Applied Mathematical Modelling 89 (2021) 1663–1677

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Bifurcation and turing instability for a predator-prey model with nonlinear reaction cross-diffusion☆

Jianzhi Cao*, Hongyan Sun, Pengmiao Hao, Peiguang Wang

College of Mathematics and Information Sciences, Key Laboratory of Machine Learning and Computational Intelligence, Hebei University, Baoding 071002, PR China

ARTICLE INFO

Article history:
Received 24 September 2019
Revised 6 August 2020
Accepted 9 August 2020
Available online 3 September 2020

Keywords:
Nonlinear cross-diffusion
Lyapunov–Schmidt reduction
Bifurcation
Turing stability

ABSTRACT

A nonlinear reaction cross-diffusion predator-prey system under Neumann boundary condition is considered. Negative diffusion coefficients with local accumulation effect of prey are introduced. Firstly, the criteria for local asymptotic stability of the positive homogeneous steady state with or without cross-diffusion are discussed. Moreover, the conditions for diffusion-driven instability are obtained and the Turing regions in the plane of cross-diffusion coefficients is achieved. Secondly, the existence and multiplicity of spatially nonhomogeneous/homogeneous steady-state solutions are studied by virtue of the Lyapunov–Schmidt reduction. Finally, to clarify the theoretical results, some numerical simulations are carried out. One of the most interesting finding is that Turing instability in the model is induced by the negative diffusion coefficients.

1. Introduction

In this paper, we consider the following predator-prey model with nonlinear cross-diffusion:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta[(1+\alpha u_2)u_1] + u_1\left(a - u_1 - \dfrac{cu_2}{1+mu_1}\right) \quad x \in \Omega, \\ \dfrac{\partial u_2}{\partial t} = \Delta[\left(\mu + \dfrac{1}{1+\beta u_1}\right)u_2] + u_2\left(b - u_2 + \dfrac{du_1}{1+mu_1}\right) \quad x \in \Omega, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0 \quad x \in \partial\Omega, \\ u_i(x,\ 0) = u_{i0}(x) \geq 0,\ x \in \Omega,\ i = 1,\ 2, \end{cases} \tag{1.1}$$

where $u_1(x,\ t)$ and $u_2(x,\ t)$ are representative of the predator density and the prey density at time t and space location x respectively; Δ denotes the Laplacian operator; $\Omega \in R^N$ ($N \geq 1$ is an integer) represents a bounded domain; a, c, d, m, μ, are positive constants and $b,\ \alpha,\ \beta \in R$. The nonlinear cross-diffusion term $\alpha\Delta u_2 u_1$ can imply different biological significance: $\alpha > 0$ ($\alpha < 0$), it depicts a tendency that prey species is far away from (resp., close to) high-density areas of predator

☆ This work was supported by the National Natural Science Foundation of China (No: 11771115, 11801128), Natural Science Foundation of Hebei Province (No: A2016201206, A2019201396), Research funding for high-level innovative talents of hebei university (No: 801260201242) and Post-graduate's Innovation Fund Project of Hebei University (No: hbu2019ss028).

* Corresponding author.
E-mail address: jzcao@hbu.edu.cn (J. Cao).

https://doi.org/10.1016/j.apm.2020.08.030
0307-904X/

species; and $\Delta\left(\frac{u_2}{1+\beta u_1}\right)$ ($\beta \geq 0$) means that the diffusion rate and the population pressure of predator species may weaken in the high density location of prey species. We refer to [1] for more background about (1.1).

Some prey populations are very clever in the natural environment. When predators flock, they do not flee but gather together in unity to form their own protective layer and close to the predator group to break it down one by one, such as Asian bumblebees [2] and Asian bees [3], especially Japanese bumblebees [4] and Japanese bees. A very large number of Japanese bees will not directly fight back but use their unique skill–kill bumblebee with a 'hot ball' when face the Bumblebee's prey. They fly into groups of bumblebees and about every 500 bees surround a bumblebee, forming one tight and hot ball after another. The bees continue to vibrate their flying muscles to emit heat. The temperature in the center of these spheres rises to 47°C in a few minutes just beyond the limit of the Bumblebee's ability to withstand heat, but not reach the upper limit of the bee. In about 20 min, Bumblebee is killed by the heat. This process has the following two phenomena: the prey population is close to the high-density area of the predator population, that implies, α may be less than 0; the rate of diffusive spread of predator populations increases in the process of forming spheres and the pressure on the predator population enlarges in the center of the sphere, which means that α and β can be smaller than 0.

Recently, much attention has been focused on Turing instability of predator-prey model by taking into account the effect cross-diffusion [5–10]. For example, in [6], Xie introduced the cross-diffusion terms in a three species food chain model, he proved that the positive equilibrium is globally asymptotically stable for the system without diffusion by Lyapunov function, and the Turing instability is driven solely form the effect of cross-diffusion. Guin [7] investigated a mathematical model of predator-prey interaction subject to self and cross-diffusion and found that the effects of self-diffusion as well as cross-diffusion play important roles in the stationary pattern formation of the model which concerns the influence of intra-species competition among. Liu et al.[9] studied the Turing instability and pattern formation in a super cross-diffusion predator-prey system with Michaelis–Menten type predator harvesting, and used some numerical simulations verify the theoretical results. From these research results, we note that the cross-diffusion could play an important role in pattern formation. However, a model very similar to (1.1) is investigated in [10], and a surprising result is that the diffusion and cross-diffusion with $\alpha > 0$ and $\beta > 0$ can not drive Turing instability.

Based on the above discussion, to our best knowledge, there are still few works devoted to take the case of Eq. (1.1) with $\alpha < 0$ or $\beta < 0$ and Neumann boundary condition into account. In particular, the existence of Turing instability and steady-state solution bifurcation plays an important role in the formation of pattern. This conclusion will be a welcoming addition to the literature. This fact motivates our work for the manuscript. The remaining part is organized as follows. Section 2 is devote to discussing the local stability of equilibria and Turing instability of the positive steady-state solutions. In Section 3, by using Lyapunov–Schmidt reduction [11], some sufficient conditions for the existence and multiplicity of non-constant stationary solutions near the trivial and non-trivial equilibrium were established. In Section 4, with the help of software Matlab-2012, some numerical simulations are executed to illustrate the main results under one-dimensional spatial domain $\Omega = (0,\ \pi)$. The paper ends with the discussion and conclusion.

2. Turing instability of the positive stationary solution

In this section, we mainly establish some criteria on the linear stability or instability of positive steady state u^* for ODEs and PDEs respectively. As is known to all, Eqs. (1.1) corresponding ODEs system is the classical predator-prey model with Holling type II functional response and has been studied extensively (see[12] and references therein). To well understand behaviors of the system (1.1), we firstly find the steady state of the spatially homogeneous system as follows.

Proposition 2.1. *For system (1.1) without diffusion, we have*

- *the equilibria $u_0 = (0,\ 0)$ and $u_{01} = (a,\ 0)$ are always unstable;*
- *if $a < bc$($a > bc$), then the equilibrium $u_{02} = (0,\ b)$ is stable(unstable);*
- *if $bc < a$, $ma \leq 1$ and $bcm + cd \leq 1$, and the unique interior equilibrium point $u^* = (u_1^*,\ u_2^*)$ is stable.*

The proof of Proposition 2.1 is simple. Here, we omit it.

