of (3.2) onto the following space of divergence-free functions:

$$H^{s,\gamma}(\mathbb{R}^3) \triangleq \{(\rho, c, u) \in (H^s(\mathbb{R}^3))^3 | \operatorname{div} u = 0\}.$$

Then problem (3.2) reduces to an ODE in the Banach space $H_k^{s,\gamma}(\mathbb{R}^3)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{k,\varepsilon}(t) = F_{k,\varepsilon}(E^{k,\varepsilon}),\ E^{k,\varepsilon}(x,0) = E_0(x)^{k,\varepsilon} = \begin{pmatrix} J_k(\rho_0^\varepsilon) \\ J_k(c_0^\varepsilon) \\ J_k(u_0^\varepsilon) \end{pmatrix},\ \text{where } \mathrm{E}^{\mathrm{k},\varepsilon} \triangleq \begin{pmatrix} \rho^{k,\varepsilon} \\ c^{k,\varepsilon} \\ u^{k,\varepsilon} \end{pmatrix} \tag{3.3}$$

and

$$F_{k,\varepsilon} = \begin{pmatrix} -J_k(J_k(u^{k,\varepsilon} * \sigma^\varepsilon)\cdot\nabla J_k\rho^{k,\varepsilon}) + \Delta J_k^2\rho^{k,\varepsilon} - \nabla\cdot J_k(J_k\rho^{k,\varepsilon}\nabla(c^{k,\varepsilon} * \sigma^\varepsilon)) - (\rho^{k,\varepsilon})^3 \\ -J_k(J_k(u^{k,\varepsilon} * \sigma^\varepsilon)\cdot\nabla J_k c^{k,\varepsilon}) + \Delta J_k^2 c^{k,\varepsilon} - J_k^2 c^{k,\varepsilon} + J_k^2\rho^{k,\varepsilon} \\ -\mathcal{P}J_k(J_k(u^{k,\varepsilon} * \sigma^\varepsilon)\cdot\nabla J_k u^{k,\varepsilon}) + \Delta J_k^2 u^{k,\varepsilon} - \mathcal{P}J_k((\rho^{k,\varepsilon}\nabla\phi) * \sigma^\varepsilon) \end{pmatrix} \triangleq \begin{pmatrix} F_{k,\varepsilon}^1 \\ F_{k,\varepsilon}^2 \\ F_{k,\varepsilon}^3 \end{pmatrix}.$$

Let us remark that problem (3.2) and problem (3.3) are the same in essence, which can be found for example in [11] for the detailed proof. □

Step 1. For each $k \in \mathbb{N}^+$, we will prove the existence and the uniqueness of solutions for (3.3). Exactly, we have the following result.

Proposition 3.2. *Let ϕ and the initial data $(\rho_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon) \in (H^s(\mathbb{R}^3))^3$ as in Proposition 3.1. There exists a unique global smooth solution $(\rho^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})(t) \in (\mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^3)))^3$ of system (3.2) for each $k \in \mathbb{N}^+$.*

Proof of Proposition 3.2. First, for each $k \geq 1$ and all $E_1^{k,\varepsilon}, E_2^{k,\varepsilon} \in H^{s,\gamma}(\mathbb{R}^3)$, it is easy to prove that

$$\|F_{k,\varepsilon}^1(E_1^{k,\varepsilon}) - F_{k,\varepsilon}^1(E_2^{k,\varepsilon})\|_{H^s} \leq C(k,\varepsilon,\|E_1^{k,\varepsilon}\|_2,\|E_2^{k,\varepsilon}\|_2)\|E_1^{k,\varepsilon} - E_2^{k,\varepsilon}\|_{H^s}, \tag{3.4}$$

$$\|F_{k,\varepsilon}^2(E_1^{k,\varepsilon}) - F_{k,\varepsilon}^2(E_2^{k,\varepsilon})\|_{H^s} \leq C(k,\varepsilon,\|E_1^{k,\varepsilon}\|_2,\|E_2^{k,\varepsilon}\|_2)\|E_1^{k,\varepsilon} - E_2^{k,\varepsilon}\|_{H^s}, \tag{3.5}$$

and

$$\|F_{k,\varepsilon}^3(E_1^{k,\varepsilon}) - F_{k,\varepsilon}^3(E_2^{k,\varepsilon})\|_{H^s} \leq C(k,\varepsilon,\|E_1^{k,\varepsilon}\|_2,\|E_2^{k,\varepsilon}\|_2,\|\nabla\phi\|_\infty)\|E_1^{k,\varepsilon} - E_2^{k,\varepsilon}\|_{H^s}. \tag{3.6}$$

The estimates (3.4)–(3.6) show that $F_{k,\varepsilon}$ maps $H^s(\mathbb{R}^3)$ into $H^s(\mathbb{R}^3)$ and $F_{k,\varepsilon}$ is locally Lipschitz continuous on $H^s(\mathbb{R}^3)$. Thanks to the Cauchy–Lipschitz theorem, we know that for every $(\rho_0, c_0, u_0) \in (H^s(\mathbb{R}^3))^3$, there exists a unique solution $(\rho^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon}) \in (\mathcal{C}^1([0,T_k), H^s(\mathbb{R}^3)))^3$ with $T_k > 0$ is the maximal existence time.

According to Heat equation theory, we can show the time differentiation. For instance, we consider the following equations:

$$u_t - \Delta u = f(x,t), \quad \text{then} \quad u_t = \Delta u - f(x,t).$$

Assume $u \in L_t^\infty H^s$ and $f \in L_t^\infty H^{s-2}$, we have $u_t \in L_t^\infty H^{s-2}$. For the arbitrariness of s, we obtain $u_t \in L_t^\infty H^s, \forall\ s > 0$. In a similar process, we conclude

$$\partial_t(u_t) - \Delta u_t = f_t$$

and $u_{tt} \in L_t^\infty H^{s-2}$. Thus, we prove the time differentiation.

Next, we need to prove that $T_k = \infty$. Since $\rho_0^\varepsilon > 0$, we obtain that $\rho^{k,\varepsilon}(t) > 0$ for all $t \in [0, T_k)$, by analyzing the first equation of (3.3). Integrating the first equation of (3.3) over $\mathbb{R}^3$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{k,\varepsilon}(t)\|_1 = -\int_{\mathbb{R}^3}(\rho^{k,\varepsilon})^3\,\mathrm{d}x,$$

which implies that

$$\|\rho^{k,\varepsilon}(t)\|_1 + \int_0^t \|\rho^{k,\varepsilon}\|_3^3\,\mathrm{d}\tau \leq \|\rho_0^{k,\varepsilon}\|_1. \tag{3.7}$$

Taking the L^2-inner product for the second equation of (3.3) with $c^{k,\varepsilon}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c^{k,\varepsilon}(t)\|_2^2 + \|\nabla J_k c^{k,\varepsilon}(t)\|_2^2 + \|J_k c^{k,\varepsilon}(t)\|_2^2$$

$$= -\int_{\mathbb{R}^3} J_k(J_k(u^{k,\varepsilon} * \sigma^\varepsilon) \cdot \nabla J_k c^{k,\varepsilon}) c^{k,\varepsilon}\, \mathrm{d}x + \int_{\mathbb{R}^3} J_k \rho^{k,\varepsilon} J_k c^{k,\varepsilon}\, \mathrm{d}x.$$

Since $\mathcal{D}(\mathbb{R}^3)$ is dense in $H^l(\mathbb{R}^3)$ for all $l > 1$, there are no boundary terms when we integrate by parts. For the first term of the right-hand side on the above equality, we conclude

$$-\int_{\mathbb{R}^3} J_k(J_k(u^{k,\varepsilon} * \sigma^\varepsilon) \cdot \nabla J_k c^{k,\varepsilon}) c^{k,\varepsilon}\, \mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^3} J_k(u^{k,\varepsilon} * \sigma^\varepsilon) \cdot \nabla (J_k c^{k,\varepsilon})^2\, \mathrm{d}x = 0.$$

Thus, the Young inequality implies

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c^{k,\varepsilon}(t)\|_2^2 + \|\nabla J_k c^{k,\varepsilon}(t)\|_2^2 + \|J_k c^{k,\varepsilon}(t)\|_2^2 \\
&\quad = \int_{\mathbb{R}^3} J_k \rho^{k,\varepsilon} J_k c^{k,\varepsilon}\, \mathrm{d}x \\
&\quad \le \|J_k \rho^{k,\varepsilon}\|_2 \|J_k c^{k,\varepsilon}\|_2 \\
&\quad \le C\|J_k \rho^{k,\varepsilon}\|_2^2 + \frac{1}{2}\|J_k c^{k,\varepsilon}\|_2^2,
\end{aligned}$$

which together with (3.7) and interpolation inequality yields that

$$\|c^{k,\varepsilon}(t)\|_2^2 + 2\int_0^t \|\nabla J_k c^{k,\varepsilon}(\tau)\|_2^2\, \mathrm{d}\tau \le \|c_0^{k,\varepsilon}\|_2^2 + \|\rho_0^{k,\varepsilon}\|_1. \tag{3.8}$$

Similarly, we can infer that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{k,\varepsilon}(t)\|_2^2 + \|\nabla J_k \rho^{k,\varepsilon}(t)\|_2^2 + \|\rho^{k,\varepsilon}(t)\|_4^4 \\
&\quad = \int_{\mathbb{R}^3} J_k \rho^{k,\varepsilon} \nabla (c^{k,\varepsilon} * \sigma^{k,\varepsilon}) \cdot \nabla J_k \rho^{k,\varepsilon}\, \mathrm{d}x \\
&\quad \le \|J_k \rho^{k,\varepsilon}\|_2 \|\nabla (c^{k,\varepsilon} * \sigma^{k,\varepsilon})\|_\infty \|\nabla J_k \rho^{k,\varepsilon}\|_2 \\
&\quad \le C(\varepsilon)\|\rho^{k,\varepsilon}\|_2^2 \|c^{k,\varepsilon}\|_2^2 + \frac{1}{4}\|\nabla J_k \rho^{k,\varepsilon}\|_2^2
\end{aligned}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^{k,\varepsilon}(t)\|_2^2 + \|\nabla J_k u^{k,\varepsilon}(t)\|_2^2 \le C\|\nabla\phi\|_\infty \|J_k \rho^{k,\varepsilon}\|_2 \|u^{k,\varepsilon}\|_2.$$

The Gronwall inequality means

$$\|\rho^{k,\varepsilon}(t)\|_2^2 + \int_0^t \|\nabla J_k \rho^{k,\varepsilon}(\tau)\|_2^2\, \mathrm{d}\tau \le \|\rho_0^{k,\varepsilon}\|_2^2 e^{C(\varepsilon)t} \tag{3.9}$$

and

$$\|u^{k,\varepsilon}(t)\|_2^2 + 2\int_0^t \|\nabla J_k u^{k,\varepsilon}(\tau)\|_2^2\, \mathrm{d}\tau \le \|u_0^{k,\varepsilon}\|_2^2 e^{C(\varepsilon)t}. \tag{3.10}$$

By the same process as used in (3.4)–(3.6), we obtain that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|E^{k,\varepsilon}(t)\|_{H^s} &\le C(k,\varepsilon,\|\rho^{k,\varepsilon}\|_2,\|c^{k,\varepsilon}\|_2,\|u^{k,\varepsilon}\|_2,\|\nabla\phi\|_\infty)\|E^{k,\varepsilon}(t)\|_{H^s} \\
&\le C(t,k,\varepsilon,\|\rho_0^{k,\varepsilon}\|_2,\|c_0^{k,\varepsilon}\|_2,\|u_0^{k,\varepsilon}\|_2,\|\nabla\phi\|_\infty,\|\rho_0^{k,\varepsilon}\|_1)\|E^{k,\varepsilon}(t)\|_{H^s},
\end{aligned}$$

where we have used estimates (3.8)–(3.10). We conclude from the Gronwall inequality that $\|E^{k,\varepsilon}(t)\|_{H^s} \le C(t,k,\varepsilon,E^{k,\varepsilon}(0))$ for all $t\in[0,T_k)$. Thus, by using the continuation property of ODEs on a Banach space, we know solutions can be continued for all time. □

Step 2. In this step, we give the uniform estimates for $(\rho^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})$ (independent of k).

