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Global well-posedness of axisymmetric solution to the 3D axisymmetric chemotaxis-Navier-Stokes equations with logistic source

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Abstract

In this paper, we investigate Cauchy problem of the 3D incompressible chemotaxis-Navier-Stokes equations with logistic source. By exploring some new *a priori* estimates and making good use of the geometry structure of axisymmetric flow without swirl, we prove the global-in-time well-posedness for the axisymmetric chemotaxis-Navier-Stokes equations.

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1. Introduction

In a process of biological individuals, it is often observed that there is tendency to move towards a chemically more favorable environment. There are many literatures [10,18,27] devoted

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to the study of chemotactic aggregation. Among them, we meet the following model proposed to describe slime mold aggregation:

$$\begin{cases} \partial_t u - \Delta u = -\nabla \cdot (u\nabla v) + f(u), \\ \partial_t v - \Delta v = -v + u, \end{cases} \tag{1.1}$$

where u represents the cell density and v the concentration of the chemotactic substance. If the force $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function generalizing the logistic source

$$f(s) = \sigma s - \mu s^2 \quad \text{for all} \quad s \geq 0 \tag{1.2}$$

with some $\sigma > 0$ and $\mu > 0$, the cell density remains bounded. Some numerical experiments, for example [19], support this phenomenon. In [18], when $f(u) = u(1-u)(u-a)(0 < a < \frac{1}{2})$, shock-type movements of interfaces are detected and cell kinetics take place much faster than the cell movement (see also [8]). Concerning existence and boundedness of solution to equations (1.1), one can refer to [27] for more details.

In this paper, we consider Cauchy problem of the 3D incompressible chemotaxis-Navier-Stokes equations with logisitic growth which take

$$\begin{cases} \partial_t n + u \cdot \nabla n - \Delta n = -\nabla \cdot (\chi(c) n \nabla c) + f(n), \\ \partial_t c + u \cdot \nabla c - \Delta c = -g(c) n, \\ \partial_t u + \kappa(u \cdot \nabla u) - \Delta u + \nabla P = -n\nabla\phi, \\ \operatorname{div} u = 0, \\ (n, c, u)|_{t=0} = (n_0, c_0, u_0). \end{cases} \tag{1.3}$$

Here $n = n(x,t)$, $c = c(x,t)$, $u(x,t)$ and $P = P(x,t)$ denote the cell density, chemical concentration, velocity field and pressure of the fluid, respectively. The constant κ is related to the strength of nonlinear fluid convection. The given potential function $\phi = \phi(x)$ represents a gravitational potential. Different functional forms of χ and f are meaningful, according to various conceivable threshold effects and saturation mechanism. In general, χ, f, g and ϕ are supposed to be sufficiently smooth given function.

System (1.3) describes a biological process, in which cells move towards a more chemically favorable environment. For example, the mechanism is a chemotactic movement of bacteria often towards higher concentration of oxygen which they consume, a gravitational effect on the motion of the fluid by the heavier bacteria, and a convective transport of both cells and oxygen through the water. Experiments can be found in [13,25]. Although the chemical substrate can be produced or consumed by the cells, we are only interested in the latter corresponds to the repulsive case of the substrate in this paper.

For the case where $f \equiv 0$, there are more progress obtained by many mathematicians. In [12], J. G. Liu and A. Lorz obtained global weak solutions to the Navier-Stokes version of (1.3) with $\kappa = 1$ and arbitrarily large initial data in $\mathbb{R}^2$ by establishing some new *a priori* estimates. M. Winkler [28] proved that system (1.3) admits a unique global classical solution in a bounded convex domain with smooth boundary in $\mathbb{R}^2$. For the whole space $\mathbb{R}^2$, we proved the almost energy solution in [35] by exploring the new estimate in Zygmund class. The convergence of solutions to a stationary state proven in [29], and its decay rate shown in [34]. As for the three-dimensional case, the global-in-time classical solutions near constant steady states were constructed for problem (1.3) with $\kappa = 1$ in [6]. Recently, M. Winkler [31] proved the global-in-time existence of weak solutions to problem (1.3) in a bounded domain with large initial data. Then H. He and Q. Zhang [7] got a similar global-in-time weak solutions to system (1.3) in the whole space $\mathbb{R}^3$. M. Winkler [32] furthermore showed that any eventual energy solution become smooth after some waiting time, and converge as t goes to infinite. If f is chosen as in (1.2), J. Lankeit [11], in a bounded smooth domain $\Omega \subset \mathbb{R}^3$, constructs weak solutions and proves that after some waiting time they become smooth and finally converge to the semi-trivial steady state $(\frac{\sigma}{\mu}, 0, 0)$. There are many other interesting results concerning chemotaxis model, one should refer to [2–5,14,20–23,26,30,33].

In this paper, we consider a simplified model, namely, choosing $\chi = \kappa = 1$, $g(c) = c$. And the logistic growth

$$f(n) = n(1-n)(n-a) \qquad \text{for} \quad 0 < a < \frac{1}{2},$$

which is originally presented by Mimura and Tsujkawa in [18]. Then system (1.3) is reduce to

$$\begin{cases} \partial_t n + u \cdot \nabla n - \Delta n = -\nabla \cdot (n \nabla c) + n(1-n)(n-a), \\ \partial_t c + u \cdot \nabla c - \Delta c = -cn, \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = -n \nabla \phi, \\ \operatorname{div} u = 0, \\ (n, c, u)|_{t=0} = (n_0, c_0, u_0). \end{cases} \tag{1.4}$$

Next, we state the main result. First of all, we assume that the initial data belong to X_0 which satisfies the following conditions throughout this paper:

(i) $n_0 \in L^1 \cap L^2,\ n_0 > 0$,
(ii) $\nabla \sqrt{c_0} \in L^2,\ c_0 \in L^1 \cap L^\infty,\ c_0 \in \dot{H}^2,\ c_0 > 0$,
(iii) $u_0 \in H^2,\ \nabla \phi \in L^\infty,\ \nabla^2 \phi \in L^\infty,\ \nabla^3 \phi \in L^\infty, \nabla \phi(0) = 0$,
(iv) u_0, ϕ are all axisymmetric without swirl and $x_1 \partial_2 \phi - x_2 \partial_1 \phi = 0$, this means that

$$u_0(x) = u_0^r(r,z) e_r + u_0^z(r,z) e_z, \quad \phi(x) = \phi^r(r,z) e_r + \phi^z(r,z) e_z$$

with $x = (x_1, x_2, z)$, $r = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}$. n_0 and c_0 are axisymmetric, that is

$$n_0(x) = n_0(r,z), \quad c_0(z) = c_0(r,z).$$

(v) $\operatorname{supp} \phi$ does not intersect the axis (Oz) and $\prod_z (\operatorname{supp} \phi)$ is a compact set, where $\prod_z$ denotes the orthogonal projector over (Oz).

Our target here is to show the global-in-time well-posedness of solutions to system (1.4) for axisymmetric without swirl initial data, which means that u_0 is assumed to be an axisymmetric vector field without swirl. Compared with the previous results concerning the local-in-time well-posedness for large initial data and global-in-time well-posedness for small initial data, we establish the global-in-time well-posedness result for large special initial data to problem (1.4) with $\kappa \neq 0$. Our result reads as follows.

Theorem 1.1. *Let the triple* $(n_0, c_0, u_0) \in X_0$ *and* $u_0 \in H^1(\mathbb{R}^3)$ *such that* $\frac{\omega_0}{r} \in L^2(\mathbb{R}^3)$ *with* $\omega_0 = \nabla \times u_0$*. Then, system* (1.4) *has a unique global-in-time solution*

$$(n, c, u) \in C\big(\mathbb{R}^+; L^2(\mathbb{R}^3)\big) \times C\big(\mathbb{R}^+; H^2(\mathbb{R}^3)\big) \times C\big(\mathbb{R}^+; H^2(\mathbb{R}^3)\big)$$

such that

$$\begin{aligned}
&n \in L^\infty\big(\mathbb{R}^+; L^1(\mathbb{R}^3)\big) \cap L^\infty_{\text{loc}}\big(\mathbb{R}^+; L^2(\mathbb{R}^3)\big) \cap L^2_{\text{loc}}\big(\mathbb{R}^+; H^1(\mathbb{R}^3)\big),\\
&c \in L^\infty\big(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^3)\big) \cap L^\infty_{\text{loc}}\big(\mathbb{R}^+; H^2(\mathbb{R}^3)\big) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^3(\mathbb{R}^3)),\\
&u \in L^\infty_{\text{loc}}\big(\mathbb{R}^+; H^1(\mathbb{R}^3)\big) \cap L^2_{\text{loc}}\big(\mathbb{R}^+; H^2(\mathbb{R}^3)\big) \ \textit{and}\\
&\frac{\omega_\theta}{r} \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^3)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^3)).
\end{aligned}$$

Remark 1.1. Let us point out that the condition $x_1\partial_2\phi - x_2\partial_1\phi = 0$ ensures that the solution is axisymmetric in the process of evolution over time.

As far as we know, most mathematician dealing with the 3D case of problem (1.3) or problem (1.4) often ignore the nonlinear convective term $u \cdot \nabla u$ appearing in the 3D incompressible Navier-Stokes equations

$$\begin{cases}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla P = 0 \qquad (x, t) \in \mathbb{R}^3 \times \mathbb{R}^+,\\
\operatorname{div} u = 0,\\
u(0, x) = u_0(x).
\end{cases} \tag{1.5}$$

In our paper, we will consider the influence of non-linear convection term in fluid flow. From the Beale-Kato-Majda criterion, we know that vorticity $\omega = \operatorname{curl} u$ is a physical quantity which plays a significant role in the global-in-time theory to the above 3D Navier-Stokes system. Since

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla u,$$

it seems difficult to get the bound of the quantity $\|\omega(t)\|_{L^\infty}$ in the lack of information concerning the manner that the vortex-stretching term $\omega \cdot \nabla u$ affects the dynamic of the fluid. This induces many mathematicians and physicists to consider the solution with the special geometry structure, such as axisymmetric without swirl (see, e.g. [1,9,16,17]).

We say that a vector field u is axisymmetric without swirl if it has the form:

$$u(t, x) = u^r(t, r, z)e_r + u^z(t, r, z)e_z, \quad x = (x_1, x_2, z), \quad r = (x_1^2 + x_2^2)^{\frac{1}{2}},$$

where (e_r, e_θ, e_z) is the cylindrical basis of $\mathbb{R}^3$ and the components u^r and u^z do not depend on the angular variable. With this structure, vorticity takes the form

$$\omega = (\partial_z u^r - \partial_r u^z) e_\theta \triangleq \omega_\theta e_\theta,$$

and satisfies

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \frac{u^r}{r} \omega.$$

Since the Laplacian operator has the form $\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz}$ in the cylindrical coordinates then the component ω_θ of the vorticity will satisfy

$$\partial_t \omega_\theta + u \cdot \nabla \omega_\theta - \Delta \omega_\theta + \frac{\omega_\theta}{r^2} = \frac{u^r}{r} \omega_\theta. \tag{1.6}$$

And then, the quantity $\Gamma \triangleq \frac{\omega_\theta}{r}$ obeys to the equation

$$\partial_t \Gamma + u \cdot \nabla \Gamma - \Delta \Gamma - \frac{2}{r} \partial_r \Gamma = 0.$$

From this, it follows that for all $p \in [1, +\infty]$,

$$\|\Gamma(t)\|_{L^p} \le \|\Gamma_0\|_{L^p}.$$

The new conservation law is strong enough to prevent the formation of singularities in finite time for axisymmetric without swirl flows.

Now we come back our model (1.4). Compared with system (1.5), we will meet some new difficulties. Let $g = n\nabla\phi = g^r e_r + g^\theta e_\theta + g^z e_z$ with

$$\begin{aligned} g^r &= \frac{x_1 n \partial_1 \phi + x_2 n \partial_2 \phi}{r} = \frac{n(x_1 \partial_1 \phi + x_2 \partial_2 \phi)}{r}, \\ g^\theta &= \frac{x_1 n \partial_2 \phi - x_2 n \partial_1 \phi}{r} = \frac{n(x_1 \partial_2 \phi - x_2 \partial_1 \phi)}{r}, \quad g^z = n \partial_3 \phi = n \partial_z \phi. \end{aligned}$$

Taking $x_1\partial_2\phi - x_2\partial_1\phi = 0$ and $n = n(r, z)$, $\phi = \phi(r, z)$, we have $g = n\nabla\phi$ is a axisymmetric vector field without swirl. Moreover,

$$\nabla \times g = \nabla \times (n\nabla\phi) = (\partial_z g^r - \partial_r g^z) e_\theta. \tag{1.7}$$

Set $x^h = (x_1, x_2)$ and $\nabla_h = (\partial_1, \partial_2)$, we observe that

$$\partial_z g^r = \frac{1}{r} \partial_z (n(x_1 \partial_1 \phi + x_2 \partial_2 \phi)) = \frac{1}{r} \partial_z n x^h \cdot \nabla_h \phi + \frac{n}{r} x^h \cdot \nabla_h (\partial_z \phi)$$

and

$$\partial_r g^z = \partial_r (n \partial_z \phi) = \partial_r n \partial_z \phi + n \partial^2_{r,z} \phi.$$

Plugging the above estimates into (1.7), we finally obtain that

$$\nabla \times g = \nabla \times (n\nabla\phi) = \left(\frac{1}{r}\partial_z n x^h \cdot \nabla_h \phi + \frac{n}{r} x^h \cdot \nabla_h(\partial_z \phi) - \partial_r n \partial_z \phi - n\partial_{r,z}^2 \phi\right) e_\theta.$$

One writes the vorticity equation as

$$\partial_t \omega_\theta + u \cdot \nabla \omega_\theta - \Delta\omega_\theta + \frac{\omega_\theta}{r^2} = \frac{u^r}{r}\omega_\theta - \frac{1}{r}\partial_z n x^h \cdot \nabla_h \phi - \frac{n}{r} x^h \cdot \nabla_h(\partial_z\phi) + \partial_r n \partial_z \phi + n\partial_{r,z}^2 \phi. \tag{1.8}$$

It follows that the evolution of the quantity $\Gamma \triangleq \frac{\omega_\theta}{r}$ is governed by the equation

$$\partial_t \Gamma + u \cdot \nabla\Gamma - \Delta\Gamma - \frac{2}{r}\partial_r \Gamma = -\frac{1}{r^2}\partial_z n x^h \cdot \nabla_h \phi - \frac{n}{r^2} x^h \cdot \nabla_h(\partial_z \phi) + \frac{\partial_r n \partial_z \phi}{r} + \frac{n\partial_{r,z}^2 \phi}{r}. \tag{1.9}$$

2. Uniform estimates for the regularized problem

This section is devoted to showing some *a priori* estimates of smooth solution to the following approximate system, which is needed for the proof of Theorem 1.1:

$$\begin{cases} \partial_t n^N + u^N \cdot \nabla n^N - \Delta n^N = -\nabla\cdot(n^N \nabla c^N) + n^N(1-n^N)(n^N - a), \quad x \in \mathbb{R}^3,\ t > 0,\\ \partial_t c^N + u^N \cdot \nabla c^N - \Delta c^N = -n^N c^N,\\ \partial_t u^N + J_N u^N \cdot \nabla u^N - \Delta u^N + \nabla P^N = -J_N(n\nabla\phi)\\ \operatorname{div} u^N = 0,\\ (n^N, c^N, u^N)|_{t=0} = J_N(n_0, c_0, u_0), \end{cases} \tag{2.1}$$

where $J_N f = \rho^{1/N} * f$ with $\rho^{1/N} = N^3 \rho(Nx)$, and ρ satisfies the following conditions

$$\rho(|x|) \in \mathcal{C}_0^\infty(\mathbb{R}^3), \quad \rho \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^3} \rho\, \mathrm{d}x = 1.$$

From Proposition A.1, we know that system (2.1) admits a unique smooth solution $(n^N,\ c^N,\ u^N)$. Next, we are going to show the uniform estimates of this solutio independent of N. We distinguish especially two kinds: the first one deals with some easy estimates that one can be obtained by energy estimates. The second one is concerned with some strong estimates which are the heart of the proof of our main result. To simplify the notations, we agree $(n, c, u) \triangleq (n^N, c^N, u^N)$ in this following part of this section.