To start Turing instability, we have to make the following assumption on the non-trivial steady-state solution u^*:

$$(H_1) \qquad bc < a,\ ma \leq 1,\ bcm + cd \leq 1.$$

When $u = u^*$, without loss of generality, for all $u = (u_1, u_2)^T$, let

$$K(u) = \left(\left(1+\alpha u_2\right)u_1,\ \left(\mu + \frac{1}{1+\beta u_1}\right)u_2\right)^T, J(u) = \left(u_1\left(a - u_1 - \frac{cu_2}{1+mu_1}\right),\ u_2\left(b - u_2 + \frac{du_1}{1+mu_1}\right)\right)^T,$$

system (1.1) may be described as

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta K(u) + J(u) \quad x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega, \\ u_i(x, 0) = u_{i0}(x) \geq 0, \quad x \in \ \Omega,\ i = 1,\ 2. \end{cases} \tag{2.1}$$

The linearization of model (2.1) at u is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta K_u(u) + J_u(u) \quad x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega, \\ u_i(x,0) = u_{i0}(x) \geq 0, \quad x \in \Omega, \ i = 1, \ 2, \end{cases} \tag{2.2}$$

where $K_u(u^*)$ is the linearization matrix of $K(u)$ at u^* and

$$K_u(u^*) = \begin{bmatrix} 1+\alpha u_2^* & \alpha u_1^* \\ \dfrac{-\beta u_2^*}{(1+\beta u_1^*)^2} & \mu + \dfrac{1}{1+\beta u_1^*} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

$$J_u(u^*) = \begin{bmatrix} \dfrac{(ma-1)u_1^* - 2m(u_1^*)^2}{1+mu_1^*} & \dfrac{-cu_1^*}{1+mu_1^*} \\ \dfrac{du_2^*}{(1+mu_1^*)^2} & -u_2^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

We perturb the model (2.1) near u^*

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1^* + \epsilon_1 \exp((k_x x + k_y y)i + \rho t) \\ u_2^* + \epsilon_2 \exp((k_x x + k_y y)i + \rho t) \end{bmatrix}, \tag{2.3}$$

$\mathbf{k} = (k_x, k_y)^T$ acts for the wave vector; ρ represents the growth rate of perturbation in time t; the Euclidean form $k = |\mathbf{k}|$ stands for the wave number of the perturbation. The following characteristic equation of model (2.1) can be obtained after replacing u with (2.3),

$$\rho^2 - \rho \operatorname{tr} L(u) + \det L(u) = 0, \tag{2.4}$$

where

$$L(u) = -k^2 K_u(u^*) + J(u^*), \ \operatorname{tr} L(u) = -(c_{11} + c_{22})k^2 + (a_{11} + a_{22}),$$

and

$$\det L(u) = Ak^4 + Bk^2 + C,$$

$$A = \det K_u(u^*) = c_{11}c_{22} - c_{12}c_{21} = (1+\alpha u_2^*)\left(\mu + \frac{1}{1+\beta u_1^*}\right) + \frac{\alpha\beta u_1^* u_2^*}{(1+\beta u_1^*)^2},$$

$$B = (1+\alpha u_2^*)u_2^* + \frac{d\alpha u_1^* u_2^*}{(1+mu_1^*)^2} + \frac{c\beta u_1^* u_2^*}{(1+mu_1^*)(1+\beta u_1^*)^2} - \frac{(1+\mu(1+\beta u_1^*))[(ma-1)u_1^* - 2m(u_1^*)^2]}{(1+\beta u_1^*)(1+mu_1^*)},$$

$$C = \det J_u(u^*).$$

In what follows, taking α as a parameter, we get the following results.

Theorem 2.1. *Suppose that H_1 is satisfied, then the cross-diffusion may exert Turing instability of the non-trivial steady-state solution u^* in system (1.1), if the following condition holds,*

$$\alpha < \alpha_T \text{ and } k > k_T,$$

where $\alpha_T = -\dfrac{[\mu(1+\beta u_1^*)+1](1+\beta u_1^*)}{u_2^*(1+\beta u_1^*)[\mu(1+\beta u_1^*)+1] + \beta u_1^* u_2^*}$, $k_T = \max\{k | Ak^4 + Bk^2 + C = 0\}$.

Proof. By Proposition 2.1, the unique interior equilibrium point $u^* = (u_1^*, \ u_2^*)$ is stable without cross-diffusion under (H1). Furthermore, by the expression of A,

$$A = (1+\alpha u_2^*)\left(\mu + \frac{1}{1+\beta u_1^*}\right) + \frac{\alpha\beta u_1^* u_2^*}{(1+\beta u_1^*)^2} = \alpha\left[u_2^*\left(\mu + \frac{1}{1+\beta u_1^*}\right) + \frac{\beta u_1^* u_2^*}{(1+\beta u_1^*)^2}\right] + \left(\mu + \frac{1}{1+\beta u_1^*}\right),$$

after a simple calculation, we have $A < 0$ when $\alpha < \alpha_T$. Since

$$\lim_{k^2 \to +\infty} \det L(u) = \lim_{k^2 \to +\infty} (Ak^4 + Bk^2 + C) = -\infty,$$

with $A < 0$, $C > 0$, by the intermediate value theorem, we obtain $\det L(u) = 0$ at least one positive root. Let $k_T = \max\{k | Ak^4 + Bk^2 + C = 0\}$, one can deduce $\det L(u) < 0$ when $k > k_T$ immediately, which imply that (2.4) has a positive root. Namely, as

α decreasing, cross-diffusion can exert Turing instability of the non-trivial steady-state solution. Until now, the theorem is completely proved. □

Remark 2.1. From Theorem 2.1, we have the following results.

- In the case $\alpha = \beta = 0$, i.e., the self-diffusion alone for system (1.1) (see [13]), from (2.4), one can obtain:

$$c_{11} = 1,\ c_{22} = \mu,\ A = 1 + \mu > 0,\ B = u_2^* - \frac{(1+\mu)[(ma-1)u_1^* - 2m(u_1^*)^2]}{1 + mu_1^*} > 0,\ C = J(u^*) > 0,$$

  which shows

$$\mathrm{tr}L(u) = -(c_{11} + c_{22})k^2 + (a_{11} + a_{22}) < 0, \quad \det L(u) = Ak^4 + Bk^2 + C > 0,\ \text{for all } k.$$

  According to Vieta theorem, (2.4) has two roots with negative real parts, which proves model (1.1) remains stable in the presence of self-diffusion but without cross-diffusion, and the Turing instability can not occur.
- In the case $\alpha > 0, \beta > 0$, after a direct calculation, one can obtain that,

$$c_{11} + c_{22} > 0,\ A > 0,\ B > 0,\ C > 0,$$

  which shows

$$\mathrm{tr}L(u) = -(c_{11} + c_{22})k^2 + (a_{11} + a_{22}) < 0, \quad \det L(u) = Ak^4 + Bk^2 + C > 0,\ \text{for all } k.$$

  Then, (2.4) has two roots with negative real parts, and system (1.1) with self-diffusion and cross-diffusion remains stable, Turing instability can not appear also.

Remark 2.2. The corresponding system with Holling I,II and III functional responses associated with Eqs. (1.1) was considered in [6]. The results obtained in [6] suggest that, if the diffusion coefficients are all positive, then diffusion-driven instability can not happen. Furthermore, from Theorem 2.1 of the present manuscript, we see that the introduction of negative diffusion coefficients play an important role in Turing instability.

