# Analysis of Acceptably Multiplicative Consistency and Consensus for Incomplete Interval-Valued Intuitionistic Fuzzy Preference Relations

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***Abstract*—This paper investigates group decision-making (GDM) problems, where the decision makers (DMs)' preference information is represented by incomplete interval-valued intuitionistic fuzzy preference relations (IVIFPRs). First, a multiplicative consistency property and an acceptably multiplicative consistency property for IVIFPRs are offered. Then, an optimization model to estimate the missing values in an incomplete IVIFPR is constructed. Subsequently, two optimization models are respectively established to derive a perfectly consistent IVIFPR and an acceptably consistent IVIFPR from a given inconsistent IVIFPR. Furthermore, a model is offered to gain the DMs' weights. Afterward, the consensus index is defined. When the consensus for IVIFPRs is unacceptable, a model is presented to reach the consensus requirement. Moreover, a novel GDM method for incomplete IVIFPRs is presented. Finally, the presented method is applied to an illustrative example that shows the feasibility of the offered method.**

**Index Terms—GDM, incomplete IVIFPR, multiplicative consistency, consensus, optimization model**

## I. Introduction

GDM usually requires a group of DMs to compare and rank alternatives. Preference relations, whose main feature is to compare alternatives pairwisely, are efficient vehicles for decision-making theory. On the basis of the characteristics of comparative judgments, traditional preference relations are divided into multiplicative preference relations (MPRs) [1] and fuzzy preference relations (FPRs) [2]. However, FPRs and MPRs only offer exact judgments, which are impractical due to the DMs' subjective vagueness. Therefore, preference relations with interval judgments are proposed among which interval FPRs [3] and interval MPRs [4] are two efficient tools that can simply express the lower and upper bounds of DMs' uncertainties. To address the situation where more than one value exists for a judgement, hesitant FPRs [5] and hesitant MPRs [6] were proposed by using hesitant fuzzy sets (HFSs) [7] and hesitant multiplicative sets (HMSs) [6], respectively. However, the elements in the aforesaid preference relations [1]-[6] are all numerical values. With the increasing diversity and complexity, DMs are challenging to give their opinions with numerical values. Thus, linguistic preference relations (LPRs) [8], which measure the preference degree between two alternatives by using linguistic term sets (LTSs) [9], [10], were proposed. However, the single linguistic term is not convenient for a DM to express the hesitant qualitative information. To break this limitation, hesitant fuzzy linguistic term sets (HFLTSs) [11] were presented to enrich the linguistic elicitation using several consecutive linguistic terms. Later, hesitant fuzzy linguistic preference relations (HFLPRs) [12], which can flexibly represent DMs' natural preferences based on HFLTSs [11], [13], were introduced. To manage the situation where various DMs may prefer to provide different linguistic terms with different importance degrees, probabilistic linguistic term sets (PLTSs) [14] were proposed. Moreover, Zhang *et al.* [15] introduced the definition of probabilistic linguistic preference relations (PLPRs), which are composed of PLTSs and are an efficient qualitative expression technique. Considering that the aforementioned preference relations [1]-[6], [8], [12], [15] only captured the DMs' positive information, Atanassov [16] proposed intuitionistic fuzzy sets (IFSs) to express the DMs' positive and negative judgments simultaneously. IFSs describe the fuzzy nature of objective things and make these things highly scientific and effective when used to deal with uncertain information. Considering the advantages of IFSs, Xu [17] introduced preference relations with IFSs called intuitionistic FPRs (IFPRs). Later, Atanassov and Gargov [18] introduced interval-valued intuitionistic fuzzy sets (IVIFSs) to express the uncertain membership and non-membership degrees. Xu and Chen [19] introduced IVIFSs into preference relations and developed the definition of IVIFPRs.

Consistency, which measures the transitivity among

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different alternatives and ensures the reasonability of the outcomes, is one of the most essential properties of preference relations. Consistency test and consistency adjustment play a key role in the decision-making process [12], [15], [20]-[24]. Zhang *et al.* [15] presented the methods to examine and ameliorate the addictive consistency of PLPRs based on the preference relation graph. Some algorithms were introduced for repairing the unacceptably consistent HFLPRs into acceptably consistent ones [12], [23]. For hesitant MPRs, Wang *et al.* [22] defined its consistency and established an algorithm for the consistency checking and inconsistency improving. Liao *et al.* [25] defined the multiplicative consistency of IVIFPRs and offered the consistency threshold using a simulation experiment. After the work of Liao *et al.* [25], many decision-making methods with IVIFPRs were proposed [26]-[36]. For example, Chu *et al.* [26] presented an additive consistency definition for IVIFPRs. Wan *et al.* [29] proposed a GDM method for additive consistent IVIFPRs. Wan *et al.* [32] defined a multiplicative consistency for IVIFPRs. Following the defined multiplicative consistency, they [32] improved the multiplicative consistency for IVIFPRs.

Group consensus coordinates the different opinions of experts to achieve sufficient consistency. To address this point, consensus reaching process is a key procedure to obtain enough consensus and reasonable results in GDM. Some work has been done to investigate the consensus of various types of preference relations [22], [24], [27], [28], [31], [37]-[40]. For example, Wang *et al.* [22] established an optimization model to address group consensus issues for hesitant MPRs. The consistency and consensus of HLFPRs in GDM was investigated in [37] and [39]. For IVIFPRs, Wan *et al.* [31] offered a GDM method that considers multiplicatively consistent IVIFPRs. Meng *et al.* [27] presented a GDM approach. Tang *et al.* [28] offered a completely additive consistency and consensus based GDM approach. The consistency improving and consensus reaching methods for IVIFPRs can be generally divided into two types, i.e., iteration-based methods [25], [27], [28], [31], [32] and optimization-based methods [30], [35], [36]. Iteration-based methods are usually convergent. However, iteration-based methods might require the DMs to set the iteration parameters in advance or to provide some adjustment advices in each iteration, so sometimes iteration-based methods are time-consuming. Therefore, if the DMs don't want to be more involved in the decision-making process and don't want to spend too much time, optimization-based methods are recommended. Optimization-based methods usually gain the adjusted preference relations with acceptable consistency and acceptable consensus by some optimization models that minimize the deviations between the original preference relations and the adjusted ones. Therefore, through optimization-based methods, we can ensure the minimum information loss. However, sometimes, the built optimization models may be very complex and difficult to be solved by some optimization software. In this case, iteration-based methods are recommended.

In some cases, DMs might not be able to compare some alternatives because of variously objective and subjective reasons. Thus, preference relations are not always complete, that is, some pairwise judgments are missed within preference relations. In this situation, it is necessary to handle GDM problems with incomplete preference relations where we need first to estimate missing values and supplement the incomplete preference relations. At present, some methods for ascertaining unknown values in incomplete preference relations were provided [27], [28], [41]-[46]. For example, Wang and Xu [44] proposed an interactive method based on a feedback mechanism for completing elements in incomplete LPRs. Xu and Cai [45] built two estimation procedures to estimate the missing values in an incomplete IVIFPR. Xu and Cai [46] developed a simple algorithm to extend each incomplete IVIFPR to a complete one based on the multiplicative consistency of existing incomplete IVIFPRs for the GDM problem. According to the additive consistency, Tang *et al.* [28] put forward a model to fulfill missing values using known information. Meng *et al.* [27] developed a constrained nonlinear optimization model, which aims for maximizing the multiplicative consistency level, to estimate the missing elements in incomplete IVIFPRs.

From the above literature review, some limitations in the existing research on IVIFPRs [25]-[34], [36], are identified:

(1) In GDM methods [25], [26], [30], [32], [33], [36], the group consensus was overlooked.

(2) Some GDM methods [25], [33], [34] overlooked the determination of DMs' weights.

(3) Some GDM methods [25], [26], [29], [30]-[34], [36] were unsuitable to cope with incomplete IVIFPRs.

(4) Some GDM methods [26], [34] cannot cope with inconsistent IVIFPRs. In general, the IVIFPRs are inconsistent. To rank alternatives from IVIFPRs logically, the consistency check and adjustment is a necessary step.

(5) Some GDM methods [26]-[30], [36] employed the complete consistency analysis. In some situations, the complete consistency requirement may be unnecessary or too strict and the consistency level to reach some degree may be sufficient.

To make up for the aforesaid limitations, this paper further studies GDM with IVIFPRs and makes the following contributions:

(1) A novel multiplicative consistency definition for IFPRs is offered, and a method to derive a multiplicatively consistent IFPR from a given intuitionistic fuzzy (IF) priority vector is presented. Then, we introduce a consistency index and present a definition of acceptably consistent IFPRs. Afterwards, a consistency property and an acceptably multiplicatively consistency property for IVIFPRs are offered.

(2) An optimization model to ascertain missing judgements in an incomplete IVIFPR is constructed that makes the completed IVIFPR have the highest consistency level. Two optimization models for receiving perfectly and acceptably multiplicatively consistent IVIFPRs are respectively established.

(3) An optimization model of maximizing the group

consensus level is set up to determine DMs' weights. For GDM with IVIFPRs, a consensus index is defined. Some methods to discern and reach the acceptable consensus of IVIFPRs in GDM are investigated.

(4) Motivated by the idea of deviation minimization, a model to find the priority vector is built. A procedure of GDM with IVIFPRs is designed that can address incompleteness, inconsistency and non-consensus.

The remainder of the paper is organized as follows. Some concepts of IFPRs and IVIFPRs are reviewed in Section II. Section III analyzes incomplete and inconsistent IVIFPRs. Several programming models to obtain missing values and derive consistent and acceptably consistent IVIFPRs are constructed, respectively. Section IV focuses on GDM and proposes a consensus measure. Next, an optimization model for reaching the consensus requirements is built. In Section V, a new GDM method about incomplete IVIFPRs is proposed. Section VI reports on case studies and covers comparative analyses. Section VII completes this paper with some conclusions.

## II. Preliminaries

To begin with, let us recall several basic concepts that are related to this study.

### A. IFPRs

*Definition 2.1 [16]:* An IFS is a set with the following form: $R=\{\langle x,\mu_R(x),\nu_R(x)\rangle | x\in U\}$, where the functions $\mu_R:U\to[0,1]$ and $\nu_R:U\to[0,1]$ define membership and non-membership degrees of the element $x$ to the set $R$, respectively. For each $x\in U$, $0\le \mu_R(x)+\nu_R(x)\le 1$.

The pair $\langle \mu_R(x),\nu_R(x)\rangle$ is called an intuitionistic fuzzy value (IFV) [47] and simply denoted by $\alpha=(\mu_\alpha,\nu_\alpha)$, where $\mu_\alpha,\nu_\alpha\in[0,1]$, and $\mu_\alpha+\nu_\alpha\le 1$.

Following the concept of IFVs, Xu [17] introduced the definition of IFPRs.

*Definition 2.2 [17]:* The matrix $A=(\alpha_{ij})_{n\times n}$ on the finite set $X=\{x_1,x_2,\cdots,x_n\}$ is called an IFPR if its elements satisfy the relationships

$$\mu_{ij},\nu_{ij}\in[0,1],\ \mu_{ij}=\nu_{ji},\ \nu_{ij}=\mu_{ji},\ \mu_{ii}=\nu_{ii}=0.5,\ \mu_{ij}+\nu_{ij}\le 1, \quad (1)$$

where $\alpha_{ij}=(\mu_{ij},\nu_{ij})$ is an IFV for all $i,j=1,2,\cdots,n$.

*Definition 2.3 [48]:* Let $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be an intuitionistic fuzzy (IF) priority weight vector of an IFPR $A$, where $\omega_i=(\omega_i^\mu,\omega_i^\nu)$ ($i=1,2,\cdots,n$) are IFVs with the conditions $\omega_i^\mu,\omega_i^\nu\in[0,1]$ and $\omega_i^\mu+\omega_i^\nu\le 1$ for $i=1,2,\cdots,n$. The vector $\omega$ is called normalized if it satisfies

$$\sum_{j=1,j\ne i}^{n}\omega_j^\mu\le\omega_i^\nu, \text{ and } \omega_i^\mu+n-2\ge\sum_{j=1,j\ne i}^{n}\omega_j^\nu, \quad (2)$$

for all $i=1,2,\cdots,n$.

*Definition 2.4 [49]:* An IFPR $A=((\mu_{ij},\nu_{ij}))_{n\times n}$ is multiplicatively consistent if it satisfies

$$\mu_{ij}\mu_{jk}\mu_{ki}=\nu_{ij}\nu_{jk}\nu_{ki}, \quad (3)$$

for all $i,j,k=1,2,\cdots,n$.

### B. IVIFPRs

*Definition 2.5 [18]:* An IVIFS $\tilde R$ in $U$ is defined as $\tilde R=\{\langle x,\tilde\mu_{\tilde R}(x),\tilde\nu_{\tilde R}(x)\rangle | x\in U\}$, where $\tilde\mu_{\tilde R}(x)=[\underline{\mu}_{\tilde R}(x),\overline{\mu}_{\tilde R}(x)]$ and $\tilde\nu_{\tilde R}(x)=[\underline{\nu}_{\tilde R}(x),\overline{\nu}_{\tilde R}(x)]$ represent the membership degree interval and non-membership degree interval of element $x$ to IVIFS $\tilde R$, respectively, satisfying $0\le\underline{\mu}_{\tilde R}(x)\le\overline{\mu}_{\tilde R}(x)\le 1$, $0\le\underline{\nu}_{\tilde R}(x)\le\overline{\nu}_{\tilde R}(x)\le 1$, and $\overline{\mu}_{\tilde R}(x)+\overline{\nu}_{\tilde R}(x)\le 1$ for any $x\in U$.

The pair $([\underline{\mu}_{\tilde R}(x),\overline{\mu}_{\tilde R}(x)],[\underline{\nu}_{\tilde R}(x),\overline{\nu}_{\tilde R}(x)])$ is called an interval-valued intuitionistic fuzzy value (IVIFV) [19] and simply denoted by $\tilde\alpha=(\tilde\mu_{\tilde\alpha},\tilde\nu_{\tilde\alpha})=([\underline{\mu}_{\tilde\alpha},\overline{\mu}_{\tilde\alpha}],[\underline{\nu}_{\tilde\alpha},\overline{\nu}_{\tilde\alpha}])$.

*Definition 2.6 [19]:* Let $\tilde\alpha=(\tilde\mu_{\tilde\alpha},\tilde\nu_{\tilde\alpha})=([\underline{\mu}_{\tilde\alpha},\overline{\mu}_{\tilde\alpha}],[\underline{\nu}_{\tilde\alpha},\overline{\nu}_{\tilde\alpha}])$ be an IVIFV. Then, its score function is defined as $s(\tilde\alpha)=\frac{1}{2}(\underline{\mu}_{\tilde\alpha}+\overline{\mu}_{\tilde\alpha}-\underline{\nu}_{\tilde\alpha}-\overline{\nu}_{\tilde\alpha})$, and its accuracy function is defined as $\gamma(\tilde\alpha)=\frac{1}{2}(\underline{\mu}_{\tilde\alpha}+\overline{\mu}_{\tilde\alpha}+\underline{\nu}_{\tilde\alpha}+\overline{\nu}_{\tilde\alpha})$.