2.1. Weak a priori estimates

Let us begin by proving the following energy estimates.

Proposition 2.1. *Let the triple $(n_0, c_0, u_0) \in X_0$, and $u_0 \in H^2$ and $\frac{\omega_0}{r} \in L^2$. Assume (n, c, u) is a smooth solution to system* (2.1)*. Then there exists a constant C independent of N such that*

$$\|n(t)\|_1 + \int_0^t \|n(\tau)\|_3^3\, \mathrm{d}\tau \leq \|n_0\|_1 e^t, \tag{2.2}$$

$$\|c(t)\|_{L^1\cap L^\infty} \le \|c_0\|_{L^1\cap L^\infty}, \tag{2.3}$$

$$\mathcal{U}(t) + \int_0^t \mathcal{V}(\tau)\,\mathrm{d}\tau \le Ce^{Ct} \tag{2.4}$$

with $\mathcal{U}(t) \triangleq \|n(t)\|_{L^1\cap L\log L} + \|\nabla\sqrt{c}(t)\|_2^2 + \|u(t)\|_2^2$,

$$\mathcal{V} \triangleq \int_{A_1} \frac{n}{n+1}|\nabla n|^2\,\mathrm{d}x + \|\Delta\sqrt{c}\|_2^2 + \|\nabla u\|_2^2 + \int_{\mathbb{R}^3} (\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x + \int_{\mathbb{R}^3} n|\nabla\sqrt{c}|^2\,\mathrm{d}x,$$

$A_\lambda \triangleq \{x \in \mathbb{R}^3 | 0 \le n(x) < \lambda\}$ *and* $A_\lambda^c \triangleq \mathbb{R}^3 \setminus A_\lambda$ *for all* $\lambda \in \mathbb{R}^+$.

Proof. Here we just give the proof for estimate (2.2) and (2.4), because the proof of (2.3) was shown in [24].

Thanks to $n \ge 0$, integrating the first equation in x, we readily have by performing the Young inequality that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|n(t)\|_1 + a\|n(t)\|_1 + \|n(t)\|_3^3 = (1+a)\|n(t)\|_2^2 \le& (1+a)\|n(t)\|_1^{\frac{1}{2}}\|n(t)\|_3^{\frac{3}{2}} \\ \le& C\|n(t)\|_1 + \frac{1}{2}\|n(t)\|_3^3.\end{aligned}$$

Moreover, by the Gronwall inequality, we obtain

$$\|n(t)\|_1 + \int_0^t \left(\|n(\tau)\|_2^2 + \|n(\tau)\|_3^3\right)\mathrm{d}\tau \le \|n_0\|_1 e^t.$$

The first equation of (2.1) can be written as

$$\partial_t(n+1) + u\cdot\nabla(n+1) - \Delta(n+1) = -\nabla\cdot((n+1)\cdot\nabla c) + \Delta c + (a+1)n^2 - an - n^3. \tag{2.5}$$

Next, multiplying (2.5) by $1+\ln(n+1)$ and integrating the resulting equality, we readily have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x + a\int_{\mathbb{R}^3} n\big(1+\log(n+1)\big)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} n^3\big(1+\ln(n+1)\big)\,\mathrm{d}x + 4\int_{\mathbb{R}^3}|\nabla\sqrt{n+1}|^2\,\mathrm{d}x \\ =& \int_{\mathbb{R}^3}\nabla(n+1)\cdot\nabla c\,\mathrm{d}x + \int_{\mathbb{R}^3}\Delta c\big(1+\ln(n+1)\big)\,\mathrm{d}x \\ &+ (1+a)\int_{\mathbb{R}^3} n^2\big(1+\ln(n+1)\big)\,\mathrm{d}x\end{aligned} \tag{2.6}$$

$$=\int_{\mathbb{R}^3} \nabla n \cdot \nabla c \,\mathrm{d}x + \int_{\mathbb{R}^3} \Delta c \ln(n+1)\,\mathrm{d}x + (1+a)\int_{\mathbb{R}^3} n^2\big(1+\ln(n+1)\big)\,\mathrm{d}x.$$

Since $\Delta c = 2|\nabla\sqrt{c}|^2 + 2\sqrt{c}\Delta\sqrt{c}$, we get by multiplying the second equation of (2.1) by $\frac{1}{2\sqrt{c}}$ that

$$\partial_t\sqrt{c} + u\cdot\nabla\sqrt{c} = (\sqrt{c})^{-1}|\nabla\sqrt{c}|^2 + \Delta\sqrt{c} - \frac{1}{2}\sqrt{c}n.$$

Then multiplying the above equation with $\Delta\sqrt{c}$ and integrating the resulting equality with respect to space variable, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\nabla\sqrt{c}(t)\big\|_2^2 + \big\|\Delta\sqrt{c}(t)\big\|_2^2\\
&=-\int_{\mathbb{R}^3}(\sqrt{c})^{-1}|\nabla\sqrt{c}|^2\cdot\Delta\sqrt{c}\,\mathrm{d}x + \int_{\mathbb{R}^3} u\cdot\nabla\sqrt{c}\cdot\Delta\sqrt{c}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} n\sqrt{c}\cdot\Delta\sqrt{c}\,\mathrm{d}x \\
&\triangleq I_1 + I_2 + I_3.
\end{aligned}\tag{2.7}$$

As for the term I_1, integrating by parts means that

$$\begin{aligned}
I_1 &= -\sum_{i,j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}(\partial_j\sqrt{c})^2\cdot\partial_{ii}\sqrt{c}\,\mathrm{d}x\\
&= -\sum_{i,j}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\partial_i\sqrt{c}|^2|\partial_j\sqrt{c}|^2\,\mathrm{d}x + 2\sum_{i,j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}\partial_i\sqrt{c}\partial_j\sqrt{c}\partial_{ij}\sqrt{c}\,\mathrm{d}x\\
&= -\sum_{i,j}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\partial_i\sqrt{c}|^2|\partial_j\sqrt{c}|^2\,\mathrm{d}x + 2\sum_{i=j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}\partial_i\sqrt{c}\partial_j\sqrt{c}\partial_{ij}\sqrt{c}\,\mathrm{d}x\\
&\quad +2\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}\partial_i\sqrt{c}\partial_j\sqrt{c}\partial_{ij}\sqrt{c}\,\mathrm{d}x.
\end{aligned}$$

We see that

$$\sum_{i=j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}\partial_i\sqrt{c}\partial_j\sqrt{c}\partial_{ij}\sqrt{c}\,\mathrm{d}x = -I_1 - \sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}(\partial_j\sqrt{c})^2\,\partial_{ii}\sqrt{c}\,\mathrm{d}x,$$

from which we obtain

$$\begin{aligned}
I_1 &= -\frac{1}{3}\sum_{i,j}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\partial_i\sqrt{c}|^2|\partial_j\sqrt{c}|^2\,\mathrm{d}x - \frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}(\partial_j\sqrt{c})^2\,\partial_{ii}\sqrt{c}\,\mathrm{d}x\\
&\quad + \frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}\partial_i\sqrt{c}\partial_j\sqrt{c}\partial_{ij}\sqrt{c}\,\mathrm{d}x.
\end{aligned}\tag{2.8}$$

Using the Cauchy-Schwarz inequality, we conclude

$$\begin{aligned}
&\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}\partial_i\sqrt{c}\partial_j\sqrt{c}\partial_{ij}\sqrt{c}\,\mathrm{d}x\\
\leq&\frac{1}{6}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\partial_i\sqrt{c}|^2|\partial_j\sqrt{c}|^2\,\mathrm{d}x+\frac{2}{3}\int_{\mathbb{R}^3}\partial_{ij}|\sqrt{c}|^2\,\mathrm{d}x.
\end{aligned}\tag{2.9}$$

Similarly we get

$$\begin{aligned}
&-\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}(\partial_j\sqrt{c})^2\,\partial_{ii}\sqrt{c}\,\mathrm{d}x\\
=&-\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}(\partial_1\sqrt{c})^2\,\partial_{22}\sqrt{c}\,\mathrm{d}x-\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}(\partial_1\sqrt{c})^2\,\partial_{33}\sqrt{c}\,\mathrm{d}x\\
&-\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}(\partial_2\sqrt{c})^2\,\partial_{11}\sqrt{c}\,\mathrm{d}x-\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}(\partial_2\sqrt{c})^2\,\partial_{33}\sqrt{c}\,\mathrm{d}x\\
&-\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}(\partial_3\sqrt{c})^2\,\partial_{11}\sqrt{c}\,\mathrm{d}x-\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^3}(\sqrt{c})^{-1}(\partial_3\sqrt{c})^2\,\partial_{22}\sqrt{c}\,\mathrm{d}x\\
\leq&\frac{1}{6}\sum_{i}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\partial_i\sqrt{c}|^4\,\mathrm{d}x+\frac{2}{3}\sum_{i}\int_{\mathbb{R}^3}|\partial_{ii}\sqrt{c}|^2\,\mathrm{d}x.
\end{aligned}\tag{2.10}$$

Plugging estimates (2.9) and (2.10) into (2.8) implies

$$\begin{aligned}
I_1\leq&-\frac{1}{3}\sum_{i,j}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\partial_i\sqrt{c}|^2|\partial_j\sqrt{c}|^2\,\mathrm{d}x+\frac{1}{6}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\partial_i\sqrt{c}|^2|\partial_j\sqrt{c}|^2\,\mathrm{d}x\\
&+\frac{2}{3}\int_{\mathbb{R}^3}\partial_{i,j}|\sqrt{c}|^2\,\mathrm{d}x+\frac{2}{3}\sum_{i}\int_{\mathbb{R}^3}|\partial_{ii}\sqrt{c}|^2\,\mathrm{d}x+\frac{1}{6}\sum_{i}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\partial_i\sqrt{c}|^4\,\mathrm{d}x\\
\leq&-\frac{1}{6}\sum_{i,j}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\partial_i\sqrt{c}|^2|\partial_j\sqrt{c}|^2\,\mathrm{d}x+\frac{2}{3}\sum_{i,j}\int_{\mathbb{R}^3}|\partial_{ij}\sqrt{c}|^2\,\mathrm{d}x.
\end{aligned}\tag{2.11}$$

As for the term I_3, we easily find that

$$\begin{aligned}
I_3=&-\frac{1}{2}\int_{\mathbb{R}^3}\nabla n\sqrt{c}\cdot\nabla\sqrt{c}\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\\
=&-\frac{1}{4}\int_{\mathbb{R}^3}\nabla n\cdot\nabla c\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}n|\nabla\sqrt{c}|^2\,\mathrm{d}x.
\end{aligned}\tag{2.12}$$

Since $\|\nabla^2\sqrt{c}\|_2 = \|\Delta\sqrt{c}\|_2$, then plugging (2.11) and (2.12) into (2.7) means that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\nabla\sqrt{c}(t)\big\|_2^2+\frac{1}{3}\|\Delta\sqrt{c}(t)\|_2^2+\frac{1}{6}\sum_{i,j}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\partial_i\sqrt{c}|^2|\partial_j\sqrt{c}|^2\,\mathrm{d}x\\&=\int_{\mathbb{R}^3}u\cdot\nabla\sqrt{c}\cdot\Delta\sqrt{c}\,\mathrm{d}x-\frac{1}{4}\int_{\mathbb{R}^3}\nabla n\cdot\nabla c\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}n|\nabla\sqrt{c}|^2\,\mathrm{d}x.\end{aligned}\tag{2.13}$$

Thus, multiplying (2.13) by 4 and adding (2.6), we conclude

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x+2\frac{\mathrm{d}}{\mathrm{d}t}\big\|\nabla\sqrt{c}(t)\big\|_2^2+\int_{\mathbb{R}^3}n\big(1+\ln(n+1)\big)\,\mathrm{d}x+\frac{4}{3}\|\Delta\sqrt{c}\|_2^2\\&+\int_{\mathbb{R}^3}n^3\big(1+\ln(n+1)\big)\,\mathrm{d}x+\frac{2}{3}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x+2\int_{\mathbb{R}^3}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\\&\le\int_{\mathbb{R}^3}\Delta c\ln(n+1)\,\mathrm{d}x+4\int_{\mathbb{R}^3}u\cdot\nabla\sqrt{c}\cdot\Delta\sqrt{c}\,\mathrm{d}x+(1+a)\int_{\mathbb{R}^3}n^2\big(1+\ln(n+1)\big)\,\mathrm{d}x.\end{aligned}\tag{2.14}$$

For the right-hand side of (2.14), we calculate

$$\begin{aligned}&\int_{\mathbb{R}^3}\Delta c\ln(n+1)\,\mathrm{d}x\\&\le\frac{1}{32}\|c_0\|_\infty\int_{\mathbb{R}^3}|\Delta c|^2\,\mathrm{d}x+C\int_{\mathbb{R}^3}|\ln(n+1)|^2\,\mathrm{d}x\\&\le\frac{1}{32}\|c_0\|_\infty\int_{\mathbb{R}^3}4|\nabla\sqrt{c}|^4\,\mathrm{d}x+\frac{1}{32}\|c_0\|_\infty\int_{\mathbb{R}^3}4|c||\Delta\sqrt{c}|^2\,\mathrm{d}x+C\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x\\&\le\frac{\|c\|_\infty}{8\|c_0\|_\infty}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x+\frac{\|c\|_\infty}{8\|c_0\|_\infty}\int_{\mathbb{R}^3}|\Delta\sqrt{c}|^2\,\mathrm{d}x+C\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x\\&\le\frac{1}{8}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x+\frac{1}{8}\int_{\mathbb{R}^3}|\Delta\sqrt{c}|^2\,\mathrm{d}x+C\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x.\end{aligned}\tag{2.15}$$