Remark 2.3. Biologically, our results suggest that if the prey is close to predator (i.e. $\alpha < 0$), or the predator species run away from the prey (i.e. $\beta < 0$) respectively, the steady state will be strongly amplified by cross-diffusion, giving rise to a spatially inhomogenous population distribution. In other words, cross-diffusion can drive the development of spatial patterns under certain conditions.

Remark 2.4. Regarding the diffusion rate of predator β as a parameter, in the same way as above, Turing instability can be discussed also.

3. The existence of non-constant steady-state solutions

In this section, the Lyapunov–Schmidt reduction method, which is known from [14], is applied to seek for the non-constant steady-states near u_0 and u^* in (1.1). The cases of other constant stationary solutions can be obtained similarly. In order to investigate whether cross-diffusion can induce the non-constant steady-states around the constant steady-state, we introduced some notations and basic facts throughout this article:

- **We always assume that** each eigenvalue λ_i $(i \in N_0 = N \cup \{0\})$ of the operator $-\Delta$ on $\overline{\Omega}$ is simple, where $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ and $n \to \infty,\ \lambda_n \to +\infty$. ϕ_i are the corresponding eigenfunction of λ_i, $\phi_0(x) > 0$ and $\{\phi_i\}_{i=0}^{\infty}$ is a complete orthonormal system in the Lebesgue space $L_2(\overline{\Omega})$ of integrable functions defined on Ω.
- $Y = L_2(\Omega),\ X = H^2(\Omega) \cap H_0^1(\Omega)$ where $H_0^1(\Omega) = \{u \in H_0^1(\Omega) | \frac{\partial u(x)}{\partial \nu} = 0,\ \forall x \in \partial\Omega\}$, $H^2(\Omega)$ represents the Sobolev space of the L_2-functions f defined on Ω. Their derivatives $\frac{d^k f}{dx^k}$ are also belong to $L^2(\Omega)$.
- The inner product formula of both space X and Y is

$$\left\langle u(x), v(x) \right\rangle = \int_{\Omega} u^T(x) v(x) dx.$$

- For each $j \in N_0$, we have

$$X_j = \left\{\varphi \in X | \int_{\Omega} \varphi(x)\phi_j(x)dx = 0\right\}, \quad Y_j = \left\{\varphi \in Y | \int_{\Omega} \varphi(x)\phi_j(x)dx = 0\right\}.$$

  For more transparent, we define the vector operator

$$F:\ X^2 \to Y^2,\ F(u) = \Big(F_1(u_1,\ u_2),\ F_2(u_1,\ u_2)\Big)^T,$$

where

$$\begin{cases} F_1(u_1,\ u_2) = \Delta\Big[(1 + \alpha u_2)u_1\Big] + u_1\Big(a - u_1 - \dfrac{cu_2}{1 + mu_1}\Big), \\ F_2(u_1,\ u_2) = \Delta\Big[(\mu + \dfrac{1}{1 + \beta u_1})u_2\Big] + u_2\Big(b - u_2 + \dfrac{du_1}{1 + mu_1}\Big). \end{cases} \tag{3.1}$$

3.1. Steady states near u_0

The Fréchet derivative of F at u_0 is given by

$$L_\eta(u_0) = \text{diag}\{\Delta + a,\ \mu\Delta + b\} = L_\eta,$$

where $\eta = (\eta_1,\ \eta_2)^T = (a,\ b)^T$. So far, we can deduce the result about the stability of u_0, which is as follows.

Remark 3.1. There is $i \in N_0$ such that $\lambda_i > a, \mu\lambda_i > b$ is fulfilled, u_0 is unstable in ODE system while stable in PDE system (1.1).

Noted that $a > 0, b > 0$, $\lambda_j (j \in N_0)$ can be regarded as the bifurcation values of a and b. In what follows, we investigate whether system (1.1) exists non-constant steady-state solutions near (u_0, η^0) with the following assumption.

(H_2): there is $\eta^0 = (\lambda_j,\ \mu\lambda_k)^T$, where λ_j, λ_k are the fixed simple eigenvalues of operator $-\Delta$. ϕ_j, ϕ_k are the corresponding character functions.

Next, we accordance with the Lyapunov–Schmidt reduction step to seek for the stationary bifurcation of u_0.

Step 1. Let $q_1 = (\phi_j,\ 0)^T,\ q_2 = (0,\ \phi_k)^T$ and $K = KerL_{\eta_0}$. From hypothesis, we have

$$L_{\eta_0} q_1 = diag\{\Delta + \lambda_j,\ \mu\Delta + \mu\lambda_k\} q_1 = \Big((-\lambda_j + \lambda_j)\phi_j,\ (\mu\Delta + \mu\lambda_k)0\Big)^T = 0,$$

$$L_{\eta_0} q_2 = diag\{\Delta + \lambda_j,\ \mu\Delta + \mu\lambda_k\} q_2 = \Big((\Delta + \lambda_j)0,\ (-\mu\lambda_k + \mu\lambda_k)\phi_k\Big)^T = 0,$$

that is, $K = span\{q_1,\ q_2\}$.

Because Y^2 is a complete space and $L_{\eta_0} : X^2 \to Y^2$ is a bounded linear operator, $dimK,\ codimRangeL_{\eta_0}$ are finite and L_{η_0} is Fredholm operator.

It is easy to verity that $\langle x,\ L_{\eta_0} y\rangle = \langle L_{\eta_0} x,\ y\rangle,\ \forall\ x,\ y \in X^2$. That is to say, L_{η_0} is a self-adjoint operator, $L^*_{\eta_0} = L_{\eta_0}$. Using Fredholm alternative theorem, we have the decompositions

$$X^2 = K \oplus X_{00},\ \ Y^2 = K \oplus Y_{00},$$

where

$$X_{00} = X_j \times X_k,\ \ Y_{00} = Y_j \times Y_k.$$

Clearly, $dimK = codimRangeL_{\eta_0},\ L_{\eta_0}$ is a Fredholm operator with index zero, and $L_{\eta_0}|_{X_{00}} :\ X_{00} \to Y_{00}$ is invertible and its inverse is bounded.

Step 2. Set P and $I - P$, P_1 and P_2 denote the projection operators from Y^2 to Y_{00} and K, from Y to Y_j and Y_k respectively, and

$$Pu = \begin{pmatrix} P_1 u_1 \\ P_2 u_2 \end{pmatrix} = \begin{pmatrix} u_1(x) - \phi_j \int_\Omega \phi_j(x) u_1(x) dx \\ u_2(x) - \phi_k \int_\Omega \phi_k(x) u_2(x) dx \end{pmatrix},\ \ \forall\ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in Y^2.$$

Therefore,

$$F(u,\ \eta) = \mathbf{0} \quad iff \qquad PF(u,\ \eta) = \mathbf{0}, \tag{3.2}$$

$$(I - P)F(u,\ \eta) = \mathbf{0}. \tag{3.3}$$

Step 3. For every $u \in X^2$, there is a unique decomposition:

$$u = z_1 q_1 + z_2 q_2 + w,\ \ z = (z_1,\ z_2)^T \in R^2,\ \ w = (w_1,\ w_2)^T \in X_{00}.$$

Replacing u of (3.2) with $u = z_1 q_1 + z_2 q_2 + w$, we have

$$PF(z_1 q_1 + z_2 q_2 + w,\ \eta) = 0.$$

More abstractly, $PF(z_1 q_1 + z_2 q_2 + w,\ \eta)$ can be viewed as a map from $K \times X_{00} \times R^2$ to Y_{00}, which satisfies two cases:

(i) $P_1 F_1(0,\ z_2,\ 0,\ w_2,\ \eta) = 0,\ \ P_2 F_2(z_1,\ 0,\ w_1,\ 0,\ \eta) = 0$;
(ii) $PF_w(0,\ 0,\ 0,\ 0,\ \eta^0) = PL_{\eta^0} = L_{\eta^0}$.