Xu and Chen [19] presented the following ranking method for IVIFVs:

(1) If $s(\tilde\alpha_1)<s(\tilde\alpha_2)$, then $\tilde\alpha_1$ is smaller than $\tilde\alpha_2$, denoted by $\tilde\alpha_1\prec\tilde\alpha_2$;

(2) If $s(\tilde\alpha_1)=s(\tilde\alpha_2)$, then

(i) If $\gamma(\tilde\alpha_1)<\gamma(\tilde\alpha_2)$, then $\tilde\alpha_1\prec\tilde\alpha_2$;

(ii) If $\gamma(\tilde\alpha_1)=\gamma(\tilde\alpha_2)$, then $\tilde\alpha_1$ is indifferent to $\tilde\alpha_2$, denoted by $\tilde\alpha_1\sim\tilde\alpha_2$.

where $\tilde\alpha_1=(\tilde\mu_{\tilde\alpha_1},\tilde\nu_{\tilde\alpha_1})$ and $\tilde\alpha_2=(\tilde\mu_{\tilde\alpha_2},\tilde\nu_{\tilde\alpha_2})$ are two IVIFVs.

Xu and Chen [19] introduced the concept of IVIFPRs whose elements are IVIFVs.

*Definition 2.7 [19]:* The matrix $\tilde A=(\tilde\alpha_{ij})_{n\times n}$ on the finite alternative set $X=\{x_1,x_2,\cdots,x_n\}$ is called an IVIFPR if its elements $\tilde\alpha_{ij}$, where $\tilde\alpha_{ij}=(\tilde\mu_{ij},\tilde\nu_{ij})$ and $i,j=1,2,\cdots,n$, are IVIFVs such that

$$\begin{cases}\tilde\mu_{ij}=[\underline{\mu}_{ij},\overline{\mu}_{ij}]\subseteq[0,1],\ \tilde\nu_{ij}=[\underline{\nu}_{ij},\overline{\nu}_{ij}]\subseteq[0,1],\\ \tilde\mu_{ij}=\tilde\nu_{ji},\ \tilde\nu_{ij}=\tilde\mu_{ji},\ \tilde\mu_{ii}=\tilde\nu_{ii}=[0.5,0.5],\ \overline{\mu}_{ij}+\overline{\nu}_{ij}\le 1,\end{cases} \quad (4)$$

for all $i,j=1,2,\cdots,n$.

*Definition 2.8:* Let $\tilde A'=((\tilde\mu'_{ij},\tilde\nu'_{ij}))_{n\times n}$ and $\tilde A''=((\tilde\mu''_{ij},\tilde\nu''_{ij}))_{n\times n}$ be two IVIFPRs, where $\tilde\mu'_{ij}=[\underline{\mu}'_{ij},\overline{\mu}'_{ij}]$,

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$\tilde{\nu}'_{ij}=\left[\underline{\nu}'_{ij},\overline{\nu}'_{ij}\right]$, $\tilde{\mu}''_{ij}=\left[\underline{\mu}''_{ij},\overline{\mu}''_{ij}\right]$, and $\tilde{\nu}''_{ij}=\left[\underline{\nu}''_{ij},\overline{\nu}''_{ij}\right]$. The deviation between $\tilde{A}'$ and $\tilde{A}''$ is defined as follows:

$$d\left(\tilde{A}',\tilde{A}''\right)=\frac{1}{2n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\underline{\mu}'_{ij}-\underline{\mu}''_{ij}\right|+\left|\overline{\mu}'_{ij}-\overline{\mu}''_{ij}\right|+\left|\underline{\nu}'_{ij}-\underline{\nu}''_{ij}\right|+\left|\overline{\nu}'_{ij}-\overline{\nu}''\right|\right).\quad(5)$$

## III. Consistency Analysis for IFPRs and IVIFPRs

This section covers a key issue of preference relations: consistency, which ensures the rationality of the ranking. We next consider the consistency for IFPRs and IVIFPRs.

### A. Consistency Analysis of IFPRs

*Theorem 3.1*: Let $A=\left(\left(\mu_{ij},\nu_{ij}\right)\right)_{n\times n}$ be an IFPR. The following two statements are equivalent:

(i) $\mu_{ij}\mu_{jk}\mu_{ki}=\nu_{ij}\nu_{jk}\nu_{ki}$, $\forall i,j,k=1,2,\cdots,n$;

(ii) $\ln\mu_{ij}+\ln\mu_{jk}+\ln\nu_{ik}=\ln\nu_{ij}+\ln\nu_{jk}+\ln\mu_{ik}$, $\forall i,j,k=1,2,\cdots,n$ and $i<j<k$.

*Proof:* It is easily proved according to six possible position cases of $i,j,k$.

The multiplicative consistency for IFPRs is redefined below:

*Definition 3.1*: An IFPR $A=\left(\left(\mu_{ij},\nu_{ij}\right)\right)_{n\times n}$ is multiplicatively consistent if it meets

$$\ln\mu_{ij}+\ln\mu_{jk}+\ln\nu_{ik}=\ln\nu_{ij}+\ln\nu_{jk}+\ln\mu_{ik},\quad(6)$$

for $i,j,k=1,2,\cdots,n$ and $i<j<k$.

The multiplicative consistency descried in *Definition 2.4* requires Eq. (3) to be hold true for all $i,j,k=1,2,\cdots,n$. Namely, if we use *Definition 2.4* to judge the consistency of an IFPR, we need to consider all IFVs in this IFPR. By contrast, the multiplicative consistency descried in *Definition 3.1* requires Eq. (6) to be hold true for all $i,j,k=1,2,\cdots,n$ with $i<j<k$. Namely, if we use *Definition 3.1* to judge the multiplicative consistency of an IFPR, we only need to consider all the IFVs in the upper triangular part of this IFPR. Thus, it is more concise and computationally simple to describe the multiplicative consistency by *Definition 3.1* than by *Definition 2.4*.

Considering the case where the completely multiplicative consistency is too strict, we further offer a consistency index to evaluate the acceptable consistency.

*Definition 3.2:* The consistency index of IFPR $A$ is defined as

$$CI(A)=\frac{1}{C_n^3}\sum_{1\le i<j<k\le n}\left|\ln\mu_{ij}+\ln\mu_{jk}+\ln\nu_{ik}-\ln\nu_{ij}-\ln\nu_{jk}-\ln\mu_{ik}\right|.\quad(7)$$

If we build the consistency index based on *Definition 2.4*, we need to calculate the all absolute deviation degrees between $\mu_{ij}\mu_{jk}\mu_{ki}$ and $\nu_{ij}\nu_{jk}\nu_{ki}$ for all $i,j,k=1,2,\cdots,n$. However, if we use *Definition 3.1* to build the consistency index, we only need to calculate the absolute deviation degrees between $\ln\mu_{ij}+\ln\mu_{jk}+\ln\nu_{ik}$ and $\ln\nu_{ij}+\ln\nu_{jk}+\ln\mu_{ik}$ for all $i,j,k=1,2,\cdots,n$ with $i<j<k$. That is to say, it is more convenient and computationally simple to build the consistency index based on *Definition 3.1* than *Definition 2.4*.

Based on Eq. (7), $A$ is completely multiplicatively consistent iff $CI(A)=0$. Furthermore, the smaller $CI(A)$ is, the more consistent the IFPR $A$. Based on the offered multiplicative consistency index, we define the acceptably multiplicatively consistent IFPRs.

*Definition 3.3:* Let $A$ be an IFPR. Given a threshold value $\overline{\mathrm{CI}}$, if $CI(A)\le\overline{\mathrm{CI}}$, then $A$ is called an IFPR with acceptably multiplicatively consistency.

By means of the normalized IF priority weight vector $\omega$, we build a new matrix $P=\left(p_{ij}\right)_{n\times n}$, where

$$p_{ij}=\left(p_{ij}^{\mu},p_{ij}^{\nu}\right)=\begin{cases}(0.5,0.5), & \text{if } i=j,\\ \left(\omega_i^{\mu}\omega_j^{\nu},\omega_i^{\nu}\omega_j^{\mu}\right), & \text{if } i\ne j.\end{cases}\quad(8)$$

*Theorem 3.2*: The matrix $P=\left(p_{ij}\right)_{n\times n}$, where $p_{ij}$ is given as Eq. (8), is a multiplicatively consistent IFPR.

*Proof:* Clearly, we have $p_{ij}^{\mu}=\omega_i^{\mu}\omega_j^{\nu}=\omega_j^{\nu}\omega_i^{\mu}=p_{ji}^{\nu}$ and $p_{ij}^{\nu}=\omega_i^{\nu}\omega_j^{\mu}=\omega_j^{\mu}\omega_i^{\nu}=p_{ji}^{\mu}$ for all $i,j=1,2,\cdots,n$. As $\omega_i^{\mu},\omega_i^{\nu}\in[0,1]$ and $\omega_i^{\mu}+\omega_i^{\nu}\le1$, it follows that $0\le p_{ij}^{\mu}=\omega_i^{\mu}\omega_j^{\nu}\le1$, $0\le p_{ij}^{\nu}=\omega_i^{\nu}\omega_j^{\mu}\le1$, and

$$p_{ij}^{\mu}+p_{ij}^{\nu}=\omega_i^{\mu}\omega_j^{\nu}+\omega_i^{\nu}\omega_j^{\mu}\le\frac{\left(\omega_i^{\mu}\right)^2+\left(\omega_j^{\nu}\right)^2}{2}+\frac{\left(\omega_i^{\nu}\right)^2+\left(\omega_j^{\mu}\right)^2}{2}$$

$$\le\frac{\omega_i^{\mu}+\omega_j^{\nu}}{2}+\frac{\omega_i^{\nu}+\omega_j^{\mu}}{2}=\frac{\left(\omega_i^{\mu}+\omega_i^{\nu}\right)+\left(\omega_j^{\mu}+\omega_j^{\nu}\right)}{2}\le\frac{1+1}{2}=1.$$

In virtue of *Definition 2.2*, $P=\left(p_{ij}\right)_{n\times n}$ is an IFPR. Next, we prove that $P=\left(p_{ij}\right)_{n\times n}$ is multiplicatively consistent. As per Eq. (8), one has

$$p_{ij}^{\mu}p_{jk}^{\mu}p_{ki}^{\mu}=\omega_i^{\mu}\omega_j^{\nu}\omega_j^{\mu}\omega_k^{\nu}\omega_k^{\mu}\omega_i^{\nu}=\omega_i^{\nu}\omega_j^{\mu}\omega_j^{\nu}\omega_k^{\mu}\omega_k^{\nu}\omega_i^{\mu}=p_{ij}^{\nu}p_{jk}^{\nu}p_{ki}^{\nu}.$$

As per *Definition 2.4*, $P$ is multiplicatively consistent.

*Theorem 3.3:* Let $A=\left(\alpha_{ij}\right)_{n\times n}$ be an IFPR with $\alpha_{ij}=\left(\mu_{ij},\nu_{ij}\right)$, if there exists a normalized IF priority weight vector, $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ with $\omega_i=\left(\omega_i^{\mu},\omega_i^{\nu}\right)$, such that

$$\alpha_{ij}=\left(\mu_{ij},\nu_{ij}\right)=\begin{cases}(0.5,0.5), & \text{if } i=j,\\ \left(\omega_i^{\mu}\omega_j^{\nu},\omega_i^{\nu}\omega_j^{\mu}\right), & \text{if } i\ne j,\end{cases}\quad(9)$$

then $A$ is a multiplicatively consistent IFPR.

Since $\mu_{ij}=\nu_{ji}$ and $\nu_{ij}=\mu_{ji}$, Eq. (9) can be easily reduced as

$$\alpha_{ij}=\left(\mu_{ij},\nu_{ij}\right)=\begin{cases}(0.5,0.5), & \text{if } i=j,\\ \left(\omega_i^{\mu}\omega_j^{\nu},\omega_i^{\nu}\omega_j^{\mu}\right), & \text{if } i<j.\end{cases}\quad(10)$$

*Theorem 3.4:* Let $A^h=\left(\alpha_{ij}^h\right)_{n\times n}$ be an IFPR with $\alpha_{ij}^h=\left(\mu_{ij}^h,\nu_{ij}^h\right)$, where $h=1,2,\cdots,m$, and let $A^c=\left(\alpha_{ij}^c\right)_{n\times n}$ be the collective IFPR, where

$\alpha_{ij}^{c}=\left(\mu_{ij}^{c},\nu_{ij}^{c}\right)=\left(\prod_{h=1}^{m}\left(\mu_{ij}^{h}\right)^{w_h},\prod_{h=1}^{m}\left(\nu_{ij}^{h}\right)^{w_h}\right)$ with $\sum_{h=1}^{m}w_h=1$ and $0\le w_h\le 1$ ( $h=1,2,\cdots,m$ ) [50]. Then, $CI\left(A^{c}\right)\le\sum_{h=1}^{m}w_h CI\left(A^{h}\right)$.

*Proof*: Following Eq. (7), we derive

$$CI\left(A^{c}\right)=\frac{1}{C_n^3}\sum_{1\le i<j<k\le n}\left|\ln\mu_{ij}^{c}+\ln\mu_{jk}^{c}+\ln\nu_{ik}^{c}-\ln\nu_{ij}^{c}-\ln\nu_{jk}^{c}-\ln\mu_{ik}^{c}\right|$$

$$=\frac{1}{C_n^3}\sum_{1\le i<j<k\le n}\left|\begin{array}{l}\sum_{h=1}^{m}w_h\ln\mu_{ij}^{h}+\sum_{h=1}^{m}w_h\ln\mu_{jk}^{h}+\sum_{h=1}^{m}w_h\ln\nu_{ik}^{h}-\\ \sum_{h=1}^{m}w_h\ln\nu_{ij}^{h}-\sum_{h=1}^{m}w_h\ln\nu_{jk}^{h}-\sum_{h=1}^{m}w_h\ln\mu_{ik}^{h}\end{array}\right|$$

$$=\frac{1}{C_n^3}\sum_{1\le i<j<k\le n}\left|\sum_{h=1}^{m}w_h\left(\ln\mu_{ij}^{h}+\ln\mu_{jk}^{h}+\ln\nu_{ik}^{h}-\ln\nu_{ij}^{h}-\ln\nu_{jk}^{h}-\ln\mu_{ik}^{h}\right)\right|$$

$$\le\frac{1}{C_n^3}\sum_{1\le i<j<k\le n}\sum_{h=1}^{m}w_h\left|\ln\mu_{ij}^{h}+\ln\mu_{jk}^{h}+\ln\nu_{ik}^{h}-\ln\nu_{ij}^{h}-\ln\nu_{jk}^{h}-\ln\mu_{ik}^{h}\right|$$

$$=\sum_{h=1}^{m}w_h CI\left(A^{h}\right).$$

*Corollary 3.1:* If all IFPRs have the multiplicative consistency or acceptably multiplicative consistency, then their collective IFPR also exhibits the multiplicative consistency or acceptably multiplicative consistency.

*Proof*: Suppose that $A^h$ is acceptably multiplicative consistent, where $h=1,2,\cdots,m$, then we have $CI\left(A^{h}\right)\le\overline{\text{CI}}$. Based on *Theorem 3.4*, we have $CI\left(A^{c}\right)\le\sum_{h=1}^{m}w_h CI\left(A^{h}\right)\le\sum_{h=1}^{m}w_h\overline{\text{CI}}=\overline{\text{CI}}$. Consequently, $A^c$ is acceptably multiplicative consistent. Moreover, let $\overline{\text{CI}}=0$, then we have that $A^c$ is multiplicatively consistent if all $A^h$ ( $h=1,2,\cdots,m$ ) are multiplicative consistent. The proof is completed.