Firstly, we obtain the L^2-estimate of solution $(\rho^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})$ from the estimates (3.8)–(3.10) that:

$$\|E^{k,\varepsilon}(t)\|_2^2 + \int_0^t \|\nabla J_k E^{k,\varepsilon}(\tau)\|_2^2 \,\mathrm{d}\tau \le C(t,\varepsilon,E^{k,\varepsilon}(0)). \tag{3.11}$$

Secondly, we will prove the H^1-estimate of solution $(\rho^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})$. Multiplying the first equation of (3.2) by $-\Delta\rho^{k,\varepsilon}$ and then integrating over $\mathbb{R}^3$ imply

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho^{k,\varepsilon}(t)\|_2^2 + \|\Delta J_k\rho^{k,\varepsilon}(t)\|_2^2 + \|\rho^{k,\varepsilon}\nabla\rho^{k,\varepsilon}\|_2^2 \\
&\quad = -\int_{\mathbb{R}^3} (\nabla J_k\rho^{k,\varepsilon})\cdot\nabla J_k(u^{k,\varepsilon}*\sigma^\varepsilon)\cdot\nabla J_k\rho^{k,\varepsilon}\,\mathrm{d}x + \int_{\mathbb{R}^3}\nabla(J_k\rho^{k,\varepsilon}\nabla(c^{k,\varepsilon}*\sigma^\varepsilon))\Delta J_k\rho^{k,\varepsilon}\,\mathrm{d}x \\
&\quad \le \|\nabla J_k(u^{k,\varepsilon}*\sigma^\varepsilon)\|_\infty\|\nabla J_k\rho^{k,\varepsilon}\|_2^2 + \|\nabla(J_k\rho^{k,\varepsilon}\nabla(c^{k,\varepsilon}*\sigma^\varepsilon))\|_2\|\Delta J_k\rho^{k,\varepsilon}\|_2 \\
&\quad \le C(\varepsilon)\|u^{k,\varepsilon}\|_2\|\nabla\rho^{k,\varepsilon}\|_2^2 + C(\varepsilon)(\|\nabla\rho^{k,\varepsilon}\|_2^2\|c^{k,\varepsilon}\|_2^2 + \|\rho^{k,\varepsilon}\|_2^2\|c^{k,\varepsilon}\|_2^2) + \frac{1}{4}\|\Delta J_k\rho^{k,\varepsilon}\|_2^2.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho^{k,\varepsilon}(t)\|_2^2 + \|\Delta J_k\rho^{k,\varepsilon}(t)\|_2^2 \\
&\quad \le C(\varepsilon)(\|\nabla\rho^{k,\varepsilon}\|_2^2(u^{k,\varepsilon}\|_2 + \|c^{k,\varepsilon}\|_2^2) + \|\rho^{k,\varepsilon}\|_2^2\|c^{k,\varepsilon}\|_2^2).
\end{aligned} \tag{3.12}$$

Similarly, we infer

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c^{k,\varepsilon}(t)\|_2^2 + \|\Delta J_k c^{k,\varepsilon}(t)\|_2^2 \\
&\quad = -\int_{\mathbb{R}^3}(\nabla J_k c^{k,\varepsilon})\cdot\nabla J_k(u^{k,\varepsilon}*\sigma^\varepsilon)\cdot\nabla J_k c^{k,\varepsilon}\,\mathrm{d}x + \int_{\mathbb{R}^3} J_k c^{k,\varepsilon}\Delta J_k c^{k,\varepsilon}\,\mathrm{d}x - \int_{\mathbb{R}^3} J_k\rho^{k,\varepsilon}\Delta J_k c^{k,\varepsilon}\,\mathrm{d}x \\
&\quad \le \|\nabla J_k(u^{k,\varepsilon}*\sigma^\varepsilon)\|_\infty\|\nabla J_k c^{k,\varepsilon}\|_2^2 + \|J_k c^{k,\varepsilon}\|_2\|\Delta J_k c^{k,\varepsilon}\|_2 + \|J_k\rho^{k,\varepsilon}\|_2\|\Delta J_k c^{k,\varepsilon}\|_2 \\
&\quad \le C(\varepsilon)(\|u^{k,\varepsilon}\|_2\|\nabla c^{k,\varepsilon}\|_2^2 + \|c^{k,\varepsilon}\|_2^2 + \|\rho^{k,\varepsilon}\|_2^2) + \frac{1}{4}\|\Delta J_k c^{k,\varepsilon}\|_2^2.
\end{aligned}$$

Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c^{k,\varepsilon}(t)\|_2^2 + \|\Delta J_k c^{k,\varepsilon}(t)\|_2^2 \le C(\varepsilon)(\|u^{k,\varepsilon}\|_2\|\nabla c^{k,\varepsilon}\|_2^2 + \|c^{k,\varepsilon}\|_2^2 + \|\rho^{k,\varepsilon}\|_2^2). \tag{3.13}$$

Operating the curl to the third equation of (3.2), then multiplying the resulting equation by $\omega^{k,\varepsilon}$ and integrating with respect to the space variable yield

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^{k,\varepsilon}(t)\|_2^2 + \|\nabla J_k\omega^{k,\varepsilon}(t)\|_2^2 &\le -\int_{\mathbb{R}^3}\operatorname{curl}(J_k((\rho^{k,\varepsilon}\nabla\phi)*\sigma^\varepsilon))\omega^{k,\varepsilon}\,\mathrm{d}x \\
&\le C(\varepsilon)\|\rho^{k,\varepsilon}\|_2^2 + \|\omega^{k,\varepsilon}\|_2^2.
\end{aligned} \tag{3.14}$$

Summing up (3.12)–(3.14), then we get from the Gronwall inequality

$$\|\nabla E^{k,\varepsilon}(t)\|_2^2 + \int_0^t \|\Delta J_k E^{k,\varepsilon}(\tau)\|_2^2\,\mathrm{d}\tau$$

$$\leq (\|\nabla E_0^{k,\varepsilon}\|_2^2 + C(\varepsilon)\int_0^t (\|\rho^{k,\varepsilon}(\tau)\|_2^2\|c^{k,\varepsilon}(\tau)\|_2^2 + \|\rho^{k,\varepsilon}(\tau)\|_2^2 + \|c^{k,\varepsilon}(\tau)\|_2^2)\,\mathrm{d}\tau)$$
$$\times\exp(C(\varepsilon)\int_0^t (\|u^{k,\varepsilon}(\tau)\|_2 + \|c^{k,\varepsilon}(\tau)\|_2^2 + 1)\,\mathrm{d}\tau).$$

This together with (3.11) gives

$$\|E^{k,\varepsilon}(t)\|_{H^1}^2 + \int_0^t \|J_k E^{k,\varepsilon}(\tau)\|_{H^2}^2 \mathrm{d}\tau \leq C\|E_0^\varepsilon\|_{H^1}\exp\exp(C(t,\varepsilon,\|\rho_0^\varepsilon\|_1,\|E_0^\varepsilon\|_2^2)). \tag{3.15}$$

Finally, we prove the H^s-estimate for solutions by utilizing the Fourier localization technique. Taking the operation Δ_q for $q \in \mathbb{Z}$ on the first equation of (3.2), we obtain

$$\partial_t\Delta_q\rho^{k,\varepsilon} + J_k\Delta_q(J_k(u^{k,\varepsilon}*\sigma^\varepsilon)\cdot\nabla J_k\rho^{k,\varepsilon}) - \Delta\Delta_q J_k^2\rho^{k,\varepsilon} = -\nabla\cdot J_k\Delta_q(J_k\rho^{k,\varepsilon}\nabla(c^{k,\varepsilon}*\sigma^\varepsilon)) - \Delta_q((\rho^{k,\varepsilon})^3).$$

Taking the L^2-inner product with the above equality by $\Delta_q\rho^{k,\varepsilon}$ and integrating by parts yield that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_q\rho^{k,\varepsilon}(t)\|_2^2 + \|\nabla\Delta_q J_k\rho^{k,\varepsilon}(t)\|_2^2\\
&= \int_{\mathbb{R}^3} \Delta_q(J_k(u^{k,\varepsilon}*\sigma^\varepsilon))J_k\rho^{k,\varepsilon})\cdot\nabla\Delta_q J_k\rho^{k,\varepsilon}\,\mathrm{d}x + \int_{\mathbb{R}^3}\Delta_q(J_k\rho^{k,\varepsilon}\nabla(c^{k,\varepsilon}*\sigma^\varepsilon))\cdot\nabla\Delta_q J_k\rho^{k,\varepsilon}\,\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^3}\Delta_q(\rho^{k,\varepsilon})^3\Delta_q\rho^{k,\varepsilon}\,\mathrm{d}x\\
&\leq \|\Delta_q(J_k(u^{k,\varepsilon}*\sigma^\varepsilon)J_k\rho^{k,\varepsilon})\|_2\|\nabla\Delta_q J_k\rho^{k,\varepsilon}\|_2 + \|\Delta_q(J_k\rho^{k,\varepsilon}\nabla(c^{k,\varepsilon}*\sigma^\varepsilon))\|_2\|\nabla\Delta_q J_k\rho^{k,\varepsilon}\|_2\\
&\leq C\|\Delta_q(J_k(u^{k,\varepsilon}*\sigma^\varepsilon)J_k\rho^{k,\varepsilon})\|_2^2 + C\|\Delta_q(J_k\rho^{k,\varepsilon}\nabla(c^{k,\varepsilon}*\sigma^\varepsilon))\|_2^2 + \frac{1}{4}\|\nabla\Delta_q J_k\rho^{k,\varepsilon}\|_2^2,
\end{aligned}$$

from which we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_q\rho^{k,\varepsilon}(t)\|_2^2 + \|\nabla\Delta_q J_k\rho^{k,\varepsilon}(t)\|_2^2\\
&\quad\leq C\|\Delta_q(J_k(u^{k,\varepsilon}*\sigma^\varepsilon)J_k\rho^{k,\varepsilon})\|_2^2 + C\|\Delta_q(J_k\rho^{k,\varepsilon}\nabla(c^{k,\varepsilon}*\sigma^\varepsilon))\|_2^2.
\end{aligned}$$

Multiplying 2^{2qs} on both sides of the above inequality, then taking the ℓ^1 norm, we conclude

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{k,\varepsilon}(t)\|_{\dot H^s}^2 + \|\nabla J_k\rho^{k,\varepsilon}(t)\|_{\dot H^s}^2\\
&\quad\leq C\|J_k(u^{k,\varepsilon}*\sigma^\varepsilon)J_k\rho^{k,\varepsilon}\|_{\dot H^s}^2 + C\|J_k\rho^{k,\varepsilon}\nabla(c^{k,\varepsilon}*\sigma^\varepsilon)\|_{\dot H^s}^2\\
&\quad\leq C(\varepsilon)(\|u^{k,\varepsilon}\|_{\dot H^s}^2\|J_k\rho^{k,\varepsilon}\|_{H^2}^2 + \|J_k u^{k,\varepsilon}\|_{H^2}^2\|\rho^{k,\varepsilon}\|_{\dot H^s}^2)\\
&\qquad+C(\varepsilon)(\|\rho^{k,\varepsilon}\|_{\dot H^s}^2 + \|J_k\rho^{k,\varepsilon}\|_{H^2}^2)\|c^{k,\varepsilon}\|_2^2.
\end{aligned}\tag{3.16}$$

In the same process, we deduce that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|c^{k,\varepsilon}(t)\|_{\dot H^s}^2 + \|\nabla J_k c^{k,\varepsilon}(t)\|_{\dot H^s}^2 &\leq C\|J_k(u^{k,\varepsilon}*\sigma^\varepsilon)J_k c^{k,\varepsilon}\|_{\dot H^s}^2 + C\|J_k c^{k,\varepsilon}\|_{\dot H^s}^2 + \|J_k\rho^{k,\varepsilon}\|_{\dot H^s}^2\\
&\leq C(\varepsilon)(\|u^{k,\varepsilon}\|_{\dot H^s}^2\|J_k c^{k,\varepsilon}\|_{H^2}^2 + \|J_k u^{k,\varepsilon}\|_{H^2}^2\|c^{k,\varepsilon}\|_{\dot H^s}^2\\
&\quad+\|c^{k,\varepsilon}\|_{\dot H^s}^2 + \|\rho^{k,\varepsilon}\|_{\dot H^s}^2).
\end{aligned}\tag{3.17}$$

Taking the operation Δ_q for $q \in \mathbb{Z}$ on the third equation of (3.2), we get

$$\partial_t\Delta_q u^{k,\varepsilon} + \mathcal{P}J_k\Delta_q(J_k(u^{k,\varepsilon}*\sigma^\varepsilon)\cdot\nabla J_k u^{k,\varepsilon}) - \Delta\Delta_q J_k^2 u^{k,\varepsilon} = -\mathcal{P}\Delta_q J_k((\rho^{k,\varepsilon}\nabla\phi)*\sigma^\varepsilon).$$