The first equal sign holds by the chain rule, and the second is because $P\,|_{Y_{00}}:\ Y_{00} \to Y_{00}$ is a identity operator.

Recall that $L_{\eta_0}\,|_{X_{00}}$ is invertible, so we apply the implicit function theorem to require a unique differentiable map $w = (w_1,\ w_2)^T$ in two open neighborhoods: δ of 0, ϵ of η^0 in R^2 respectively, which satisfies

$$w_1(0,\ z,\ \eta) = w_2(z,\ 0,\ \eta) = 0,\ \ \forall (z,\ \eta) \in \delta \times \epsilon.$$

Substituting $u = z_1 q_1 + z_2 q_2 + w(z_1,\ z_2,\ \eta)$ into (3.3), we attain

$$PF(z_1 q_1 + z_2 q_2 + w(z_1,\ z_2,\ \eta),\ \eta) = \mathbf{0}. \tag{3.4}$$

Step 4. In this step, we solve the specific expressions of $w(z_1,z_2,\eta)$. Let

$$w_{ls}(z_1,\ z_2) = \frac{\partial^{l+s} w(z_1,\ z_2)}{\partial z_l^l \partial z_2^s} \quad \forall\ l,\ s \in N_0,$$

and $d^n F_z,\ n \in Z^+$ denote the n-order differential operators of F subject to z. For example, dF_z expresses the Jacobian matrix of F subject to z. By the chain rule of matrix, $d(G \circ F)_z = dG_{F(z)} dF_z$, and differentiate (3.4) with respect to z, obtaining

$$d(PF)_z = PL_{\eta^0} \cdot \begin{bmatrix} \varphi_j + w_1^1 & w_2^1 \\ w_1^2 & \varphi_k + w_2^2 \end{bmatrix} = PL_{\eta^0} \cdot \begin{bmatrix} w_1^1 & w_2^1 \\ w_1^2 & w_2^2 \end{bmatrix} = L_{\eta^0} \cdot \begin{bmatrix} w_1^1 & w_2^1 \\ w_1^2 & w_2^2 \end{bmatrix}.$$

The first equality results from that the sequence of differentiate operations and projection operator P is commutative, namely, $d^n(PF)_z = Pd^n F_z,\ n \in Z^+$. The second equality is because $K = span\{q_1,\ q_2\}$. $L_{\eta_0}\ |_{X_{00}}$ is invertible, so

$$dw_z = \begin{bmatrix} w_1^1 & w_2^1 \\ w_1^2 & w_2^2 \end{bmatrix} = \mathbf{0}.$$

This proof is also mentioned roughly by [15].

The Taylor expansion of $w(z_1,z_2,\eta)$ is

$$w(z_1,\ z_2,\ \eta) = \frac{1}{2} w_{20} z_1^2 + w_{11} z_1 z_2 + \frac{1}{2} w_{02} z_1^2 + \cdots.$$

Continue to differentiate dF_z subject to z to derive d^2F_z, go on differentiating d^2F_z with respect to z to derive d^3F_z, the rest higher derivative can be got in the same manner.

$\forall \zeta = (\zeta_1,\ \zeta_2)^T,\ \xi = (\xi_1,\ \xi_2)^T,\ \varepsilon = (\varepsilon_1,\ \varepsilon_2)^T \in K,\ d^2F_z(\zeta,\xi)$ and $d^3F_z(\zeta,\xi,\varepsilon)$ are respectively the following two matrixes:

$$\left(\begin{matrix} -2\zeta_1\xi_1 + (\alpha\Delta - c)(\zeta_2\xi_1 + \zeta_1\xi_2) \\ (d - \beta\Delta)(\zeta_2\xi_1 + \zeta_1\xi_2) - 2\zeta_2\xi_2 \end{matrix}\right), \left(\begin{matrix} 2cm(\zeta_1\xi_2\varepsilon_1 + \zeta_2\xi_1\varepsilon_1 + \zeta_1\xi_1\varepsilon_2) \\ (-2md + 2\beta^2\Delta)(\zeta_1\xi_2\varepsilon_1 + \zeta_2\xi_1\varepsilon_1 + \zeta_1\xi_1\varepsilon_2) \end{matrix}\right).$$

Furthermore, we have

$$w_{20} = -(L_{\eta^0})^{-1} Pd^2F_z(q_1,\ q_1) = (w_{20}^1,\ w_{20}^2)^T = \left(\frac{2\phi_j(\phi_j - \int_\Omega \phi_j^3(x)dx)}{\Delta + \lambda_j},\ 0\right)^T,$$

$$w_{11} = -(L_{\eta^0})^{-1} Pd^2F_z(q_1,\ q_2) = \begin{pmatrix} w_{11}^1 \\ w_{11}^2 \end{pmatrix} = \begin{pmatrix} \dfrac{\phi_j(\alpha\Delta - c)(-\phi_k + \int_\Omega \phi_j^2(x)\phi_k(x)dx)}{\Delta + \lambda_j} \\ \dfrac{\phi_k(d - \beta\Delta)(-\phi_j + \int_\Omega \phi_j(x)\phi_k^2(x)dx)}{\mu\Delta + \mu\lambda_k} \end{pmatrix},$$

$$w_{02} = -(L_{\eta^0})^{-1} Pd^2F_z(q_2,\ q_2) = (w_{02}^1,\ w_{02}^2)^T = \left(0,\ \frac{2\phi_k(\phi_k - \int_\Omega \phi_k^3(x)dx)}{\mu\Delta + \mu\lambda_k}\right)^T,\ \cdots\cdots,$$

where (q_i,q_j) by $q_i \cdot q_j$ computing.

Substituting $u = z_1q_1 + z_2q_2 + w(z_1,\ z_2,\ \eta)$ into (3.3) leads to

$$\Phi(z_1,\ z_2,\ \eta) = (I - P)F(z_1q_1 + z_2q_2 + w(z_1,\ z_2,\ \eta),\ \eta) = \mathbf{0}. \tag{3.5}$$

Until now, the original bifurcation problem is reduced to looking for solutions of $\Phi(z_1,z_2,\eta)$. Moreover,

$$\Phi(0,\ 0,\ \eta) = 0,\ \ \Phi_{z_1}(0,\ 0,\ \eta^0) = 0,\ \ \Phi_{z_2}(0,\ 0,\ \eta^0) = 0.$$

From the consequence of Lyapunov–Schmidt reduction, there is a vicinity $V \in R^2 \times R^2$ of $(0,0,\eta^0)$ satisfying that each solution of $\Phi(z_1,z_2,\eta)$ one-to-one corresponds to a solution of $F(u,\ \eta) = \mathbf{0}$.