## B. Consistency Analysis of IVIFPRs

For an IVIFPR $\tilde{A}=\left(\tilde{\alpha}_{ij}\right)_{n\times n}=\left(\left(\tilde{\mu}_{ij},\tilde{\nu}_{ij}\right)\right)_{n\times n}$ with $\tilde{\mu}_{ij}=\left[\underline{\mu}_{ij},\overline{\mu}_{ij}\right]$ and $\tilde{\nu}_{ij}=\left[\underline{\nu}_{ij},\overline{\nu}_{ij}\right]$, Wan *et al.* [32] defined its lower IFPR $A^{-}=\left(\alpha_{ij}^{-}\right)_{n\times n}$ and its upper IFPR $A^{+}=\left(\alpha_{ij}^{+}\right)_{n\times n}$ as follows:

$$\alpha_{ij}^{-}=\left(\mu_{ij}^{-},\nu_{ij}^{-}\right)=\begin{cases}\left(\underline{\mu}_{ij},\overline{\nu}_{ij}\right), & \text{if } i<j,\\ \left(0.5,0.5\right), & \text{if } i=j,\\ \left(\overline{\mu}_{ij},\underline{\nu}_{ij}\right), & \text{if } i>j,\end{cases}\tag{11}$$

$$\alpha_{ij}^{+}=\left(\mu_{ij}^{+},\nu_{ij}^{+}\right)=\begin{cases}\left(\overline{\mu}_{ij},\underline{\nu}_{ij}\right), & \text{if } i<j,\\ \left(0.5,0.5\right), & \text{if } i=j,\\ \left(\underline{\mu}_{ij},\overline{\nu}_{ij}\right), & \text{if } i>j.\end{cases}\tag{12}$$

The multiplicatively consistency and acceptably multiplicatively consistency for an IVIFPR are defined by using its lower and upper IFPRs.

*Definition 3.4 [32]:* An IVIFPR $\tilde{A}$ is multiplicatively consistent or acceptably multiplicatively consistent if both $A^-$ and $A^+$ are multiplicatively consistent or acceptably multiplicatively consistent.

From Eqs. (6) and (7), we derive the following assertions.

*Theorem 3.5:* An IVIFPR $\tilde{A}$ is multiplicatively consistent iff $\ln\underline{\mu}_{ij}+\ln\underline{\mu}_{jk}+\ln\overline{\nu}_{ik}=\ln\overline{\nu}_{ij}+\ln\overline{\nu}_{jk}+\ln\underline{\mu}_{ik}$ and $\ln\overline{\mu}_{ij}+\ln\overline{\mu}_{jk}+\ln\underline{\nu}_{ik}=\ln\underline{\nu}_{ij}+\ln\underline{\nu}_{jk}+\ln\overline{\mu}_{ik}$, for all $i,j,k=1,2,\cdots,n$ and $i<j<k$.

*Theorem 3.6:* An IVIFPR $\tilde{A}$ is acceptably multiplicatively consistent iff

$$\sum_{1\le i<j<k\le n}\left|\ln\underline{\mu}_{ij}+\ln\underline{\mu}_{jk}+\ln\overline{\nu}_{ik}-\ln\overline{\nu}_{ij}-\ln\overline{\nu}_{jk}-\ln\underline{\mu}_{ik}\right|\le\overline{\text{CI}}\cdot C_n^3,\tag{13}$$

and

$$\sum_{1\le i<j<k\le n}\left|\ln\overline{\mu}_{ij}+\ln\overline{\mu}_{jk}+\ln\underline{\nu}_{ik}-\ln\underline{\nu}_{ij}-\ln\underline{\nu}_{jk}-\ln\overline{\mu}_{ik}\right|\le\overline{\text{CI}}\cdot C_n^3.\tag{14}$$

Let $\tilde{A}$ be an IVIFPR whose priority weight vector is $\tilde{\omega}=\left(\tilde{\omega}_1,\tilde{\omega}_2,\cdots,\tilde{\omega}_n\right)^T$, where $\tilde{\omega}_i=\left(\left[\underline{\omega}_i^{\mu},\overline{\omega}_i^{\mu}\right],\left[\underline{\omega}_i^{\nu},\overline{\omega}_i^{\nu}\right]\right)$ ( $i=1,2,\cdots,n$ ) are IVIFVs. Let the lower and upper IFPRs of $\tilde{A}$ are $A^-$ and $A^+$, whose IF priority weight vectors are $\omega^{-}=\left(\omega_1^{-},\omega_2^{-},\cdots,\omega_n^{-}\right)^T$ and $\omega^{+}=\left(\omega_1^{+},\omega_2^{+},\cdots,\omega_n^{+}\right)^T$, respectively, where $\omega_i^{-}=\left(\omega_i^{\mu-},\omega_i^{\nu-}\right)$ and $\omega_i^{+}=\left(\omega_i^{\mu+},\omega_i^{\nu+}\right)$ ( $i=1,2,\cdots,n$ ) are IFVs. Wan *et al.* [32] derived the interval-valued intuitionistic fuzzy (IVIF) priority weight vector $\tilde{\omega}$ of $\tilde{A}$ by the following formulas:

$$\begin{cases}\underline{\omega}_i^{\mu}=\min\left\{\omega_i^{\mu-},\omega_i^{\mu+}\right\}, & \overline{\omega}_i^{\mu}=\max\left\{\omega_i^{\mu-},\omega_i^{\mu+}\right\},\\ \underline{\omega}_i^{\nu}=\min\left\{\omega_i^{\nu-},\omega_i^{\nu+}\right\}, & \overline{\omega}_i^{\nu}=\max\left\{\omega_i^{\nu-},\omega_i^{\nu+}\right\}.\end{cases}\tag{15}$$

## C. Models for Estimating Missing Values

Let $\tilde{A}=\left(\left(\tilde{\mu}_{ij},\tilde{\nu}_{ij}\right)\right)_{n\times n}$ be an incomplete IVIFPR. We establish an IVIFPR $\tilde{A}'=\left(\left(\tilde{\mu}'_{ij},\tilde{\nu}'_{ij}\right)\right)_{n\times n}$ with $\tilde{\mu}'_{ij}=\left[\underline{\mu}'_{ij},\overline{\mu}'_{ij}\right]$ and $\tilde{\nu}'_{ij}=\left[\underline{\nu}'_{ij},\overline{\nu}'_{ij}\right]$, where

$$\underline{\mu}'_{ij}=\begin{cases}\underline{\mu}_{ij}, & \text{if } \underline{\mu}_{ij} \text{ is known},\\ \underline{\rho}_{ij}, & \text{if } \underline{\mu}_{ij} \text{ is unknown},\end{cases}\quad \overline{\mu}'_{ij}=\begin{cases}\overline{\mu}_{ij}, & \text{if } \overline{\mu}_{ij} \text{ is known},\\ \overline{\rho}_{ij}, & \text{if } \overline{\mu}_{ij} \text{ is unknown},\end{cases}$$

$$\underline{\nu}'_{ij}=\begin{cases}\underline{\nu}_{ij}, & \text{if } \underline{\nu}_{ij} \text{ is known},\\ \underline{\sigma}_{ij}, & \text{if } \underline{\nu}_{ij} \text{ is unknown},\end{cases}\quad \overline{\nu}'_{ij}=\begin{cases}\overline{\nu}_{ij}, & \text{if } \overline{\nu}_{ij} \text{ is known},\\ \overline{\sigma}_{ij}, & \text{if } \overline{\nu}_{ij} \text{ is unknown}.\end{cases}\tag{16}$$

Because the estimated values make the multiplicative consistency level of IVIFPR $\tilde{A}'$ the larger the better, we build a model to estimate the unknown judgments.

$$F=\min\delta$$
$$\text{s.t.}\begin{cases}\sum\limits_{1\le i<j<k\le n}\left|\ln\underline{\mu}'_{ij}+\ln\underline{\mu}'_{jk}+\ln\bar{v}'_{ik}-\ln\bar{v}'_{ij}-\ln\bar{v}_{jk}-\ln\underline{\mu}'_{ik}\right|\le\delta\cdot C_n^3,\\ \sum\limits_{1\le i<j<k\le n}\left|\ln\bar{\mu}'_{ij}+\ln\bar{\mu}'_{jk}+\ln\underline{v}'_{ik}-\ln\underline{v}'_{ij}-\ln\underline{v}'_{jk}-\ln\bar{\mu}'_{ik}\right|\le\delta\cdot C_n^3,\\ 0\le\underline{\mu}'_{ij}\le\bar{\mu}'_{ij}\le1,\ i,j=1,2,\cdots,n,\ i<j,\\ 0\le\underline{v}'_{ij}\le\bar{v}'_{ij}\le1,\ i,j=1,2,\cdots,n,\ i<j,\\ \bar{\mu}'_{ij}+\bar{v}'_{ij}\le1,\ i,j=1,2,\cdots,n,\ i<j.\end{cases}\tag{M-1}$$

By deleting the absolute value symbols, model (M-1) is transformed as

$$F=\min\delta$$
$$\text{s.t.}\begin{cases}\sum\limits_{1\le i<j<k\le n}\left(\zeta_{ijk}+\gamma_{ijk}\right)\le\delta\cdot C_n^3,\\ \sum\limits_{1\le i<j<k\le n}\left(\varsigma_{ijk}+\chi_{ijk}\right)\le\delta\cdot C_n^3,\\ \ln\underline{\mu}'_{ij}+\ln\underline{\mu}'_{jk}+\ln\bar{v}'_{ik}-\ln\bar{v}'_{ij}-\ln\bar{v}_{jk}-\ln\underline{\mu}'_{ik}-\zeta_{ijk}+\gamma_{ijk}=0,\\ i,j,k=1,2,\cdots,n,\ i<j<k,\\ \ln\bar{\mu}'_{ij}+\ln\bar{\mu}'_{jk}+\ln\underline{v}'_{ik}-\ln\underline{v}'_{ij}-\ln\underline{v}'_{jk}-\ln\bar{\mu}'_{ik}-\varsigma_{ijk}+\chi_{ijk}=0,\\ i,j,k=1,2,\cdots,n,\ i<j<k,\\ 0\le\underline{\mu}'_{ij}\le\bar{\mu}'_{ij}\le1,\ i,j=1,2,\cdots,n,\ i<j,\\ 0\le\underline{v}'_{ij}\le\bar{v}'_{ij}\le1,\ i,j=1,2,\cdots,n,\ i<j,\\ \bar{\mu}'_{ij}+\bar{v}'_{ij}\le1,\ i,j=1,2,\cdots,n,\ i<j,\\ \zeta_{ijk},\gamma_{ijk},\varsigma_{ijk},\chi_{ijk}\ge0,\ i,j,k=1,2,\cdots,n,\ i<j<k.\end{cases}\tag{M-2}$$

Solving the model (M-2), we can acquire the optimal objective function value $F^*$ and a complete IVIFPR $\tilde{A}'$. If $F^*\le\overline{\text{CI}}$, then $\tilde{A}'$ has acceptable consistency. If $F^*>\overline{\text{CI}}$, then $\tilde{A}'$ has unacceptable consistency.

*D. Models for Deriving a Perfectly Multiplicatively Consistent IVIFPR or an Acceptably Multiplicatively Consistent IVIFPR*

For an inconsistent IVIFPR $\tilde{A}'=\left(\left(\tilde{\mu}'_{ij},\tilde{v}'_{ij}\right)\right)_{n\times n}$ with $\tilde{\mu}'_{ij}=\left[\underline{\mu}'_{ij},\bar{\mu}'_{ij}\right]$ and $\tilde{v}'_{ij}=\left[\underline{v}'_{ij},\bar{v}'_{ij}\right]$, the next crucial step is to find a perfectly or acceptably multiplicatively consistent IVIFPR $\tilde{A}''=\left(\left(\tilde{\mu}''_{ij},\tilde{v}''_{ij}\right)\right)_{n\times n}$ with $\tilde{\mu}''_{ij}=\left[\underline{\mu}''_{ij},\bar{\mu}''_{ij}\right]$ and $\tilde{v}''_{ij}=\left[\underline{v}''_{ij},\bar{v}''_{ij}\right]$, which preserves the original preferences as much as possible.

*(1) Derive a perfectly multiplicatively consistent IVIFPR based on an inconsistent IVIFPR*

In what follows, a programming model is offered to acquire a perfectly multiplicatively consistent IVIFPR $\tilde{A}''$ for an inconsistent IVIFPR $\tilde{A}'$:

$$Q=\min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\underline{\mu}'_{ij}-\underline{\mu}''_{ij}\right|+\left|\bar{\mu}'_{ij}-\bar{\mu}''_{ij}\right|+\left|\underline{v}'_{ij}-\underline{v}''_{ij}\right|+\left|\bar{v}'_{ij}-\bar{v}''_{ij}\right|\right)$$
$$\text{s.t.}\begin{cases}\ln\left(\underline{\mu}''_{ij}\right)+\ln\left(\underline{\mu}''_{jk}\right)+\ln\left(\bar{v}''_{ik}\right)=\ln\left(\bar{v}''_{ij}\right)+\ln\left(\bar{v}''_{jk}\right)+\ln\left(\underline{\mu}''_{ik}\right),\\ i,j,k=1,2,\cdots,n,\ i<j<k,\\ \ln\left(\bar{\mu}''_{ij}\right)+\ln\left(\bar{\mu}''_{jk}\right)+\ln\left(\underline{v}''_{ik}\right)=\ln\left(\underline{v}''_{ij}\right)+\ln\left(\underline{v}''_{jk}\right)+\ln\left(\bar{\mu}''_{ik}\right),\\ i,j,k=1,2,\cdots,n,\ i<j<k,\\ 0<\underline{\mu}''_{ij}\le\bar{\mu}''_{ij}<1,\ i,j=1,2,\cdots,n,\ i<j,\\ 0<\underline{v}''_{ij}\le\bar{v}''_{ij}<1,\ i,j=1,2,\cdots,n,\ i<j,\\ \bar{\mu}''_{ij}+\bar{v}''_{ij}\le1,\ i,j=1,2,\cdots,n,\ i<j,\end{cases}\tag{M-3}$$

where $\underline{\mu}''_{ij},\bar{\mu}''_{ij},\underline{v}''_{ij},\bar{v}''_{ij}$ ($i,j=1,2,\cdots,n,i<j$) are all decision variables. The first constraint and the second constraint are the multiplicative consistency condition.

Obviously, the optimal solutions of model (M-3) can construct a multiplicatively consistent IVIFPR. Especially, the original IVIFPR is perfectly multiplicatively consistent when the optimal objective function value of model (M-3) is equal to zero.