In the same way, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|u^{k,\varepsilon}(t)\|_{\dot{H}^s}^2+\|\nabla J_k u^{k,\varepsilon}(t)\|_{\dot{H}^s}^2\\
&\le C\|J_k(u^{k,\varepsilon}*\sigma^\varepsilon)J_k u^{k,\varepsilon}\|_{\dot{H}^s}^2+C\|(\rho^{k,\varepsilon}\nabla\phi)*\sigma^\varepsilon\|_{\dot{H}^s}^2+\|u^{k,\varepsilon}\|_{\dot{H}^s}^2\\
&\le C(\varepsilon)\|u^{k,\varepsilon}\|_{\dot{H}^s}^2\|J_k u^{k,\varepsilon}\|_{H^2}^2+C(\varepsilon)\|\rho^{k,\varepsilon}\|_2^2+\|u^{k,\varepsilon}\|_{\dot{H}^s}^2.
\end{aligned}\tag{3.18}$$

Summing up (3.16)–(3.18) and using the Gronwall inequality yield

$$\begin{aligned}
&\|E^{k,\varepsilon}(t)\|_{\dot{H}^s}^2+\int_0^t\|J_k E^{k,\varepsilon}(\tau)\|_{\dot{H}^{s+1}}^2\,\mathrm{d}\tau\\
&\le (\|E^{k,\varepsilon}(0)\|_{\dot{H}^s}^2+C(\varepsilon)\int_0^t(\|J_k\rho^{k,\varepsilon}\|_{H^2}^2\|c^{k,\varepsilon}\|_2^2+\|\rho^{k,\varepsilon}\|_2^2)\,\mathrm{d}\tau)\\
&\times\exp(C(\varepsilon)\int_0^t(\|J_k\rho^{k,\varepsilon}\|_{H^2}^2+\|J_k u^{k,\varepsilon}\|_{H^2}^2+\|c^{k,\varepsilon}\|_2^2+\|J_k c^{k,\varepsilon}\|_{H^2}^2+1)\,\mathrm{d}\tau).
\end{aligned}$$

Then, combining with (3.11), we obtain

$$\begin{aligned}
&\|E^{k,\varepsilon}(t)\|_{H^s}^2+\int_0^t\|J_k E^{k,\varepsilon}(\tau)\|_{H^{s+1}}^2\,\mathrm{d}\tau\\
&\le C\|E_0^\varepsilon\|_{H^s}\exp\exp\exp(C(t,\varepsilon,\|\rho_0^\varepsilon\|_1,\|E_0^\varepsilon\|_2^2)).
\end{aligned}\tag{3.19}$$

Step 3. This step is to prove that $(\rho^{k,\varepsilon},c^{k,\varepsilon},u^{k,\varepsilon})$ converge to a limit $(\rho^\varepsilon,c^\varepsilon,u^\varepsilon)$ satisfying system (3.1) in the sense of distribution.

From estimate (3.19), we conclude that the family $(\partial_t\rho^{k,\varepsilon},\partial_t c^{k,\varepsilon},\partial_t u^{k,\varepsilon})$ belongs to $(L^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{R}^3)))^3$. Suppose $\{\chi_l\}_{l\in\mathbb{N}}$ is a sequence of $C_0^\infty(\mathbb{R}^3)$ cutoff functions supported in the ball $B(0,l+1)$ of $\mathbb{R}^3$ and equal to 1 in a neighborhood of $B(0,l)$, namely

$$\chi_l=\begin{cases}1, & \text{in } B(0,l),\\ 0, & \text{in } B^c(0,l+1).\end{cases}$$

Thus, we have $\|\chi_l\|_{L^\infty}\le 1$ and $\|\chi_l\|_{W^{s,\infty}}\le C$ with C is independent of l.

The Moser estimate enables us to conclude that, for any $l\in\mathbb{N}$,

$$\|\chi_l\rho^{k,\varepsilon}\|_{H^{s-1}}\le\|\chi_l\|_{L^\infty}\|\rho^{k,\varepsilon}\|_{H^{s-1}}+\|\chi_l\|_{W^{s-1,\infty}}\|\rho^{k,\varepsilon}\|_{L^2}\le C(\|\rho^{k,\varepsilon}\|_{H^{s-1}}+\|\rho^{k,\varepsilon}\|_{L^2}),$$

where C is independent of l, which implies that $\{(\chi_l n^{k,\varepsilon},\chi_l c^{k,\varepsilon},\chi_l u^{k,\varepsilon})\}_{k\in\mathbb{N}}$ is uniformly bounded.

On the other hand,

$$\begin{aligned}
&\|\chi_l\rho^{k,\varepsilon}(s)-\chi_l\rho^{k,\varepsilon}(t)\|_{H^{s-1}}=\big\|\chi_l\int_s^t\partial_t\rho^{k,\varepsilon}(\tau)\mathrm{d}\tau\big\|_{H^{s-1}}\\
&\le\int_s^t\|\chi_l\partial_t\rho^{k,\varepsilon}(\tau)\|_{H^{s-1}}\mathrm{d}\tau\le\big(\int_s^t\mathrm{d}\tau\big)^{\frac12}\big(\int_s^t\|\chi_l\partial_t\rho^{k,\varepsilon}(\tau)\|_{H^{s-1}}^2\mathrm{d}\tau\big)^{\frac12}\\
&\le(t-s)^{\frac12}\big(\int_s^t\|\chi_l\partial_t\rho^{k,\varepsilon}(\tau)\|_{H^{s-1}}^2\mathrm{d}\tau\big)^{\frac12},
\end{aligned}$$

from which it follows that $\{(\chi_l\rho^{k,\varepsilon},\chi_l c^{k,\varepsilon},\chi_l u^{k,\varepsilon})\}_{k\in\mathbb{N}}$ is equicontinuous in $(\mathcal{C}(\mathbb{R}^+,H^{s-1}(\mathbb{R}^3)))^3$.

Notice that the application $\rho^{k,\varepsilon} \mapsto \chi_l \rho^{k,\varepsilon}$ is compact from H^s into $H^{s'}$ as $s > s'$. By applying the Lions–Aubin lemma to the family $\{(\chi_l\rho^{k,\varepsilon}, \chi_l c^{k,\varepsilon}, \chi_l u^{k,\varepsilon})\}_{k\in\mathbb{N}}$ on the time interval $[0,l]$, and using Cantor's diagonal process, it finally reduces a distribution $(\rho^\varepsilon, c^\varepsilon, u^\varepsilon)$ belonging to $(\mathcal{C}(\mathbb{R}^+, H^{s'}(\mathbb{R}^3)))^3$ and a subsequence (which we still denote by $\{(\rho^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})\}_{k\in\mathbb{N}}$) such that, for all $l \in \mathbb{N}$, we have

$$(\chi_l\rho^{k,\varepsilon}, \chi_l c^{k,\varepsilon}, \chi_l u^{k,\varepsilon}) \to_{k\to\infty} (\chi_l\rho^{\varepsilon}, \chi_l c^{\varepsilon}, \chi_l u^{\varepsilon}) \text{ in } (\mathcal{C}([0,l], H^{s'}(\mathbb{R}^3)))^3.$$

This obviously ensures that $(\rho^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})$ tends to $(\rho^\varepsilon, c^\varepsilon, u^\varepsilon)$ in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^3)$. According to the Fatou lemma, we conclude

$$(\rho^\varepsilon, c^\varepsilon, u^\varepsilon) \in (L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{R}^3))^3.$$

Now we show the continuity of $(\rho^\varepsilon, c^\varepsilon, u^\varepsilon)$ in $H^s\times H^s\times H^s$. Taking the operator $\Delta_q (q\geq 0)$ to the first equation of (3.1) yields

$$\partial_t\Delta_q\rho^\varepsilon - \Delta\Delta_q\rho^\varepsilon = -\Delta_q((u^\varepsilon * \sigma^\varepsilon)\cdot\nabla\rho^\varepsilon) - \nabla\cdot\Delta_q(\rho^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon)) - \Delta_q(\rho^\varepsilon)^3.$$

Operating the L^2-inner product of the above equation with $\Delta_q\rho^\varepsilon$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_q\rho^\varepsilon(t)\|_2^2 + C2^{2q}\|\Delta_q\rho^\varepsilon(t)\|_2^2\\
&\quad= -\langle\Delta_q((u^\varepsilon*\sigma^\varepsilon)\cdot\nabla\rho^\varepsilon), \Delta_q\rho^\varepsilon\rangle - \langle\nabla\cdot\Delta_q(\rho^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon)), \Delta_q\rho^\varepsilon\rangle - \langle\Delta_q(\rho^\varepsilon)^3, \Delta_q\rho^\varepsilon\rangle\\
&\quad\leq C\|\Delta_q(u^\varepsilon\rho^\varepsilon)\|_2^2 + C\|\Delta_q(\rho^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon))\|_2^2 + \frac{C}{4}2^{2q}\|\Delta_q\rho^\varepsilon\|_2^2\\
&\qquad+C2^{-2q}\|\Delta_q(\rho^\varepsilon)^3\|_2^2 + \frac{C}{4}2^{2q}\|\Delta_q\rho^\varepsilon\|_2^2,
\end{aligned}$$

from which it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_q\rho^\varepsilon(t)\|_2^2 + 2^{2q}\|\Delta_q\rho^\varepsilon(t)\|_2^2 \leq C(\|\Delta_q(u^\varepsilon\rho^\varepsilon)\|_2^2 + \|\Delta_q(\rho^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon))\|_2^2 + 2^{-2q}\|\Delta_q(\rho^\varepsilon)^3\|_2^2).$$

Integrating in time t yields

$$\begin{aligned}
&\|\Delta_q\rho^\varepsilon(t)\|_{L^\infty_t L^2}^2 + \int_0^t 2^{2q}\|\Delta_q\rho^\varepsilon(\tau)\|_2^2\,\mathrm{d}\tau\\
&\quad\leq \|\Delta_q\rho_0^\varepsilon\|_2^2 + C\int_0^t(\|\Delta_q(u^\varepsilon\rho^\varepsilon)(\tau)\|_2^2 + \|\Delta_q(\rho^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon))(\tau)\|_2^2 + 2^{-2q}\|\Delta_q(\rho^\varepsilon)^2(\tau)\|_2^2)\,\mathrm{d}\tau.
\end{aligned}$$

Multiplying the above inequality by 2^{2qs} and computing the ℓ^2-norm result in

$$\begin{aligned}
&\Big(\sum_{q\geq 0}2^{2qs}\|\Delta_q\rho^\varepsilon\|_{L^\infty_t L^2}^2\Big)^{\frac12} + \Big(\sum_{q\geq 0}2^{2q(s+1)}\|\Delta_q\rho^\varepsilon\|_{L^2_t L^2}^2\Big)^{\frac12}\\
&\quad\leq \|\rho_0^\varepsilon\|_{H^s} + C\|u^\varepsilon\rho^\varepsilon\|_{\tilde{L}^1_t H^s} + C\|\rho^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon)\|_{\tilde{L}^1_t H^s} + C\|(\rho^\varepsilon)^3\|_{\tilde{L}^1_t H^{s-1}}\\
&\quad\leq \|\rho_0^\varepsilon\|_{H^s} + C\|u^\varepsilon\|_{L^2_t H^s}\|\rho^\varepsilon\|_{L^2_t H^s} + C(\varepsilon)\|\rho^\varepsilon\|_{L^2_t H^s}\|c^\varepsilon\|_{L^2_t L^2} + C\|\rho^\varepsilon\|_{L^\infty_t H^s}\|\rho^\varepsilon\|_{L^2_t H^s}^2 < \infty.
\end{aligned}$$

This combining with the fact $\rho^\varepsilon(t)\in L^\infty_t H^s$ gives us

$$\Big(\sum_{q\geq -1}2^{2qs}\|\Delta_q\rho^\varepsilon\|_{L^\infty_t L^2}^2\Big)^{\frac12} < \infty.$$

Namely, the sequence $\{S_N\rho^\varepsilon\}_{N\in\mathbb{Z}^+}$ converges uniformly to ρ^ε in $L^\infty_t H^s$. Moreover, we can infer from the fact $\partial_t\rho^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s-1}(\mathbb{R}^3))$ that $\rho^\varepsilon\in\mathcal{C}(\mathbb{R}^+; H^{s'}(\mathbb{R}^3))$ with $s' < s$. It implies that $S_N\rho^\varepsilon \in \mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^3))$ for a fixed $N\in\mathbb{Z}^+$. Thus, we know $\rho^\varepsilon\in\mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^3))$. Similarly, we also have that $c^\varepsilon\in\mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^3))$ and $u^\varepsilon\in\mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^3))$.

Step 4. Uniqueness.