Step 5. Calculate the inner product of (3.5) with q_1 and q_2, deriving $G(z_1,\ z_2,\ \eta) = 0$, where $G = (G_1,\ G_2)^T$ is given by

$$G_s = \Big\langle\ q_s,\ F(z_1q_1 + z_2q_2 + w(z_1,\ z_2,\ \eta),\ \eta)\Big\rangle\ \ s = 1,\ 2.$$

It follows from (3.4) and the definition of F, we have for all $z \in R$,

$$G_1(0,\ z,\ \eta) = G_2(z,\ 0,\ \eta) = 0,$$

and

$$dG_z(0,\ 0,\ \eta^0) = diag(a - \lambda_j,\ b - \mu\lambda_k).$$

Thus, we can rewrite $G(z_1,z_2,\eta)$ as $\Big(z_1g_1(z_1,\ z_2,\ \eta),\ z_2g_2(z_1,\ z_2,\ \eta)\Big)^T$ and

$$\begin{pmatrix} g_1(z_1,\ z_2,\ \eta) \\ g_2(z_1,\ z_2,\ \eta) \end{pmatrix} = \begin{pmatrix} a - \lambda_j + J_{11}z_1 + J_{12}z_2 + \dfrac{1}{6}(b_{120}z_1^2 + 3b_{111}z_1z_2 + 3b_{102}z_2^2) + \cdots \\ b - \mu\lambda_k + J_{21}z_1 + J_{22}z_2 + \dfrac{1}{6}(3b_{220}z_1^2 + 3b_{211}z_1z_2 + b_{202}z_2^2) + \cdots \end{pmatrix},$$

where

$$J_{11} = \Big\langle q_1,\ d^2F_z(q_1,\ q_1)\Big\rangle = -2\int_\Omega \phi_j^3(x)dx,$$
$$J_{12} = \Big\langle q_1,\ d^2F_z(q_1,\ q_2)\Big\rangle = -(c+\alpha\lambda_j)\int_\Omega \phi_k(x)\phi_j^2(x)dx,$$
$$J_{21} = \Big\langle q_2,\ d^2F_z(q_1,\ q_2)\Big\rangle = (d+\beta\lambda_k)\int_\Omega \phi_k^2(x)\phi_j(x)dx,$$
$$J_{22} = \Big\langle q_2,\ d^2F_z(q_2,\ q_2)\Big\rangle = -2\int_\Omega \phi_k^3(x)dx,$$

$$b_{120} = \Big\langle q_1,\ d^3F_z(q_1,\ q_1,\ q_1) + 3d^2F_z(q_1,\ w_{20})\Big\rangle = \frac{12\Big[(\int_\Omega \phi_j^3(x)dx)^2 - \int_\Omega \phi_j^4(x)dx\Big]}{\Delta+\lambda_j},$$
$$b_{202} = \Big\langle q_2,\ d^3F_z(q_2,\ q_2,\ q_2) + 3d^2F_z(q_2,\ w_{02})\Big\rangle = \frac{12\Big[(\int_\Omega \phi_k^3(x)dx)^2 - \int_\Omega \phi_k^4(x)dx\Big]}{\mu\Delta+\mu\lambda_k},$$

$$\begin{aligned} b_{102} &= \Big\langle q_1,\ d^3F_z(q_1,\ q_2,\ q_2) + 2d^2F_z(q_2,\ w_{11}) + d^2F_z(q_1,\ w_{02})\Big\rangle \\ &= (\alpha\Delta - c)\Big[2\int_\Omega \phi_j(x)\phi_k(x)w_{11}^1dx + \int_\Omega \phi_j^2(x)w_{02}^2dx\Big], \\ b_{211} &= \Big\langle q_2,\ d^3F_z(q_1,\ q_2,\ q_2) + 2d^2F_z(q_2,\ w_{11}) + d^2F_z(q_1,\ w_{02})\Big\rangle \\ &= (d-\beta\Delta)\Big[2\int_\Omega \phi_k^2(x)w_{11}^1dx + \int_\Omega \phi_k(x)\phi_j(x)w_{02}^2dx\Big] - 4\int_\Omega \phi_k^2(x)w_{11}^2dx, \end{aligned}$$

$$\begin{aligned} b_{220} &= \Big\langle q_2,\ d^3F_z(q_1,\ q_1,\ q_2) + d^2F_z(q_2,\ w_{20}) + 2d^2F_z(q_1,\ w_{11})\Big\rangle \\ &= (d-\beta\Delta)\Big[2\int_\Omega \phi_k(x)\phi_j(x)w_{11}^2dx + \int_\Omega \phi_k^2(x)w_{20}^1dx\Big] + (2md-\beta^2\Delta)\int_\Omega \phi_j^2(x)\phi_k^2(x)dx, \\ b_{111} &= \Big\langle q_1,\ d^3F_z(q_1,\ q_1,\ q_2) + 2d^2F_z(q_1, w_{11}) + d^2F_z(q_2, w_{20})\Big\rangle \\ &= (\alpha\Delta - c)\Big[2\int_\Omega \phi_j^2(x)w_{11}^2dx + \int_\Omega \phi_j(x)\phi_k(x)w_{20}^1dx\Big] + 2cm\int_\Omega \phi_j^3(x)\phi_k(x)dx - 4\int_\Omega \phi_j^2(x)w_{11}^1dx. \end{aligned}$$

We next discuss the existence and multiplicity near u_0 by form. As we all know, the non-constant steady-states are generally trivial, semi-trivial and non-trivial.

Obviously, $g(0,\ 0,\ \eta^0) = 0$, $F(z_1, z_2, \eta)$ has trivial steady-state solution.

$F(z_1, z_2, \eta)$ may have non-constant semi-trivial steady-states that take the form of $(z_1, 0)^T$ or $(0, z_2)^T$ in a neighborhood of u_0, where z_1, z_2 satisfy $z_1z_2 \neq 0$ and $F_1(z_1,\ 0,\ \eta) = 0$, $F_2(0,\ z_2,\ \eta) = 0$. The previous equations are equivalent respectively to $g_1(z_1,\ 0,\ \eta) = 0$, $g_2(0,\ z_2,\ \eta) = 0$.

What we are going to do is to observe the existence of $(z_1, 0)^T$, and owing to the symmetry of J_{11} and J_{22}, b_{120} and b_{202}, the questions whether and how $(0, z_2)^T$ exists can be solved by the same way.

Let ε_1 and ε_2 represent two positive constants and

$$J = dg_z = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}; \quad \begin{cases} U_i = \{\eta_i \in R \mid 0 < \mid \eta_i - \eta_i^0 \mid < \varepsilon_i\}, \quad i = 1,\ 2; \\ U_i^\pm = \{\eta_i^\pm \in R \mid 0 < \mid \eta_i^\pm \mp \eta_i^0 \mid < \varepsilon_i\}, \ \ i = 1,\ 2, \end{cases}$$

where η_i^0 are mentioned at the beginning part of the subsection.