We have with Young's inequality that

$$
\begin{aligned}
4\int_{\mathbb{R}^3}(u\cdot\nabla)\sqrt{c}\cdot\Delta\sqrt{c}\,\mathrm{d}x &= -4\int_{\mathbb{R}^3}\nabla u\cdot|\nabla\sqrt{c}|^2\,\mathrm{d}x = -4\int_{\mathbb{R}^3}\sqrt{c}\nabla u\cdot(\sqrt{c})^{-1}|\nabla\sqrt{c}|^2\,\mathrm{d}x\\
&\le 8\int_{\mathbb{R}^3}|\sqrt{c}\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x\\
&\le 8\|c\|_\infty\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x\\
&\le 8\|c_0\|_\infty\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x\\
&\le C\|\nabla u\|_2^2+\frac{1}{2}\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x.
\end{aligned}
\tag{2.16}
$$

Plugging (2.15) and (2.16) into (2.14), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x+2\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\sqrt{c}(t)\|_2^2+\int_{\mathbb{R}^3}(an+n^3)\big(1+\ln(n+1)\big)\,\mathrm{d}x\\
&+\|\Delta\sqrt{c}\|_2^2+\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x+\int_{\mathbb{R}^3}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\\
\le{}& C\|\nabla u\|_2^2+C(1+a)\int_{\mathbb{R}^3}n^2\big(1+\ln(n+1)\big)\,\mathrm{d}x.
\end{aligned}
\tag{2.17}
$$

Operating the L^2-inner product with the third equation of (2.1) by u, we get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2+\|\nabla u(t)\|_2^2=-\int_{\mathbb{R}^3}J_N(n\nabla\phi)u\,\mathrm{d}x.
$$

Therefore we obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2+\|\nabla u(t)\|_2^2\le\|\nabla\phi\|_\infty\|n\|_2\|u\|_2.
\tag{2.18}
$$

Moreover, adding (2.18) to (2.17), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x+\|\nabla\sqrt{c}\|_2^2+\|u\|_2^2\Big)+\int_{\mathbb{R}^3}(an+n^3)\big(2+\ln(n+1)\big)\,\mathrm{d}x\\
&+\|\nabla u\|_2^2+\|\Delta\sqrt{c}\|_2^2+\int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x+\int_{\mathbb{R}^3}n|\nabla\sqrt{c}|^2\,\mathrm{d}x
\end{aligned}
\tag{2.19}
$$

$$\leq C \int_{\mathbb{R}^3} (n+1)\ln(n+1)\,\mathrm{d}x + C\|n\|_2\|u\|_2 + (1+a)\int_{\mathbb{R}^3} n^2\big(1+\ln(n+1)\big)\,\mathrm{d}x.$$

For the third term in the last line of (2.19), we observe that

$$\begin{aligned}\int_{\mathbb{R}^3} n^2\big(1+\ln(n+1)\big)\,\mathrm{d}x =& \int_{A_{2(1+a)}} n^2\big(2+\ln(n+1)\big)\,\mathrm{d}x + \int_{A^c_{2(1+a)}} n^2\big(2+\ln(n+1)\big)\,\mathrm{d}x \\ \leq& 4\|n\|_1 + \frac{1}{2(1+a)}\int_{\mathbb{R}^3} n^3\big(2+\ln(n+1)\big)\,\mathrm{d}x.\end{aligned}$$

Inserting this estimate into (2.19) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x + \|\nabla\sqrt{c}\|_2^2 + \|u\|_2^2\Big) + \|\nabla u\|_2^2 \\ &+ \frac{1}{2}\int_{\mathbb{R}^3}(an+n^3)\big(1+\ln(n+1)\big)\,\mathrm{d}x + \|\Delta\sqrt{c}\|_2^2 \\ &+ \int_{\mathbb{R}^3}(\sqrt{c})^{-2}|\nabla\sqrt{c}|^4\,\mathrm{d}x + \int_{\mathbb{R}^3} n|\nabla\sqrt{c}|^2\,\mathrm{d}x \\ \leq& C\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x + C\|n\|_2\|u\|_2 + C\|n\|_1.\end{aligned} \tag{2.20}$$

Next we show that $\|(n+1)\ln(n+1)\|_1$ and $\|n\|_{L^1\cap L\log L}$ are equivalent. For one thing, we get

$$\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x \geq \int_{\mathbb{R}^3} n\ln(n+1)\,\mathrm{d}x = \|n\|_{L\log L}.$$

For another thing, we have

$$\int_{\mathbb{R}^3}(n+1)\ln(n+1)\,\mathrm{d}x \leq \int_{\mathbb{R}^3} n\ln(n+1)\,\mathrm{d}x + \int_{\mathbb{R}^3}\ln(n+1)\,\mathrm{d}x \leq \|n\|_{L\log L} + \|n\|_1.$$

Finally, performing the Gronwall inequality to (2.20) gives

$$\mathcal{U}(t) + \int_0^t \mathcal{V}(\tau)\,\mathrm{d}\tau + \int_0^t\int_{\mathbb{R}^3} n|\nabla\sqrt{c}|^2\,\mathrm{d}x\mathrm{d}\tau - \int_0^t\int_{\mathbb{R}^3} n|\nabla\sqrt{c}|^2\,\mathrm{d}x\mathrm{d}\tau \leq Ce^{Ct}. \tag{2.21}$$

This inequality together with (2.21) ensures the required estimate (2.4). □

Corollary 2.1. *Let the triple $(n_0, c_0, u_0) \in X_0$. Assume that (n, c, u) is a smooth solution to equations (2.1). Then there exists a constant C dependent of the initial data such that*

$$\|c\|_{L_t^\infty H_x^1} + \|c\|_{L_t^2 H_x^2} \le Ce^{Ct}.$$

Proof. Firstly, we see that

$$\|\nabla c\|_{L_t^\infty L_x^2} \le 2\|\sqrt{c}\nabla\sqrt{c}\|_{L_t^\infty L_x^2} \le C\|\sqrt{c}\|_{L_t^\infty L_x^\infty}\|\nabla\sqrt{c}\|_{L_t^\infty L_x^2} \le C\|c\|^{\frac{1}{2}}_{L_t^\infty L_x^\infty}\|\nabla\sqrt{c}\|_{L_t^\infty L_x^2},$$

and

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^3} |\Delta c|^2 \,\mathrm{d}x\mathrm{d}\tau &\le \int_0^t\int_{\mathbb{R}^3} 4|\nabla\sqrt{c}|^4 \,\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_{\mathbb{R}^3} 4|c||\Delta\sqrt{c}|^2 \,\mathrm{d}x\mathrm{d}\tau \\ &\le \|c\|_\infty \int_0^t\int_{\mathbb{R}^3} (\sqrt{c})^{-2}|\nabla\sqrt{c}|^4 \,\mathrm{d}x\mathrm{d}\tau + \|c\|_\infty \int_0^t\int_{\mathbb{R}^3} |\Delta\sqrt{c}|^2 \,\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

The above two inequalities together with (2.3) and (2.4) imply the desired result. □

2.2. Strong a priori estimates

The task is now to show some global-in-time estimates about a better regularity of solutions of (2.1). The estimates developed below will be the basic ingredient of the proof of Theorem 1.1.

Proposition 2.2. *Let the triple $(n_0, c_0, u_0) \in X_0$, $\frac{\omega_0}{r} \in L^2$. Then every smooth solution (n, c, u) of system (2.1) satisfies*

$$\left\|\frac{\omega}{r}(t)\right\|_2^2 + \int_0^t \left\|\nabla\left(\frac{\omega}{r}\right)(\tau)\right\|_2^2 \mathrm{d}\tau \le Ce^{Ct}, \tag{2.22}$$

$$\|u(t)\|_{H^1}^2 + \int_0^t \|u(\tau)\|_{H^2}^2 \,\mathrm{d}\tau \le Ce^{e^{Ct}}, \tag{2.23}$$

$$\|\nabla c(t)\|_{\dot{H}^1}^2 + \int_0^t \|\nabla c(\tau)\|_{\dot{H}^2}^2 \,\mathrm{d}\tau \le Ce^{e^{Ct}}, \tag{2.24}$$

$$\|n(t)\|_2^2 + \int_0^t \|\nabla n(\tau)\|_2^2 \mathrm{d}\tau + \int_0^t \|n(\tau)\|_4^4 \,\mathrm{d}\tau \le Ce^{e^{Ct}}. \tag{2.25}$$

Proof. According to equation (1.8), it is clear that $\frac{\omega_\theta}{r}$ satisfies the following equation

$$\begin{aligned}&\partial_t\frac{\omega_\theta}{r}+J_N u\cdot\nabla(\frac{\omega_\theta}{r})-\Delta\frac{\omega_\theta}{r}-\frac{2}{r}\partial_r\Big(\frac{\omega_\theta}{r}\Big)\\ =&\Big(-\frac{1}{r^2}\partial_z n x^h\cdot\nabla_h\phi-\frac{n}{r^2}x^h\cdot\nabla_h(\partial_z\phi)+\frac{\partial_r n\partial_z\phi}{r}+\frac{n\partial^2_{r,z}\phi}{r}\Big)*\rho^{1/N}\\ &+\Big(\frac{u^r}{r^2}(\omega*\rho^{1/N})e_\theta-\frac{u^r*\rho^{1/N}}{r^2}\omega_\theta\Big).\end{aligned}\tag{2.26}$$

Now taking the L^2-inner product of (2.26) with $\frac{\omega_\theta}{r}$ yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\|\frac{\omega_\theta}{r}\Big\|_2^2+\Big\|\nabla\frac{\omega_\theta}{r}(t)\Big\|_2^2\\ \le&-\int\limits_{\mathbb{R}^3}\frac{1}{r^2}\partial_z n x^h\cdot\nabla_h\phi\Big(\frac{\omega_\theta}{r}*\rho^{1/N}\Big)\,\mathrm{d}x-\int\limits_{\mathbb{R}^3}\frac{n}{r^2}x^h\cdot\nabla_h(\partial_z\phi)\Big(\frac{\omega_\theta}{r}*\rho^{1/N}\Big)\,\mathrm{d}x\\ &+\int\limits_{\mathbb{R}^3}\frac{\partial_r n\partial_z\phi}{r}\Big(\frac{\omega_\theta}{r}*\rho^{1/N}\Big)\,\mathrm{d}x+\int\limits_{\mathbb{R}^3}\frac{n\partial^2_{r,z}\phi}{r}\Big(\frac{\omega_\theta}{r}*\rho^{1/N}\Big)\,\mathrm{d}x\\ &+\int\limits_{\mathbb{R}^3}\Big(\frac{u^r}{r^2}(\omega*\rho^{1/N})e_\theta-\frac{u^r*\rho^{1/N}}{r^2}\omega_\theta\Big)\frac{\omega_\theta}{r}\mathrm{d}x\\ \triangleq&II_1+II_2+II_3+II_4+II_5.\end{aligned}\tag{2.27}$$

Integrating by parts, we easily find that

$$\begin{aligned}II_1=&\int\limits_{\mathbb{R}^3}\frac{1}{r^2}nx^h\partial_z(\nabla_h\phi)\Big(\frac{\omega_\theta}{r}*\rho^{1/N}\Big)\,\mathrm{d}x+\int\limits_{\mathbb{R}^3}\frac{1}{r^2}nx^h\nabla_h\phi\partial_z\Big(\frac{\omega_\theta}{r}*\rho^{1/N}\Big)\,\mathrm{d}x\\ =&\int\limits_{\mathbb{R}^3}n\frac{x_h}{r}\frac{\partial_z(\nabla_h\phi)-\partial_z(\nabla_h\phi(0,0,x_3))}{r}\Big(\frac{\omega_\theta}{r}*\rho^{1/N}\Big)\,\mathrm{d}x\\ &+\int\limits_{\mathbb{R}^3}n\frac{x_h}{r}\frac{\nabla_h\phi-\nabla_h\phi(0,0,x_3)}{r}\partial_z\Big(\frac{\omega_\theta}{r}*\rho^{1/N}\Big)\,\mathrm{d}x.\end{aligned}$$

By the Hölder inequality, one has

$$\begin{aligned}II_1\le&\|n\|_2\Big\|\frac{x_h}{r}\Big\|_\infty\|\nabla^3\phi\|_\infty\Big\|\frac{\omega_\theta}{r}*\rho^{1/N}\Big\|_2+\|n\|_2\Big\|\frac{x_h}{r}\Big\|_\infty\|\nabla^2\phi\|_\infty\Big\|\nabla\frac{\omega_\theta}{r}*\rho^{1/N}\Big\|_2\\ \le&C\|n\|_2^2+\Big\|\frac{\omega_\theta}{r}\Big\|_2^2+\frac{1}{8}\Big\|\nabla\frac{\omega_\theta}{r}\Big\|_2^2.\end{aligned}\tag{2.28}$$

Similarly,

$$II_2 \leq \|n\|_2 \left\|\frac{x_h}{r}\right\|_\infty \|\nabla^3\phi\|_\infty \left\|\frac{\omega_\theta}{r}\right\|_2 \leq C\|n\|_2^2 + \left\|\frac{\omega_\theta}{r}\right\|_2^2. \tag{2.29}$$

According to the coordinate transformation, we have

$$\begin{aligned} II_3 &= \int_{-\infty}^{+\infty}\int_0^{+\infty} \frac{\partial_r n \partial_z \phi}{r}\left(\frac{\omega_\theta}{r} * \rho^{1/N}\right) r\,\mathrm{d}r\mathrm{d}z \\ &= -\int_{\mathbb{R}^3} n\frac{\partial_r\partial_z\phi}{r}\left(\frac{\omega_\theta}{r} * \rho^{1/N}\right)\mathrm{d}x - \int_{\mathbb{R}^3} n\frac{\partial_z\phi}{r}\partial_r\left(\frac{\omega_\theta}{r} * \rho^{1/N}\right)\mathrm{d}x. \end{aligned}$$

Moreover, we have by the Hölder inequality that

$$\begin{aligned} II_3 \leq& \|n\|_2\|\nabla^3\phi\|_\infty \left\|\frac{\omega_\theta}{r} * \rho^{1/N}\right\|_2 + \|n\|_2\|\nabla^2\phi\|_\infty \left\|\nabla\frac{\omega_\theta}{r} * \rho^{1/N}\right\|_2 \\ \leq& C\|n\|_2^2 + \left\|\frac{\omega_\theta}{r}\right\|_2^2 + \frac{1}{8}\left\|\nabla\frac{\omega_\theta}{r}\right\|_2^2. \end{aligned} \tag{2.30}$$

For the fourth term, we get by the Hölder inequality and the Young inequality that

$$II_4 \leq \|n\|_2\|\nabla^3\phi\|_\infty \left\|\frac{\omega_\theta}{r}\right\|_2 \leq C\|n\|_2^2 + \left\|\frac{\omega_\theta}{r}\right\|_2^2. \tag{2.31}$$

Next we estimate the last term

$$\begin{aligned} II_5 =& \int_{\mathbb{R}^3}\left(\frac{u^r}{r^2}(\omega * \rho^{1/N})e_\theta - \frac{u^r * \rho^{1/N}}{r^2}\omega_\theta\right)\frac{\omega_\theta}{r}\mathrm{d}x \\ =& \int_{\mathbb{R}^3}\frac{u^r}{r^2}\left(\int_{\mathbb{R}^3}\omega(y)\rho^{1/N}(x-y)\mathrm{d}y e_\theta\right)\frac{\omega_\theta}{r}\mathrm{d}x - \int_{\mathbb{R}^3}\frac{u^r * \rho^{1/N}}{r^2}\omega_\theta\frac{\omega_\theta}{r}\mathrm{d}x \\ =& \int_{\mathbb{R}^3}\omega_\theta(y)\int_{\mathbb{R}^3}\frac{u^r}{r^2}(x)\rho^{1/N}(y-x)\,\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3}\frac{u^r * \rho^{1/N}}{r^2}\omega_\theta\frac{\omega_\theta}{r}\mathrm{d}x = 0. \end{aligned} \tag{2.32}$$

Plugging both estimates (2.28)-(2.32) into (2.27) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\omega_\theta}{r}(t)\right\|_2^2 + \left\|\nabla\left(\frac{\omega_\theta}{r}\right)(t)\right\|_2^2 \leq C\|n\|_2^2 + C\left\|\frac{\omega_\theta}{r}\right\|_2^2, \tag{2.33}$$

from which we have

$$\left\|\frac{\omega_\theta}{r}(t)\right\|_2^2 + \int_0^t \left\|\nabla\left(\frac{\omega_\theta}{r}\right)(\tau)\right\|_2^2 \mathrm{d}\tau \leq Ce^{Ct}. \tag{2.34}$$

This gives (2.22).