Now, let us look at the difference equations to prove the uniqueness of the solutions of (3.1). Assume that $(\rho_1^\varepsilon, c_1^\varepsilon, u_1^\varepsilon)$ and $(\rho_2^\varepsilon, c_2^\varepsilon, u_2^\varepsilon)$ are two solutions of (3.1) with the same initial data $(\rho_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon)$. Let us for simplicity set $\delta\rho^\varepsilon = \rho_1^\varepsilon - \rho_2^\varepsilon$, $\delta c^\varepsilon = c_1^\varepsilon - c_2^\varepsilon$ and $\delta u^\varepsilon = u_1^\varepsilon - u_2^\varepsilon$. Then $(\delta\rho^\varepsilon, \delta c^\varepsilon, \delta u^\varepsilon)$ satisfies

$$\begin{cases} \partial_t \delta\rho^\varepsilon + (\delta u^\varepsilon * \sigma^\varepsilon)\cdot\nabla\rho_1^\varepsilon + (u_2^\varepsilon * \sigma^\varepsilon)\cdot\nabla\delta\rho^\varepsilon = \Delta\delta\rho^\varepsilon - \nabla\cdot(\delta\rho^\varepsilon\nabla(c_1^\varepsilon*\sigma^\varepsilon)) \\ -\nabla\cdot(\rho_2^\varepsilon\nabla(\delta c^\varepsilon * \sigma^\varepsilon)) - \delta\rho^\varepsilon((\rho_1^\varepsilon)^2 + \rho_1^\varepsilon\rho_2^\varepsilon + (\rho_2^\varepsilon)^2), \\ \partial_t\delta c^\varepsilon + (\delta u^\varepsilon * \sigma^\varepsilon)\cdot\nabla c_1^\varepsilon + (u_2^\varepsilon*\sigma^\varepsilon)\cdot\nabla\delta c^\varepsilon = \Delta\delta c^\varepsilon - \delta c^\varepsilon + \delta\rho^\varepsilon, \\ \partial_t\delta u^\varepsilon + (\delta u^\varepsilon*\sigma^\varepsilon)\cdot\nabla u_1^\varepsilon + (u_2^\varepsilon*\sigma^\varepsilon)\cdot\nabla\delta u^\varepsilon + \nabla(P_1^\varepsilon - P_2^\varepsilon) = \Delta\delta u^\varepsilon - (\delta\rho^\varepsilon\nabla\phi)*\sigma^\varepsilon. \end{cases}$$

Making the L^2-inner product of the first equation with $\delta\rho^\varepsilon$ gives

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\rho^\varepsilon(t)\|_2^2 + \|\nabla\delta\rho^\varepsilon(t)\|_2^2 + \int\limits_{\mathbb{R}^3}\big((\rho_1^\varepsilon)^2 + \rho_1^\varepsilon\rho_2^\varepsilon + (\rho_2^\varepsilon)^2\big)(\delta\rho^\varepsilon)^2\,\mathrm{d}x \\ &= -\int\limits_{\mathbb{R}^3}(\delta u^\varepsilon*\sigma^\varepsilon)\cdot\nabla\rho_1^\varepsilon\delta\rho^\varepsilon\,\mathrm{d}x - \int\limits_{\mathbb{R}^3}\nabla\cdot(\delta\rho^\varepsilon\nabla(c_1^\varepsilon*\sigma^\varepsilon))\delta\rho^\varepsilon\,\mathrm{d}x \\ &\quad -\int\limits_{\mathbb{R}^3}\nabla\cdot(\rho_2^\varepsilon\nabla(\delta c^\varepsilon*\sigma^\varepsilon))\delta\rho^\varepsilon\,\mathrm{d}x \\ &\le \|\delta u^\varepsilon*\sigma^\varepsilon\|_2\|\rho_1^\varepsilon\|_\infty\|\nabla\delta\rho^\varepsilon\|_2 + \|\delta\rho^\varepsilon\|_2\|\nabla(c_1^\varepsilon*\sigma^\varepsilon)\|_\infty\|\nabla\delta\rho^\varepsilon\|_2 \\ &\quad + \|\rho_2^\varepsilon\|_2\|\nabla(\delta c^\varepsilon*\sigma^\varepsilon)\|_\infty\|\nabla\delta\rho^\varepsilon\|_2 \\ &\le C\|\rho_1^\varepsilon\|_{H^s}^2\|\delta u^\varepsilon\|_2^2 + C(\varepsilon)(\|\delta\rho^\varepsilon\|_2^2\|c_1^\varepsilon\|_2^2 + \|\rho_2^\varepsilon\|_2^2\|\delta c^\varepsilon\|_2^2) + \frac{1}{2}\|\nabla\delta\rho^\varepsilon\|_2^2. \end{aligned}$$

In a similar process, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta c^\varepsilon(t)\|_2^2 + \|\nabla\delta c^\varepsilon(t)\|_2^2 \\ &= -\int\limits_{\mathbb{R}^3}(\delta u^\varepsilon*\sigma^\varepsilon)\cdot\nabla c_1^\varepsilon\delta c^\varepsilon\,\mathrm{d}x - \int\limits_{\mathbb{R}^3}\delta c^\varepsilon\delta c^\varepsilon\,\mathrm{d}x + \int\limits_{\mathbb{R}^3}\delta\rho^\varepsilon\delta c^\varepsilon\,\mathrm{d}x \\ &\le \|\delta u^\varepsilon*\sigma^\varepsilon\|_2\|c_1^\varepsilon\|_\infty\|\nabla\delta c^\varepsilon\|_2 + \|\delta c^\varepsilon\|_2^2 + \|\delta\rho^\varepsilon\|_2\|\delta c^\varepsilon\|_2 \\ &\le C\|c_1^\varepsilon\|_{H^s}^2\|\delta u^\varepsilon\|_2^2 + C\|\delta c^\varepsilon\|_2^2 + \|\delta\rho^\varepsilon\|_2^2 + \frac{1}{2}\|\nabla\delta c^\varepsilon\|_2^2 \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u^\varepsilon(t)\|_2^2 + \|\nabla\delta u^\varepsilon(t)\|_2^2 \\ &= -\int\limits_{\mathbb{R}^3}(\delta u^\varepsilon*\sigma^\varepsilon)\cdot\nabla u_1^\varepsilon\delta u^\varepsilon\,\mathrm{d}x - \int\limits_{\mathbb{R}^3}((\delta\rho^\varepsilon\nabla\phi)*\sigma^\varepsilon)\delta u^\varepsilon\,\mathrm{d}x \\ &\le \|\delta u^\varepsilon*\sigma^\varepsilon\|_2\|u_1^\varepsilon\|_\infty\|\nabla\delta u^\varepsilon\|_2 + \|(\delta\rho^\varepsilon\nabla\phi)*\sigma^\varepsilon\|_2\|\delta u^\varepsilon\|_2 \\ &\le C\|u_1^\varepsilon\|_{H^s}^2\|\delta u^\varepsilon\|_2^2 + C\|\delta\rho^\varepsilon\|_2^2 + \|\delta u^\varepsilon\|_2^2 + \frac{1}{2}\|\nabla\delta u^\varepsilon\|_2^2. \end{aligned}$$

Summing up the above estimates yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\delta\rho^\varepsilon(t)\|_2^2 + \|\delta c^\varepsilon(t)\|_2^2 + \|\delta u^\varepsilon(t)\|_2^2) + \|\nabla\delta\rho^\varepsilon(t)\|_2^2 + \|\nabla\delta c^\varepsilon(t)\|_2^2 + \|\nabla\delta u^\varepsilon(t)\|_2^2 \\ &\le C\|\rho_1^\varepsilon\|_{H^s}^2\|\delta u^\varepsilon\|_2^2 + C(\varepsilon)(\|\delta\rho^\varepsilon\|_2^2\|c_1^\varepsilon\|_2^2 + \|\rho_2^\varepsilon\|_2^2\|\delta c^\varepsilon\|_2^2) \\ &\quad + C\|c_1^\varepsilon\|_{H^s}^2\|\delta u^\varepsilon\|_2^2 + C\|\delta c^\varepsilon\|_2^2 + \|\delta\rho^\varepsilon\|_2^2 + C\|u_1^\varepsilon\|_{H^s}^2\|\delta u^\varepsilon\|_2^2 + C\|\delta\rho^\varepsilon\|_2^2 + \|\delta u^\varepsilon\|_2^2. \end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\delta\rho^\varepsilon(t)\|_2^2+\|\delta c^\varepsilon(t)\|_2^2+\|\delta u^\varepsilon(t)\|_2^2)$$
$$\leq(\|\delta\rho^\varepsilon\|_2^2+\|\delta c^\varepsilon\|_2^2+\|\delta u^\varepsilon\|_2^2)(\|\rho_1^\varepsilon\|_{H^s}^2+\|\rho_2^\varepsilon\|_{H^s}^2+\|c_1^\varepsilon\|_{H^s}^2+\|u_1^\varepsilon\|_{H^s}^2+1).$$

The Gronwall inequality ensures that $\rho_1^\varepsilon=\rho_2^\varepsilon$, $c_1^\varepsilon=c_2^\varepsilon$, $u_1^\varepsilon=u_2^\varepsilon$ on time interval $[0,T]$.

Next we show the positivity of $n^\varepsilon>0$ and $c^\varepsilon>0$ for all $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$. Let

$$(n^\varepsilon)^-\triangleq\min\{n^\varepsilon,0\}.$$

Multiplying the first equation of (3.1) by $(n^\varepsilon)^-$ and integrating in space variable x, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(n^\varepsilon)^-\|_{L^2}^2+\|\nabla(n^\varepsilon)^-\|_{L^2}^2=-\int_{\mathbb{R}^3}\nabla\cdot(n^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon))(n^\varepsilon)^-\mathrm{d}x-\int_{\mathbb{R}^3}(n^\varepsilon)^3(n^\varepsilon)^-\mathrm{d}x.$$

Integrating by parts and utilizing Höder's inequality imply

$$-\int_{\mathbb{R}^3}\nabla\cdot(n^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon))(n^\varepsilon)^-\mathrm{d}x=\int_{\mathbb{R}^3}n^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon)\nabla(n^\varepsilon)^-\mathrm{d}x$$
$$\leq C(\varepsilon)\|c^\varepsilon\|_{L^2}^2\|(n^\varepsilon)^-\|_{L^2}^2+\frac{1}{4}\|\nabla(n^\varepsilon)^-\|_{L^2}^2.$$

On the other hand, we get

$$-\int_{\mathbb{R}^3}(n^\varepsilon)^3(n^\varepsilon)^-\mathrm{d}x\leq C\|n^\varepsilon\|_{H^s}^2\|(n^\varepsilon)^-\|_{L^2}^2.$$

Collecting the above inequalities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(n^\varepsilon)^-\|_{L^2}^2+\|\nabla(n^\varepsilon)^-\|_{L^2}^2\leq C(\varepsilon)(\|c^\varepsilon\|_{L^2}^2+\|n^\varepsilon\|_{H^s}^2)\|(n^\varepsilon)^-\|_{L^2}^2.$$

Using Gronwall's inequality, we have for any $t>0$

$$\|(n^\varepsilon)^-(t)\|_{L^2}\leq\|(n_0^\varepsilon)^-\|_{L^2}C(t)=0,$$

which gives that $n^\varepsilon\geq 0$ for almost everywhere $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$. Since $s>1$, we infer $H^1(\mathbb{R}^3)\hookrightarrow\mathcal{C}_b(\mathbb{R}^3)$. Thus, we conclude that $n^\varepsilon\geq 0$ for all $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$. By the same process, we can obtain the positivity of $c^\varepsilon>0$ for all $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$. Hence, we complete the proof of Proposition 3.1.

4. Uniform estimates for the regularized problem

In this section, we show uniform estimates of smooth solutions $(\rho^\varepsilon,c^\varepsilon,u^\varepsilon)$ to the regularized problem (3.1) which are independent neither of $\varepsilon>0$ nor the mollifier σ^ε. To simplify the notation, we set $(\rho,c,u)\triangleq(\rho^\varepsilon,c^\varepsilon,u^\varepsilon)$ in the following part of this section.