*(2) Derive an acceptably multiplicatively consistent IVIFPR based on an unacceptably multiplicatively consistent IVIFPR*

Considering the case where the completely multiplicative consistency may be too strict, a model is presented to acquire an acceptably multiplicatively consistent IVIFPR $\tilde{A}''$ from an unacceptably multiplicatively consistent IVIFPR $\tilde{A}'$, which is shown as follows:

$$V=\min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\underline{\mu}'_{ij}-\underline{\mu}''_{ij}\right|+\left|\bar{\mu}'_{ij}-\bar{\mu}''_{ij}\right|+\left|\underline{v}'_{ij}-\underline{v}''_{ij}\right|+\left|\bar{v}'_{ij}-\bar{v}''_{ij}\right|\right)$$
$$\text{s.t.}\begin{cases}\sum\limits_{1\le i<j<k\le n}\left|\ln\left(\underline{\mu}''_{ij}\right)+\ln\left(\underline{\mu}''_{jk}\right)+\ln\left(\bar{v}''_{ik}\right)-\ln\left(\bar{v}''_{ij}\right)-\ln\left(\bar{v}''_{jk}\right)-\ln\left(\underline{\mu}''_{ik}\right)\right|\le\overline{\text{CI}}\cdot C_n^3,\\ \sum\limits_{1\le i<j<k\le n}\left|\ln\left(\bar{\mu}''_{ij}\right)+\ln\left(\bar{\mu}''_{jk}\right)+\ln\left(\underline{v}''_{ik}\right)-\ln\left(\underline{v}''_{ij}\right)-\ln\left(\underline{v}''_{jk}\right)-\ln\left(\bar{\mu}''_{ik}\right)\right|\le\overline{\text{CI}}\cdot C_n^3,\\ 0<\underline{\mu}''_{ij}\le\bar{\mu}''_{ij}<1,\ 0<\underline{v}''_{ij}\le\bar{v}''_{ij}<1,\ i,j=1,2,\cdots,n,\ i<j,\\ \bar{\mu}''_{ij}+\bar{v}''_{ij}\le1,\ i,j=1,2,\cdots,n,\ i<j,\end{cases}\tag{M-4}$$

where $\underline{\mu}''_{ij},\bar{\mu}''_{ij},\underline{v}''_{ij},\bar{v}''_{ij}$ ($i,j=1,2,\cdots,n,i<j$) are decision variables. Solving the model (M-4), an acceptably multiplicatively consistent IVIFPR $\tilde{A}''$ can be obtained. Especially, (M-4) is equivalent to (M-3) when $\overline{\text{CI}}=0$.

Clearly, the adjusted IVIFPR $\tilde{A}''$ derived by model (M-3) or model (M-4) is closest to the initial complete IVIFPR $\tilde{A}'$ on the premise of perfectly multiplicative consistency or acceptably multiplicative consistency.

## IV. Consensus Analysis for IVIFPRs

In this section, we discuss the consensus. Let $E=\{e_1,e_2,\cdots,e_m\}$ be a set of DMs, who are required to compare alternatives in $X=\{x_1,x_2,\cdots,x_n\}$. Let $\tilde{A}^h=\left(\tilde{\alpha}_{ij}^h\right)_{n\times n}$ be the individual incomplete IVIFPR provided by the DM $e_h$, where $\tilde{\alpha}_{ij}^h=\left(\tilde{\mu}_{ij}^h,\tilde{v}_{ij}^h\right)=\left(\left[\underline{\mu}_{ij}^h,\bar{\mu}_{ij}^h\right],\left[\underline{v}_{ij}^h,\bar{v}_{ij}^h\right]\right)$ and $h=1,2,\cdots,m$. Let $\tilde{A}'^h=\left(\tilde{\alpha}'^h_{ij}\right)_{n\times n}$ be the complete IVIFPR obtained by the model (M-2), where

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$\tilde{\alpha}_{ij}^{\prime h}=\left(\tilde{\mu}_{ij}^{\prime h},\tilde{v}_{ij}^{\prime h}\right)=\left(\left[\underline{\mu}_{ij}^{\prime h},\overline{\mu}_{ij}^{\prime h}\right],\left[\underline{v}_{ij}^{\prime h},\overline{v}_{ij}^{\prime h}\right]\right)$. Let $\tilde{A}^{\prime\prime h}=\left(\tilde{\alpha}_{ij}^{\prime\prime h}\right)_{n\times n}$ be the perfectly multiplicatively consistent or acceptably multiplicatively consistent IVIFPR obtained by the model (M-3) or (M-4), where $\tilde{\alpha}_{ij}^{\prime\prime h}=\left(\tilde{\mu}_{ij}^{\prime\prime h},\tilde{v}_{ij}^{\prime\prime h}\right)=\left(\left[\underline{\mu}_{ij}^{\prime\prime h},\overline{\mu}_{ij}^{\prime\prime h}\right],\left[\underline{v}_{ij}^{\prime\prime h},\overline{v}_{ij}^{\prime\prime h}\right]\right)$.

Let $\tilde{A}^{c}=\left(\tilde{\alpha}_{ij}^{c}\right)_{n\times n}$ be a collective IVIFPR [32], where

$$\tilde{\alpha}_{ij}^{c}=\left(\tilde{\mu}_{ij}^{c},\tilde{v}_{ij}^{c}\right)=\left(\left[\underline{\mu}_{ij}^{c},\overline{\mu}_{ij}^{c}\right],\left[\underline{v}_{ij}^{c},\overline{v}_{ij}^{c}\right]\right)$$
$$=\left(\left[\prod_{h=1}^{m}\left(\underline{\mu}_{ij}^{\prime\prime h}\right)^{w_h},\prod_{h=1}^{m}\left(\overline{\mu}_{ij}^{\prime\prime h}\right)^{w_h}\right],\left[\prod_{h=1}^{m}\left(\underline{v}_{ij}^{\prime\prime h}\right)^{w_h},\prod_{h=1}^{m}\left(\overline{v}_{ij}^{\prime\prime h}\right)^{w_h}\right]\right),\quad(17)$$

where $w=\left(w_1,w_2,\cdots,w_m\right)^T$ is the DMs' weight vector such that $\sum_{h=1}^{m}w_h=1$ and $w_h\ge 0$ for all $h=1,2,\cdots,m$.

*Theorem 4.1:* If $\tilde{A}^{\prime\prime h}$ ($h=1,2,\cdots,m$) are all perfectly multiplicatively consistent or acceptably multiplicatively consistent, then the collective IVIFPR $\tilde{A}^{c}$ obtained from Eq. (17) are perfectly multiplicatively consistent or acceptably multiplicatively consistent.

*Proof:* Let $A^{\prime\prime h-}=\left(\alpha_{ij}^{\prime\prime h-}\right)_{n\times n}$ and $A^{\prime\prime h+}=\left(\alpha_{ij}^{\prime\prime h+}\right)_{n\times n}$ be the lower and upper IFPRs of $\tilde{A}^{\prime\prime h}$, respectively. Based on Eqs. (11) and (12), we have

$$\alpha_{ij}^{\prime\prime h-}=\left(\mu_{ij}^{\prime\prime h-},v_{ij}^{\prime\prime h-}\right)=\begin{cases}\left(\underline{\mu}_{ij}^{\prime\prime h},\overline{v}_{ij}^{\prime\prime h}\right), & \text{if } i<j,\\ (0.5,0.5), & \text{if } i=j,\\ \left(\overline{\mu}_{ij}^{\prime\prime h},\underline{v}_{ij}^{\prime\prime h}\right), & \text{if } i>j,\end{cases}\quad(18)$$

$$\alpha_{ij}^{\prime\prime h+}=\left(\mu_{ij}^{\prime\prime h+},v_{ij}^{\prime\prime h+}\right)=\begin{cases}\left(\overline{\mu}_{ij}^{\prime\prime h},\underline{v}_{ij}^{\prime\prime h}\right), & \text{if } i<j,\\ (0.5,0.5), & \text{if } i=j,\\ \left(\underline{\mu}_{ij}^{\prime\prime h},\overline{v}_{ij}^{\prime\prime h}\right), & \text{if } i>j.\end{cases}\quad(19)$$

To prove the acceptably multiplicative consistency of $\tilde{A}^{c}$, we need to prove its lower matrix $A^{c-}=\left(\alpha_{ij}^{c-}\right)_{n\times n}$ with $\alpha_{ij}^{c-}=\left(\mu_{ij}^{c-},v_{ij}^{c-}\right)$ and its upper matrix $A^{c+}=\left(\alpha_{ij}^{c+}\right)_{n\times n}$ with $\alpha_{ij}^{c+}=\left(\mu_{ij}^{c+},v_{ij}^{c+}\right)$ are acceptably multiplicatively consistent. Following Eqs. (17)-(19), we derive

$$\left(\mu_{ij}^{c-},v_{ij}^{c-}\right)=\begin{cases}\left(\underline{\mu}_{ij}^{c},\overline{v}_{ij}^{c}\right), & \text{if } i<j\\ (0.5,0.5), & \text{if } i=j\\ \left(\overline{\mu}_{ij}^{c},\underline{v}_{ij}^{c}\right), & \text{if } i>j\end{cases}=\begin{cases}\left(\prod_{h=1}^{m}\left(\underline{\mu}_{ij}^{\prime\prime h}\right)^{w_h},\prod_{h=1}^{m}\left(\overline{v}_{ij}^{\prime\prime h}\right)^{w_h}\right), & \text{if } i<j,\\ (0.5,0.5), & \text{if } i=j,\\ \left(\prod_{h=1}^{m}\left(\overline{\mu}_{ij}^{\prime\prime h}\right)^{w_h},\prod_{h=1}^{m}\left(\underline{v}_{ij}^{\prime\prime h}\right)^{w_h}\right), & \text{if } i>j,\end{cases}$$

$$\left(\mu_{ij}^{c+},v_{ij}^{c+}\right)=\begin{cases}\left(\overline{\mu}_{ij}^{c},\underline{v}_{ij}^{c}\right), & \text{if } i<j\\ (0.5,0.5), & \text{if } i=j\\ \left(\underline{\mu}_{ij}^{c},\overline{v}_{ij}^{c}\right), & \text{if } i>j\end{cases}=\begin{cases}\left(\prod_{h=1}^{m}\left(\overline{\mu}_{ij}^{\prime\prime h}\right)^{w_h},\prod_{h=1}^{m}\left(\underline{v}_{ij}^{\prime\prime h}\right)^{w_h}\right), & \text{if } i<j,\\ (0.5,0.5), & \text{if } i=j,\\ \left(\prod_{h=1}^{m}\left(\underline{\mu}_{ij}^{\prime\prime h}\right)^{w_h},\prod_{h=1}^{m}\left(\overline{v}_{ij}^{\prime\prime h}\right)^{w_h}\right), & \text{if } i>j.\end{cases}$$

Obviously, $A^{c-}$ is a combination of $A^{\prime\prime h-}$ ($h=1,2,\cdots m$), and $A^{c+}$ is a combination of $A^{\prime\prime h+}$ ($h=1,2,\cdots m$).

Because $\tilde{A}^{\prime\prime h}$ is perfectly multiplicatively consistent or acceptably multiplicatively consistent, $A^{\prime\prime h-}$ and $A^{\prime\prime h+}$ are perfectly multiplicatively consistent or acceptably multiplicatively consistent following *Definition 3.4*. Then, as per *Theorem 3.4*, $A^{c-}$ and $A^{c+}$ are perfectly multiplicatively consistent or acceptably multiplicatively consistent. Furthermore, following *Definition 3.4*, we conclude that $\tilde{A}^{c}$ is perfectly multiplicatively consistent or acceptably multiplicatively consistent, which completes the proof.

*Definition 4.1:* Let $\tilde{A}^{h}$ be an incomplete IVIFPR, let $\tilde{A}^{\prime h}$ be its complete IVIFPR, and let $\tilde{A}^{\prime\prime h}$ be the adjusted multiplicatively consistent or acceptably multiplicatively consistent IVIFPR, where $h=1,2,\cdots,m$. Furthermore, let $\tilde{A}^{c}$ be the collective IVIFPR obtained from Eq. (17). The consensus index of $\tilde{A}^{\prime\prime h}$ is defined below:

$$GCI\left(\tilde{A}^{\prime\prime h}\right)=\frac{1}{2n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\ln\left(\underline{\mu}_{ij}^{\prime\prime h}\right)-\ln\left(\underline{\mu}_{ij}^{c}\right)\right|+\left|\ln\left(\overline{\mu}_{ij}^{\prime\prime h}\right)-\ln\left(\overline{\mu}_{ij}^{c}\right)\right|+\left|\ln\left(\underline{v}_{ij}^{\prime\prime h}\right)-\ln\left(\underline{v}_{ij}^{c}\right)\right|+\left|\ln\left(\overline{v}_{ij}^{\prime\prime h}\right)-\ln\left(\overline{v}_{ij}^{c}\right)\right|\right),\quad(20)$$

where $h=1,2,\cdots,m$.

Following Eq. (20), we derive $GCI\left(\tilde{A}^{\prime\prime h}\right)\ge 0$ for any IVIFPR $\tilde{A}^{\prime\prime h}$ ($h=1,2,\cdots,m$).

In the procedure of calculating the collective IVIFPR, the DMs' weight vector is used. In the setting of GDM, the weights of DMs are usually unknown. Therefore, we first need to determine the DMs' weights. Following Eq. (20), we next build a model to generate the DMs' weight vector by minimizing the consensus index.

$$Z=\min\sum_{h=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\ln\left(\underline{\mu}_{ij}^{\prime\prime h}\right)-\sum_{h=1}^{m}w_h\ln\left(\underline{\mu}_{ij}^{\prime\prime h}\right)\right|+\left|\ln\left(\overline{\mu}_{ij}^{\prime\prime h}\right)-\sum_{h=1}^{m}w_h\ln\left(\overline{\mu}_{ij}^{\prime\prime h}\right)\right|+\left|\ln\left(\underline{v}_{ij}^{\prime\prime h}\right)-\sum_{h=1}^{m}w_h\ln\left(\underline{v}_{ij}^{\prime\prime h}\right)\right|+\left|\ln\left(\overline{v}_{ij}^{\prime\prime h}\right)-\sum_{h=1}^{m}w_h\ln\left(\overline{v}_{ij}^{\prime\prime h}\right)\right|\right)\quad\text{(M-5)}$$
$$\text{s.t.}\begin{cases}w_h\in[0,1],\ h=1,2,\cdots,m,\\ \sum_{h=1}^{m}w_h=1.\end{cases}$$

Furthermore, we introduce several positive slack variables $\vartheta_{ij}^{h+},\vartheta_{ij}^{h-},\theta_{ij}^{h+},\theta_{ij}^{h-},\xi_{ij}^{h+},\xi_{ij}^{h-},\eta_{ij}^{h+},\eta_{ij}^{h-}$ to remove the symbol of absolute value in model (M-5). Then, model (M-5) is transformed as

$$Z=\min\sum_{h=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\vartheta_{ij}^{h+}+\vartheta_{ij}^{h-}+\theta_{ij}^{h+}+\theta_{ij}^{h-}+\xi_{ij}^{h+}+\xi_{ij}^{h-}+\eta_{ij}^{h+}+\eta_{ij}^{h-}\right)$$
$$\text{s.t.}\begin{cases}\ln\left(\underline{\mu}_{ij}^{\prime\prime h}\right)-\sum_{h=1}^{m}w_h\ln\left(\underline{\mu}_{ij}^{\prime\prime h}\right)-\vartheta_{ij}^{h+}+\vartheta_{ij}^{h-}=0,\ i,j=1,2,\cdots,n,\ i<j,\ h=1,2,\cdots,m,\\ \ln\left(\overline{\mu}_{ij}^{\prime\prime h}\right)-\sum_{h=1}^{m}w_h\ln\left(\overline{\mu}_{ij}^{\prime\prime h}\right)-\theta_{ij}^{h+}+\theta_{ij}^{h-}=0,\ i,j=1,2,\cdots,n,\ i<j,\ h=1,2,\cdots,m,\\ \ln\left(\underline{v}_{ij}^{\prime\prime h}\right)-\sum_{h=1}^{m}w_h\ln\left(\underline{v}_{ij}^{\prime\prime h}\right)-\xi_{ij}^{h+}+\xi_{ij}^{h-}=0,\ i,j=1,2,\cdots,n,\ i<j,\ h=1,2,\cdots,m,\\ \ln\left(\overline{v}_{ij}^{\prime\prime h}\right)-\sum_{h=1}^{m}w_h\ln\left(\overline{v}_{ij}^{\prime\prime h}\right)-\eta_{ij}^{h+}+\eta_{ij}^{h-}=0,\ i,j=1,2,\cdots,n,\ i<j,\ h=1,2,\cdots,m,\\ \sum_{h=1}^{m}w_h=1,\ w_h\in[0,1],\ h=1,2,\cdots,m,\\ \vartheta_{ij}^{h+},\vartheta_{ij}^{h-},\theta_{ij}^{h+},\theta_{ij}^{h-},\xi_{ij}^{h+},\xi_{ij}^{h-},\eta_{ij}^{h+},\eta_{ij}^{h-}\ge 0,\ i,j=1,2,\cdots,n,\ i<j,\ h=1,2,\cdots,m.\end{cases}$$
$$\text{(M-6)}$$

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Solving model (M-6) yields the DMs' weight vector $w^* = \left(w_1^*, w_2^*, \cdots, w_m^*\right)^T$, and the optimal slack variables $\vartheta_{ij}^{*h+}, \vartheta_{ij}^{*h-}, \theta_{ij}^{*h+}, \theta_{ij}^{*h-}, \xi_{ij}^{*h+}, \xi_{ij}^{*h-}, \eta_{ij}^{*h+}, \eta_{ij}^{*h-}$. Then, by using Eq. (20), we obtain the consensus index of $\tilde{A}^{\prime\prime h}$ as $GCI\left(\tilde{A}^{\prime\prime h}\right) = \frac{1}{2n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\vartheta_{ij}^{*h+} + \vartheta_{ij}^{*h-} + \theta_{ij}^{*h+} + \theta_{ij}^{*h-} + \xi_{ij}^{*h+} + \xi_{ij}^{*h-} + \eta_{ij}^{*h+} + \eta_{ij}^{*h-}\right)$.