For (2.23), taking the L^2-inner product of (1.8) with ω_θ, we immediately get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_\theta(t)\|_2^2+\|\nabla\omega_\theta(t)\|_2^2+\left\|\frac{\omega_\theta}{r}(t)\right\|_2^2\\ \le& C\left\|\frac{u^r}{r}\right\|_\infty\|\omega_\theta\|_2^2+C\|\nabla\phi\|_\infty\|n\|_2\|\nabla\omega\|_2.\end{aligned}\tag{2.35}$$

Since

$$\|\omega\|_2=\|\omega_\theta\|_2 \quad\text{and}\quad \|\nabla\omega\|_2^2=\|\nabla\omega_\theta\|_2^2+\left\|\frac{\omega_\theta}{r}\right\|_2^2,$$

we get by the Cauchy-Schwarz inequality that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_2^2+\|\nabla\omega(t)\|_2^2\le C\left\|\frac{u^r}{r}\right\|_\infty\|\omega\|_2^2+C\|n\|_2^2+\frac{1}{2}\|\nabla\omega\|_2^2,\tag{2.36}$$

from which we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_2^2+\|\nabla\omega(t)\|_2^2\le C\left\|\frac{u^r}{r}\right\|_\infty\|\omega\|_2^2+C\|n\|_2^2.\tag{2.37}$$

The Gronwall inequality implies

$$\|\omega(t)\|_2^2+\int_0^t\|\nabla\omega(\tau)\|_2^2\,\mathrm{d}\tau\le\Big(\|\omega_0\|_2^2+C\int_0^t\|n(\tau)\|_2^2\,\mathrm{d}\tau\Big)e^{C\int_0^t\|\frac{u^r}{r}(\tau)\|_\infty\,\mathrm{d}\tau}.\tag{2.38}$$

Employing the magic formula established in [16] and the Hölder inequality, we have

$$\begin{aligned}\int_0^t\left\|\frac{u^r}{r}(\tau)\right\|_\infty\mathrm{d}\tau\le& C\int_0^t\left\|\frac{\omega_\theta}{r}(\tau)\right\|_2^{\frac{1}{2}}\left\|\frac{\omega_\theta}{r}(\tau)\right\|_{\dot H^1}^{\frac{1}{2}}\mathrm{d}\tau\\ \le& C\Big(\int_0^t 1\,\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_0^t\left\|\frac{\omega_\theta}{r}(\tau)\right\|_2^2\mathrm{d}\tau\Big)^{\frac{1}{4}}\Big(\int_0^t\left\|\frac{\omega_\theta}{r}(\tau)\right\|_{\dot H^1}^2\mathrm{d}\tau\Big)^{\frac{1}{4}}\\ \le& C\Big(t+\int_0^t\left\|\frac{\omega_\theta}{r}(\tau)\right\|_2^2\mathrm{d}\tau+\int_0^t\left\|\frac{\omega_\theta}{r}(\tau)\right\|_{\dot H^1}^2\mathrm{d}\tau\Big)\le Ce^{Ct}.\end{aligned}\tag{2.39}$$

Plugging (2.39) into (2.38) implies

$$\|\omega(t)\|_2^2+\int_0^t\|\nabla\omega(\tau)\|_2^2\,\mathrm{d}\tau\le Ce^{e^{Ct}},\tag{2.40}$$

which together with (2.4) means (2.23).

For (2.24), taking ∂_i on both side of the second equation to system (2.1), we immediately have

$$\partial_t \partial_i c + u \cdot \nabla \partial_i c - \Delta \partial_i c = -\partial_i (nc) - \partial_i u \cdot \nabla c. \tag{2.41}$$

Multiplying (2.41) by $\partial_i c$ yields

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_i c(t)\right\|_2^2 + \int_{\mathbb{R}^3} |\nabla \partial_i c(x,t)|^2 \,\mathrm{d}x \\ =& -\int_{\mathbb{R}^3} \partial_i (nc)\partial_i c \,\mathrm{d}x - \sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_i u^j \partial_j c \partial_i c \,\mathrm{d}x \triangleq III_1 + III_2. \end{aligned} \tag{2.42}$$

For the term III_1, we see by the Young inequality that

$$III_1 = \int_{\mathbb{R}^3} nc\, \partial_{ii}^2 c \,\mathrm{d}x \le C\|n\|_2 \|c\|_\infty \left\|\partial_{ii}^2 c\right\|_2. \tag{2.43}$$

Next we tackle with III_2. It can be bounded as follows

$$III_2 = \sum_{j=1}^{3}\int_{\mathbb{R}^3} u^j \partial_{ij}^2 c \partial_i c \,\mathrm{d}x + \sum_{j=1}^{3}\int_{\mathbb{R}^3} u^j \partial_j c \partial_{ii}^2 c \,\mathrm{d}x \le 2\|u\|_\infty \|\nabla c\|_2 \|\Delta c\|_2. \tag{2.44}$$

Putting (2.43) and (2.44) into (2.42), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c(t)\|_2^2 + \int_{\mathbb{R}^3} |\nabla^2 c(x,t)|^2 \,\mathrm{d}x \\ \le& C\|n\|_2 \|c\|_\infty \|\Delta c\|_2 + C\|u\|_\infty \|\nabla c\|_2 \|\Delta c\|_2 \\ \le& C\|n\|_2^2 \|c\|_\infty^2 + C\|u\|_\infty^2 \|\nabla c\|_2^2 + \frac{1}{4}\|\Delta c\|_2^2. \end{aligned} \tag{2.45}$$

From this, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c(t)\|_2^2 + \int_{\mathbb{R}^3} |\nabla^2 c(x,t)|^2 \,\mathrm{d}x \le C\|n\|_2^2 \|c\|_\infty^2 + C\|u\|_\infty^2 \|\nabla c\|_2^2. \tag{2.46}$$

The Gronwall inequality implies

$$\begin{aligned} \|\nabla c(t)\|_2^2 + \int_0^t \|\nabla^2 c(\tau)\|_2^2 \,\mathrm{d}\tau \le& C\left(\|\nabla c_0\|_2 + \|c_0\|_\infty \int_0^t \|n(\tau)\|_2^2 \,\mathrm{d}\tau\right) e^{C\int_0^t \|u(\tau)\|_\infty \,\mathrm{d}\tau} \\ \le& Ce^{Ct} e^{C\int_0^t \|u(\tau)\|_{H^2} \,\mathrm{d}\tau} \le Ce^{e^{Ct}}. \end{aligned} \tag{2.47}$$

Next, we turn to show estimate of $\|\nabla c\|_3$.

$$\begin{aligned}&\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_i c(t)\right\|_3^3+\int_{\mathbb{R}^3}\partial_i c(x,t)|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x\\&=-\int_{\mathbb{R}^3}\partial_i\big(nc\big)|\partial_i c|\partial_i c\,\mathrm{d}x-\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_i u^j\partial_j c|\partial_i c|\partial_i c\,\mathrm{d}x.\end{aligned}\tag{2.48}$$

Integrating by parts and then using the Hölder inequality, we see that

$$\begin{aligned}-\int_{\mathbb{R}^3}\partial_i\big(nc\big)|\partial_i c|\partial_i c\,\mathrm{d}x=&2\int_{\mathbb{R}^3}nc|\partial_i c|\partial_{ii}^2 c\,\mathrm{d}x\\\le&\|c\|_\infty\|n\|_3\|\partial_i c\|_3^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}|\partial_i c(x,t)|\,|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\\\le&\|c\|_\infty^2\|n\|_3^2\|\partial_i c\|_3+\frac{1}{4}\int_{\mathbb{R}^3}|\partial_i c(x,t)|\,|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x.\end{aligned}\tag{2.49}$$

Integrating by parts and performing the Hölder inequality, one has

$$\begin{aligned}-\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_i u^j\partial_j c|\partial_i c|\partial_i c\,\mathrm{d}x=&2\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_i u^j c|\partial_i c|\partial_{ij}^2 c\,\mathrm{d}x\\\le&2\|c\|_\infty\|\nabla u\|_3\|\partial_i c\|_3^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}|\partial_i c(x,t)|\,|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\\\le&2\|c\|_\infty^2\|\nabla u\|_3^2\|\partial_i c\|_3+\frac{1}{3}\int_{\mathbb{R}^3}|\partial_i c(x,t)|\,|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x\\\le&2\|c\|_\infty^2\|\nabla u\|_2\|\Delta u\|_2\|\partial_i c\|_3+\frac{1}{4}\int_{\mathbb{R}^3}|\partial_i c(x,t)|\,|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x\end{aligned}$$

Plugging this estimate and (2.49) into (2.48), we obtain

$$\begin{aligned}&\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_i c(t)\right\|_3^3+\int_{\mathbb{R}^3}\partial_i c(x,t)|\nabla\partial_i c(x,t)|^2\,\mathrm{d}x\\\le&\|c\|_\infty^2\|n\|_3^2\left\|\partial_i c(t)\right\|_3+\|c\|_\infty^2\|\nabla u\|_2\|\Delta u\|_2\left\|\partial_i c(t)\right\|_3.\end{aligned}\tag{2.50}$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla c(t)\right\|_3^2\le\|c\|_\infty^2\|n\|_3^2+\|c\|_\infty^2\|\nabla u\|_2\|\Delta u\|_2.$$

Integrating with respect to time t yields

$$\left\|\nabla c(t)\right\|_3 \leq C\left\|\nabla c_0\right\|_3 e^{Ct}.$$

Inserting this estimate into (2.50), we finally obtain

$$\left\|\nabla c(t)\right\|_3^3 + \int_0^t \left\|\nabla|\nabla c(\tau)|^{\frac{3}{2}}\right\|_2^2 \mathrm{d}\tau \leq Ce^{Ct}.$$

For (2.25), taking the L^2-inner product of the first equation of (2.1) with n, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n(t)\|_2^2 + \|\nabla n(t)\|_2^2 + a\|n(t)\|_2^2 + \|n(t)\|_4^4 = -\int_{\mathbb{R}^3} \nabla\cdot\big(n\nabla c\big)\cdot n\,\mathrm{d}x + (a+1)\|n(t)\|_3^3. \tag{2.51}$$

Integrating by parts and using the Hölder inequality, one has

$$\begin{aligned}
-\int_{\mathbb{R}^3} \nabla\cdot\big(n\nabla c\big)\cdot n\,\mathrm{d}x =& \int_{\mathbb{R}^3} n\nabla c\nabla n\,\mathrm{d}x \\
\leq& C\|n\|_{L^3}\left\||\nabla c|^{\frac{3}{2}}\right\|_4^{\frac{2}{3}}\|\nabla n\|_2 \\
\leq& C\|n\|_{L^2}^{\frac{1}{2}}\left\||\nabla c|^{\frac{3}{2}}\right\|_2^{\frac{1}{6}}\left\||\nabla c|^{\frac{3}{2}}\right\|_6^{\frac{1}{2}}\|\nabla n\|_2^{\frac{3}{2}} \\
\leq& C\,\|\nabla c\|_3\left\|\nabla|\nabla c|^{\frac{3}{2}}\right\|_2^2\|n\|_{L^2}^2 + \frac{1}{4}\|\nabla n\|_2^2.
\end{aligned}$$

Plugging this estimate into (2.51) and integrating the resulting inequality in time t entails

$$\begin{aligned}
&\|n(t)\|_2^2 + \int_0^t \|\nabla n(\tau)\|_2^2\,\mathrm{d}\tau + a\int_0^t \|n(\tau)\|_2^2\,\mathrm{d}\tau + \int_0^t \|n(\tau)\|_4^4\,\mathrm{d}\tau \\
\leq& \|n_0\|_{L^2}^2 + C\int_0^t \|\nabla c(\tau)\|_3\left\|\nabla|\nabla c|^{\frac{3}{2}}(\tau)\right\|_2^2\|n(\tau)\|_{L^2}^2\,\mathrm{d}\tau + (1+a)\int_0^t \|n(\tau)\|_3^3\,\mathrm{d}\tau.
\end{aligned} \tag{2.52}$$

The Gronwall inequality implies

$$\|n(t)\|_2^2 + \int_0^t \|\nabla n(\tau)\|_2^2\,\mathrm{d}\tau + a\int_0^t \|n(\tau)\|_2^2\,\mathrm{d}\tau + \int_0^t \|n(\tau)\|_4^4\,\mathrm{d}\tau \leq Ce^{Ct}.$$

Lastly, we show estimate of $\|c\|_{H^2}$. Performing Δ to the second equation of (2.1) gives

$$\partial_t \Delta c - \Delta\Delta c = -\Delta(u\cdot\nabla c) - \Delta(cn).$$

Multiplying the above equality with Δu and integrating in spaces, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta c\|_2^2 + \|\nabla\Delta c\|_2^2 &= -\int_{\mathbb{R}^3}\Delta(u\cdot\nabla c)\Delta c\,\mathrm{d}x - \int_{\mathbb{R}^3}\Delta(cn)\Delta c\,\mathrm{d}x\\ &\triangleq V_1 + V_2.\end{aligned}\tag{2.53}$$