Proposition 4.1. *Let the initial $(\rho_0,c_0,u_0)\in X\cap(H^{s,\gamma})^3$, and $u_0\in H^2$, $\frac{w_0}{r}\in L^2$. Suppose (ρ,c,u) is a smooth solution of system (3.1). Then there exists a constant C independent of ε such that*

$$\|\rho(t)\|_1+\int_0^t\|\rho(\tau)\|_3^3\,\mathrm{d}\tau\leq\|\rho_0\|_1,\tag{4.1}$$

$$\|u(t)\|_2^2+\int_0^t\|\nabla u(\tau)\|_2^2\,\mathrm{d}\tau\leq Ce^{Ct},\tag{4.2}$$

$$\|c(t)\|_2^2+\int_0^t\|\nabla c(\tau)\|_2^2\,\mathrm{d}\tau\le\|c_0\|_2^2+\|\rho_0\|_1. \tag{4.3}$$

Proof. For (4.1), integrating the first equation of (3.1) in x over $\mathbb{R}^3$, then integrating the resulting equation with respect to time t yields

$$\|\rho(t)\|_1+\int_0^t\|\rho(\tau)\|_3^3\,\mathrm{d}\tau=\|\rho_0\|_1. \tag{4.4}$$

Operating the L^2-inner product for the third equation of (3.1) with u, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2+\|\nabla u\|_2^2&=-\int_{\mathbb{R}^3}(\rho\nabla\phi)*\sigma^\varepsilon u\,\mathrm{d}x\\&\le\|\nabla\phi\|_\infty\|\rho\|_2\|u\|_2.\end{aligned}$$

Performing the Gronwall inequality to the above inequality and using the interpolation inequality, we obtain

$$\|u(t)\|_2^2+\int_0^t\|\nabla u(\tau)\|_2^2\,\mathrm{d}\tau\le Ce^{Ct}. \tag{4.5}$$

Similarly, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c\|_2^2+\|\nabla c\|_2^2+\|c\|_2^2&=\int_{\mathbb{R}^3}\rho c\,\mathrm{d}x\\&\le\|\rho\|_2\|c\|_2\\&\le C\|\rho\|_2^2+\frac{1}{2}\|c\|_2^2.\end{aligned}$$

Then, integrating in time together with (4.4) gives us that

$$\|c(t)\|_2^2+\int_0^t\|\nabla c(\tau)\|_2^2\,\mathrm{d}\tau\le\|c_0\|_2^2+\|\rho_0\|_1. \tag{4.6}$$

This completes the proof of Proposition 4.1. □

Proposition 4.2. *Let the initial* $(\rho_0,c_0,u_0)\in X\cap(H^{s,\gamma})^3$, $\frac{\omega_0}{r}\in L^2$. *Then every solution* (ρ,c,u) *of system* (3.1) *satisfies*

$$\|\frac{\omega_\theta}{r}(t)\|_2^2+\int_0^t\|\nabla(\frac{\omega_\theta}{r})(\tau)\|_2^2\,\mathrm{d}\tau\le Ce^{Ct}, \tag{4.7}$$

$$\|u(t)\|_{H^1}^2+\int_0^t\|u(\tau)\|_{H^2}^2\,\mathrm{d}\tau\le Ce^{e^{Ct}}, \tag{4.8}$$

$$\|\nabla c(t)\|_2^2+\int_0^t\|\nabla^2c(\tau)\|_2^2\,\mathrm{d}\tau\le Ce^{e^{e^{Ct}}}, \tag{4.9}$$

$$\|\rho(t)\|_2^2+\int_0^t(\|\nabla\rho(\tau)\|_2^2+\|\rho(\tau)\|_4^4)\,\mathrm{d}\tau\le Ce^{Ct}. \tag{4.10}$$

Proof. Performing the same process of deducing (1.5), it is clear that $\frac{\omega_\theta}{r}$ satisfies the following equation

$$\begin{aligned}
&\partial_t\left(\frac{\omega_\theta}{r}\right)+(u*\sigma^\varepsilon)\cdot\nabla\left(\frac{\omega_\theta}{r}\right)-\Delta\left(\frac{\omega_\theta}{r}\right)-\frac{2}{r}\partial_r\left(\frac{\omega_\theta}{r}\right)\\
&=\bigg(-\frac{1}{r^2}\partial_z\rho x^h\cdot\nabla_h\phi-\frac{n}{r^2}x^h\cdot\nabla_h(\partial_z\phi)+\frac{\partial_r\rho\partial_z\phi}{r}+\frac{\rho\partial^2_{r,z}\phi}{r}\bigg)*\sigma^\varepsilon\\
&\quad+\bigg(\frac{u^r}{r^2}(\omega*\sigma^\varepsilon)e_\theta-\frac{u^r*\sigma^\varepsilon}{r^2}\omega_\theta\bigg).
\end{aligned}$$

Taking the L^2-inner product of the above equality with $\frac{\omega_\theta}{r}$ gives

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\|\frac{\omega_\theta}{r}\Big\|_2^2+\Big\|\nabla\frac{\omega_\theta}{r}(t)\Big\|_2^2\\
&\le-\int_{\mathbb{R}^3}\frac{1}{r^2}\partial_z\rho x^h\cdot\nabla_h\phi\Big(\frac{\omega_\theta}{r}*\sigma^\varepsilon\Big)\,\mathrm{d}x-\int_{\mathbb{R}^3}\frac{\rho}{r^2}x^h\cdot\nabla_h(\partial_z\phi)\Big(\frac{\omega_\theta}{r}*\sigma^\varepsilon\Big)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\frac{\partial_r\rho\partial_z\phi}{r}\Big(\frac{\omega_\theta}{r}*\sigma^\varepsilon\Big)\,\mathrm{d}x+\int_{\mathbb{R}^3}\frac{\rho\partial^2_{r,z}\phi}{r}\Big(\frac{\omega_\theta}{r}*\sigma^\varepsilon\Big)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\Big(\frac{u^r}{r^2}(\omega*\sigma^\varepsilon)e_\theta-\frac{u^r*\sigma^\varepsilon}{r^2}\omega_\theta\Big)\frac{\omega_\theta}{r}\,\mathrm{d}x\\
&=I_1+I_2+I_3+I_4+I_5.
\end{aligned}\tag{4.11}$$

Next, we estimate the right-hand side of (4.11). An integration by parts yields

$$\begin{aligned}
I_1&=\int_{\mathbb{R}^3}\frac{1}{r^2}\rho x^h\partial_z(\nabla_h\phi)(\frac{\omega_\theta}{r}*\sigma^\varepsilon)\,\mathrm{d}x+\int_{\mathbb{R}^3}\frac{1}{r^2}\rho x^h\nabla_h\phi\partial_z(\frac{\omega_\theta}{r}*\sigma^\varepsilon)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^3}\rho\frac{x^h}{r}\frac{\partial_z(\nabla_h\phi)-\partial_z(\nabla_h\phi(0,0,x_3))}{r}(\frac{\omega_\theta}{r}*\sigma^\varepsilon)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\rho\frac{x^h}{r}\frac{(\nabla_h\phi)-(\nabla_h\phi(0,0,x_3))}{r}\partial_z(\frac{\omega_\theta}{r}*\sigma^\varepsilon)\,\mathrm{d}x\\
&\le\|\rho\|_2\Big\|\frac{x^h}{r}\Big\|_\infty\|\nabla^3\phi\|_\infty\Big\|\frac{\omega_\theta}{r}*\sigma^\varepsilon\Big\|_2+\|\rho\|_2\Big\|\frac{x^h}{r}\Big\|_\infty\|\nabla^2\phi\|_\infty\Big\|\nabla\frac{\omega_\theta}{r}*\sigma^\varepsilon\Big\|_2\\
&\le C\|\rho\|_2^2+\Big\|\frac{\omega_\theta}{r}\Big\|_2^2+\frac{1}{8}\Big\|\nabla\frac{\omega_\theta}{r}\Big\|_2^2.
\end{aligned}$$

Similarly,

$$I_2\le\|\rho\|_2\Big\|\frac{x^h}{r}\Big\|_\infty\|\nabla^3\phi\|_\infty\Big\|\frac{\omega_\theta}{r}\Big\|_2\le C\|\rho\|_2^2+\Big\|\frac{\omega_\theta}{r}\Big\|_2^2.$$

Employing coordinate transformation, we compute I_3 and I_4, respectively,

$$\begin{aligned}
I_3&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\partial_r\rho\partial_z\phi}{r}\left(\frac{\omega_\theta}{r}*\sigma^\varepsilon\right)r\,\mathrm{d}r\mathrm{d}z\\
&=-\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\rho\partial_r\partial_z\phi(\frac{\omega_\theta}{r}*\sigma^\varepsilon)\,\mathrm{d}r\mathrm{d}z-\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\rho\partial_z\phi\partial_r(\frac{\omega_\theta}{r}*\sigma^\varepsilon)\,\mathrm{d}r\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^3} \rho \frac{\partial_r \partial_z \phi}{r} (\frac{\omega_\theta}{r} * \sigma^\varepsilon) \, \mathrm{d}x - \int_{\mathbb{R}^3} \rho \frac{\partial_z \phi}{r} \partial_r (\frac{\omega_\theta}{r} * \sigma^\varepsilon) \, \mathrm{d}x \\
&\le \|\rho\|_2 \|\nabla^3 \phi\|_\infty \left\| \frac{\omega_\theta}{r} * \sigma^\varepsilon \right\|_2 + \|\rho\|_2 \|\nabla^2 \phi\|_\infty \left\| \nabla \frac{\omega_\theta}{r} * \sigma^\varepsilon \right\|_2 \\
&\le C\|\rho\|_2^2 + \left\| \frac{\omega_\theta}{r} \right\|_2^2 + \frac{1}{8} \left\| \nabla \frac{\omega_\theta}{r} \right\|_2^2
\end{aligned}
$$

and

$$
I_4 \le \|\rho\|_2 \|\nabla^3 \phi\|_\infty \left\| \frac{\omega_\theta}{r} \right\|_2 \le C\|\rho\|_2^2 + \left\| \frac{\omega_\theta}{r} \right\|_2^2 .
$$

For the last term, we have

$$
\begin{aligned}
I_5 &= \int_{\mathbb{R}^3} \left(\frac{u^r}{r^2} (\omega * \sigma^\varepsilon) e_\theta - \frac{u^r * \sigma^\varepsilon}{r^2} \omega_\theta \right) \frac{\omega_\theta}{r} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{u^r}{r^2} \left(\int_{\mathbb{R}^3} \omega(y) \sigma^\varepsilon(x-y) \mathrm{d}y e_\theta \right) \frac{\omega_\theta}{r} \, \mathrm{d}x - \int_{\mathbb{R}^3} \frac{u^r * \sigma^\varepsilon}{r^2} \omega_\theta \frac{\omega_\theta}{r} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{u^r}{r^2} \left(\int_{\mathbb{R}^3} \omega_\theta(y) e_\theta \sigma^\varepsilon(x-y) \mathrm{d}y e_\theta \right) \frac{\omega_\theta}{r} \, \mathrm{d}x - \int_{\mathbb{R}^3} \frac{u^r * \sigma^\varepsilon}{r^2} \omega_\theta \frac{\omega_\theta}{r} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{u^r}{r^2} \left(\int_{\mathbb{R}^3} \omega_\theta(y) \sigma^\varepsilon(x-y) \mathrm{d}y \right) \frac{\omega_\theta}{r} \, \mathrm{d}x - \int_{\mathbb{R}^3} \frac{u^r * \sigma^\varepsilon}{r^2} \omega_\theta \frac{\omega_\theta}{r} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{u^r}{r^2} \left(\int_{\mathbb{R}^3} \omega_\theta(y) \sigma^\varepsilon(x-y) \frac{\omega_\theta}{r} \mathrm{d}y \right) \mathrm{d}x - \int_{\mathbb{R}^3} \frac{u^r * \sigma^\varepsilon}{r^2} \omega_\theta \frac{\omega_\theta}{r} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \omega_\theta(y) \int_{\mathbb{R}^3} \frac{u^r}{r^2}(x) \sigma^\varepsilon(y-x) \frac{\omega_\theta}{r} \mathrm{d}x \, \mathrm{d}y - \int_{\mathbb{R}^3} \frac{u^r * \sigma^\varepsilon}{r^2} \omega_\theta \frac{\omega_\theta}{r} \, \mathrm{d}x = 0.
\end{aligned}
$$

Combining the estimates above yields

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left\| \frac{\omega_\theta}{r}(t) \right\|_2^2 + \left\| \nabla (\frac{\omega_\theta}{r})(t) \right\|_2^2 \le C\|\rho\|_2^2 + C \left\| \frac{\omega_\theta}{r} \right\|_2^2 ,
$$

from which (4.7) follows.