Let $\tau$ be the given threshold of consensus. If the consensus level does not satisfy this threshold, namely, $GCI\left(\tilde{A}^{\prime\prime h}\right) > \tau$, then we improve the consensus level. Considering that the influences of different judgments are different, their adjustments should be different too. For all $i, j = 1, 2, \cdots, n$ with $i < j$, let

$$\begin{cases} \underline{\mu}_{ij}^{\prime\prime\prime h} = \left(\underline{\mu}_{ij}^{\prime\prime h}\right)^{\left(1-\underline{\lambda}_{ij}^h\right)} \times \left(\underline{\mu}_{ij}^c\right)^{\underline{\lambda}_{ij}^h}, \quad \overline{\mu}_{ij}^{\prime\prime\prime h} = \left(\overline{\mu}_{ij}^{\prime\prime h}\right)^{\left(1-\overline{\lambda}_{ij}^h\right)} \times \left(\overline{\mu}_{ij}^c\right)^{\overline{\lambda}_{ij}^h}, \\ \underline{\nu}_{ij}^{\prime\prime\prime h} = \left(\underline{\nu}_{ij}^{\prime\prime h}\right)^{\left(1-\underline{\varepsilon}_{ij}^h\right)} \times \left(\underline{\nu}_{ij}^c\right)^{\underline{\varepsilon}_{ij}^h}, \quad \overline{\nu}_{ij}^{\prime\prime\prime h} = \left(\overline{\nu}_{ij}^{\prime\prime h}\right)^{\left(1-\overline{\varepsilon}_{ij}^h\right)} \times \left(\overline{\nu}_{ij}^c\right)^{\overline{\varepsilon}_{ij}^h}, \end{cases} \tag{21}$$

where $\underline{\lambda}_{ij}^h, \overline{\lambda}_{ij}^h, \underline{\varepsilon}_{ij}^h, \overline{\varepsilon}_{ij}^h \in [0,1]$.

As per Eq. (21), we have

$$\begin{cases} \ln\left(\underline{\mu}_{ij}^{\prime\prime\prime h}\right) = \left(1-\underline{\lambda}_{ij}^h\right)\ln\left(\underline{\mu}_{ij}^{\prime\prime h}\right) + \underline{\lambda}_{ij}^h \ln\left(\underline{\mu}_{ij}^c\right), \\ \ln\left(\overline{\mu}_{ij}^{\prime\prime\prime h}\right) = \left(1-\overline{\lambda}_{ij}^h\right)\ln\left(\overline{\mu}_{ij}^{\prime\prime h}\right) + \overline{\lambda}_{ij}^h \left(\overline{\mu}_{ij}^c\right), \\ \ln\left(\underline{\nu}_{ij}^{\prime\prime\prime h}\right) = \left(1-\underline{\varepsilon}_{ij}^h\right)\ln\left(\underline{\nu}_{ij}^{\prime\prime h}\right) + \underline{\varepsilon}_{ij}^h \ln\left(\underline{\nu}_{ij}^c\right), \\ \ln\left(\overline{\nu}_{ij}^{\prime\prime\prime h}\right) = \left(1-\overline{\varepsilon}_{ij}^h\right)\ln\left(\overline{\nu}_{ij}^{\prime\prime h}\right) + \overline{\varepsilon}_{ij}^h \ln\left(\overline{\nu}_{ij}^c\right). \end{cases} \tag{22}$$

Moreover, the adjusted IVIFPRs should achieve several goals: (i) The adjusted IVIFPRs should have acceptable consistency; (ii) The adjusted IVIFPRs should have acceptable consensus; (iii) The adjusted IVIFPRs should have the smallest deviations from the original IVIFPRs. Hence, we present the following model:

$$Y = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\underline{\lambda}_{ij}^h + \overline{\lambda}_{ij}^h + \underline{\varepsilon}_{ij}^h + \overline{\varepsilon}_{ij}^h\right)$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Big( \Big|\left(1-\underline{\lambda}_{ij}^h\right)\ln\left(\underline{\mu}_{ij}^{\prime\prime h}\right) + \underline{\lambda}_{ij}^h \ln\left(\underline{\mu}_{ij}^c\right) - \\ w_h\left(\left(1-\underline{\lambda}_{ij}^h\right)\ln\left(\underline{\mu}_{ij}^{\prime\prime h}\right) + \underline{\lambda}_{ij}^h \ln\left(\underline{\mu}_{ij}^c\right)\right) - \\ \sum_{z=1, z\neq h}^{m} w_z \ln\left(\underline{\mu}_{ij}^{\prime\prime z}\right)\Big| + \Big|\left(1-\overline{\lambda}_{ij}^h\right)\ln\left(\overline{\mu}_{ij}^{\prime\prime h}\right) + \\ \overline{\lambda}_{ij}^h\left(\overline{\mu}_{ij}^c\right) - w_h\left(\left(1-\overline{\lambda}_{ij}^h\right)\ln\left(\overline{\mu}_{ij}^{\prime\prime h}\right) + \overline{\lambda}_{ij}^h\left(\overline{\mu}_{ij}^c\right)\right) - \\ \sum_{z=1, z\neq h}^{m} w_z \ln\left(\overline{\mu}_{ij}^{\prime\prime z}\right)\Big| + \Big|\left(1-\underline{\varepsilon}_{ij}^h\right)\ln\left(\underline{\nu}_{ij}^{\prime\prime h}\right) + \underline{\varepsilon}_{ij}^h \ln\left(\underline{\nu}_{ij}^c\right) \\ -w_h\left(\left(1-\underline{\varepsilon}_{ij}^h\right)\ln\left(\underline{\nu}_{ij}^{\prime\prime h}\right) + \underline{\varepsilon}_{ij}^h \ln\left(\underline{\nu}_{ij}^c\right)\right) \\ -\sum_{z=1, z\neq h}^{m} w_z \ln\left(\underline{\nu}_{ij}^{\prime\prime z}\right)\Big| + \Big|\left(1-\overline{\varepsilon}_{ij}^h\right)\ln\left(\overline{\nu}_{ij}^{\prime\prime h}\right) + \overline{\varepsilon}_{ij}^h \ln\left(\overline{\nu}_{ij}^c\right) \\ -w_h\left(\left(1-\overline{\varepsilon}_{ij}^h\right)\ln\left(\overline{\nu}_{ij}^{\prime\prime h}\right) + \overline{\varepsilon}_{ij}^h \ln\left(\overline{\nu}_{ij}^c\right)\right) - \\ \sum_{z=1, z\neq h}^{m} w_z \ln\left(\overline{\nu}_{ij}^{\prime\prime z}\right)\Big|\Big) \le 2n(n-1)\tau, \\ \underline{\lambda}_{ij}^h, \overline{\lambda}_{ij}^h, \underline{\varepsilon}_{ij}^h, \overline{\varepsilon}_{ij}^h \in [0,1],\ i, j = 1, 2, \cdots, n,\ i < j. \end{cases} \tag{M-7}$$

From model (M-7), the optimal solutions $\underline{\lambda}_{ij}^{h*}$, $\overline{\lambda}_{ij}^{h*}$, $\underline{\varepsilon}_{ij}^{h*}, \overline{\varepsilon}_{ij}^{h*}$ are obtained. Then, by inserting these optimal solutions into Eq. (21), the upper triangular components $\underline{\mu}_{ij}^{\prime\prime\prime h*}, \overline{\mu}_{ij}^{\prime\prime\prime h*}, \underline{\nu}_{ij}^{\prime\prime\prime h*}, \overline{\nu}_{ij}^{\prime\prime\prime h*}$ ( $i, j = 1, 2, \cdots, n,\ i < j$ ) are obtained. Then, a modified IVIFPR $\tilde{A}^{\prime\prime\prime h*} = \left(\tilde{\alpha}_{ij}^{\prime\prime\prime h*}\right)_{n\times n}$ with acceptable consensus is obtained via Eq. (4) below:

$$\tilde{\alpha}_{ij}^{\prime\prime\prime h*} = \left(\tilde{\mu}_{ij}^{\prime\prime\prime h*}, \tilde{\nu}_{ij}^{\prime\prime\prime h*}\right) = \begin{cases} \left(\left[\underline{\mu}_{ij}^{\prime\prime\prime h*}, \overline{\mu}_{ij}^{\prime\prime\prime h*}\right], \left[\underline{\nu}_{ij}^{\prime\prime\prime h*}, \overline{\nu}_{ij}^{\prime\prime\prime h*}\right]\right), & i < j, \\ \left([0.5, 0.5], [0.5, 0.5]\right), & i = j, \\ \left(\left[\underline{\nu}_{ji}^{\prime\prime\prime h*}, \overline{\nu}_{ji}^{\prime\prime\prime h*}\right], \left[\underline{\mu}_{ji}^{\prime\prime\prime h*}, \overline{\mu}_{ji}^{\prime\prime\prime h*}\right]\right), & i > j. \end{cases} \tag{23}$$

## V. GDM With Incomplete IVIFPRs

In this part, we study the derivation of the priority weights based on the collective IVIFPR. Then, a GDM method is presented.

### *A. Derivation of the Priority Weights*

Assume that the IVIF priority vector of $\tilde{A}^c$ is $\tilde{\omega} = \left(\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n\right)^T$, where $\tilde{\omega}_i = \left(\left[\underline{\omega}_i^\mu, \overline{\omega}_i^\mu\right], \left[\underline{\omega}_i^\nu, \overline{\omega}_i^\nu\right]\right)$ are IVIFVs. Let $A^{c-} = \left(\alpha_{ij}^{c-}\right)_{n\times n} = \left(\left(\mu_{ij}^{c-}, \nu_{ij}^{c-}\right)\right)_{n\times n}$ and $A^{c+} = \left(\alpha_{ij}^{c+}\right)_{n\times n} = \left(\left(\mu_{ij}^{c+}, \nu_{ij}^{c+}\right)\right)_{n\times n}$ be the lower and upper IFPRs of $\tilde{A}^c$. Let the IF priority weight vectors of $A^{c-}$ and $A^{c+}$ be $\omega^- = \left(\omega_1^-, \omega_2^-, \cdots, \omega_n^-\right)^T$ and $\omega^+ = \left(\omega_1^+, \omega_2^+, \cdots, \omega_n^+\right)^T$, respectively, where $\omega_i^- = \left(\omega_i^{\mu-}, \omega_i^{\nu-}\right)$ and $\omega_i^+ = \left(\omega_i^{\mu+}, \omega_i^{\nu+}\right)$ are IFVs.

Based on *Definition 3.4* and *Theorem 3.3*, IVIFPR $\tilde{A}^c$ is multiplicatively consistent iff $A^{c-}$ and $A^{c+}$ are multiplicatively consistent iff there exist two normalized IF priority weight vectors $\omega^-$ and $\omega^+$ such that

$$\alpha_{ij}^{c-} = \left(\mu_{ij}^{c-}, \nu_{ij}^{c-}\right) = \begin{cases} (0.5, 0.5), & \text{if } i = j, \\ \left(\omega_i^{\mu-}\omega_j^{\nu-}, \omega_i^{\nu-}\omega_j^{\mu-}\right), & \text{if } i < j, \end{cases} \tag{24}$$

and

$$\alpha_{ij}^{c+} = \left(\mu_{ij}^{c+}, \nu_{ij}^{c+}\right) = \begin{cases} (0.5, 0.5), & \text{if } i = j, \\ \left(\omega_i^{\mu+}\omega_j^{\nu+}, \omega_i^{\nu+}\omega_j^{\mu+}\right), & \text{if } i < j. \end{cases} \tag{25}$$

Furthermore, Eqs. (24) and (25) are simplified as

$$\mu_{ij}^{c-} = \omega_i^{\mu-}\omega_j^{\nu-},\ \nu_{ij}^{c-} = \omega_i^{\nu-}\omega_j^{\mu-},\ \mu_{ij}^{c+} = \omega_i^{\mu+}\omega_j^{\nu+},\ \nu_{ij}^{c+} = \omega_i^{\nu+}\omega_j^{\mu+}, \tag{26}$$

where $i, j = 1, 2, \cdots, n,\ i < j$.