When $J_{11} = dg_{1z_1} \neq 0$, recall that $g_1(z_1,\ 0,\ \eta) = 0$, and apply the implicit function theorem for $g_1(z_1, 0, \eta)$ subject to z_1. We obtain a constant ε_1 and continuously differentiable map $\eta_1 \to z_{1\eta_1} : U_1 \to R$ satisfying that $g_1(z_1, 0, \eta)$ has a zero point $z_{1\eta_1}$, which is given by

$$z_{1\eta_1} = \frac{-(a-\lambda_j)}{J_{11}} + o(|\eta_1 - \lambda_j|). \tag{3.6}$$

It is clearly that $z_{1\eta_1}$ relies on η_1 and trends to 0 as η_1 trends to λ_j. Moreover, if $(a-\lambda_j)J_{11} < 0$ (resp., > 0), then $z_{1\eta_1} > 0$ (resp., $z_{1\eta_1} < 0$). Particularly, if $j = 0$, $J_{11} < 0$, then $z_{1\eta_1} > 0$.

Meanwhile, $F(u_1, u_2, \eta)$ with $\eta_1 \in U_1$ has a semi-trivial steady-states $(u_{1\eta_1},\ 0)$, where

$$u_{1\eta_1} = z_{1\eta_1}\phi_j + w_1(z_{1\eta_1},\ 0,\ \eta_1). \tag{3.7}$$

We can see that $u_{1\eta_1}$ depends on $z_{1\eta_1}$, so $u_{1\eta_1}$ relies on η_1 indirectly and also has the quality that $u_{1\eta_1} \to 0$, as $\eta_1 \to \lambda_j$. Particularly, as $j = 0$, $\eta_1 J_{11} < 0$, $z_{1\eta_1} > 0$, $u_{1\eta_1}$ is positive.

When $J_{11} = 0$, by a directly calculation, we have

$$\int_\Omega \phi_j^3(x)dx = 0, \quad b_{120} = \frac{-12\int_\Omega \phi_j^4(x)dx}{a-\lambda_j}, \quad b_{120}(a-\lambda_j) = -12\int_\Omega \phi_j^4(x)dx < 0.$$

Then there are two continuously differentiable maps $\eta_1^\pm \to z_{1\eta_1}^\pm$ from $U_1^\pm$ to R satisfying (1.1) with $\eta_1^\pm \in U_1^\pm$ has two non-constant steady-state solutions $u_{1\eta_1}^\pm$ and

$$u_{1\eta_1}^\pm = z_{1\eta_1}^\pm \phi_j + w(z_{1\eta_1}^\pm, \ \eta_1), \quad z_{1\eta_1}^\pm = \sqrt{\frac{(a-\lambda_j)^2}{2\int_\Omega \phi_j^4(x)dx}} + o(|\eta_1 - \eta_1^0|). \tag{3.8}$$

Analogously, we can get the result of $(0, z_2)^T$.

When $J_{22} \neq 0$, $g_2(0, z_2, \eta)$ has a zero $z_{2\eta_2}$ depending on η_2, which is

$$z_{2\eta_2} = \frac{-(b-\mu\lambda_k)}{J_{22}} + o(|\eta_2 - \eta_2^0|). \tag{3.9}$$

$z_{2\eta_2}$ approaches to 0 as η_2 gets close to λ_k. Moreover, if $(b-\mu\lambda_k)J_{22} < 0$ (resp., > 0), then $z_{2\eta_2} > 0$ (resp., $z_{2\eta_2} < 0$). Particularly, as $k = 0$, $J_{22} < 0$, then $z_{2\eta_2} > 0$.

Meanwhile, $F(u_1, u_2, \eta)$ with $\eta_2 \in U_2$ has a zero $(0, \ u_{2\eta_2})$, with

$$u_{2\eta_2} = z_{2\eta_2}\phi_k + w_2(0, \ z_{2\eta_2}, \ \eta_2), \tag{3.10}$$

which relies on η_2 indirectly and has the same trend with $z_{2\eta_2}$. Moreover, $u_{2\eta_2}$ is positive(resp., negative) if $(\eta_2 - \lambda_k)J_{22} < 0$ (resp., > 0). Particularly, as $k = 0$, $\eta_2 J_{22} < 0$, $z_{2\eta_2} > 0$, $u_{2\eta_2}$ is positive.

When $J_{22} = 0$, there are two continuously differentiable maps $\eta_2 \to z_{2\eta_2}^\pm$ from $U_2^\pm$ to R satisfying that (1.1) with $\eta_2 \in U_2$ has two non-constant steady-state solutions $u_{2\eta 2}^\pm$ and

$$u_{2\eta_2}^\pm = z_{2\eta_2}^\pm \phi_k + w(z_{2\eta_2}^\pm, \ \eta_2), \quad z_{2\eta_2}^\pm = \sqrt{\frac{(b-\mu\lambda_k)^2}{2\int_\Omega \phi_k^4(x)dx}} + o(|\eta_2 - \eta_2^0|). \tag{3.11}$$

Therefore, the following assertions hold.

Theorem 3.1. *Suppose the H_2 holds, then system (1.1) possesses different forms of non-constant semi-trivial steady-state solution, which are splited into four circumstances:*

(i) *system (1.1) with $\eta_1 \in U_1$ has the solution taking the form of $(u_{1\eta_1}, 0)^T$ if $J_{11} \neq 0$, which is revealed by (3.6) and (3.7). Moreover, $u_{1\eta_1}$ is positive as $j = 0$, $J_{11} < 0$.*

(ii) *If $J_{11} = 0$, then system (1.1) with $\eta_1^\pm \in U_1^\pm$ owns two solutions $(u_{1\eta_1}^\pm, \ 0)^T$, which is displayed by (3.8), and the sign of $u_{1\eta_1}^\pm$ allows positive as well as negative.*

(iii) *system (1.1) with $\eta_2 \in U_2$ has a solution taking the form of $(0, \ u_{2\eta_2})^T$ if $J_{22} = dg_{2z_2} \neq 0$, which is showed by (3.9) and (3.10). Moreover, $u_{2\eta_2}$ is positive as $j = 0$, $J_{22} < 0$.*

(iv) *If $J_{22} = 0$, then system (1.1) with $\eta_2^\pm \in U_2^\pm$ owns two solutions $(0, \ u_{2\eta_2}^\pm)^T$, which is exhibited by (3.11), and the sign of $u_{2\eta_2}^\pm$ allows positive as well as negative.*

Remark 3.2. When η is close to some critical values, there is one or two semi-trivial steady-state solutions in a neighbourhood of u_0. Both the situation (ii) and (iv) of the Theorem 3.1 show that the sign of non-constant steady-state solutions can be negative, but the negative steady-states are meaningless in biology models.

Lastly, in a vicinity of u_0, $F(z_1, z_2, \eta)$ can exist positive non-constant steady-state solutions taking the form of $(z_1, z_2)^T$, and z_1, z_2 satisfy $F_1(z_1, \ z_2, \ \eta) = 0$, $F_2(z_1, \ z_2, \ \eta) = 0$, namely, $g_1(z_1, \ z_2, \ \eta) = 0$, $g_2(z_1, \ z_2, \ \eta) = 0$.