For the term V_1, we have

$$\begin{aligned}V_1 =& \int_{\mathbb{R}^3}\nabla u\cdot\nabla c\cdot\nabla\Delta c\,\mathrm{d}x + \int_{\mathbb{R}^3} u\cdot\nabla^2 c\cdot\nabla\Delta c\,\mathrm{d}x\\ \le& \|\nabla u\|_\infty\|\nabla c\|_2\|\nabla\Delta c\|_2 + \|u\|_\infty\|\nabla^2 c\|_2\|\nabla\Delta c\|_2\\ \le& C\|\nabla u\|_\infty^2\|\nabla c\|_2^2 + C\|u\|_\infty^2\|\nabla^2 c\|_2^2 + \frac{1}{4}\|\nabla\Delta c\|_2.\end{aligned}\tag{2.54}$$

For the term V_2, we conclude

$$\begin{aligned}V_2 =& \int_{\mathbb{R}^3}\nabla cn\cdot\nabla\Delta c\,\mathrm{d}x + \int_{\mathbb{R}^3} c\nabla n\cdot\nabla\Delta c\,\mathrm{d}x\\ \le& \|\nabla c\|_6\|n\|_3\|\nabla\Delta c\|_2 + \|c\|_\infty\|\nabla n\|_2\|\nabla\Delta c\|_2\\ \le& C\|\Delta c\|_2^2\|n\|_3^2 + C\|c\|_\infty^2\|\nabla n\|_2^2 + \frac{1}{4}\|\nabla\Delta c\|_2^2.\end{aligned}\tag{2.55}$$

Plugging (2.54) and (2.55) into (2.53) and using Gronwall's inequality, we have (2.25). This completes the proof of Proposition 2.2. □

3. Proof of the main result

This section is devoted to showing Theorem 1.1. Now we first focus on the existence statement. We build up the following approximation scheme:

$$\begin{cases}\partial_t n^N + u^N\cdot\nabla n^N - \Delta n^N = -\nabla\cdot(n^N\nabla c^N) + n^N(1-n^N)(n^N-a), & x\in\mathbb{R}^3,\ t>0,\\ \partial_t c^N + u^N\cdot\nabla c^N - \Delta c^N = -n^N c^N, & \\ \partial_t u^N + J_N u^N\cdot\nabla u^N - \Delta u^N + \nabla P^N = -J_N(n\nabla\phi) & \\ \operatorname{div} u^N = 0, & \\ (n^N, c^N, u^N)|_{t=0} = J_N(n_0, c_0, u_0). & \end{cases}\tag{3.1}$$

By the property of mollifier, we see that $(n_0^N, c_0^N, u_0^N)\in X_0\cap(H^\infty)^3$ with $H^\infty\triangleq\cap_{s\ge0}H^s$. According to Proposition A.1 we notice that the system (3.1) admits a unique global smooth solution. Moreover, Proposition 2.1 and Proposition 2.2 ensure the following bounds is uniform in N:

$$\begin{aligned}
&n^N \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^3)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^3)),\\
&c^N \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^3(\mathbb{R}^3)),\\
&u^N \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^3(\mathbb{R}^3)),\\
&\frac{\omega_\theta^N}{r} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^3)).
\end{aligned}$$

Next our task is to show that $(\partial_t n^N, \partial_t c^N, \partial_t u^N)$ is bounded uniformly. The first equation of system (3.1) can be written as

$$\partial_t n^N = \Delta n^N - u^N \cdot \nabla n^N - \nabla \cdot (n^N \nabla c^N) + (a+1)(n^N)^2 - a n^N - (n^N)^3.$$

Simple calculations yield

$$\begin{aligned}
\|\partial_t n^N\|_{L_t^{\frac{4}{3}} H^{-1}} \leq &\|\Delta n^N\|_{L_t^{\frac{4}{3}} H^{-1}} + \|u^N \cdot \nabla n^N\|_{L_t^{\frac{4}{3}} H^{-1}} + \|\nabla \cdot (n^N \nabla c^N)\|_{L_t^{\frac{4}{3}} H^{-1}}\\
&+ (a+1)\|(n^N)^2\|_{L_t^{\frac{4}{3}} H^{-1}} + a\|n^N\|_{L_t^2 H^{-1}} + \|(n^N)^3\|_{L_t^{\frac{4}{3}} H^{-1}}\\
\leq &\|\Delta n^N\|_{L_t^{\frac{4}{3}} H^{1}} + \|u^N\|_{L_t^\infty L^4} \|n^N\|_{L_t^{\frac{4}{3}} L^4} + \|n^N\|_{L_t^2 L^4} \|\nabla c^N\|_{L_t^4 L^4}\\
&+ (a+1)\|n^N\|^2_{L_t^{\frac{8}{3}} L^{\frac{12}{5}}} + a\|n^N\|_{L_t^2 L^{\frac{6}{5}}} + \|n^N\|^3_{L_t^4 L^{\frac{18}{5}}} \leq C(t).
\end{aligned}$$

This implies that $\partial_t n^N$ is bounded in $L^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}^+; H^{-1})$. Similarly, we can deduce that $\partial_t c^N$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^+; H^{-1})$ and $\partial_t u^N$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^+; H^{-1})$. Meanwhile, we know that L^2 local compactly embeds in H^s and H^s continuously embed in H^{-1} with $s \in (-1, 0)$. Repeating the process used in proof of Proposition A.1, we infer that the sequence (n^N, c^N, u^N) tends to (n, c, u) in $\mathcal{C}(\mathbb{R}^+; H^s)$ with $s < 0$ and the solution (n, c, u) satisfies equation (1.4) in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^3)$. From Fatou's lemma, we conclude that

$$\begin{aligned}
&n \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^3)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^3)),\\
&c \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^3(\mathbb{R}^3)),\\
&u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^3)).
\end{aligned}$$

For more details, one can refer to *Step 3* in the proof of Proposition A.1 in Appendix A.

Now we turn to show the uniqueness. Let (n_1, c_1, u_1) and (n_2, c_2, u_2) are solution of system with the same initial data, then the different $(\delta n, \delta c, \delta u) := (n_1 - n_2, c_1 - c_2, u_1 - u_2)$ solves

$$\begin{cases}
\partial_t \delta n + u_1 \cdot \nabla \delta n - \Delta \delta n = -\nabla \cdot (n_1 \nabla \delta c) - \nabla \cdot (\delta n \nabla c_2) - \delta u \cdot \nabla n_2 + f(n_1) - f(n_2),\\
\partial_t \delta c + u_1 \cdot \nabla \delta c - \Delta \delta c = -\delta n c_2 - n_1 \delta c - \delta u \cdot \nabla c_2,\\
\partial_t \delta u + u_1 \cdot \nabla \delta u - \Delta \delta u + \nabla \delta p = -\nabla \phi \delta n - \delta u \cdot \nabla n_2,\\
(\delta n, \delta c, \delta u)|_{t=0} = (0, 0, 0).
\end{cases} \tag{3.2}$$

Taking the standard L^2-estimate of $(\delta n, \delta c, \delta u)$.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta n(t)\|_2^2+\|\nabla\delta n(t)\|_2^2=-\int\limits_{\mathbb{R}^3}\nabla\cdot(n_1\nabla\delta c)\delta n\,\mathrm{d}x-\int\limits_{\mathbb{R}^3}\nabla\cdot(\delta n\cdot\nabla c_2)\delta n\,\mathrm{d}x$$
$$+\int\limits_{\mathbb{R}^3}\big(f(n_1)-f(n_2)\big)\delta n\,\mathrm{d}x-\int\limits_{\mathbb{R}^3}\delta u\cdot\nabla n_2\delta n\,\mathrm{d}x.$$

Integrating by parts, we find that

$$\begin{aligned}-\int\limits_{\mathbb{R}^3}\nabla\cdot(n_1\nabla\delta c)\delta n\,\mathrm{d}x=&-\int\limits_{\mathbb{R}^3}n_1\nabla\delta c\nabla\delta n\,\mathrm{d}x\\ \le&\|n_1\|_3\|\nabla\delta c\|_6\|\nabla\delta n\|_2\\ \le&C\|n_1\|_3^2\|\nabla^2\delta c\|_2^2+\frac{1}{8}\|\nabla\delta n\|_2^2.\end{aligned}$$

Similarly, we can infer that

$$\begin{aligned}-\int\limits_{\mathbb{R}^3}\nabla\cdot(\delta n\nabla c_2)\delta n\,\mathrm{d}x=&\int\limits_{\mathbb{R}^3}\delta n\cdot\nabla c_2\nabla\delta n\,\mathrm{d}x\\ \le&\|\delta n\|_2\|\nabla c_2\|_\infty\|\nabla\delta n\|_2\\ \le&\|\nabla c_2\|_2^{\frac14}\|\nabla^3 c_2\|_2^{\frac34}\|\delta n\|_2\|\nabla\delta n\|_2\\ \le&\|\nabla c_2\|_2^{\frac12}\left\|\nabla^3 c_2\right\|_2^{\frac32}\|\delta n\|_2^2+\frac{1}{8}\left\|\nabla\delta n\right\|_2^2.\end{aligned}$$

A simple calculation gives

$$\begin{aligned}&\int\limits_{\mathbb{R}^3}\big(f(n_1)-f(n_2)\big)\delta n\,\mathrm{d}x\\ =&-(1+a)\|\delta n\|_2^2-\int\limits_{\mathbb{R}^3}\Big(n_1^2+n_1n_2+n_2^2\Big)(\delta n)^2\,\mathrm{d}x+\int\limits_{\mathbb{R}^3}(n_1+n_2)(\delta n)^2\,\mathrm{d}x.\end{aligned}$$

Since the first two terms of the right hand-side of the above equality are negative, we can ignore them. For the third term, by the Hölder inequality, we immediately obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^3}(n_1+n_2)(\delta n)^2\,\mathrm{d}x\le&\big(\|n_1\|_3+\|n_2\|_3\big)\|\delta n\|_6\|\delta n\|_2\\ \le&C\big(\|n_1\|_3+\|n_2\|_3\big)\|\nabla\delta n\|_2\|\delta n\|_2\\ \le&C\big(\|n_1\|_3+\|n_2\|_3\big)^2\|\delta n\|_2^2+\frac{1}{8}\|\nabla\delta n\|_2^2.\end{aligned}$$

Performing the Hölder inequality and the Young inequality, one has

$$-\int_{\mathbb{R}^3} \delta u \cdot \nabla n_2 \delta n \,\mathrm{d}x = \int_{\mathbb{R}^3} n_2 \delta u \cdot \nabla \delta n \,\mathrm{d}x$$
$$\le C\|n_2\|_3\|\delta u\|_6\|\nabla\delta n\|_2$$
$$\le C\|n_2\|_3^2\|\nabla\delta u\|_2^2 + \frac{1}{8}\|\nabla\delta n\|_2^2.$$

Collecting these estimates, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta n(t)\|_2^2 + \|\nabla\delta n(t)\|_2^2$$
$$\le C\bigg(\|\nabla c_2\|_2^{\frac{1}{2}}\left\|\nabla^3 c_2\right\|_2^{\frac{3}{2}}\|\delta n\|_2^2 + \big(\|n_1\|_3 + \|n_2\|_3\big)^2\|\delta n\|_2^2 + \|n_1\|_3^2\|\nabla^2\delta c\|_2^2 + \|n_2\|_3^2\|\nabla\delta u\|_2^2\bigg). \tag{3.3}$$

Next, we begin to show H^2-estimate for δc.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta c(t)\|_2^2 + \|\nabla\delta c(t)\|_2^2 = -\int_{\mathbb{R}^3} \delta n c_2 \delta c \,\mathrm{d}x - \int_{\mathbb{R}^3} n_1 \delta c \delta c \,\mathrm{d}x - \int_{\mathbb{R}^3} \delta u \cdot \nabla c_2 \delta c \,\mathrm{d}x. \tag{3.4}$$

By the Hölder inequality, one has

$$-\int_{\mathbb{R}^3} \delta n c_2 \delta c \,\mathrm{d}x \le \|c_2\|_\infty\|\delta n\|_2\|\delta c\|_2 \le C(\|\delta n\|_2^2 + \|\delta c\|_2^2). \tag{3.5}$$

In the same way as above, one can conclude that

$$-\int_{\mathbb{R}^3} n_1 \delta c \delta c \,\mathrm{d}x \le \|n_1\|_3\|\delta c\|_2\|\delta c\|_6 \le C\|n_1\|_3^2\|\delta c\|_2^2 + \frac{1}{4}\|\nabla\delta c\|_2^2.$$

Also

$$-\int_{\mathbb{R}^3} \delta u \cdot \nabla c_2 \delta c \,\mathrm{d}x = \int_{\mathbb{R}^3} c_2 \delta u \cdot \nabla\delta c \,\mathrm{d}x \le \|c_2\|_\infty\|\delta u\|_2\|\nabla\delta c\|_2$$
$$\le C\|c_2\|_\infty^2\|\delta u\|_2^2 + \frac{1}{4}\|\nabla\delta c\|_2^2$$

Therefore, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta c(t)\|_2^2 + \|\nabla\delta c(t)\|_2^2 \le \bigg(\|\delta n\|_2^2 + \|\delta c\|_2^2 + \|n_1\|_3^2\|\delta c\|_2^2 + \|c_2\|_\infty^2\|\delta u\|_2^2\bigg). \tag{3.6}$$

For H^2-estimate of δc.