However, there may exist some deviations between the ideal judgments and the real judgments. Moreover, the smaller the deviations are, and the more consistent the collective IVIFPR $\tilde{A}^c$ is. Thus, by minimizing the deviations, the following model is offered to acquire the normalized IF priority weights $\omega^-$ and $\omega^+$.

$$\varepsilon = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left( \begin{array}{l} \left| \mu_{ij}^{c-} - \omega_i^{\mu-} \omega_j^{\nu-} \right| + \left| \nu_{ij}^{c-} - \omega_i^{\nu-} \omega_j^{\mu-} \right| + \\ \left| \mu_{ij}^{c+} - \omega_i^{\mu+} \omega_j^{\nu+} \right| + \left| \nu_{ij}^{c+} - \omega_i^{\nu+} \omega_j^{\mu+} \right| \end{array} \right)$$

$$\text{s.t.} \begin{cases} 0 \le \omega_i^{\mu-}, \omega_i^{\nu-}, \omega_i^{\mu+}, \omega_i^{\nu+} \le 1, \ i = 1,2,\cdots,n, \\ \omega_i^{\mu-} + \omega_i^{\nu-} \le 1, \ \omega_i^{\mu+} + \omega_i^{\nu+} \le 1, \ i = 1,2,\cdots,n, \\ \sum_{j=1, j\ne i}^{n} \omega_j^{\mu-} \le \omega_i^{\nu-}, \ \omega_i^{\mu-} + n - 2 \ge \sum_{j=1, j\ne i}^{n} \omega_j^{\nu-}, \ i = 1,2,\cdots,n, \\ \sum_{j=1, j\ne i}^{n} \omega_j^{\mu+} \le \omega_i^{\nu+}, \ i = 1,2,\cdots,n, \\ \omega_i^{\mu+} + n - 2 \ge \sum_{j=1, j\ne i}^{n} \omega_j^{\nu+}, \ i = 1,2,\cdots,n, \\ \max\left\{\omega_i^{\mu-}, \omega_i^{\mu+}\right\} + \max\left\{\omega_i^{\nu-}, \omega_i^{\nu+}\right\} \le 1, \ i = 1,2,\cdots,n, \end{cases} \quad \text{(M-8)}$$

where $\omega_i^{\mu-}$, $\omega_i^{\nu-}$, $\omega_i^{\mu+}$ and $\omega_i^{\nu+}$ ($i = 1,2,\cdots,n$) are decision variables. The first three constraint conditions is to derive IF priority weights. The fourth to seventh constraints are the normalization constraints imposed on the IF priority vectors $\omega^-$ and $\omega^+$.

Let $\max\left\{\omega_i^{\mu-}, \omega_i^{\mu+}\right\} = s_i$ and let $\max\left\{\omega_i^{\nu-}, \omega_i^{\nu+}\right\} = t_i$, where $i = 1,2,\cdots,n$. Then, model (M-8) is further transformed as follows:

$$\varepsilon = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left( \chi_{ij}^1 + \chi_{ij}^2 + \chi_{ij}^3 + \chi_{ij}^4 + \chi_{ij}^5 + \chi_{ij}^6 + \chi_{ij}^7 + \chi_{ij}^8 \right)$$

$$\text{s.t.} \begin{cases} \mu_{ij}^{c-} - \omega_i^{\mu-} \omega_j^{\nu-} - \chi_{ij}^1 + \chi_{ij}^2 = 0, \ i,j = 1,2,\cdots,n, \ i < j, \\ \nu_{ij}^{c-} - \omega_i^{\nu-} \omega_j^{\mu-} - \chi_{ij}^3 + \chi_{ij}^4 = 0, \ i,j = 1,2,\cdots,n, \ i < j, \\ \mu_{ij}^{c+} - \omega_i^{\mu+} \omega_j^{\nu+} - \chi_{ij}^5 + \chi_{ij}^6 = 0, \ i,j = 1,2,\cdots,n, \ i < j, \\ \nu_{ij}^{c+} - \omega_i^{\nu+} \omega_j^{\mu+} - \chi_{ij}^7 + \chi_{ij}^8 = 0, \ i,j = 1,2,\cdots,n, \ i < j, \\ \chi_{ij}^1, \chi_{ij}^2, \chi_{ij}^3, \chi_{ij}^4, \chi_{ij}^5, \chi_{ij}^6, \chi_{ij}^7, \chi_{ij}^8 \ge 0, \ i,j = 1,2,\cdots,n, \ i < j, \\ 0 \le \omega_i^{\mu-}, \omega_i^{\nu-}, \omega_i^{\mu+}, \omega_i^{\nu+} \le 1, \ i = 1,2,\cdots,n, \\ \omega_i^{\mu-} + \omega_i^{\nu-} \le 1, \ \omega_i^{\mu+} + \omega_i^{\nu+} \le 1, \ i = 1,2,\cdots,n, \\ \sum_{j=1, j\ne i}^{n} \omega_j^{\mu-} \le \omega_i^{\nu-}, \ \omega_i^{\mu-} + n - 2 \ge \sum_{j=1, j\ne i}^{n} \omega_j^{\nu-}, \ i = 1,2,\cdots,n, \\ \sum_{j=1, j\ne i}^{n} \omega_j^{\mu+} \le \omega_i^{\nu+}, \ \omega_i^{\mu+} + n - 2 \ge \sum_{j=1, j\ne i}^{n} \omega_j^{\nu+}, \ i = 1,2,\cdots,n, \\ s_i + t_i \le 1, \ i = 1,2,\cdots,n, \\ \omega_i^{\mu-}, \omega_i^{\mu+} \le s_i, \ \omega_i^{\nu-}, \omega_i^{\nu+} \le t_i, \ i = 1,2,\cdots,n, \\ s_i, t_i \ge 0, \ i = 1,2,\cdots,n. \end{cases} \quad \text{(M-9)}$$

The optimal solution $\omega_i^{*\mu-}$, $\omega_i^{*\nu-}$, $\omega_i^{*\mu+}$ and $\omega_i^{*\nu+}$ ($i = 1,2,\cdots,n$) of model (M-9) can be applied to construct an IVIF priority weight $\tilde{\omega}_i$ ($i = 1,2,\cdots,n$) via Eq. (15).

### *B. An Algorithm*

A GDM method for incomplete IVIFPRs is put forward as follows:

*Step 1:* Let $\tilde{A}^h$, where $h = 1,2,\cdots,m$, be any $m$ IVIFPRs. If all of them are complete, then set $\tilde{A}'^h = \tilde{A}^h$ and go to *Step 2*. Otherwise, model (M-2) is adopted to estimate missing judgements, which is denoted as $\tilde{A}'^h$.

*Step 2:* Let $\overline{\text{CI}}$ be a predetermined acceptably multiplicative consistency threshold. For each complete IVIFPR $\tilde{A}'^h$, *Definition 3.4* is adopted to check its multiplicative consistency or acceptably multiplicative consistency. If all of them are perfectly multiplicatively consistent or acceptably multiplicatively consistent, then let $\tilde{A}''^h = \tilde{A}'^h$, where $h = 1,2,\cdots,m$, and go to *Step 4*. Otherwise, go to *Step 3*.

*Step 3:* Substitute $\tilde{A}'^h$ into model (M-3) or (M-4). If (M-3) or (M-4) has solutions, then a perfectly multiplicatively consistent or acceptably multiplicatively consistent IVIFPR $\tilde{A}''^h$ is derived by using model (M-3) or (M-4). If (M-3) or (M-4) has no solutions, then reset the consistency threshold $\overline{\text{CI}}$ and return to *Step 2*.

*Step 4:* Using model (M-6), the DMs' weight vector is obtained, where $w^* = \left(w_1^*, w_2^*, \cdots, w_m^*\right)^T$. Eq. (17) is applied to calculate the collective IVIFPR $\tilde{A}^c$.

*Step 5:* According to Eq. (20), compute the consensus index of $\tilde{A}''^h$, denoted by $GCI\left(\tilde{A}''^h\right)$ ($h = 1,2,\cdots,m$).

*Step 6:* Predetermine the consensus threshold $\tau$. Judge whether or not all IVIFPRs are of acceptable consensus. If yes, go to *Step 8*. Otherwise, let $GCI\left(\tilde{A}''^h\right) = \max_{1 \le l \le m}\left\{GCI\left(\tilde{A}''^l\right)\right\}$ and go to *Step 7*.

*Step 7:* Plug $\tilde{A}''^h$ into model (M-7) and solve this model. If model (M-7) has solutions, then the revised IVIFPR $\tilde{A}'''^h$ is obtained by using model (M-7) and Eq. (23). And then we set $\tilde{A}''^h = \tilde{A}'''^h$ and return to *Step 4*. If model (M-7) has no solutions, then reset the consensus threshold $\tau$ and return to *Step 6*.

*Step 8:* Obtain the normalized IF priority weight vector $\omega^-$ from $A^{c-}$ and the normalized IF priority weight vector $\omega^+$ from $A^{c+}$ by the model (M-9), respectively.

*Step 9:* Based on $\omega^-$ and $\omega^+$, the IVIF priority weight $\tilde{\omega}_i$ is obtained via Eq. (15), where $i = 1,2,\cdots,n$.

*Step 10:* Compute the score value $s\left(\tilde{\omega}_i\right)$ and accuracy value $\gamma\left(\tilde{\omega}_i\right)$ of IVIF priority weight $\tilde{\omega}_i$ via *Definition 2.6*, where $i = 1,2,\cdots,n$.

*Step 11:* Rank alternatives $x_i$ ($i = 1,2,\cdots,n$) based on $\tilde{\omega}_i$ ($i = 1,2,\cdots,n$).

A flowchart of the presented GDM procedure with incomplete IVIFPRs is described in Fig. 1.

## VI. Applications and Comparisons

In this section, we evaluate the applicability and effectiveness of our algorithm using a numerical example adopted from [28] and compare its performance with those of several existing methods.

### *A. Numerical Example*

*Example 6.1 [28]:* Consider a case study concerning the introduction of talents in a Chinese "double first-class" university. To select the best talents, the university constructs a committees, comprised of four senior experts $e_h$ ($h=1,2,3,4$), to evaluate all the candidates in this introduction. In this case study, there are four candidates $x_1$, $x_2$, $x_3$ and $x_4$. Four incomplete IVIFPRs $\tilde{A}^h$ ($h=1,2,3,4$) are constructed by the experts and are listed in Example 5 of [28].

To rank four candidates via the proposed method, the following process is realized:

*Step 1:* Four complete IVIFPRs $\tilde{A}'^h$ ($h=1,2,3,4$) based on model (M-2) are derived and their lower IFPRs $A'^{h-}$ ($h=1,2,3,4$) and upper IFPRs $A'^{h+}$ ($h=1,2,3,4$) based on Eqs. (11) and (12) are obtained.

*Steps 2 and 3:* Let $\overline{\mathrm{CI}}=0.1$. It can be computed via Eq. (7) that $CI\left(A'^{1-}\right)=0.06$, $CI\left(A'^{1+}\right)=0.76$, $CI\left(A'^{2-}\right)=0.14$, $CI\left(A'^{2+}\right)=0.29$, $CI\left(A'^{3-}\right)=0.18$, $CI\left(A'^{3+}\right)=0.18$, $CI\left(A'^{4-}\right)=0.18$, and $CI\left(A'^{4+}\right)=0.18$, meaning that the IVIFPR $\tilde{A}'^h$ is of unacceptable consistency, where $h=1,2,3,4$. With respect to each complete IVIFPR, the acceptably multiplicatively consistent IVIFPRs $\tilde{A}''^h$ ($h=1,2,3,4$) following model (M-4) are obtained.

*Step 4:* Following the model (M-6), the weight vector of DMs is $w^*=\left(0.13,0.53,0.28,0.06\right)^T$. The collective IVIFPR $\tilde{A}''^c$ using Eq. (17) is derived.

*Steps 5, 6 and 7:* Let $\tau=0.1$ be the acceptable consensus threshold. Based on Eq. (20), we derive $GCI\left(\tilde{A}''^1\right)=0.21$, $GCI\left(\tilde{A}''^2\right)=0.1$, $GCI\left(\tilde{A}''^3\right)=0.18$, and $GCI\left(\tilde{A}''^4\right)=0.17$. Based on model (M-7), the acceptably multiplicatively consistent IVIFPRs $\tilde{A}'''^h$ ($h=1,2,3,4$) with the given consensus level can be obtained.

Furthermore, the collective IVIFPR $\tilde{A}'''^c$ is obtained by Eq. (17).

*Step 8:* Following the collective IVIFPR $\tilde{A}'''^c$, its lower IFPR $A'''^{c-}$ and upper IFPR $A'''^{c+}$ are derived via Eqs. (11) and (12). Furthermore, by solving model (M-9), the IF priority weight vectors of $A'''^{c-}$ and $A'''^{c+}$ are derived as follows:

$$\omega^-=\left((0.24,0.76),(0.39,0.61),(0,1),(0.38,0.62)\right)^T,$$

$$\omega^+=\left((0.24,0.76),(0.39,0.61),(0,1),(0.38,0.62)\right)^T.$$

*Step 9:* Using Eq. (15), the IVIF priority weights are identified as

$$\tilde{\omega}_1=\left([0.24,0.24],[0.76,0.76]\right),$$

$$\tilde{\omega}_2=\left([0.39,0.39],[0.61,0.61]\right),$$

$$\tilde{\omega}_3=\left([0,0],[1,1]\right),$$

$$\tilde{\omega}_4=\left([0.38,0.38],[0.62,0.62]\right).$$

*Steps 10 and 11:* The scores based on *Definition 2.6* are $s\left(\tilde{\omega}_1\right)=-0.52$, $s\left(\tilde{\omega}_2\right)=-0.22$, $s\left(\tilde{\omega}_3\right)=-1$, and $s\left(\tilde{\omega}_4\right)=-0.24$. Thus, the ranking is $x_2 \succ x_4 \succ x_1 \succ x_3$, namely, the second candidate is the most suitable choice.

*B. Comparative Analyses with Previous Methods*

This section covers the comparative analyses with the methods presented in [27] and [28].

(1) We compare our method with Tang *et al.*'s method [28]. In this example, the ranking is $x_2 \succ x_4 \succ x_1 \succ x_3$ based on our method that is slightly different from the obtained ranking $x_2 \succ x_1 \succ x_4 \succ x_3$ using Tang *et al.*'s method [28]. However, both of them show that $x_2$ is the best choice. Compared with Tang *et al.*'s method [28], there are several merits of our method:

(i) Our method adopts the acceptably multiplicative consistency analysis, while Tang *et al.*'s method [28] employed the completely additive consistency analysis. Due to the differences in professional competence and thinking among experts, it takes lots of time and efforts to reach complete consistency of all experts, and sometimes it may be even impossible to attain the complete consistency of a group. Therefore, in real life, the result of GDM is supposed to be acceptable if the consistency degree achieves a certain level. Under this view, an acceptably multiplicative consistency concept is more suitable. It is noticeable that when we let the acceptably multiplicative consistency threshold be equal to 1, we derived decision-making methods with IVIFPRs following the completely multiplicative consistency analysis.

(ii) Tang *et al.*'s method [28] improves the consensus level by an iterative method that is based on more adjustments. By contrast, our method adjusts the consensus degree by a model that is based on once adjustment for judgments in the procedure of the consistency and consensus analysis. It is worth stressing that the presented model can be easily handled using MATLAB or LINGO.

(2) When our method is adopted to manage Example 6 considered in [27], the ranking is $x_2 \succ x_1 \succ x_4 \succ x_3$, which shows that $x_2$ is the best option. When Meng *et al.*'s method [27] is used to handle this example, the ranking is $x_1 \succ x_2 \succ x_3 \succ x_4$. This result indicates that different rankings and best options may be derived following two different methods. Compared with Meng *et al.*'s method [27], some merits and differences between ours and Meng *et al.*'s method [27] are shown as:

(i) Meng *et al.*'s method [27] used completely multiplicatively consistent IVIFPRs, while our method uses the acceptably multiplicative consistency analysis following *Definition 3.1*.

(ii) Meng *et al.*'s method [27] employed an iterative method to reach the consensus requirement, and all IVIFVs in IVIFPRs are adjusted by the same proportion. The new method researches the acceptably multiplicative consistency and consensus simultaneously, which can ensure two goals: (a) the smallest total adjustment is guaranteed; (b) it permits the adjustment to be different for different IVIFVs in IVIFPRs.

(iii) The fully multiplicative consistency and consensus of IVIFPRs were checked and reached by a threshold and a control parameter in [27], and how to select a suitable threshold and a control parameter could take much time. The proposed method can simultaneously check and reach the acceptably multiplicative consistency and consensus of the given IVIFPRs only by the optimization models without the help of the predetermined control parameter.