By directly computing, there is

$$\det J = 4\int_\Omega \phi_j^3(x)dx \int_\Omega \phi_k^3(x)dx + (c+\alpha\lambda_j)(d+\beta\lambda_k)\int_\Omega \phi_k(x)\phi_j^2(x)dx \int_\Omega \phi_k^2(x)\phi_j(x)dx.$$

If $\det J \neq 0$, from the implicit function theorem, then there is a constant $\delta > 0$ and a unique differentiable continuously map $z_\eta = (z_{1\eta}, \ z_{2\eta})^T$ from $N(\eta^0, \ \delta) = \{\eta \in R^2 \mid |\eta - \eta^0| < \delta\}$ to R^2 meeting that $g(z_\eta, \ \eta) = 0$ for all $\eta \in N(\eta^0, \delta)$. Moreover, we have

$$\begin{cases} z_{1\eta} = \dfrac{J_{12}(b-\mu\lambda_k) - J_{22}(a-\lambda_j)}{\det J} + o(|\eta - \eta^0|), \\ z_{2\eta} = \dfrac{J_{21}(a-\lambda_j) - J_{11}(b-\mu\lambda_k)}{\det J} + o(|\eta - \eta^0|), \end{cases} \tag{3.12}$$

and $\lim_{\eta\to\eta^0} z_\eta = 0$. At the same time, $F(u_1, u_2, \eta)$ with $\eta \in N(\eta^0, \delta)$ have only one non-constant positive steady-states $u_\eta = (u_{1\eta},\ u_{2\eta})^T$ in a vicinity of u_0, with

$$\begin{cases} u_{1\eta} = z_{1\eta}\phi_j + \omega_1(z_{1\eta},\ z_{2\eta},\ \eta), \\ u_{2\eta} = z_{2\eta}\phi_k + \omega_2(z_{1\eta},\ z_{2\eta},\ \eta), \end{cases} \tag{3.13}$$

and $\lim_{\eta\to\eta^0} u_\eta = 0$.

If $\det J = 0$, then there are $\vartheta_s \in R^2$ and $\vartheta_s^* \in R^{2*}$ $(s = 0,\ 1)$ satisfying $J\vartheta_0 = \mathbf{0}$, $\vartheta_0^* J = \mathbf{0}$, and $\vartheta_l^*\vartheta_s = \delta_{ls},\ l,\ s = 0,\ 1$, where R^{2*} represents the Euclidian space of two-dimensional row vectors. Hence, $g(z,\ \eta) = \mathbf{0}$ if and only if for all $x_1, y \in R$, we have

$$\vartheta_0^* g(x_1\vartheta_0 + y\vartheta_1,\ \eta) = 0, \tag{3.14}$$

$$\vartheta_1^* g(x_1\vartheta_0 + y\vartheta_1,\ \eta) = 0. \tag{3.15}$$

Note that

$$\vartheta_1^* g(0,\ \eta) = 0, \quad \vartheta_1^* g_y(0,\ \eta) = \vartheta_1^* J\vartheta_1 \neq \mathbf{0},$$

from the implicit function theorem, there is a positive constant σ and a unique continuously differentiable map $y(x, \eta)$ from $N(0, \sigma) \times U$ to R, where $N(0,\ \delta) = \{x_1 \in R \,||\, x_1 \,| < \sigma\}$ and $U = \{\eta \in R^2 \,|\| \eta - \eta^0 \| < \delta\}$. $y(x_1, \eta)$ satisfies that for every $(x_1, \eta) \in N(0, \sigma) \times U$, there are

$$y(0,\ \eta^0) = 0, \quad \vartheta_1^* g\Big(x_1\vartheta_0 + y(x_1,\ \eta^0)\vartheta_1,\ \eta\Big) = 0.$$

Substituting $y(x_1, \eta)$ into (3.14), we have

$$\begin{aligned} &\vartheta_0^* g\Big(x_1\vartheta_0 + y(x_1,\ \eta)\vartheta_1,\ \eta\Big) \\ &= (a - \lambda_j)\vartheta_{01}^* + (b - \mu\lambda_k)\vartheta_{02}^* + \frac{1}{12}\Big(x_1\vartheta_0 + y(x_1,\ \eta)\vartheta_1\Big)^T D_\eta \Big(x_1\vartheta_0 + y(x_1,\ \eta)\vartheta_1\Big) \\ &= (a - \lambda_j)\vartheta_{01}^* + (b - \mu\lambda_k)\vartheta_{02}^* + \frac{1}{12} x_1^2 \vartheta_0^T D_\eta \vartheta_0 + o\Big(x_1^3 + |\, x_1^2(\eta - \eta^0)\,|\Big) = 0, \end{aligned}$$

where

$$D_\eta = \begin{bmatrix} 2(\vartheta_{01}^* b_{120} + \vartheta_{02}^* b_{220}) & 3(\vartheta_{01}^* b_{111} + \vartheta_{02}^* b_{211}) \\ 3(\vartheta_{01}^* b_{111} + \vartheta_{02}^* b_{211}) & 6\vartheta_{01}^* b_{102} \end{bmatrix}.$$

Let

$$S = \Big[(a - \lambda_j)\vartheta_{01}^* + (b - \mu\lambda_k)\Big]\vartheta_0^T D_{\eta^0}\vartheta_0.$$

It is not hard to find that if $S > 0$, then $r(x_1, \eta)$ has no zeros, otherwise, two nontrivial zeros $x_{1\eta}^{\pm}$. Moreover,

$$x_{1\eta}^{\pm} = -\frac{(a - \lambda_j)\vartheta_{01}^* + (b - \mu\lambda_k)}{\vartheta_0^T D_{\eta^0}} + o\Big(x_1^3 + |\, x_1^2(\eta - \eta^0)\,|\Big),\ z_\eta^{\pm} = x_{1\eta}^{\pm}\vartheta_0 + r(x_{1\eta}^{\pm},\ \eta)\vartheta_1. \tag{3.16}$$

Correspondingly, system (1.1) has no non-constant positive steady-state solutions near u_0 if $S > 0$, and otherwise, two non-constant positive solutions: $u(z_\eta^{\pm})$, where

$$u(z_\eta^{\pm}) = z_{1\eta_1}^{\pm} q_1 + z_{2\eta_2}^{\pm} q_2 + w(z_{1\eta_1}^{\pm},\ z_{2\eta_2}^{\pm},\ \eta). \tag{3.17}$$

Thus, we obtain the following results.

Theorem 3.2. *Suppose the H_2 holds, then system (1.1) exists different forms of the non-constant positive steady-states, which takes part in two kinds of circumstances:*

(i) *$F(u_1, u_2, \eta)$ with $\eta \in N(\eta^0, \delta)$ has only one non-constant positive steady-state solution $(u_{1\eta}, u_{2\eta})^T$ in a vicinity of u_0 if $\det J \neq 0$, which is displayed by the combination of (3.12) and (3.13). This means that only one bifurcation* ***solution*** *occurs in a vicinity of u_0.*

(ii) *If $\det J = 0$, $\vartheta_0^T D_{\eta^0}\vartheta_0 \neq 0$, then the non-constant solutions near u^* of system (1.1) undergo a bifurcation as η near η^0. More precisely, if $\vartheta_0^T D_{\eta^0}\vartheta_0 > 0$ (resp., < 0), two nontrivial solutions $u(z_\eta^{\pm})$ exist for η with $S < 0$ (resp., $S > 0$), which are described by the combination of (3.16) and (3.17). Moreover, the two non-boundary solutions coalesce into zero solution as η goes to η^0.*

Remark 3.3. From now on, we have used the Lyapunov-Schmidt reduction method to obtain the existence and multiplicity of non-constant steady-state solutions near the origan of (1.1). Our results showed the non-constant stationary will occur as the parameters value take for a certain range. Compare [1], we can see our process is more detailed and our results about of non-constant stationary solution more comprehensive.