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\delta c(t)\|_2^2+\|\nabla\Delta\delta c(t)\|_2^2=&-\int_{\mathbb{R}^3}\Delta\big(n_1\delta c\big)\Delta\delta c\,\mathrm{d}x-\int_{\mathbb{R}^3}\Delta\big(\delta n c_2\big)\Delta\delta c\,\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\Delta\big(\delta u\cdot\nabla c_2\big)\Delta\delta c\,\mathrm{d}x-\int_{\mathbb{R}^3}\Delta u_1\cdot\nabla\delta c\Delta\delta c\,\mathrm{d}x\\
&-2\int_{\mathbb{R}^3}\nabla u_1\cdot\nabla\nabla\delta c\Delta\delta c\,\mathrm{d}x.
\end{aligned}
$$

A simple calculation yields

$$
\begin{aligned}
-\int_{\mathbb{R}^3}\Delta\big(n_1\delta c\big)\Delta\delta c\,\mathrm{d}x=&-\int_{\mathbb{R}^3}n_1\Delta\delta c\Delta\delta c\,\mathrm{d}x-\int_{\mathbb{R}^3}\Delta n_1\delta c\Delta\delta c\,\mathrm{d}x\\
&-2\int_{\mathbb{R}^3}\nabla\delta c\nabla n_1\Delta\delta c\,\mathrm{d}x\\
=&-\int_{\mathbb{R}^3}n_1|\Delta\delta c|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla n_1\delta c\nabla\Delta\delta c\,\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\delta c\nabla n_1\Delta\delta c\,\mathrm{d}x.
\end{aligned}
$$

For the first term on the right hand-side of the above equality, we know

$$
\begin{aligned}
-\int_{\mathbb{R}^3}n_1|\Delta\delta c|^2\,\mathrm{d}x\le&\|n_1\|_3\|\Delta\delta c\|_2\|\Delta\delta c\|_6\\
\le&\|n_1\|_3^2\|\Delta\delta c\|_2^2+\frac{1}{4}\|\nabla\Delta\delta c\|_2^2
\end{aligned}
$$

By the Hölder inequality, one has

$$
\begin{aligned}
\int_{\mathbb{R}^3}\nabla n_1\delta c\nabla\Delta\delta c\,\mathrm{d}x\le&\|\nabla n_1\|_2\|\delta c\|_\infty\|\nabla\Delta\delta c\|_2\\
\le&C\|\nabla n_1\|_2^2\|\delta c\|_{H^2}^2+\frac{1}{4}\|\nabla\Delta\delta c\|_2^2.
\end{aligned}
$$

In the similar fashion, we get

$$
\begin{aligned}
-\int_{\mathbb{R}^3}\nabla\delta c\nabla n_1\Delta\delta c\,\mathrm{d}x\le&\|\nabla n_1\|_2\|\nabla\delta c\|_3\|\Delta\delta c\|_6\\
\le&C\|\nabla n_1\|_2^2\|\delta c\|_{H^2}^2+\frac{1}{4}\|\nabla\Delta\delta c\|_2^2.
\end{aligned}
$$

Integrating by parts,

$$
\begin{aligned}
-\int_{\mathbb{R}^3} \Delta\big(\delta n c_2\big)\Delta\delta c\,\mathrm{d}x &= \int_{\mathbb{R}^3} \nabla\big(\delta n c_2\big)\nabla\Delta\delta c\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \big(\nabla\delta n c_2\big)\nabla\Delta\delta c\,\mathrm{d}x + \int_{\mathbb{R}^3} \big(\delta n\nabla c_2\big)\nabla\Delta\delta c\,\mathrm{d}x.
\end{aligned} \tag{3.7}
$$

By the Hölder inequality, we easily find

$$
\begin{aligned}
\int_{\mathbb{R}^3} \big(\nabla\delta n c_2\big)\nabla\Delta\delta c\,\mathrm{d}x &\le \|c_2\|_\infty\|\nabla\delta n\|_2\|\nabla\Delta\delta c\|_2 \\
&\le \|c_2\|_\infty^2\|\nabla\delta n\|_2^2 + \frac{1}{4}\|\nabla\Delta\delta c\|_2^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} \big(\delta n\nabla c_2\big)\nabla\Delta\delta c\,\mathrm{d}x &\le \|\nabla c_2\|_\infty\|\delta n\|_2\|\nabla\Delta\delta c\|_2 \\
&\le \|c_2\|_{H^3}^2\|\delta n\|_2^2 + \frac{1}{4}\|\nabla\Delta\delta c\|_2^2.
\end{aligned} \tag{3.8}
$$

Integration by parts,

$$
\begin{aligned}
-\int_{\mathbb{R}^3} \Delta\big(\delta u\cdot\nabla c_2\big)\Delta\delta c\,\mathrm{d}x &= \int_{\mathbb{R}^3} \nabla\big(\delta u\cdot\nabla c_2\big)\nabla\Delta\delta c\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \big(\nabla\delta u\cdot\nabla c_2\big)\nabla\Delta\delta c\,\mathrm{d}x + \int_{\mathbb{R}^3} \big(\delta u\cdot\nabla\nabla c_2\big)\nabla\Delta\delta c\,\mathrm{d}x.
\end{aligned} \tag{3.9}
$$

By the Hölder inequality, we see that

$$
\begin{aligned}
\int_{\mathbb{R}^3} \big(\nabla\delta u\cdot\nabla c_2\big)\nabla\Delta\delta c\,\mathrm{d}x &\le \|\nabla\delta u\|_2\|\nabla c_2\|_\infty\|\nabla\Delta\delta c\|_2 \\
&\le \|c_2\|_{H^3}^2\|\nabla\delta u\|_2^2 + \frac{1}{4}\|\nabla\Delta\delta c\|_2^2.
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} \big(\delta u\cdot\nabla\nabla c_2\big)\nabla\Delta\delta c\,\mathrm{d}x &\le \|\delta u\|_6\|\nabla^2 c_2\|_3\|\nabla\Delta\delta c\|_2 \\
&\le \|c_2\|_{H^3}^2\|\nabla\delta u\|_2^2 + \frac{1}{4}\|\nabla\Delta\delta c\|_2^2.
\end{aligned}
$$

By the Hölder inequality, one has

$$-\int\limits_{\mathbb{R}^3} \Delta u_1 \cdot \nabla\delta c \Delta\delta c \,\mathrm{d}x \leq \|\Delta u_1\|_3 \|\nabla\delta c\|_6 \|\Delta\delta c\|_2$$
$$\leq C\|u_1\|_{H^3}^2 \|\Delta\delta c\|_2^2.$$

By the Young inequality, one has

$$-2\int\limits_{\mathbb{R}^3} \nabla u_1 \cdot \nabla\nabla\delta c \Delta\delta c \,\mathrm{d}x \leq 2\|\nabla u_1\|_3 \|\nabla^2\delta c\|_2 \|\Delta\delta c\|_6$$
$$\leq C\|u_1\|_{H^2}^2 \|\nabla^2\delta c\|_2^2 + \frac{1}{4}\|\nabla\Delta\delta c\|_2^2.$$

So, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\delta c(t)\|_2^2 + \|\nabla\Delta\delta c(t)\|_2^2 \\
\leq&(\|n_1\|_3^2 + \|u_1\|_{H^3}^2)\|\Delta\delta c\|_2^2 + \|c_2\|_{H^3}^2 \|\nabla\delta u\|_2^2 + \|c_2\|_{H^3}^2 \|\delta n\|_2^2 \\
&+ C\|\nabla n_1\|_2^2 \|\delta c\|_{H^2}^2 + C\|u_1\|_{H^2}^2 \|\nabla^2\delta c\|_2^2 + \|c_2\|_\infty^2 \|\nabla\delta n\|_2^2.
\end{aligned} \tag{3.10}$$

Lastly, H^1-estimate for δu. We start with L^2-estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u(t)\|_2^2 + \|\nabla\delta u(t)\|_2^2 = -\int\limits_{\mathbb{R}^3} \delta n \nabla\phi \delta u \,\mathrm{d}x - \int\limits_{\mathbb{R}^3} \delta u \cdot \nabla u_2 \delta u \,\mathrm{d}x.$$

By the Hölder inequality, we have

$$-\int\limits_{\mathbb{R}^3} \delta n \nabla\phi \delta u \,\mathrm{d}x \leq \|\nabla\phi\|_\infty \|\delta n\|_2 \|\delta u\|_2 \leq C(\|\delta n\|_2^2 + \|\delta u\|_2^2) \tag{3.11}$$

and

$$-\int\limits_{\mathbb{R}^3} \delta u \cdot \nabla u_2 \delta u \,\mathrm{d}x \leq \|\nabla u_2\|_3 \|\delta u\|_6 \|\delta u\|_2 \leq \|u_2\|_{H^2}^2 \|\delta u\|_2^2 + \frac{1}{4}\|\nabla\delta u\|_2^2 \tag{3.12}$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u(t)\|_2^2 + \|\nabla\delta u(t)\|_2^2 \leq C(\|u_2\|_{H^2}^2 \|\delta u\|_2^2 + \|\delta n\|_2^2 + \|\delta u\|_2^2). \tag{3.13}$$

Now we turn t show H^1-estimate. The standard argument enables us to infer that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\delta u(t)\|_2^2+\|\nabla^2\delta u(t)\|_2^2=-\int_{\mathbb{R}^3}\nabla\big(\delta n\nabla\phi\big)\nabla\delta u\,\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\big(\delta u\cdot\nabla u_2\big)\nabla\delta u\,\mathrm{d}x$$
$$-\int_{\mathbb{R}^3}\nabla\delta u\cdot\nabla u_2\nabla\delta u\,\mathrm{d}x.$$

Integrating by parts means,

$$-\int_{\mathbb{R}^3}\nabla\big(\delta n\nabla\phi\big)\nabla\delta u\,\mathrm{d}x=\int_{\mathbb{R}^3}\delta n\nabla\phi\nabla^2\delta u\,\mathrm{d}x\le\|\nabla\phi\|_\infty\|\delta n\|_2\|\nabla^2\delta u\|_2$$
$$\le\|\nabla\phi\|_\infty^2\|\delta n\|_2^2+\frac{1}{4}\|\nabla^2\delta u\|_2^2.$$

By the Hölder inequality, we obtain

$$-\int_{\mathbb{R}^3}\nabla\big(\delta u\cdot\nabla u_2\big)\nabla\delta u\,\mathrm{d}x=\int_{\mathbb{R}^3}\big(\delta u\cdot\nabla u_2\big)\nabla^2\delta u\,\mathrm{d}x\le C\|\nabla u_2\|_3\|\delta u\|_6\|\nabla^2\delta u\|_2$$
$$\le C\|u_2\|_{H^2}^2\|\nabla\delta u\|_2^2+\frac{1}{4}\|\nabla^2\delta u\|_2^2.$$

By the Hölder inequality and the Young inequality, one has

$$-\int_{\mathbb{R}^3}\nabla\delta u\cdot\nabla u_2\nabla\delta u\,\mathrm{d}x\le\|\nabla u_2\|_3\|\nabla\delta u\|_6\|\nabla\delta u\|_2$$
$$\le C\|u_2\|_{H^2}^2\|\nabla\delta u\|_2^2+\frac{1}{4}\|\nabla^2\delta u\|_2^2.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\delta u(t)\|_2^2+\|\nabla^2\delta u(t)\|_2^2\le C\|u_2\|_{H^2}^2\|\nabla\delta u\|_2^2+\|\nabla\phi\|_\infty^2\|\delta n\|_2^2. \tag{3.14}$$

Collecting estimates (3.3), (3.6), (3.10), (3.13), (3.14), we finally get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\delta n(t)\|_2^2+\|\delta c(t)\|_{H^2}^2+\big\|\delta u(t)\big\|_{H^1}^2\Big)\\&\le C(\|\delta n(t)\|_2^2+\|\delta c(t)\|_{H^2}^2+\big\|\delta u(t)\big\|_{H^1}^2)\Big(\|\nabla c_2\|_2^{\frac{1}{2}}\big\|\nabla^3c_2\big\|_2^{\frac{3}{2}}+\big(\|n_1\|_3+\|n_2\|_3\big)^2\\&\quad+\|u_2\|_{H^2}^2+\|u_1\|_{H^3}^2+\|c_2\|_{H^3}^2+\|\nabla n_1\|_2^2+1\Big).\end{aligned}\tag{3.15}$$

By the Gronwall inequality and the continuity argument, we eventually conclude that

$$(\delta n,\delta c,\delta u)\equiv(0,0,0)$$

for all $t>0$.

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Appendix A

In appendix, we shall prove the global well-posedness for (2.1) using energy estimates and classical compactness arguments.

Proposition A.1. *Let $\nabla\phi \in L^\infty(\mathbb{R}^3)$ and the initial data $(n_0, c_0, u_0) \in (H^s(\mathbb{R}^3))^3$ with $s > \frac{5}{2}$. Suppose that the initial data n_0 and c_0 are positive. Then the regularized system (2.1) admits a unique global solution $(n^N, c^N, u^N) \in \big(\mathcal{C}([0,\infty); H^s(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}([0,\infty); H^{s+1}(\mathbb{R}^3))\big)^3$. Moreover, $n^N(x,t) > 0$ and $c^N(x,t) > 0$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$.*

Proof. Let us denote

$$H^s_M(\mathbb{R}^3) \triangleq \left\{ f \in H^s(\mathbb{R}^3) \mid \|f\|_{H^s(\mathbb{R}^3)} \le M \right\}$$

for $s > \frac{3}{2}$. Our task in now is to establish the global-in-time solution to the following system in the space $H^s_M(\mathbb{R}^3)$

$$\begin{cases} \partial_t n^{N,\varepsilon} + \mathcal{K}_\varepsilon\left(\mathcal{K}_\varepsilon u^{N,\varepsilon}\cdot\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}\right) = \Delta\mathcal{K}_\varepsilon^2 n^{N,\varepsilon} - \mathcal{K}_\varepsilon\nabla\cdot(\mathcal{K}_\varepsilon n^{N,\varepsilon}\nabla\mathcal{K}_\varepsilon c^{N,\varepsilon}) - a\mathcal{K}_\varepsilon^2 n^{N,\varepsilon} \\ \qquad\qquad +(a+1)\mathcal{K}_\varepsilon(\mathcal{K}_\varepsilon n^{N,\varepsilon})^2 - \mathcal{K}_\varepsilon(\mathcal{K}_\varepsilon n^{N,\varepsilon})^3, \\ \partial_t c^{N,\varepsilon} + \mathcal{K}_\varepsilon\left(\mathcal{K}_\varepsilon u^{N,\varepsilon}\cdot\nabla\mathcal{K}_\varepsilon c^{N,\varepsilon}\right) = \Delta\mathcal{K}_\varepsilon^2 c^{N,\epsilon} - \mathcal{K}_\varepsilon(\mathcal{K}_\varepsilon c^{N,\varepsilon}\mathcal{K}_\varepsilon n^{N,\varepsilon}), \\ \partial_t u^{N,\varepsilon} + \mathcal{K}_\varepsilon\left(J_N u^{N,\varepsilon}\cdot\nabla\mathcal{K}_\varepsilon u^{N,\varepsilon}\right) + \nabla\mathcal{K}_\varepsilon P^{N,\varepsilon} = \Delta\mathcal{K}_\varepsilon^2 u^{N,\varepsilon} - J_N(\mathcal{K}_\varepsilon n^{N,\varepsilon}\nabla\phi), \\ \operatorname{div} u^{N,\varepsilon} = 0, \\ (n^{N,\varepsilon}, c^{N,\varepsilon}, u^{N,\varepsilon})|_{t=0} = J_N(n_0, c_0, u_0), \end{cases} \tag{A.1}$$

where $\mathcal{K}_\varepsilon f = \rho^\varepsilon * f$ with $\rho^\varepsilon = \frac{1}{\varepsilon^3}\rho(\frac{x}{\varepsilon})$. Using Leray's method and the incompressibility condition $\operatorname{div} u^{N,\varepsilon} = 0$, we eliminate the pressure $P^{N,\varepsilon}$ by projecting the third equation of the above system onto the space of divergence-free functions:

$$H^{s,\sigma}(\mathbb{R}^3) \triangleq \left\{ (n,c,u) \in (H^s(\mathbb{R}^3))^3 \mid \operatorname{div} u = 0 \right\}.$$

Then, problem (A.1) becomes to an ODE in the Banach space $H^{s,\sigma}_M$:

$$\frac{\mathrm{d}}{\mathrm{d}t} E^{N,\varepsilon}(x,t) = F_{N,\varepsilon}(E^{N,\varepsilon}), \quad E^{N,\varepsilon}(x,0) = E_0^{N,\varepsilon}(x) = \begin{pmatrix} J_N n_0 \\ J_N c_0 \\ J_N u_0 \end{pmatrix}, \tag{A.2}$$

where

$$F^1_{N,\varepsilon}(E^{N,\varepsilon}) = \Delta\mathcal{K}^2_\varepsilon n^{N,\varepsilon} - \mathcal{K}_\varepsilon\nabla\cdot(\mathcal{K}_\varepsilon n^{N,\varepsilon}\nabla\mathcal{K}_\varepsilon c^{N,\varepsilon}) - a\mathcal{K}^2_\varepsilon n^{N,\varepsilon}$$
$$+ (a+1)\mathcal{K}_\varepsilon(\mathcal{K}_\varepsilon n^{N,\varepsilon})^2 - \mathcal{K}_\varepsilon(\mathcal{K}_\varepsilon n^{N,\varepsilon})^3,$$
$$F^2_{N,\varepsilon}(E^{N,\varepsilon}) = \Delta\mathcal{K}^2_\varepsilon c^{N,\epsilon} - \mathcal{K}_\varepsilon(\mathcal{K}_\varepsilon c^{N,\varepsilon}\mathcal{K}_\varepsilon n^{N,\varepsilon}),$$
$$F^3_{N,\varepsilon}(E^{N,\varepsilon}) = -\mathcal{P}\mathcal{K}_\varepsilon\left(J_N u^{N,\varepsilon}\cdot\nabla\mathcal{K}_\varepsilon u^{N,\varepsilon}\right) + \Delta\mathcal{K}^2_\varepsilon u^{N,\varepsilon} - \mathcal{P}J_N(\mathcal{K}_\varepsilon n^{N,\varepsilon}\nabla\phi),$$

with $\mathcal{P}$ is Leray projector. And (A.1) and (A.2) are equivalent, see for example [15].