(iv) For improving the consensus of an IVIFPR, Meng *et al.*'s method [27] may need several iteration times, while the proposed method can improve the consensus of an IVIFPR only by solving a programming model. Thus, our method is time-saving.

$$\tilde{A}^1=\begin{Bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.2,0.3]) & ([-,-],[-,-]) & ([0.55,0.6],[0.3,0.35]) \\ ([0.2,0.3],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.35,0.55],[0.25,0.3]) & ([-,-],[-,-]) \\ ([-,-],[-,-]) & ([0.25,0.3],[0.35,0.55]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.3,0.45]) \\ ([0.3,0.35],[0.55,0.6]) & ([-,-],[-,-]) & ([0.3,0.45],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) \end{Bmatrix},$$

$$\tilde{A}^2=\begin{Bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.3,0.4],[0.4,0.45]) & ([0.4,-],[-,0.45]) & ([-,0.5],[0.2,-]) \\ ([0.4,0.45],[0.3,0.4]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.45],[0.35,0.4]) & ([0.3,0.35],[0.25,0.45]) \\ ([-,0.45],[0.4,-]) & ([0.35,0.4],[0.4,0.45]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,-],[-,0.5]) \\ ([0.2,-],[-,0.5]) & ([0.25,0.45],[0.3,0.35]) & ([-,0.5],[0.3,-]) & ([0.5,0.5],[0.5,0.5]) \end{Bmatrix},$$

$$\tilde{A}^3=\begin{Bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.35,0.4],[0.4,0.5]) & ([0.3,0.4],[0.4,0.6]) & ([-,-],[-,-]) \\ ([0.4,0.5],[0.35,0.4]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.35,0.45]) & ([0.5,0.65],[0.25,0.3]) \\ ([0.4,0.6],[0.3,0.4]) & ([0.35,0.45],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([-,-],[-,-]) \\ ([-,-],[-,-]) & ([0.25,0.3],[0.5,0.65]) & ([-,-],[-,-]) & ([0.5,0.5],[0.5,0.5]) \end{Bmatrix},$$

$$\tilde{A}^4=\begin{Bmatrix} ([0.5,0.5],[0.5,0.5]) & ([-,-],[-,-]) & ([0.25,0.45],[0.3,0.4]) & ([0.3,0.45],[0.25,-]) \\ ([-,-],[-,-]) & ([0.5,0.5],[0.5,0.5]) & ([-,-],[-,-]) & ([0.3,0.55],[0.35,0.4]) \\ ([0.3,0.4],[0.25,0.45]) & ([-,-],[-,-]) & ([0.5,0.5],[0.5,0.5]) & ([-,0.45],[0.25,0.4]) \\ ([0.25,-],[0.3,0.45]) & ([0.35,0.4],[0.3,0.55]) & ([0.25,0.4],[-,0.45]) & ([0.5,0.5],[0.5,0.5]) \end{Bmatrix},$$

$$\tilde{A}'^1=\begin{Bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.2,0.3]) & ([0.61,0.61],[0.39,0.39]) & ([0.55,0.6],[0.3,0.35]) \\ ([0.2,0.3],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.35,0.55],[0.25,0.3]) & ([0.51,0.51],[0.49,0.49]) \\ ([0.39,0.39],[0.61,0.61]) & ([0.25,0.3],[0.35,0.55]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.3,0.45]) \\ ([0.3,0.35],[0.55,0.6]) & ([0.49,0.49],[0.51,0.51]) & ([0.3,0.45],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) \end{Bmatrix},$$

$$\tilde{A}'^2=\begin{Bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.3,0.4],[0.4,0.45]) & ([0.4,0.4],[0.27,0.45]) & ([0.27,0.5],[0.2,0.5]) \\ ([0.4,0.45],[0.3,0.4]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.45],[0.35,0.4]) & ([0.3,0.35],[0.25,0.45]) \\ ([0.27,0.45],[0.4,0.4]) & ([0.35,0.4],[0.4,0.45]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.44],[0.4,0.5]) \\ ([0.2,0.5],[0.27,0.5]) & ([0.25,0.45],[0.3,0.35]) & ([0.4,0.5],[0.3,0.44]) & ([0.5,0.5],[0.5,0.5]) \end{Bmatrix},$$

$$\tilde{A}'^3=\begin{Bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.35,0.4],[0.4,0.5]) & ([0.3,0.4],[0.4,0.6]) & ([0.53,0.61],[0.3,0.4]) \\ ([0.4,0.5],[0.35,0.4]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.35,0.45]) & ([0.5,0.65],[0.25,0.3]) \\ ([0.4,0.6],[0.3,0.4]) & ([0.35,0.45],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.67,0.67],[0.33,0.33]) \\ ([0.3,0.4],[0.53,0.61]) & ([0.25,0.3],[0.5,0.65]) & ([0.33,0.33],[0.67,0.67]) & ([0.5,0.5],[0.5,0.5]) \end{Bmatrix},$$

$$\tilde{A}'^4=\begin{Bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.53,0.56],[0.41,0.44]) & ([0.25,0.45],[0.3,0.4]) & ([0.3,0.45],[0.25,0.41]) \\ ([0.41,0.44],[0.53,0.56]) & ([0.5,0.5],[0.5,0.5]) & ([0.42,0.47],[0.53,0.53]) & ([0.3,0.55],[0.35,0.4]) \\ ([0.3,0.4],[0.25,0.45]) & ([0.53,0.53],[0.42,0.47]) & ([0.5,0.5],[0.5,0.5]) & ([0.39,0.45],[0.25,0.4]) \\ ([0.25,0.41],[0.3,0.45]) & ([0.35,0.4],[0.3,0.55]) & ([0.25,0.4],[0.39,0.45]) & ([0.5,0.5],[0.5,0.5]) \end{Bmatrix},$$

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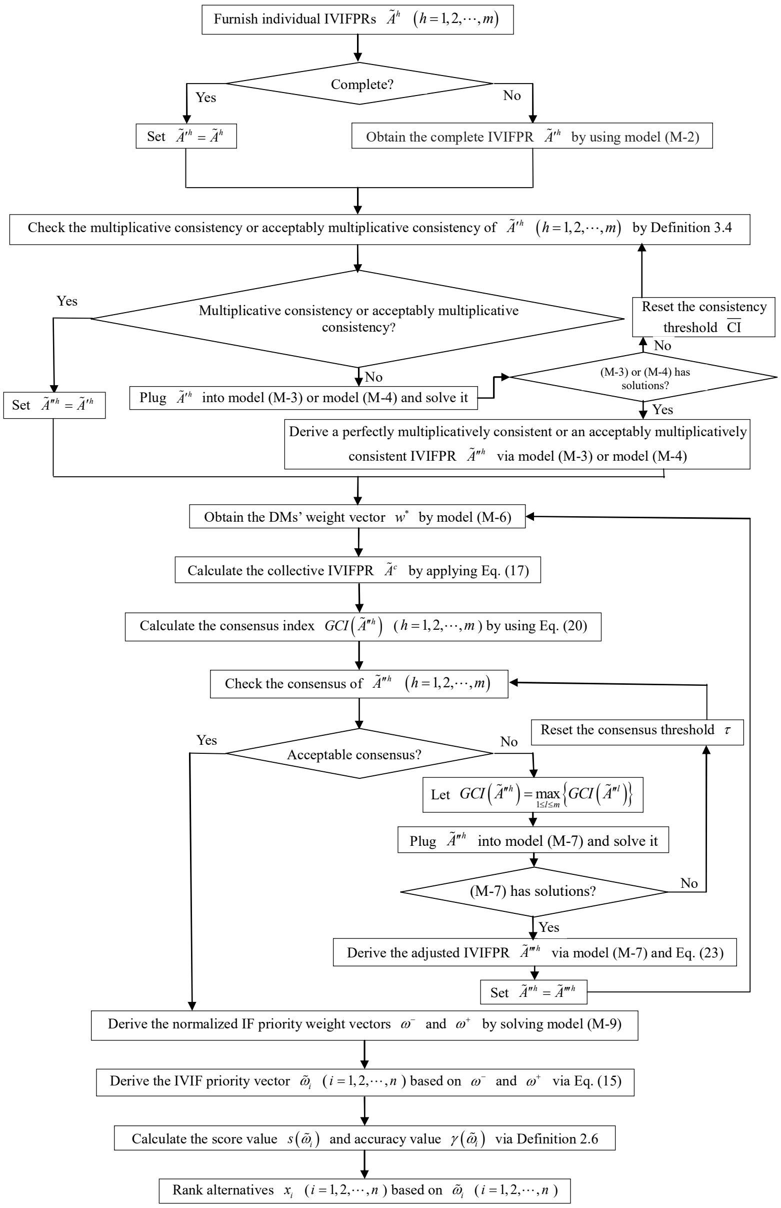


Fig. 1. The flowchart of the proposed GDM algorithm based on incomplete IVIFPRs

$$\tilde{A}''^1 = \left\{ \begin{array}{cccc} ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.28,0.3]) & ([0.61,0.61],[0.39,0.39]) & ([0.55,0.6],[0.3,0.35]) \\ ([0.28,0.3],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.35,0.35],[0.32,0.32]) & ([0.51,0.51],[0.46,0.49]) \\ ([0.39,0.39],[0.61,0.61]) & ([0.32,0.32],[0.35,0.35]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.4,0.45]) \\ ([0.3,0.35],[0.55,0.6]) & ([0.46,0.49],[0.51,0.51]) & ([0.4,0.45],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) \end{array} \right\},$$

$$\tilde{A}''^2 = \left\{ \begin{array}{cccc} ([0.5,0.5],[0.5,0.5]) & ([0.33,0.4],[0.4,0.45]) & ([0.4,0.4],[0.27,0.45]) & ([0.27,0.5],[0.29,0.5]) \\ ([0.4,0.45],[0.33,0.4]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.45],[0.35,0.4]) & ([0.3,0.35],[0.25,0.45]) \\ ([0.27,0.45],[0.4,0.4]) & ([0.35,0.4],[0.4,0.45]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.44],[0.4,0.5]) \\ ([0.29,0.5],[0.27,0.5]) & ([0.25,0.45],[0.3,0.35]) & ([0.4,0.5],[0.3,0.44]) & ([0.5,0.5],[0.5,0.5]) \end{array} \right\},$$

$$\tilde{A}''^{3}=\left\{\begin{array}{cccc}([0.5,0.5],[0.5,0.5]) & ([0.35,0.35],[0.4,0.5]) & ([0.33,0.4],[0.4,0.6]) & ([0.53,0.61],[0.3,0.4]) \\ ([0.4,0.5],[0.35,0.35]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.36,0.45]) & ([0.5,0.65],[0.25,0.28]) \\ ([0.4,0.6],[0.33,0.4]) & ([0.36,0.45],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.67,0.67],[0.33,0.33]) \\ ([0.3,0.4],[0.53,0.61]) & ([0.25,0.28],[0.5,0.65]) & ([0.33,0.33],[0.67,0.67]) & ([0.5,0.5],[0.5,0.5])\end{array}\right\},$$

$$\tilde{A}''^{4}=\left\{\begin{array}{cccc}([0.5,0.5],[0.5,0.5]) & ([0.53,0.56],[0.41,0.44]) & ([0.3,0.45],[0.31,0.4]) & ([0.3,0.45],[0.21,0.41]) \\ ([0.41,0.44],[0.53,0.56]) & ([0.5,0.5],[0.5,0.5]) & ([0.41,0.47],[0.53,0.53]) & ([0.3,0.55],[0.35,0.4]) \\ ([0.31,0.4],[0.3,0.45]) & ([0.53,0.53],[0.41,0.47]) & ([0.5,0.5],[0.5,0.5]) & ([0.39,0.45],[0.25,0.4]) \\ ([0.21,0.41],[0.3,0.45]) & ([0.35,0.4],[0.3,0.55]) & ([0.25,0.4],[0.39,0.45]) & ([0.5,0.5],[0.5,0.5])\end{array}\right\},$$

$$\tilde{A}''^{c}=\left\{\begin{array}{cccc}([0.5,0.5],[0.5,0.5]) & ([0.35,0.4],[0.38,0.44]) & ([0.39,0.43],[0.32,0.47]) & ([0.36,0.54],[0.29,0.44]) \\ ([0.38,0.44],[0.35,0.4]) & ([0.5,0.5],[0.5,0.5]) & ([0.39,0.45],[0.36,0.41]) & ([0.37,0.45],[0.28,0.4]) \\ ([0.32,0.47],[0.39,0.43]) & ([0.36,0.41],[0.39,0.45]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.37,0.43]) \\ ([0.29,0.44],[0.36,0.54]) & ([0.28,0.4],[0.37,0.45]) & ([0.37,0.43],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5])\end{array}\right\},$$

$$\tilde{A}'''^{1}=\left\{\begin{array}{cccc}([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.38,0.44]) & ([0.39,0.43],[0.39,0.39]) & ([0.36,0.6],[0.3,0.35]) \\ ([0.38,0.44],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.35,0.45],[0.32,0.34]) & ([0.37,0.51],[0.28,0.49]) \\ ([0.39,0.39],[0.39,0.43]) & ([0.32,0.34],[0.35,0.45]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.4,0.45]) \\ ([0.3,0.35],[0.36,0.6]) & ([0.28,0.49],[0.37,0.51]) & ([0.4,0.45],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5])\end{array}\right\},$$

$$\tilde{A}'''^{2}=\left\{\begin{array}{cccc}([0.5,0.5],[0.5,0.5]) & ([0.33,0.4],[0.4,0.45]) & ([0.4,0.4],[0.27,0.45]) & ([0.27,0.5],[0.29,0.5]) \\ ([0.4,0.45],[0.33,0.4]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.45],[0.35,0.4]) & ([0.3,0.51],[0.25,0.45]) \\ ([0.27,0.45],[0.4,0.4]) & ([0.35,0.4],[0.4,0.45]) & ([0.5,0.5],[0.5,0.5]) & ([0.35,0.44],[0.4,0.5]) \\ ([0.29,0.5],[0.27,0.5]) & ([0.25,0.45],[0.3,0.51]) & ([0.4,0.5],[0.35,0.44]) & ([0.5,0.5],[0.5,0.5])\end{array}\right\},$$

$$\tilde{A}'''^{3}=\left\{\begin{array}{cccc}([0.5,0.5],[0.5,0.5]) & ([0.35,0.35],[0.4,0.5]) & ([0.33,0.4],[0.4,0.44]) & ([0.34,0.61],[0.3,0.4]) \\ ([0.4,0.5],[0.35,0.35]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.36,0.45]) & ([0.36,0.51],[0.25,0.42]) \\ ([0.4,0.44],[0.33,0.4]) & ([0.36,0.45],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.67],[0.33,0.33]) \\ ([0.3,0.4],[0.34,0.61]) & ([0.25,0.42],[0.36,0.51]) & ([0.33,0.33],[0.4,0.67]) & ([0.5,0.5],[0.5,0.5])\end{array}\right\},$$