Next we prove (4.8), applying the curl to the third equation of (3.1) yields

$$
\partial_t \omega + (u * \sigma^\varepsilon) \cdot \nabla \omega - \Delta \omega = \frac{u^r}{r} (\omega * \sigma^\varepsilon) - \mathrm{curl}(\rho \nabla \phi) * \sigma^\varepsilon. \tag{4.12}
$$

Taking the L^2-inner product with ω for (4.12), we get

$$
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\omega(t)\|_2^2 + \|\nabla \omega(t)\|_2^2 \le C \left\| \frac{u^r}{r} \right\|_\infty \left\| \omega \right\|_2^2 + C\|\rho\|_2^2 + \frac{1}{2} \|\nabla \omega\|_2^2 ,
$$

from which we conclude

$$
\frac{\mathrm{d}}{\mathrm{d}t} \|\omega(t)\|_2^2 + \|\nabla \omega(t)\|_2^2 \le C \left\| \frac{u^r}{r} \right\|_\infty \|\omega\|_2^2 + C\|\rho\|_2^2 .
$$

Using the Gronwall inequality, we infer

$$
\|\omega(t)\|_2^2 + \int_0^t \|\nabla \omega(t)\|_2^2 \, \mathrm{d}\tau \le \left(\|\omega_0\|_2^2 + C \int_0^t \|\rho(\tau)\|_2^2 \mathrm{d}\tau \right) e^{C \int_0^t \| \frac{u^r}{r} \|_\infty \mathrm{d}\tau} . \tag{4.13}
$$

By Lemma 2.2, we have

$$\begin{aligned}\int_0^t \Big\|\frac{u^r}{r}\Big\|_\infty \mathrm{d}\tau &\le C\int_0^t \Big\|\frac{\omega_\theta}{r}(\tau)\Big\|_2^{\frac12}\Big\|\frac{\omega_\theta}{r}(\tau)\Big\|_{\dot H^1}^{\frac12}\mathrm{d}\tau\\ &\le C\Big(\int_0^t 1\mathrm{d}\tau\Big)^{\frac12}\Big(\int_0^t \|\frac{\omega_\theta}{r}(\tau)\|_2^2\mathrm{d}\tau\Big)^{\frac14}\Big(\int_0^t \|\frac{\omega_\theta}{r}(\tau)\|_{\dot H^1}^2\mathrm{d}\tau\Big)^{\frac14}\\ &\le C\Big(t+\int_0^t \Big\|\frac{\omega_\theta}{r}(\tau)\Big\|_2^2\mathrm{d}\tau+\int_0^t \Big\|\frac{\omega_\theta}{r}(\tau)\Big\|_{\dot H^1}^2\mathrm{d}\tau\Big)\\ &\le Ce^{Ct}.\end{aligned}$$

Plugging the above inequality into (4.13), we obtain

$$\|\omega(t)\|_2^2+\int_0^t \|\nabla\omega(t)\|_2^2\,\mathrm{d}\tau \le Ce^{e^{Ct}},$$

which together with (4.2) yields (4.8).

Taking ∂_i on both sides of the second equation of (3.1) implies

$$\partial_t\partial_i c+(u*\sigma^\varepsilon)\cdot\nabla\partial_i c-\Delta\partial_i c=-\partial_i c+\partial_i\rho-\partial_i(u*\sigma^\varepsilon)\cdot\nabla c.$$

Multiplying the above equality by $\partial_i c$ and integrating with respect to space variable yield

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i c(t)\|_2^2+\int_{\mathbb{R}^3}|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x\\ &=-\int_{\mathbb{R}^3}\partial_i c\partial_i c\,\mathrm{d}x+\int_{\mathbb{R}^3}\partial_i\rho\partial_i c\,\mathrm{d}x-\sum_{j=1}^3\int_{\mathbb{R}^3}\partial_i(u^j*\sigma^\varepsilon)\partial_j c\partial_i c\,\mathrm{d}x.\end{aligned}\tag{4.14}$$

For the last term, we have

$$\begin{aligned}&-\int_{\mathbb{R}^3}\partial_i(u^j*\sigma^\varepsilon)\partial_j c\partial_i c\,\mathrm{d}x\\ &=\sum_{j=1}^3\int_{\mathbb{R}^3}(u^j*\sigma^\varepsilon)\partial_{ij}^2 c\partial_i c\,\mathrm{d}x+\sum_{j=1}^3\int_{\mathbb{R}^3}(u^j*\sigma^\varepsilon)\partial_j c\partial_{ii}^2 c\,\mathrm{d}x\\ &\le 2\|u\|_\infty\|\nabla c\|_2\|\Delta c\|_2.\end{aligned}$$

Insert the above estimate into (4.14), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c\|_2^2+\int_{\mathbb{R}^3}|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x\\ &\le C\|u\|_\infty\|\nabla c\|_2\|\Delta c\|_2+\|\nabla c\|_2^2+\|\rho\|_2\|\Delta c\|_2\\ &\le C\|u\|_\infty^2\|\nabla c\|_2^2+\|\nabla c\|_2^2+C\|\rho\|_2^2+\frac14\|\Delta c\|_2^2.\end{aligned}$$

By the Gronwall inequality, we get

$$\|\nabla c(t)\|_2^2+\int_0^t \|\nabla^2 c(\tau)\|_2^2\,\mathrm{d}\tau\le Ce^{e^{e^{Ct}}},$$

which implies (4.9).

Next, we turn to estimate the term of $\|\nabla c\|_3$.

$$\begin{aligned}\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\partial_i c(t)\big\|_3^3 + \int\limits_{\mathbb{R}^3} \partial_i c(x,t)|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x \\ = -\int\limits_{\mathbb{R}^3} \partial_i c|\partial_i c|\partial_i c\mathrm{d}x + \int\limits_{\mathbb{R}^3} \partial_i\rho|\partial_i c|\partial_i c\,\mathrm{d}x - \sum_{j=1}^{3}\int\limits_{\mathbb{R}^3} \partial_i(u^j * \sigma^\varepsilon)\partial_j c|\partial_i c|\partial_i c\,\mathrm{d}x.\end{aligned} \tag{4.15}$$

By the Hölder inequality, we have

$$\begin{aligned}\int\limits_{\mathbb{R}^3} \partial_i\rho|\partial_i c|\partial_i c\,\mathrm{d}x =& -2\int\limits_{\mathbb{R}^3} \rho|\partial_i c|\partial_{ii}^2 c\,\mathrm{d}x \\ &\le C\|\rho\|_3\|\partial_i c\|_3^{\frac{1}{2}}\Big(\int\limits_{\mathbb{R}^3} |\partial_i c(x,t)|\,|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \\ &\le C\|\rho\|_3^2\|\partial_i c\|_3 + \frac{1}{4}\int\limits_{\mathbb{R}^3} |\partial_i c(x,t)|\,|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x.\end{aligned} \tag{4.16}$$

Similarly, we obtain

$$\begin{aligned}-\sum_{j=1}^{3}\int\limits_{\mathbb{R}^3} \partial_i(u^j * \rho^\varepsilon)\partial_j c|\partial_i c|\partial_i c\,\mathrm{d}x =& 2\sum_{j=1}^{3}\int\limits_{\mathbb{R}^3} \partial_i(u^j * \rho^\varepsilon)c|\partial_i c|\partial_{ij}^2 c\,\mathrm{d}x \\ &\le 2\|c\|_\infty\|\nabla u\|_3\|\partial_i c\|_3^{\frac{1}{2}}\Big(\int\limits_{\mathbb{R}^3} |\partial_i c(x,t)|\,|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \\ &\le 2\|c\|_\infty^2\|\nabla u\|_3^2\|\partial_i c\|_3 + \frac{1}{3}\int\limits_{\mathbb{R}^3} |\partial_i c(x,t)|\,|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x \\ &\le 2\|c\|_\infty^2\|\nabla u\|_2\|\Delta u\|_2\|\partial_i c\|_3 + \frac{1}{4}\int\limits_{\mathbb{R}^3} |\partial_i c(x,t)|\,|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x.\end{aligned}$$

Plugging the above estimate and (4.16) into (4.15), we get

$$\begin{aligned}\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\partial_i c(t)\big\|_3^3 + \int\limits_{\mathbb{R}^3} \partial_i c(x,t)|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x \\ \le \|\partial_i c\|_3^3 + \|c\|_\infty^2\|n\|_3^2\big\|\partial_i c(t)\big\|_3 + \|c\|_\infty^2\|\nabla u\|_2\|\Delta u\|_2\big\|\partial_i c(t)\big\|_3.\end{aligned} \tag{4.17}$$

This means

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\nabla c(t)\big\|_3^2 \le \|\nabla c\|_3^2 + \|c\|_\infty^2\|n\|_3^2 + \|c\|_\infty^2\|\nabla u\|_2\|\Delta u\|_2.$$

Using Gronwall's inequality yields

$$\big\|\nabla c(t)\big\|_3 \le C\big\|\nabla c_0\big\|_3 e^{e^{Ct}}.$$

Plugging this estimate into (4.17), we finally have

$$\big\|\nabla c(t)\big\|_3^3 + \int\limits_0^t \Big\|\nabla|\nabla c(\tau)|^{\frac{3}{2}}\Big\|_2^2\,\mathrm{d}\tau \le Ce^{e^{Ct}}.$$

For (4.10), making the L^2-inner product of the first equation of (3.1) with ρ, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(t)\|_2^2+\|\nabla\rho(t)\|_2^2+\|\rho(t)\|_4^4=-\int_{\mathbb{R}^3}\nabla\cdot\big(\rho\nabla c\big)\cdot\rho\,\mathrm{d}x. \tag{4.18}$$

Integrating by parts and using the Hölder inequality, we conclude

$$\begin{aligned}-\int_{\mathbb{R}^3}\nabla\cdot\big(\rho\nabla c\big)\cdot\rho\,\mathrm{d}x=&\int_{\mathbb{R}^3}\rho\nabla c\nabla\rho\,\mathrm{d}x\\ \le& C\|\rho\|_3\left\||\nabla c|^{\frac{3}{2}}\right\|_4^{\frac{2}{3}}\|\nabla\rho\|_2\\ \le& C\|\rho\|_2^{\frac{1}{2}}\left\||\nabla c|^{\frac{3}{2}}\right\|_2^{\frac{1}{6}}\left\||\nabla c|^{\frac{3}{2}}\right\|_6^{\frac{1}{2}}\|\nabla\rho\|_2^{\frac{3}{2}}\\ \le& C\,\|\nabla c\|_3\left\|\nabla|\nabla c|^{\frac{3}{2}}\right\|_2^2\|\rho\|_2^2+\frac{1}{4}\|\nabla\rho\|_2^2.\end{aligned}$$

Inserting this estimate into (4.18) and integrating the resulting inequality in time t give

$$\begin{aligned}&\|\rho(t)\|_2^2+\int_0^t\|\nabla\rho(\tau)\|_2^2\,\mathrm{d}\tau+\int_0^t\|\rho(\tau)\|_4^4\,\mathrm{d}\tau\\ &\le\|\rho_0\|_{L^2}^2+C\int_0^t\|\nabla c(\tau)\|_3\left\|\nabla|\nabla c|^{\frac{3}{2}}(\tau)\right\|_2^2\|\rho(\tau)\|_{L^2}^2\,\mathrm{d}\tau.\end{aligned} \tag{4.19}$$

Using Gronwall's inequality means

$$\|\rho(t)\|_2^2+\int_0^t\|\nabla\rho(\tau)\|_2^2\,\mathrm{d}\tau+\int_0^t\|\rho(\tau)\|_4^4\,\mathrm{d}\tau\le Ce^{Ct}.$$

Thus, we have finished the proof of Proposition 4.2. □

Proposition 4.3. *Let the initial* $(\rho_0,c_0,u_0)\in X\cap(H^{s,\gamma})^3$. *Then every solution* (ρ,c,u) *of system* (3.1) *satisfies*

$$\|\Delta u(t)\|_2^2+\int_0^t\|\nabla\Delta u(\tau)\|_2^2\,\mathrm{d}\tau\le \mathrm{Ce}^{\mathrm{e}^{\mathrm{Ct}}}, \tag{4.20}$$

$$\|\Delta c(t)\|_2^2+\int_0^t\|\nabla\Delta c(\tau)\|_2^2\,\mathrm{d}\tau\le \mathrm{Ce}^{\mathrm{e}^{\mathrm{Ct}}}, \tag{4.21}$$

$$\|\nabla\rho\|_2+\int_0^t\|\Delta\rho(\tau)\|_2^2\mathrm{d}\tau\le Ce^{e^{Ct}}. \tag{4.22}$$

Proof. Taking the Δ to the third equation of (3.1) yields

$$\partial_t\Delta u-\Delta\Delta u+\nabla\Delta P=-\Delta((u*\sigma^\varepsilon)\cdot\nabla u)-\Delta((\rho\nabla\phi)*\sigma^\varepsilon). \tag{4.23}$$

Multiplying (4.23) with Δu and integrating in spaces, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_2^2+\|\nabla\Delta u\|_2^2=-\int_{\mathbb{R}^3}\Delta((u*\sigma^\varepsilon)\cdot\nabla u)\Delta u\,\mathrm{d}x-\int_{\mathbb{R}^3}\Delta((\rho\nabla\phi)*\sigma^\varepsilon)\Delta u\,\mathrm{d}x. \tag{4.24}$$