Step 1: For each $N \in \mathbb{N}^+$, we shall prove the existence and the uniqueness of solutions to problem (A.1).

Proposition A.2. *Let $\nabla\phi \in L^\infty(\mathbb{R}^3)$ and $(n_0, c_0, u_0) \in (H^s(\mathbb{R}^3))^3$ with $s > 3/2$. Then, for each $N \in \mathbb{N}^+$, system (A.1) has a unique global smooth solution*

$$\left(n^{N,\varepsilon}, c^{N,\varepsilon}, u^{N,\varepsilon}\right)(t) \in \left(\mathcal{C}(\mathbb{R}^+;\ H^s(\mathbb{R}^3))\right)^3.$$

Proof of Proposition A.2. For every $N \geq 1$, it is easy to check that

$$\begin{aligned}
&\|F^1_{N,\varepsilon}(E^{N,\varepsilon}_1(t)) - F^1_{N,\varepsilon}(E^{N,\varepsilon}_2(t))\|_{H^s} \\
&\quad \leq C\big(N, \varepsilon, \|E^{N,\varepsilon}_1(t)\|_2, \|E^{N,\varepsilon}_2(t)\|_2\big)\|E^{N,\varepsilon}_1(t) - E^{N,\varepsilon}_2(t)\|_{H^s}, \\
&\|F^2_{N,\varepsilon}(E^{N,\varepsilon}_1(t)) - F^2_{N,\varepsilon}(E^{N,\varepsilon}_2(t))\|_{H^s} \\
&\quad \leq C\big(N, \varepsilon, \|E^{N,\varepsilon}_1(t)\|_2, \|E^{N,\varepsilon}_2(t)\|_2\big)\|E^{N,\varepsilon}_1(t) - E^{N,\varepsilon}_2(t)\|_{H^s},
\end{aligned}$$

and

$$\begin{aligned}
&\|F^3_{N,\varepsilon}(E^{N,\varepsilon}_1(t)) - F^3_{N,\varepsilon}(E^{N,\varepsilon}_2(t))\|_{H^s} \\
\leq & C\big(N, \varepsilon, \|E^{N,\varepsilon}_1(t)\|_2, \|E^{N,\varepsilon}_2(t)\|_2, \|\nabla\phi\|_\infty\big)\|E^{N,\varepsilon}_1(t) - E^{N,\varepsilon}_2(t)\|_{H^s}.
\end{aligned}$$

This means that $F_{N,\epsilon}$ maps $H^s(\mathbb{R}^3)$ into $H^s(\mathbb{R}^3)$ and $F_{N,\epsilon}$ is locally Lipschitz continuous on $H^s(\mathbb{R}^3)$. Hence the Cauchy-Lipschitz theorem ensures that for every $(n_0, c_0, u_0) \in (H^s(\mathbb{R}^3))^3$, there exists a unique solution $(n^{N,\varepsilon}, c^{N,\varepsilon}, u^{N,\varepsilon}) \in \left(\mathcal{C}^1([0, T_N), H^s_N(\mathbb{R}^3))\right)^3$ with $T_N > 0$ is the maximal existence time.

According to heat equation theory, we can show the time differentiation. For example, we consider the ihomogeneous heat equations: $u_t - \Delta u = f(x,t)$, which can rewrite as

$$u_t = \Delta u - f(x,t).$$

Assume $u \in L^\infty_t H^s(\mathbb{R}^3)$ and $f \in L^\infty_t H^{s-2}(\mathbb{R}^3)$, we have $u_t \in L^\infty_t H^{s-2}(\mathbb{R}^3)$. For the arbitrariness of s, we obtain

$$u_t \in L^\infty_t H^s(\mathbb{R}^3) \quad \text{for each } s > 0.$$

In a similar process, we conclude

$$(\partial_t - \Delta)u_t = f_t,$$

and then we have $u_{tt} \in L_t^\infty H^{s-2}(\mathbb{R}^3)$. Thus we prove the time differentiation.

Now, we show that $T_N = \infty$. Since $J_N n_0 > 0$, we deduce that $n^{N,\varepsilon}(t) > 0$ for all $t \in [0, T_N)$, by utilizing the first equation of equations (A.1). Taking the L^2-inner product with the second equation of (A.1) by $c^{N,\varepsilon}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c^{N,\varepsilon}(t)\|_2^2 + \|\nabla\mathcal{K}_\varepsilon c^{N,\varepsilon}(t)\|_2^2 = -\int_{\mathbb{R}^3} \mathcal{K}_\varepsilon(\mathcal{K}_\varepsilon c^{N,\varepsilon}\mathcal{K}_\varepsilon n^{N,\varepsilon})c^{N,\varepsilon}\,\mathrm{d}x \le 0,$$

which gives that

$$\|c^{N,\varepsilon}(t)\|_2^2 + 2\int_0^t \|\nabla\mathcal{K}_\varepsilon c^{N,\varepsilon}(\tau)\|_2^2\,\mathrm{d}\tau \le \left\|c_0^{N,\varepsilon}\right\|_2^2. \tag{A.3}$$

Also we have

$$\|c^{N,\varepsilon}(t)\|_p \le \left\|c_0\right\|_p \quad \text{for each } p \in [2, \infty].$$

Since $\mathcal{D}(\mathbb{R}^3)$ is dense in $H^\ell(\mathbb{R}^3)$ for all $\ell > 1$, there are no boundary terms when we integrate by parts in the following calculation. In the same process, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n^{N,\varepsilon}(t)\|_2^2 + a\left\|\mathcal{K}_\varepsilon n^{N,\varepsilon}(t)\right\|_2^2 + \left\|\mathcal{K}_\varepsilon n^{N,\varepsilon}(t)\right\|_4^4 + \left\|\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}(t)\right\|_2^2\\
=&\int_{\mathbb{R}^3} \mathcal{K}_\varepsilon n^{N,\varepsilon}\nabla\mathcal{K}_\varepsilon c^{N,\varepsilon}\cdot\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}\,\mathrm{d}x + (1+a)\int_{\mathbb{R}^3}(\mathcal{K}_\varepsilon n^{N,\varepsilon})^3\mathrm{d}x\\
\le& C(\varepsilon)\|n^{N,\varepsilon}\|_2^2\|c^{N,\varepsilon}\|_2^2 + \frac{1}{4}\|\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}\|_2^2 + C\|n^{N,\varepsilon}\|_2^2 + \frac{1}{2}\|\mathcal{K}_\varepsilon n^{N,\varepsilon}\|_4^4
\end{aligned}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^{N,\varepsilon}(t)\|_2^2 + \left\|\nabla\mathcal{K}_\varepsilon u^{N,\varepsilon}(t)\right\|_2^2 \le C\|\nabla\phi\|_\infty\|n^{N,\varepsilon}\|_2\|u^{N,\varepsilon}\|_2.$$

We deduce by using Gronwall's inequality that

$$\|n^{N,\varepsilon}(t)\|_2^2 + 2\int_0^t \|\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}(\tau)\|_2^2\,\mathrm{d}\tau \le \|n_0^{N,\varepsilon}\|_2^2 e^{C(\varepsilon)t} \tag{A.4}$$

and

$$\|u^{N,\varepsilon}(t)\|_2^2 + 2\int_0^t \|\nabla\mathcal{K}_\varepsilon u^{N,\varepsilon}(\tau)\|_2^2\,\mathrm{d}\tau \le \|u_0^{N,\varepsilon}\|_2^2 e^{C(\varepsilon)t}. \tag{A.5}$$

By the same argument as used in (A.3)-(A.5), we can show that for $s > \frac{3}{2}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|E^{N,\varepsilon}(t)\|_{H^s} \le C\left(t, N, \varepsilon, \|n_0\|_2, \|c_0\|_2, \|u_0\|_2, \|\nabla\phi\|_\infty\right)\|E^{N,\varepsilon}(t)\|_{H^s}.$$

Then we conclude by Gronwall's inequality that

$$\|E^{N,\varepsilon}(t)\|_{H^s} \le C\big(t, N, \varepsilon, E(0)\big) \quad \text{for all } t \in [0, T_N).$$

Here and what in follows, $E(0) = (n_0, c_0, u_0)$. Therefore, solutions can be continued for all time by invoking the continuation property of ODEs on a Banach space. □

Step 2: In this step, we give the uniform boundedness for solution $(n^{N,\varepsilon}, c^{N,\varepsilon}, u^{N,\varepsilon})$ (independent of ε).

First we show the L^2-estimate of solution $(n^{N,\varepsilon}, c^{N,\varepsilon}, u^{N,\varepsilon})$. As in proving (A.3)-(A.5), we get

$$\|E^{N,\varepsilon}(t)\|_2^2 + \int_0^t \|\nabla\mathcal{K}_\varepsilon E^{N,\varepsilon}(\tau)\|_2^2\,\mathrm{d}\tau \le C\big(t, \|E(0)\|_2\big). \tag{A.6}$$

Now we turn to show H^1-norm of solutions. Thanks to estimate (A.6), we easily get that the triple $(J_N n^{N,\varepsilon}, J_N c^{N,\varepsilon}, J_N u^{N,\varepsilon})$ belong to $\big(\mathcal{C}^1_{\mathrm{loc}}([0,\infty), H^\ell(\mathbb{R}^3))\big)^3$ for all $\ell > 1$. Hence, we have enough regularity to justify all calculations below. First, performing $H^s(\mathbb{R}^3)$ to $u^{N,\varepsilon}$, we can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^{N,\varepsilon}(t)\|_{\dot{H}^s}^2 + \|\mathcal{K}_\varepsilon u^{N,\varepsilon}(t)\|_{\dot{H}^{s+1}}^2 \le C(N)\|u^{N,\varepsilon}\|_2\|u^{N,\varepsilon}\|_{H^s}^2 + C(N)\|n^{N,\varepsilon}\|_2\|u^{N,\varepsilon}\|_{H^s}.$$

From this, it follows that

$$\|u^{N,\varepsilon}(t)\|_{\dot{H}^s}^2 + \int_0^t \|\mathcal{K}_\varepsilon u^{N,\varepsilon}(\tau)\|_{\dot{H}^{s+1}}^2\,\mathrm{d}\tau \le C\left(t, N, E_0\right). \tag{A.7}$$

Multiplying the second equation of (A.1) by $-\Delta c^{N,\varepsilon}$ and integrating in space variable x yield that for $s > \frac{3}{2}$,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c^{N,\varepsilon}(t)\|_2^2 + \|\Delta\mathcal{K}_\varepsilon c^{N,\varepsilon}(t)\|_2^2\\
\le{}& C\|\mathcal{K}_\varepsilon u^{N,\varepsilon}\|_{H^s}\|\nabla c^{N,\varepsilon}\|_2^2 + C\|n^{N,\varepsilon}\|_\infty^2\|c^{N,\varepsilon}\|_{H^{s+1}}^2 + \frac{1}{4}\|\Delta\mathcal{K}_\varepsilon c^{N,\varepsilon}\|_2^2.
\end{aligned}$$

Then we have

$$\|\nabla c^{N,\varepsilon}(t)\|_2^2 + \int_0^t \|\Delta\mathcal{K}_\varepsilon c^{N,\varepsilon}(\tau)\|_2^2\,\mathrm{d}\tau \le C\left(t, N, E_0\right). \tag{A.8}$$

Similarly, we can show that for $s > \frac{5}{2}$,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta c^{N,\varepsilon}(t)\|_2^2+\|\nabla\Delta\mathcal{K}_\varepsilon c^{N,\varepsilon}(t)\|_2^2\\
\leq& C\big(\|\mathcal{K}_\varepsilon u^{N,\varepsilon}\|_{H^s}^2\|\nabla c^{N,\varepsilon}\|_2^2+\|n^{N,\varepsilon}\|_3^2\|c^{N,\varepsilon}\|_{H^2}^2+\|\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}\|_2^2\|c^{N,\varepsilon}\|_\infty^2\big)+\frac{1}{4}\|\nabla\Delta\mathcal{K}_\varepsilon c^{N,\varepsilon}\|_2^2.
\end{aligned}
$$

Moreover, we have

$$
\|\Delta c^{N,\varepsilon}(t)\|_2^2+\int_0^t\|\nabla\Delta\mathcal{K}_\varepsilon c^{N,\varepsilon}(\tau)\|_2^2\,\mathrm{d}\tau\leq C\,(t,N,E_0)\,.
$$

Multiplying the first equation of (A.1) by $-\Delta n^{N,\varepsilon}$ and integrating in x yield