$$\tilde{A}'''^{4}=\left\{\begin{array}{cccc}([0.5,0.5],[0.5,0.5]) & ([0.39,0.56],[0.41,0.44]) & ([0.3,0.45],[0.31,0.4]) & ([0.3,0.45],[0.28,0.41]) \\ ([0.41,0.44],[0.39,0.56]) & ([0.5,0.5],[0.5,0.5]) & ([0.41,0.47],[0.36,0.52]) & ([0.3,0.55],[0.35,0.4]) \\ ([0.31,0.4],[0.3,0.45]) & ([0.36,0.52],[0.41,0.47]) & ([0.5,0.5],[0.5,0.5]) & ([0.39,0.45],[0.36,0.4]) \\ ([0.28,0.41],[0.3,0.45]) & ([0.35,0.4],[0.3,0.55]) & ([0.36,0.4],[0.39,0.45]) & ([0.5,0.5],[0.5,0.5])\end{array}\right\},$$

$$\tilde{A}'''^{c}=\left\{\begin{array}{cccc}([0.5,0.5],[0.5,0.5]) & ([0.37,0.45],[0.4,0.45]) & ([0.36,0.42],[0.34,0.42]) & ([0.32,0.54],[0.29,0.41]) \\ ([0.4,0.45],[0.37,0.45]) & ([0.5,0.5],[0.5,0.5]) & ([0.39,0.46],[0.34,0.41]) & ([0.33,0.52],[0.28,0.45]) \\ ([0.34,0.42],[0.36,0.42]) & ([0.34,0.41],[0.39,0.46]) & ([0.5,0.5],[0.5,0.5]) & ([0.38,0.5],[0.38,0.43]) \\ ([0.29,0.41],[0.32,0.54]) & ([0.28,0.45],[0.33,0.52]) & ([0.38,0.43],[0.38,0.5]) & ([0.5,0.5],[0.5,0.5])\end{array}\right\},$$

$$A'''^{c-}=\left\{\begin{array}{cccc}(0.5,0.5) & (0.37,0.45) & (0.36,0.42) & (0.32,0.41) \\ (0.45,0.37) & (0.5,0.5) & (0.39,0.41) & (0.33,0.45) \\ (0.42,0.36) & (0.41,0.39) & (0.5,0.5) & (0.38,0.43) \\ (0.41,0.32) & (0.45,0.33) & (0.43,0.38) & (0.5,0.5)\end{array}\right\},$$

$$A'''^{c+}=\left\{\begin{array}{cccc}(0.5,0.5) & (0.45,0.4) & (0.42,0.34) & (0.54,0.29) \\ (0.4,0.45) & (0.5,0.5) & (0.46,0.34) & (0.52,0.28) \\ (0.34,0.42) & (0.34,0.46) & (0.5,0.5) & (0.5,0.38) \\ (0.29,0.54) & (0.28,0.52) & (0.38,0.5) & (0.5,0.5)\end{array}\right\}.$$

## VII. Conclusions

Considering that IVIFPRs are useful to describe the uncertain judgments of DMs, we have further researched the utilization of IVIFPRs. The main original facets of the study are summarized below:

(1) A multiplicative consistency of IFPRs was defined to guarantee the ranking accurately. Using this new consistency concept, an approach of establishing the multiplicative consistent IFPR from the given IF priority weight vector was presented. Then, by splitting an IVIFPR into two IFPRs, two consistency properties for IVIFPRs were proposed in accordance with that of these two IFPRs.

(2) Considering the situation where the IVIFPRs provided by DMs are often incomplete and inconsistent, several mathematical programming models for determining missing values and deriving multiplicatively consistent and acceptably multiplicatively consistent IVIFPRs were established, respectively.

(3) For GDM with incomplete IVIFPRs, in order to reach maximum group support degree, a programming model was presented to gain the DMs' weights. Moreover, individual IVIFPRs were integrated into the collective IVIFPR.

(4) For GDM with IVIFPRs, we defined a consensus index. When the consensus fails to fulfil the requirement, a model was presented to achieve the consensus requirement, ensure the acceptably multiplicative consistency, and try to retain the original information.

(5) A model of achieving the priority weight vector was established. A GDM method with incomplete IVIFPRs was designed. A problem was presented to display the developed method, followed by some comparison analysis and discussion.

A question on how to extend our method to deal with other uncertain information [51]-[53] becomes an interesting issue deserving further investigations.

## References

[1] T. L. Saaty, "A scaling method for priorities in hierarchical structures," *Journal of Mathematical Psychology*, vol. 15, no. 3, pp. 234–281, 1977.
[2] S. Orlovsky, "Decision-making with a fuzzy preference relation," *Fuzzy Sets and Systems*, vol. 1, no. 3, pp. 155–167, 1978.
[3] Z. Xu, "On compatibility of interval fuzzy preference matrices," *Fuzzy Optimization and Decision Making*, vol. 3, no. 3, pp. 217–225, 2004.
[4] T. L. Saaty and L. G. Vargas, "Uncertainty and rank order in the analytic hierarchy process," *European Journal of Operational Research*, vol. 32, no. 1, pp. 107–117, Oct. 1987.
[5] B. Zhu and Z. Xu, "Regression methods for hesitant fuzzy preference relations," *Technological and Economic Development of Economy*, vol. 19, no. Supp1, pp. S214–S227, Jan. 2014.
[6] B. Zhu and Z. Xu, "Analytic hierarchy process-hesitant group decision making," *European Journal of Operational Research*, vol. 239, no. 3, pp. 794–801, Dec. 2014.
[7] V. Torra, "Hesitant fuzzy sets," *International Journal of Intelligent Systems*, vol. 25, no. 6, pp. 529–539, 2010.
[8] F. Herrera, E. Herrera-Viedma and J. L. Verdegay, "A rational consensus model in group decision making using linguistic assessments," *Fuzzy Sets and Systems*, vol. 88, no. 1, pp. 31–49, 1997.
[9] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning–Part I," *Information Sciences*, vol. 8, no. 3, pp. 199–249, 1975.
[10] Z. S. Chen, Y. Yang, X. J. Wang, K. S. Chin and K. L.Tsui, "Fostering linguistic decision-making under uncertainty: a proportional interval type-2 hesitant fuzzy TOPSIS approach based on Hamacher aggregation operators and andness optimization models," *Information Sciences*, vol. 500, pp. 229–258, 2019.
[11] R. M. Rodriguez, L. Martinez and F. Herrera, "Hesitant fuzzy linguistic term sets for decision making," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 1, pp. 109–119, 2012.
[12] B. Zhu and Z. Xu, "Consistency measures for hesitant fuzzy linguistic preference relations," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 1, pp. 35–45, 2014.
[13] Z. S. Chen, K. S. Chin, L. Martínez and K. L. Tsui, "Customizing semantics for individuals with attitudinal HFLTS possibility distributions," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 6, pp. 3452–3466, 2018.
[14] Q. Pang, H. Wang and Z. Xu, "Probabilistic linguistic term sets in multi-attribute group decision making," *Information Sciences*, vol. 369, pp. 128–143, 2016.
[15] Y. Zhang, Z. Xu, H. Wang and H. Liao, "Consistency-based risk assessment with probabilistic linguistic preference relation," *Applied Soft Computing*, vol. 49, pp. 817–833, 2016.
[16] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.
[17] Z. Xu, "Intuitionistic preference relations and their application in group decision making," *Information Sciences*, vol. 177, no. 11, pp. 2363–2379, 2007.
[18] K. Atanassov, and G. Gargov, "Interval-valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 31, no. 3, pp. 343–349, 1989.
[19] Z. Xu and J. Chen, "An approach to group decision making based on interval-valued intuitionistic judgment matrices," *System Engineering and Theory Practice*, vol. 27, no. 4, pp. 126–133, 2007.
[20] F. Y. Meng, J. Tang and H. Fujita, "Consistency-based algorithms for decision making with interval fuzzy preference relations," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 10, pp. 2052–2066, 2019.
[21] Z. J. Wang and J. Lin, "Consistency and optimized priority weight analytical solutions of interval multiplicative preference relations," *Information Sciences*, vol. 482, pp. 105–122, 2019.
[22] R. Wang, B. Shuai, Z. S. Chen, K. S. Chin and J. H. Zhu, "Revisiting the role of hesitant multiplicative preference relations in group decision making with novel consistency improving and consensus reaching processes," *International Journal of Computational Intelligence Systems*, vol. 12, no. 2, pp.1029–1046, 2019.
[23] P. Wu, J. Zhu, L. Zhou and H. Chen, "Automatic iterative algorithm with local revised strategies to improve the consistency of hesitant fuzzy linguistic preference relations," *International Journal of Fuzzy Systems*, vol. 21, no. 7, pp. 2283–2298, 2019.
[24] G. L. Xu, S. P. Wan, F. Wang, J. Y. Dong and Y. F. Zeng, "Mathematical programming methods for consistency and consensus in group decision making with intuitionistic fuzzy preference relations," *Knowledge-Based Systems*, vol. 98, pp. 30–43, 2016.
[25] H. Liao, Z. Xu and M. Xia, "Multiplicative consistency of interval-valued intuitionistic fuzzy preference relation," *Journal of Intelligent & Fuzzy Systems*, vol. 27, no. 6, pp. 2969–2985, 2014.
[26] J. Chu, X. Liu, L. Wang and Y. Wang, "A group decision making approach based on newly defined additively consistent interval-valued intuitionistic preference relations," *International Journal of Fuzzy Systems*, vol. 20, no. 3, pp. 1027–1046, 2018.
[27] F. Meng, J. Tang, P. Wang and X. Chen, "A programming-based algorithm for interval-valued intuitionistic fuzzy group decision making," *Knowledge-Based Systems*, vol. 144, pp. 122–143, 2018.
[28] J. Tang, F. Meng, and Y. Zhang, "Decision making with interval-valued intuitionistic fuzzy preference relations based on additive consistency analysis," *Information Sciences*, vol. 467, pp. 115–134, 2018.
[29] S. P. Wan, F. Wang and J. Y. Dong, "Additive consistent interval-valued Atanassov intuitionistic fuzzy preference relation and likelihood comparison algorithm based group decision making," *European Journal of Operational Research*, vol. 263, no. 2, pp. 571–582, 2017.
[30] S. P. Wan, F. Wang and J. Y. Dong, "A three-phase method for group decision making with interval-valued intuitionistic fuzzy preference relations," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 2, pp. 998–1010, 2017.
[31] S. P. Wan, F. Wang and J. Y. Dong, "A group decision-making method considering both the group consensus and multiplicative consistency of interval-valued intuitionistic fuzzy preference relations," *Information Sciences*, vol. 466, pp. 109–128, 2018.

[32] S. P. Wan, G. L. Xu and J. Y. Dong, "A novel method for group decision making with interval-valued Atanassov intuitionistic fuzzy preference relations," *Information Sciences*, vol. 372, pp. 53–71, 2016.

[33] J. Wu and F. Chiclana, "Non-dominance and attitudinal prioritisation methods for intuitionistic and interval-valued intuitionistic fuzzy preference relations," *Expert Systems with Applications*, vol. 39, no. 18, pp. 13409–134165, 2012.

[34] Z. Xu and R. R. Yager, "Intuitionistic and interval-valued intutionistic fuzzy preference relations and their measures of similarity for the evaluation of agreement within a group," *Fuzzy Optimization and Decision Making*, vol. 8, no. 2, pp. 123–139, 2009.

[35] S. Zhang and F. Meng, "Analysis of the consistency and consensus for group decision-making with interval-valued intuitionistic fuzzy preference relations," *Computational and Applied Mathematics*, vol. 39, no. 3, 147, 2020.

[36] H. Zhuang, "Additively consistent interval-valued intuitionistic fuzzy preference relations and their application to group decision making," *Information*, vol. 9, no. 260, pp. 1–20, 2018.

[37] N. Liu, Y. He and Z. Xu, "A new approach to deal with consistency and consensus issues for hesitant fuzzy linguistic preference relations," *Applied Soft Computing*, vol. 76, pp. 400–415, 2019.

[38] Y. Xu, X. Liu and L. Xu, "A dynamic expert contribution-based consensus model for hesitant fuzzy group decision making with an application to water resources allocation selection," *Soft Computing*, vol. 24, no. 6, pp. 4693–4708, 2020.

[39] Z. Zhang and S. M. Chen, "A consistency and consensus-based method for group decision making with hesitant fuzzy linguistic preference relations," *Information. Sciences*, vol. 501, pp. 317–336, 2019.

[40] Y. Zhang, Z. Xu and H. Liao, "A consensus process for group decision making with probabilistic linguistic preference relations," *Information Sciences*, vol. 414, pp. 260–275, 2017.

[41] P. Liu, P. Wang and W. Pedrycz, "Consistency- and consensus-based group decision-making method with incomplete probabilistic linguistic preference relations," *IEEE Transactions on Fuzzy Systems*, https://doi.org/10.1109/TFUZZ.2020.3003501.

[42] F. Liu, W. G. Zhang and Z. X. Wang, "A goal programming model for incomplete interval multiplicative preference relations and its application in group decision-making," *European Journal of Operational Research*, vol. 218, no. 3, pp. 747–754, 2012.

[43] P. Ren, Z. Hao, X. Wang, X. J. Zeng and Z. Xu, "Decision-making models based on incomplete hesitant fuzzy linguistic preference relation with application to site selection of hydropower stations," *IEEE Transactions on Engineering Management*, https://doi.org/10.1109/TEM.2019.2962180.

[44] H. Wang and Z. Xu, "Interactive algorithms for improving incomplete linguistic preference relations based on consistency measures," *Applied Soft Computing*, vol. 42, pp. 66–79, 2016.

[45] Z. Xu and X. Cai, "Incomplete interval-valued intuitionistic fuzzy preference relations," *International Journal of General Systems*, vol. 38, no. 8, pp. 871–886, 2009.

[46] Z. Xu and X. Cai, "Group decision making with incomplete interval-valued intuitionistic preference relations," *Group Decision Negotiation*, vol. 24, no. 2, pp. 193–215, 2015.

[47] Z. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *International Journal of General Systems*, vol. 35, no. 4, pp. 417–433, 2006.

[48] Z. J. Wang, "Derivation of intuitionistic fuzzy weights based on intuitionistic fuzzy preference relations," *Applied Mathematical Modelling*, vol. 37, no. 9, pp. 6377–6388, 2013.

[49] H. Liao and Z. Xu, "Priorities of intuitionistic fuzzy preference relation based on multiplicative consistency," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 6, pp. 1669–1681, 2014.

[50] H. Liao and Z. Xu, "Consistency of the fused intuitionistic fuzzy preference relation in group intuitionistic fuzzy analytic hierarchy process," *Applied Soft Computing*, vol. 35, pp. 812–826, 2015.

[51] H. Gao, L. Ran, G. Wei, C. Wei and J. Wu, "VIKOR method for MAGDM based on q-rung interval-valued orthopair fuzzy information and its application to supplier selection of medical consumption products," *International Journal of Environmental Research and Public Health*, vol. 17, no. 2, 525, 2020.

[52] H. Gao, J. Wu, C. Wei and G. Wei, "MADM method with interval-valued Bipolar uncertain linguistic information for evaluating the computer network security," *IEEE Access*, vol. 7, pp. 151506–151524, 2019.

[53] L. Wu, G. Wei, J. Wu and C. Wei, "Some interval-valued intuitionistic fuzzy Dombi Heronian mean operators and their application for evaluating the ecological value of forest ecological tourism demonstration areas," *International Journal of Environmental Research and Public Health*, vol. 17, no. 3, 829, 2020.

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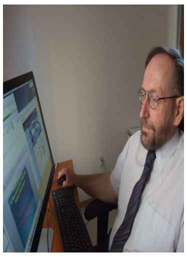

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