For the first term, we get

$$\begin{aligned}-\int_{\mathbb{R}^3}\Delta((u*\sigma^\varepsilon)\cdot\nabla u)\Delta u\,\mathrm{d}x &= -\int_{\mathbb{R}^3}\Delta(u*\sigma^\varepsilon\cdot\nabla u\Delta u\,\mathrm{d}x - 2\int_{\mathbb{R}^3}\nabla(u*\sigma^\varepsilon)\cdot\nabla^2 u\Delta u\,\mathrm{d}x\\ &\le C\|\nabla u\|_2\|\Delta u\|_4^2 \le C\|\nabla u\|_2\|\Delta u\|_2^{\frac{1}{2}}\|\nabla\Delta u\|_2^{\frac{3}{2}}\\ &\le C\|\nabla u\|_2^4\|\Delta u\|_2^2+\frac{1}{4}\|\nabla\Delta u\|_2^2.\end{aligned}$$

For the second term, integration by parts implies

$$\begin{aligned}-\int_{\mathbb{R}^3}\Delta((\rho\nabla\phi)*\sigma^\varepsilon)\Delta u\,\mathrm{d}x &= \int_{\mathbb{R}^3}\nabla((\rho\nabla\phi)*\sigma^\varepsilon)\nabla\Delta u\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}(\nabla\rho\nabla\phi)*\sigma^\varepsilon\nabla\Delta u\,\mathrm{d}x+\int_{\mathbb{R}^3}(\rho\nabla^2\phi)*\sigma^\varepsilon\nabla\Delta u\,\mathrm{d}x\\ &\le \|\nabla\rho\|_2\|\nabla\phi\|_\infty\|\nabla\Delta u\|_2+\|\rho\|_2\|\nabla^2\phi\|_\infty\|\nabla\Delta u\|_2\\ &\le C(\|\nabla\rho\|_2^2+\|\rho\|_2^2)+\frac{1}{4}\|\nabla\Delta u\|_2^2.\end{aligned}$$

Inserting the above inequalities to (4.24), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_2^2+\|\nabla\Delta u\|_2^2 \le C\|\nabla u\|_2^4\|\Delta u\|_2^2+C(\|\nabla\rho\|_2^2+\|\rho\|_2^2).$$

By Gronwall's inequality, we derive (4.20).
Taking the Δ to the second equation of (3.1) yields

$$\partial_t\Delta c-\Delta\Delta c=-\Delta((u*\sigma^\varepsilon)\cdot\nabla c)-\Delta c+\Delta\rho. \tag{4.25}$$

Multiplying (4.25) with Δc and integrating in spaces give

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta c\|_2^2+\|\nabla\Delta c\|_2^2 &= -\int_{\mathbb{R}^3}\Delta((u*\sigma^\varepsilon)\cdot\nabla c)\Delta c\,\mathrm{d}x-\int_{\mathbb{R}^3}\Delta c\Delta c\,\mathrm{d}x+\int_{\mathbb{R}^3}\Delta\rho\Delta c\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}\nabla((u*\sigma^\varepsilon)\cdot\nabla c)\nabla\Delta c\,\mathrm{d}x-\int_{\mathbb{R}^3}\Delta c\Delta c\,\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\rho\nabla\Delta c\,\mathrm{d}x\\ &\le \|\nabla u\|_\infty\|\nabla c\|_2\|\nabla\Delta c\|_2+\|u\|_\infty\|\nabla^2 c\|_2\|\nabla\Delta c\|_2+\|\Delta c\|_2^2+\|\nabla\rho\|_2\|\nabla\Delta c\|_2\\ &\le C\|\nabla u\|_\infty^2\|\nabla c\|_2^2+C\|u\|_\infty^2\|\nabla^2 c\|_2^2+\|\Delta c\|_2^2+C\|\nabla\rho\|_2^2+\frac{1}{2}\|\nabla\Delta c\|_2^2.\end{aligned}$$

Using the Gronwall inequality, we obtain (4.21).
Taking ∇ to the first equation of (3.1) yields

$$\partial_t\nabla\rho+\nabla(u\cdot\nabla\rho)-\nabla\Delta\rho=-\nabla\nabla\cdot(\rho\nabla c)-\nabla(\rho^3).$$

Multiplying the above equality with $\nabla\rho$ and integrating with respect to space variable imply

$$\begin{aligned}&\frac{1}{2}\|\nabla\rho\|_2^2+\|\Delta\rho\|_2^2+3\|\rho\nabla\rho\|_2^2\\ &= \int_{\mathbb{R}^3}\nabla u\cdot\nabla\rho\nabla\rho\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\nabla\cdot(\rho\nabla c)\nabla\rho\mathrm{d}x-\int_{\mathbb{R}^3}\nabla(\rho^3)\nabla\rho\mathrm{d}x\\ &\le \|\nabla u\|_\infty\|\nabla\rho\|_2^2+\|\nabla\rho\|_2\|\nabla c\|_\infty\|\Delta\rho\|_2+\|\rho\|_6\|\Delta c\|_3\|\Delta\rho\|_2\\ &\le \|u\|_{H^3}\|\nabla\rho\|_2^2+C\|\nabla\rho\|_2^2\|c\|_{H^3}^2+\|\nabla\rho\|_2^2\|\Delta c\|_2\|\nabla\Delta c\|_2+\frac{1}{2}\|\Delta c\|_2^2,\end{aligned}$$

from which it follows that

$$\begin{aligned}&\frac{1}{2}\|\nabla\rho\|_2^2+\|\Delta\rho\|_2^2+3\|\rho\nabla\rho\|_2^2\\&\quad\le\|u\|_{H^3}\|\nabla\rho\|_2^2+C\|\nabla\rho\|_2^2\|c\|_{H^3}^2+\|\nabla\rho\|_2^2\|\Delta c\|_2\|\nabla\Delta c\|_2.\end{aligned}$$

Using Gronwall's inequality and combining with (4.20)–(4.21) means (4.22). This completes the proof of Proposition 4.3. □

5. Proof of the main result

This section is devoted to presenting the proof of Theorem 1.1. We first prove the existence. Considering the following approximation scheme:

$$\begin{cases}\partial_t\rho^\varepsilon+(u^\varepsilon*\sigma^\varepsilon)\cdot\nabla\rho^\varepsilon-\Delta\rho^\varepsilon=-\nabla\cdot(\rho^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon))-(\rho^\varepsilon)^3,\\ \partial_t c^\varepsilon+(u^\varepsilon*\sigma^\varepsilon)\cdot\nabla c^\varepsilon-\Delta c^\varepsilon=-c^\varepsilon+\rho^\varepsilon,\\ \partial_t u^\varepsilon+(u^\varepsilon*\sigma^\varepsilon)\cdot\nabla u^\varepsilon-\Delta u^\varepsilon+\nabla P^\varepsilon=-(\rho^\varepsilon\nabla\phi)*\sigma^\varepsilon,\\ \mathrm{div}\,\mathrm{u}^\varepsilon=0,\\ (\rho^\varepsilon,c^\varepsilon,u^\varepsilon)|_{t=0}=(\rho_0*\sigma^\varepsilon,c_0*\sigma^\varepsilon,u_0*\sigma^\varepsilon).\end{cases}\tag{5.1}$$

The property of the mollifier σ^ε and $(\rho_0,c_0,u_0)\in X$ enable us to know that $(\rho_0^\varepsilon,c_0^\varepsilon,u_0^\varepsilon)\in X_0\cap(H^\infty)^3$ with $H^\infty\triangleq\cap_{s\ge0}H^s$. Proposition 3.1 ensures us that system (5.1) admits a unique globally smooth solution. In addition, from Sect. 4, we have the following bounds uniform in ε:

$$\begin{aligned}&\rho^\varepsilon\in L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^3))\cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{R}^3))\cap L^2_{\mathrm{loc}}(\mathbb{R}^+;H^2(\mathbb{R}^3)),\\&c^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^2(\mathbb{R}^3))\cap L^2_{\mathrm{loc}}(\mathbb{R}^+;H^3(\mathbb{R}^3)),\\&u^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^2(\mathbb{R}^3))\cap L^2_{\mathrm{loc}}(\mathbb{R}^+;H^3(\mathbb{R}^3)).\end{aligned}$$

Now in order to apply the Aubin–Lions compactness lemma, it is suffices to prove the uniform boundedness for $\partial_t\rho^k$, $\partial_t c^k$ and $\partial_t u^k$. From the first equation of (5.1), we get

$$\begin{aligned}\|\partial_t\rho^\varepsilon\|_{L^2_tH^{-1}}&=\|\Delta\rho^\varepsilon\|_{L^2_tH^{-1}}+\|(u^\varepsilon*\sigma^\varepsilon)\cdot\nabla\rho^\varepsilon\|_{L^2_tH^{-1}}+\|\nabla\cdot(\rho^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon))\|_{L^2_tH^{-1}}+\|(\rho^\varepsilon)^3\|_{L^2_tH^{-1}}\\&\le\|\rho^\varepsilon\|_{L^2_tH^1}+\|u^\varepsilon\cdot\rho^\varepsilon\|_{L^2_tL^2}+\|\rho^\varepsilon\nabla(c^\varepsilon*\sigma^\varepsilon)\|_{L^2_tL^2}+\|(\rho^\varepsilon)^3\|_{L^2_tL^2}\\&\le\|\rho^\varepsilon\|_{L^2_tH^1}+\|u^\varepsilon\|_{L^\infty_tL^\infty}\|\rho^\varepsilon\|_{L^2_tL^2}+\|\rho^\varepsilon\|_{L^\infty_tL^2}\|\nabla c^\varepsilon\|_{L^2_tL^\infty}+\|\rho^\varepsilon\|^2_{L^\infty_tL^6}\|\rho^\varepsilon\|_{L^2_tL^6}\\&\le\|\rho^\varepsilon\|_{L^2_tH^1}+\|u^\varepsilon\|_{L^\infty_tH^2}\|\rho^\varepsilon\|_{L^2_tL^2}+\|\rho^\varepsilon\|_{L^\infty_tL^2}\|c^\varepsilon\|_{L^2_tH^3}+\|\rho^\varepsilon\|^2_{L^\infty_tH^1}\|\rho^\varepsilon\|_{L^2_tH^2}\\&\le C.\end{aligned}$$

This means that $\partial_t\rho^\varepsilon$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^+;H^{-1})$. Similarly, we can also deduce that $\partial_t c^\varepsilon$ and $\partial_t u^\varepsilon$ are bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^+;H^{-1})$, respectively. Notice that L^2 is locally compactly embedded in H^s and H^s continuously embedded in H^{-1} with $s\in(-1,0)$. Repeating the process used in the proof of Proposition 3.1, we conclude that the sequence $(\rho^\varepsilon,c^\varepsilon,u^\varepsilon)$ converges to (ρ,c,u) in $\mathcal{C}(\mathbb{R}^+;H^s)$ with $s<0$ and the solution (ρ,c,u) satisfies (1.4) in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^3)$. According to Fatou's lemma, we obtain

$$\begin{aligned}&\rho\in L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^3))\cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{R}^3))\cap L^2_{\mathrm{loc}}(\mathbb{R}^+;H^2(\mathbb{R}^3)),\\&c\in L^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^2(\mathbb{R}^3))\cap L^2_{\mathrm{loc}}(\mathbb{R}^+;H^3(\mathbb{R}^3)),\\&u\in L^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^2(\mathbb{R}^3))\cap L^2_{\mathrm{loc}}(\mathbb{R}^+;H^3(\mathbb{R}^3)).\end{aligned}$$

Next, we prove the uniqueness. Let us consider the two solutions (ρ_1,c_1,u_1) and (ρ_2,c_2,u_2) of system (1.4) associated with the same initial data (ρ_0,c_0,u_0), then the different $(\delta\rho,\delta c,\delta u)$ solve the following difference equations.

$$
\begin{cases}
\partial_t \delta\rho + \delta u\cdot\nabla\rho_1 + u_2\cdot\nabla\delta\rho = \Delta\delta\rho - \nabla\cdot(\delta\rho\nabla c_1) - \nabla\cdot(\rho_2\nabla\delta c) - \delta\rho(\rho_1^2+\rho_1\rho_2+\rho_2^2),\\
\partial_t \delta c + \delta u\cdot\nabla c_1 + u_2\cdot\nabla\delta c = \Delta\delta c - \delta c + \delta\rho,\\
\partial_t \delta u + \delta u\cdot\nabla u_1 + u_2\cdot\nabla\delta u + \nabla(P_1-P_2) = \Delta\delta u - \delta\rho\nabla\phi.
\end{cases} \tag{5.2}
$$