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n^{N,\varepsilon}(t)\|_2^2+\|\Delta\mathcal{K}_\varepsilon n^{N,\varepsilon}(t)\|_2^2+a\|\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}\|_2^2+3\big\|\mathcal{K}_\varepsilon n^{N,\varepsilon}\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}\big\|_2^2\\
=&-\int_{\mathbb{R}^3}(\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon})\cdot\nabla\mathcal{K}_\varepsilon u^{N,\varepsilon}\cdot\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\left(\mathcal{K}_\varepsilon n^{N,\varepsilon}\nabla\mathcal{K}_\varepsilon c^{N,\varepsilon}\right)\Delta\mathcal{K}_\varepsilon n^{N,\varepsilon}\,\mathrm{d}x\\
&+2(1+a)\int_{\mathbb{R}^3}\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}\cdot(\mathcal{K}_\varepsilon n^{N,\varepsilon}\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon})\,\mathrm{d}x\\
\leq& C\|\nabla\mathcal{K}_\varepsilon u^{N,\varepsilon}\|_3^2\|\nabla n^{N,\varepsilon}\|_2^2+C\left(\|\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}\|_2^2\|\nabla\mathcal{K}_\varepsilon c^{N,\varepsilon}\|_\infty^2+\|\nabla n^{N,\varepsilon}\|_2^2\|\Delta\mathcal{K}_\varepsilon c^{N,\varepsilon}\|_3^2\right)\\
&+C\|\nabla n^{N,\varepsilon}\|_2^2+\frac{1}{2}\|\mathcal{K}_\varepsilon n^{N,\varepsilon}\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}\|_2^2+\frac{1}{4}\|\Delta\mathcal{K}_\varepsilon n^{N,\varepsilon}\|_2^2.
\end{aligned}
$$

Thus, we obtain

$$
\|\nabla n^{N,\varepsilon}(t)\|_2^2+\int_0^t\|\Delta\mathcal{K}_\varepsilon n^{N,\varepsilon}(\tau)\|_2^2\mathrm{d}\tau\leq C\,(t,N,E_0)\,.
$$

Lastly, we shall give H^s-norm for $n^{N,\varepsilon}$ and $c^{N,\varepsilon}$ by the theory for heat equation. Firstly, we have that for $s>\frac{5}{2}$,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c^{N,\varepsilon}(t)\|_{H^s}^2+\|\nabla\mathcal{K}_\varepsilon c^{N,\varepsilon}(t)\|_{H^s}^2\\
\leq& C\|u^{N,\varepsilon}\|_{H^s}^2\|c^{N,\varepsilon}\|_{H^s}^2+C\|n^{N,\varepsilon}\|_\infty\|c^{N,\varepsilon}\|_{H^s}\|n^{N,\varepsilon}\|_{H^s}+C\|c^{N,\varepsilon}\|_\infty\|n^{N,\varepsilon}\|_{H^s}^2\\
&+\frac{1}{4}\|\mathcal{K}_\varepsilon c^{N,\varepsilon}\|_{H^{s+1}}^2.
\end{aligned}
\tag{A.9}
$$

Similarly,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n^{N,\varepsilon}(t)\|_{H^s}^2+\|\nabla\mathcal{K}_\varepsilon n^{N,\varepsilon}(t)\|_{H^s}^2+\|\mathcal{K}_\varepsilon n^{N,\varepsilon}\|_{H^s}^2\\
\leq & C\|u^{N,\varepsilon}\|_{H^s}^2\|n^{N,\varepsilon}\|_{H^s}^2+C\|n^{N,\varepsilon}\|_\infty\|\mathcal{K}_\varepsilon n^{N,\varepsilon}\|_{H^{s+1}}^2+C\|\nabla\mathcal{K}_\varepsilon c^{N,\varepsilon}\|_\infty\|n^{N,\varepsilon}\|_{H^s}^2\\
&+\frac{1}{4}\|\mathcal{K}_\varepsilon n^{N,\varepsilon}\|_{H^{s+1}}^2.
\end{aligned}$$

This estimate together with (A.7) and (A.9) yields

$$\|E^{N,\varepsilon}(t)\|_{H^s}^2+\int_0^t\|\mathcal{K}_\varepsilon E^{N,\varepsilon}(\tau)\|_{H^{s+1}}^2\,\mathrm{d}\tau\leq C(t,N,E_0). \tag{A.10}$$

Step 3: This step is to prove that $(n^{N,\varepsilon},\ c^{N,\varepsilon},\ u^{N,\varepsilon})$ converge to a limit $(n^N,\ c^N,\ u^N)$ satisfying system (2.1) in the sense of distribution.

Estimate (A.10) ensures that the family $(\partial_t n^{N,\varepsilon},\ \partial_t c^{N,\varepsilon},\ \partial_t u^{N,\varepsilon})\in(L^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{R}^3)))^3$. Let $\{\chi_l\}_{l\in\mathbb{N}}$ be a sequence of $C_0^\infty(\mathbb{R}^3)$ cut-off functions supported in the ball $B(0,l+1)$ of $\mathbb{R}^3$ and equal to 1 in a neighborhood of $B(0,l)$, namely

$$\chi_l=\begin{cases}1, & \text{in}\quad B(0,l),\\ 0, & \text{in}\quad B^c(0,l+1).\end{cases}$$

Thus we have $\|\chi_l\|_{L^\infty}\leq 1$ and $\|\chi_l\|_{W^{s,\infty}}\leq C$ with C is independent of l.

By the Leibinz estimate, we get that, for any $l\in\mathbb{N}$,

$$\|\chi_l n^{N,\varepsilon}\|_{H^{s-1}}\leq\|\chi_l\|_{L^\infty}\|n^{N,\varepsilon}\|_{H^{s-1}}+\|\chi_l\|_{W^{s-1,\infty}}\|n^{N,\varepsilon}\|_{L^2}\leq C(\|n^{N,\varepsilon}\|_{H^{s-1}}+\|n^{N,\varepsilon}\|_{L^2}),$$

where C is independent of l, which implies that $\{(\chi_l n^{N,\varepsilon},\chi_l c^{N,\varepsilon},\chi_l u^{N,\varepsilon})\}$ is uniformly bounded.

On the other hand,

$$\|\chi_l n^{N,\varepsilon}(s)-\chi_l n^{N,\varepsilon}(t)\|_{H^{s-1}}\leq(t-s)^{\frac{1}{2}}\left(\int_s^t\|\chi_l\partial_t n^{N,\varepsilon}(\tau)\|_{H^{s-1}}^2\mathrm{d}\tau\right)^{\frac{1}{2}},$$

from which it follows that $\{(\chi_l n^{N,\varepsilon},\chi_l c^{N,\varepsilon},\chi_l u^{N,\varepsilon})\}$ is equicontinuous in $(\mathcal{C}(\mathbb{R}^+,H^{s-1}(\mathbb{R}^3)))^3$. Notice that the application $n^{N,\varepsilon}\mapsto\chi_l n^{N,\varepsilon}$ is compact from H^s into $H^{s'}$ as $s>s'$. We use Lions-Aubin lemma to the family $\big\{(\chi_l n^{N,\varepsilon},\chi_l c^{N,\varepsilon},\chi_l u^{N,\varepsilon})\big\}$ on the time internal $[0,l]$, and then use Cantor's diagonal process. This finally reduces a distribution (n^N,c^N,u^N) belonging to $\big(\mathcal{C}(\mathbb{R}^+;H^{s'}(\mathbb{R}^3))\big)^3$ and a subsequence (which we still denote by $\big\{(n^{N,\varepsilon},c^{N,\varepsilon},u^{N,\varepsilon})\big\}$) such that, for all $l\in\mathbb{N}$, we get

$$(\chi_l n^{N,\varepsilon},\chi_l c^{N,\varepsilon},\chi_l u^{N,\varepsilon})\xrightarrow{\varepsilon\to 0}(\chi_l n^N,\chi_l c^N,\chi_l u^N)\quad\text{in}\quad\big(\mathcal{C}([0,l];H^{s'}(\mathbb{R}^3))\big)^3.$$

This obviously means that $(n^{N,\varepsilon},c^{N,\varepsilon},u^{N,\varepsilon})$ converges to (n^N,c^N,u^N) in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^3)$. Fatou lemma entails that

$$(n^N,c^N,u^N)\in\big(L^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s)\cap L^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s+1}(\mathbb{R}^3))\big)^3.$$

Next, we show that (n^N, c^N, u^N) is continuous in $H^s \times H^s \times H^s$. By Littlewood-Paley, we can show that

$$\Big(\sum_{q\geq 0} 2^{2qs}\|\Delta_q n^N\|^2_{L^\infty_t L^2}\Big)^{1/2} \leq \|n_0\|_{H^s} + C\|u^N\|_{L^2_t H^s}\|n^N\|_{L^2_t H^s} + C\|n^N\|_{L^2_t H^s}\|c^N\|_{L^2_t L^2} + C\|n^N\|^2_{L^2_t H^s} + C\|n^N\|_{L^\infty_t H^s}\|n^\varepsilon\|^2_{L^2_t H^s} < \infty.$$

Combining the above estimate with the fact $n^N(t) \in L^\infty_t H^s$ implies

$$\Big(\sum_{q\geq -1} 2^{2qs}\|\Delta_q n^N\|^2_{L^\infty_t L^2}\Big)^{1/2} < \infty.$$

Then, the sequence $\{S_M n^N\}_{M\in\mathbb{Z}^+}$ tends uniformly to n^N in $L^\infty_t H^s$ with S_M is the low-frequency cut-off operator. Besides, the fact $\partial_t n^N \in L^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s-1}(\mathbb{R}^3))$ gives rise to $n^N \in \mathcal{C}(\mathbb{R}^+; H^{s'}(\mathbb{R}^3))$ with $s' < s$. This reduces that $S_M n^N \in \mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^3))$ for a fixed $M \in \mathbb{Z}^+$. Thus, we have $n^N \in \mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^3))$. In the same process, we can obtain that $c^N \in \mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^3))$ and $u^N \in \mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^3))$.

Step 4: We show the uniqueness of solutions. Suppose that (n_i^N, c_i^N, u_i^N), $i = 1, 2$, are two solutions of the system (2.1) with the same initial data. Then the difference $(\delta n^N, \delta c^N, \delta u^N)$ satisfies

$$\begin{cases} \partial_t \delta n^N + u_1^N \cdot \nabla \delta n^\varepsilon - \Delta \delta n^N = -\nabla\cdot(n_1^N \nabla \delta c^N) - \nabla\cdot(\delta n^N \nabla c_2^N) \\ \qquad\qquad -\delta u^N \cdot \nabla n_2^N + f(n_1^N) - f(n_2^N), \\ \partial_t \delta c^N + u_1^N \cdot \nabla \delta c^N - \Delta \delta c^N = -\delta n^N c_2^N - n_1^N \delta c^N - \delta u^N \cdot \nabla c_2^N, \\ \partial_t \delta u^N + (J_N u_1^N)\cdot\nabla \delta u^N - \Delta \delta u^N + \nabla \delta p^N = -J_N(\nabla\phi\,\delta n^N) - (J_N \delta u^N)\cdot \nabla u_2^N, \\ (\delta^N n, \delta^N c, \delta^N u)|_{t=0} = (0,0,0), \end{cases} \tag{A.11}$$

where $f(n^N) = (a+1)(n^N)^2 - a n^N - (n^N)^3$.

Making the L^2-inner product with the first equation of the above system with δn^N, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta n^N\|_2^2 + \|\nabla \delta n^\varepsilon\|_2^2 + a\|\delta n^\varepsilon\|_2^2 + \int_{\mathbb{R}^3}\big((n_1^N)^2 + n_1^N n_2^N + (n_2^N)^2\big)(\delta n^N)^2\,\mathrm{d}x \\ \leq& C\|(n^N, c^N, u^N)\|^2_{H^s}\|(\delta n^N, \delta c^N, \delta u^N)\|^2_{H^s} + \frac{1}{8}\|\nabla \delta n^N\|_2^2, \end{aligned}$$

where

$$\|(n^N, c^N, u^N)\|_{H^s} = \|(n_1^N, c_1^N, u_1^N)\|^2_{H^s} + \|(n_2^N, c_2^N, u_2^N)\|_{H^s}.$$

In a similar way, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta c^N\|_2^2 + \|\nabla \delta c^N\|_2^2 \leq C\|(n^N, c^N, u^N)\|^2_{H^s}\|(\delta n^N, \delta c^N, \delta u^N)\|^2_{H^s} + \|\delta n^N\|_2^2 + \frac{1}{8}\|\nabla \delta c^N\|_2^2,$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u^N\|_2^2+\|\nabla\delta u^N\|_2^2\le C\|(n^N,c^N,u^N)\|_{H^s}^2\|(\delta n^N,\delta c^N,\delta u^N)\|_{H^s}^2+\frac{1}{8}\|\nabla\delta u^\varepsilon\|_2^2.$$

Summing up the above estimates gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\delta n^N,\delta c^N,\delta u^N)\|_{H^s}^2\le C\|(n^N,c^N,u^N)\|_{H^s}^2\|(\delta n^N,\delta c^N,\delta u^N)\|_{H^s}^2.$$

The Gronwall inequality implies $n_1^N\equiv n_2^N,\ c_1^N\equiv c_2^N,\ u_1^N\equiv u_2^N$ on time interval $[0,T]$.

Next we show the positivity of $n^N>0$ and $c^N>0$ for all $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$. Let us denote

$$(n^N)^-\triangleq\min\{n^N,0\}.$$

Multiplying the first equation of (2.1) by $(n^N)^-$ and integrating in space variable x, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(n^N)^-\|_{L^2}^2+\|\nabla(n^N)^-\|_{L^2}^2=&-\int_{\mathbb{R}^3}\nabla\cdot(n^N\nabla c^N)(n^N)^-\mathrm{d}x+(a+1)\int_{\mathbb{R}^3}(n^N)^2(n^N)^-\mathrm{d}x\\&-a\int_{\mathbb{R}^3}n^N(n^N)^-\mathrm{d}x-\int_{\mathbb{R}^3}(n^N)^3(n^N)^-\mathrm{d}x.\end{aligned}$$

Integrating by parts and utilizing Höder's inequality imply

$$-\int_{\mathbb{R}^3}\nabla\cdot(n^N\nabla c^N)(n^N)^-\mathrm{d}x\le C\|\nabla c^N\|_\infty^2\|(n^N)^-\|_2^2+\frac{1}{4}\|\nabla(n^N)^-\|_2^2.$$

On the other hand, we get

$$(a+1)\int_{\mathbb{R}^3}(n^N)^2(n^N)^-\mathrm{d}x\le C\|n^N\|_\infty\|(n^N)^-\|_2^2,\quad -a\int_{\mathbb{R}^3}n^N(n^N)^-\mathrm{d}x\le C\|(n^\varepsilon)^-\|_2^2$$

and

$$-\int_{\mathbb{R}^3}(n^N)^3(n^N)^-\mathrm{d}x\le\|n^N\|_\infty^2\|(n^N)^-\|_2^2.$$

Collecting the above inequalities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(n^N)^-\|_{L^2}^2+\|\nabla(n^N)^-\|_{L^2}^2\le C(t,N,E_0)\|(n^N)^-\|_{L^2}^2.$$

Using Gronwall's inequality, we have for any $t>0$

$$\|(n^N)^-(t)\|_{L^2}\le\|(n_0^N)^-\|_{L^2}C(t)=0,$$

which gives that $n^\varepsilon \geq 0$ for almost everywhere $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$. Since $s > 1$, we infer

$$H^1(\mathbb{R}^3) \hookrightarrow \mathcal{C}_b(\mathbb{R}^3).$$

Thus we conclude that $n^N \geq 0$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$. By the same process, we can obtain the positivity of $c^N > 0$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$. Therefore, we complete the proof. □

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