Operating the L^2-inner product of the first equation with $\delta\rho$ gives

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\rho(t)\|_2^2 + \|\nabla\delta\rho(t)\|_2^2 &= -\int_{\mathbb{R}^3}(\delta u\cdot\nabla\rho_1)\delta\rho\,\mathrm{d}x - \int_{\mathbb{R}^3}\nabla\cdot(\delta\rho\nabla c_1)\delta\rho\,\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^3}\nabla\cdot(\rho_2\nabla\delta c)\delta\rho\,\mathrm{d}x - \int_{\mathbb{R}^3}(\rho_1^2+\rho_1\rho_2+\rho_2^2)\delta\rho\delta\rho\,\mathrm{d}x\\
&\triangleq V_1+V_2+V_3+V_4.
\end{aligned} \tag{5.3}
$$

For IV_1, by Young's inequality, we get

$$
V_1 = \int_{\mathbb{R}^3}\rho_1\delta u\cdot\nabla\delta\rho\,\mathrm{d}x \le \|\delta u\|_6\|\rho_1\|_3\|\nabla\delta\rho\|_2 \le C\|\nabla\delta u\|_2^2\|\rho_1\|_3^2 + \frac{1}{8}\|\nabla\delta\rho\|_2^2. \tag{5.4}
$$

By the Hölder inequality, we have

$$
\begin{aligned}
V_2 &= \int_{\mathbb{R}^3}\delta\rho\nabla c_1\cdot\nabla\delta\rho\,\mathrm{d}x\\
&\le \|\delta\rho\|_2\|\nabla c_1\|_\infty\|\nabla\delta\rho\|_2\\
&\le \|\delta\rho\|_2^2\|c_1\|_{H^3}^2 + \frac{1}{8}\|\nabla\delta\rho\|_2^2.
\end{aligned} \tag{5.5}
$$

In a similar process,

$$
V_3 = \int_{\mathbb{R}^3}\rho_2\nabla\delta c\cdot\nabla\delta\rho\,\mathrm{d}x \le \|\rho_2\|_3\|\nabla\delta c\|_6\|\nabla\delta\rho\|_2 \le C\|\rho_2\|_3^2\|\nabla^2\delta c\|_2^2 + \frac{1}{8}\|\nabla\delta\rho\|_2^2. \tag{5.6}
$$

Then V_4 can be ignored for it is negative. Substituting (5.4)–(5.6) into (5.3), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\rho(t)\|_2^2 + \|\nabla\delta\rho(t)\|_2^2 &\le C(\|\nabla\delta u\|_2^2\|\rho_1\|_3^2 + \|\delta\rho\|_2^2\|c_1\|_{H^3}^2\\
&\quad + \|\rho_2\|_3^2\|\nabla^2\delta c\|_2^2 + (\|\rho_1\|_3+\|\rho_2\|_3)^2\|\delta\rho\|_2^2).
\end{aligned} \tag{5.7}
$$

Taking the L^2-inner product of the second equation of (5.2) with δc yields

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta c(t)\|_2^2 + \|\nabla\delta c(t)\|_2^2 &= -\int_{\mathbb{R}^3}(\delta u\cdot\nabla c_1)\delta c\,\mathrm{d}x - \int_{\mathbb{R}^3}\delta c\delta c\,\mathrm{d}x + \int_{\mathbb{R}^3}\delta\rho\delta c\,\mathrm{d}x\\
&= \int_{\mathbb{R}^3}c_1\delta u\cdot\nabla\delta c\,\mathrm{d}x - \int_{\mathbb{R}^3}\delta c\delta c\,\mathrm{d}x + \int_{\mathbb{R}^3}\delta\rho\delta c\,\mathrm{d}x\\
&\le \|\delta u\|_2\|c_1\|_\infty\|\nabla\delta c\|_2 + \|\delta c\|_2^2 + \|\delta\rho\|_2\|\delta c\|_2\\
&\le \|\delta u\|_2^2\|c_1\|_\infty^2 + \frac{1}{2}\|\nabla\delta c\|_2^2 + \|\delta c\|_2^2 + C(\|\delta\rho\|_2^2+\|\delta c\|_2^2),
\end{aligned}
$$

from which we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\delta c(t)\|_2^2 + \|\nabla\delta c(t)\|_2^2 \le C(\|\delta u\|_2^2\|c_1\|_\infty^2 + \|\delta\rho\|_2^2 + \|\delta c\|_2^2). \tag{5.8}
$$

For H^2-estimate of δc,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\delta c(t)\|_2^2+\|\nabla\Delta\delta c(t)\|_2^2\\
&=-\int_{\mathbb{R}^3}\Delta(\delta u\cdot\nabla c_1)\Delta\delta c\,\mathrm{d}x-\int_{\mathbb{R}^3}\Delta u_2\cdot\nabla\delta c\Delta\delta c\,\mathrm{d}x-2\int_{\mathbb{R}^3}\nabla u_2\cdot\nabla\nabla\delta c\Delta\delta c\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}u_2\cdot\nabla\Delta\delta c\Delta\delta c\,\mathrm{d}x-\int_{\mathbb{R}^3}\Delta\delta c\Delta\delta c\,\mathrm{d}x+\int_{\mathbb{R}^3}\Delta\delta\rho\Delta\delta c\,\mathrm{d}x\\
&\triangleq IV_1+IV_2+IV_3+IV_4+IV_5+IV_6.
\end{aligned}\tag{5.9}$$

For IV_1, by the Hölder inequality and the Young inequality, we obtain

$$\begin{aligned}
IV_1&=\int_{\mathbb{R}^3}\nabla(\delta u\cdot\nabla c_1)\nabla\Delta\delta c\,\mathrm{d}x\\
&=\int_{\mathbb{R}^3}(\nabla\delta u\cdot\nabla c_1)\nabla\Delta\delta c\,\mathrm{d}x+\int_{\mathbb{R}^3}\delta u\cdot\nabla\nabla c_1\nabla\Delta\delta c\,\mathrm{d}x\\
&\le\|\nabla\delta u\|_2\|\nabla c_1\|_\infty\|\nabla\Delta\delta c\|_2+\|\delta u\|_6\|\nabla^2c_1\|_3\|\nabla\Delta\delta c\|_2\\
&\le\|\nabla\delta u\|_2\|\nabla c_1\|_\infty\|\nabla\Delta\delta c\|_2+\|\nabla\delta u\|_2\|\nabla^2c_1\|_3\|\nabla\Delta\delta c\|_2\\
&\le\|\nabla\delta u\|_2^2\|c_1\|_{H^3}^2+\frac{1}{8}\|\nabla\Delta\delta c\|_2^2.
\end{aligned}\tag{5.10}$$

For the second term, we have by the Hölder inequality

$$IV_2\le C\|\Delta u_2\|_3\|\nabla\delta c\|_6\|\Delta\delta c\|_2\le C\|u_2\|_{H^3}^2\|\Delta\delta c\|_2^2.\tag{5.11}$$

By the Young inequality, one has

$$IV_3\le 2\|\nabla u_2\|_3\|\nabla^2\delta c\|_2\|\Delta\delta c\|_6\le C\|u_2\|_{H^2}^2\|\Delta\delta c\|_2^2+\frac{1}{8}\|\nabla\Delta\delta c\|_2^2.\tag{5.12}$$

By the Hölder inequality,

$$IV_4\le\|u_2\|_\infty\|\nabla\Delta\delta c\|_2\|\Delta\delta c\|_2\le C\|u_2\|_{H^2}^2\|\Delta\delta c\|_2^2+\frac{1}{8}\|\nabla\Delta\delta c\|_2^2\tag{5.13}$$

and

$$IV_6=-\int_{\mathbb{R}^3}\nabla\delta\rho\nabla\Delta\delta c\,\mathrm{d}x\le C\|\nabla\delta\rho\|_2^2+\frac{1}{8}\|\nabla\Delta\delta c\|_2^2.\tag{5.14}$$

With the estimates (5.10)–(5.14), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\delta c(t)\|_2^2+\|\Delta\delta c(t)\|_2^2\\
&\le\|\Delta\delta c\|_2^2(\|u_2\|_{H^3}^2+\|u_2\|_{H^2}^2)+\|\nabla\delta u\|_2^2\|c_1\|_{H^3}^2+\|\nabla\delta\rho\|_2^2.
\end{aligned}\tag{5.15}$$

Taking the L^2-inner product of the third equation of (5.2) with δu, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u(t)\|_2^2+\|\nabla\delta u(t)\|_2^2&=-\int_{\mathbb{R}^3}(\delta u\cdot\nabla u_1)\cdot\delta u\,\mathrm{d}x-\int_{\mathbb{R}^3}\delta\rho\nabla\phi\cdot\delta u\mathrm{d}x\\
&\le\|\delta u\|_6\|\nabla u_1\|_3\|\delta u\|_2+\|\nabla\phi\|_\infty\|\delta\rho\|_2\|\delta u\|_2\\
&\le C\|\delta u\|_2^2\|u_1\|_{H^2}^2+\frac{1}{2}\|\nabla\delta u\|_2^2+C(\|\delta\rho\|_2^2+\|\delta u\|_2^2).
\end{aligned}$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u(t)\|_2^2+\|\nabla\delta u(t)\|_2^2\le C(\|\delta u\|_2^2\|u_1\|_{H^2}^2+\|\delta\rho\|_2^2+\|\delta u\|_2^2). \tag{5.16}$$

For the H^1-estimate of δu,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\delta u(t)\|_2^2+\|\nabla^2\delta u(t)\|_2^2\\
&=-\int_{\mathbb{R}^3}\nabla(\delta u\cdot\nabla u_1)\nabla\delta u\,\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\delta u\cdot\nabla u_1\nabla\delta u\,\mathrm{d}x-\int_{\mathbb{R}^3}\nabla(\delta\rho\nabla\phi)\nabla\delta u\mathrm{d}x\\
&=-\int_{\mathbb{R}^3}(\delta u\cdot\nabla u_1)\nabla^2\delta u\,\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\delta u\cdot\nabla u_1\nabla\delta u\,\mathrm{d}x-\int_{\mathbb{R}^3}\delta\rho\nabla\phi\nabla^2\delta u\mathrm{d}x\\
&\le C\|\delta u\|_6\|\nabla u_1\|_3\|\nabla^2\delta u\|_2+\|\nabla\delta u\|_6\|\nabla u_1\|_3\|\nabla\delta u\|_2+\|\nabla\phi\|_\infty\|\delta\rho\|_2\|\nabla^2\delta u\|_2\\
&\le C\|\nabla\delta u\|_2^2\|u_1\|_{H^2}^2+\|\nabla\phi\|_\infty^2\|\delta\rho\|_2^2+\frac{1}{2}\|\nabla^2\delta u\|_2^2.
\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\delta u(t)\|_2^2+\|\nabla^2\delta u(t)\|_2^2\le C\|\nabla\delta u\|_2^2\|u_1\|_{H^2}^2+\|\nabla\phi\|_\infty^2\|\delta\rho\|_2^2. \tag{5.17}$$

Collecting (5.7), (5.8), (5.15), (5.16) and (5.17), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\delta\rho(t)\|_2^2+\|\delta c(t)\|_{H^2}^2+\|\delta u(t)\|_{H^1}^2)+\|\nabla\delta\rho\|_2^2+\|\nabla\delta c\|_2^2+\|\nabla\delta c\|_{H^2}^2+\|\nabla\delta u\|_{H^1}^2\\
&\quad\le C(\|\nabla\delta u\|_2^2\|\rho_1\|_3^2+\|\delta\rho\|_2^2\|c_1\|_{H^3}^2+\|\rho_2\|_3^2\|\nabla^2\delta c\|_2^2+\|\nabla\delta u\|_2^2\|u_1\|_{H^2}^2+\|\nabla\phi\|_\infty^2\|\delta\rho\|_2^2\\
&\qquad+\|\Delta\delta c\|_2^2(\|u_2\|_{H^3}^2+\|u_2\|_{H^2}^2)+\|\nabla\delta u\|_2^2\|c_1\|_{H^3}^2\\
&\qquad+\|\delta u\|_2^2\|u_1\|_{H^2}^2+\|\delta\rho\|_2^2+\|\delta u\|_2^2+\|\delta u\|_2^2\|c_1\|_\infty^2+\|\delta\rho\|_2^2+\|\delta c\|_2^2).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\delta\rho(t)\|_2^2+\|\delta c(t)\|_{H^2}^2+\|\delta u(t)\|_{H^1}^2)\\
&\quad\le CF(t)(\|\delta\rho(t)\|_2^2+\|\delta c(t)\|_{H^2}^2+\|\delta u(t)\|_{H^1}^2),
\end{aligned}$$

where

$$F(t)=\|\rho_1\|_3^2+\|\rho_2\|_3^2+\|u_1\|_{H^2}^2+\|u_2\|_{H^3}^2+\|c_1\|_{H^3}^2+1.$$

We know from Sect. 4 that $F(t)$ is integrable. Using the Gronwall inequality, we can obtain the uniqueness.

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