Remark 3.4. Compare with [16–18], although the contents all contain Turing instability, the cross-diffusion terms of these models are different from each other, the degree of nonlinear for the cross-diffusion terms in this paper looks higher.

3.2. Steady states near u^*

In this section, we consider the existence of non-constant steady-state solutions of u^* under the condition H_1. Recall that the Fréchet derivative of F at u^* has been given by $K_u(u^*)\Delta + J_u(u^*)$, which reads

$$R_\alpha := \begin{bmatrix} (1+\alpha u_2^*)\Delta + a - 2u_1^* - \dfrac{cu_2^*}{(1+mu_1^*)^2} & \alpha u_1^*\Delta - \dfrac{cu_1^*}{1+mu_1^*} \\ \dfrac{-\beta u_2^*}{(1+\beta u_1^*)^2}\Delta + \dfrac{du_1^*}{(1+mu_1^*)^2} & \left(\mu + \dfrac{1}{1+\beta u_1^*}\right)\Delta - u_2^* \end{bmatrix}.$$

It is not hard to deduce that the adjoint operator R_α^* of R_α is

$$\begin{bmatrix} (1+\alpha u_2^*)\Delta + a - 2u_1^* - \dfrac{cu_2^*}{(1+mu_1^*)^2} & \dfrac{-\beta u_2^*}{(1+\beta u_1^*)^2}\Delta + \dfrac{du_1^*}{(1+mu_1^*)^2} \\ \alpha u_1^*\Delta - \dfrac{cu_1^*}{1+mu_1^*} & \left(\mu + \dfrac{1}{1+\beta u_1^*}\right)\Delta - u_2^* \end{bmatrix}.$$

$u = (u_1, u_2)^T \in KerR_\alpha$ if and only if

$$\begin{cases} (1+\alpha u_2^*)\Delta u_1 + \dfrac{(ma-1)u_1^* - 2m(u_1^*)^2}{1+mu_1^*}u_1 + \alpha u_1^*\Delta u_2 - \dfrac{cu_1^*}{1+mu_1^*}u_2 = 0, \\ \dfrac{-\beta u_2^*}{(1+\beta u_1^*)^2}\Delta u_1 + \dfrac{du_2^*}{(1+mu_1^*)^2}u_1 + (\mu + \dfrac{1}{1+\beta u_1^*})\Delta u_2 - u_2^* u_2 = 0, \end{cases} \tag{3.18}$$

that is equivalent to

$$\begin{cases} c_{11}\Delta u_1 + a_{11}u_1 + c_{12}\Delta u_2 + a_{12}u_2 = 0, \\ c_{21}\Delta u_1 + a_{21}u_1 + c_{22}\Delta u_2 + a_{22}u_2 = 0. \end{cases} \tag{3.19}$$

Substitute $u = \sum v_j\phi_j,\ j = 0,\ 1,\ 2,\ \ldots,\ v_j \in R^2$ into (3.19), from the definition of $\{\phi_j\}_{n=0}^\infty$, then we obtain the following algebraic equations

$$\begin{bmatrix} -c_{11}\lambda_j + a_{11} & -c_{12}\lambda_j + a_{12} \\ -c_{21}\lambda_j + a_{21} & -c_{22}\lambda_j + a_{22} \end{bmatrix} v_j = Q(\lambda_j)v_j = 0,\ \forall\ j \in N_0. \tag{3.20}$$

Therefore, (3.19) has a nontrivial solution if and only if (3.20) has a nontrivial solution v_j for some $j \in N^0$, namely exists j satisfying $\det Q(\lambda_j) = 0$. It is easy to see that

$$\det Q(\lambda_j) = A\lambda_j^2 + B\lambda_j + C = \det L_\eta(u^*).$$

From Section 2, we have the following results.

Lemma 3.1. *If one of conditions of Theorem (2.2) holds, then for all* $j \in N^0$, *we have* $\det Q(\lambda_j) > 0$, $KerR_\alpha = \emptyset$. *If one of conditions of Theorem (2.3) holds, then there is* $j \in N^0$ *such that* $\det Q(\lambda_j) = 0$, *which yields that* $KerR_\alpha \neq \emptyset$.

Next step, we still consider α as bifurcation parameter to investigate the existence of non-constant steady-state solutions near u^* with the following hypothesis.

$$(H_{i_2}^*):\quad \alpha < \alpha_T,\ \det Q(\lambda_{i_2},\ \alpha^0) = 0,\ \det Q(\lambda_s,\ \alpha^0) \neq 0,\ \forall\ s \in N_0 \setminus \{i_2\},$$

from (3.20),

$$\alpha^0 = -\frac{c_{22}\lambda_{i_2}^2 + (a_{12}c_{21} - a_{11}c_{22} - a_{22})\lambda_{i_2} + C}{(c_{22}u_2^* + c_{21}u_1^*)\lambda_{i_2}^2 + (a_{21}u_1^* - a_{22}u_2^*)\lambda_{i_2}}.$$

Equally, use the Lyapunov–Schmidt reduction to derive the stationary bifurcation.

Set $K = KerR_\alpha,\ K^* = KerR_\alpha^*$, from Eq. (3.20), we have $K = span\{q_{i_2}\},\ K^* = span\{p_{i_2}\}$, where

$$q_{i_2} = v_{i_2}\phi_{i_2},\ \ p_{i_2} = v_{i_2}^*\phi_{i_2},$$

$v_{i_2},\ v_{i_2}^* \in R^2 \setminus \{\mathbf{0}\}$ satisfying

$$Q(\lambda_{i_2})v_{i_2} = 0,\ Q^*(\lambda_{i_2})v_{i_2}^* = 0,\ v_{i_2}^* v_{i_2} = 1.$$

We have the following decompositions

$$X^2 = K \oplus X_{i_2},\ Y^2 = K^* \oplus Y_{i_2}.$$

The represent meanings of $\phi_{i_2},\ X_{i_2},\ Y_{i_2}$ are not changed. Due to the same reason of L_{η_0} in the Section 3.1, $R_\alpha : X^2 \to Y^2$ is Fredholm operator with zero index, and $R_\alpha|_{X_{i_2}} :\ X_{i_2} \to Y_{i_2}$ is invertible and has a bounded inverse. Let M and $I - M$ denote the projection operators from Y^2 to Y_{i_2} and K^* respectively, then naturally for every $u(x) \in Y^2$, we have

$$Mu = u(x) - \nu_{i_2}\phi_{i_2}\int_\Omega \nu_{i_2}^*\phi_{i_2}(x)u(x)dx = u(x) - \phi_{i_2}\int_\Omega \phi_{i_2}(x)u(x)dx.$$

Furthermore,

$$F(u,\ \alpha) = \mathbf{0} \qquad iff \qquad MF(u,\ \alpha) = \mathbf{0}, \tag{3.21}$$

$$(I - M)F(u,\ \alpha) = \mathbf{0}. \tag{3.22}$$

Noticed, for every $u \in X^2$, there is a unique decomposition

$$u = u^* + zq_{i_2} + w.$$

Hence, (3.19) can be rewritten as

$$MF(u^* + zq_{i_2} + w,\ \alpha) = \mathbf{0}.$$

Similarly, it can be considered as a map $MF :\ K \times X_{i_2} \times R \to Y_{i_2}$, which satisfies the conditions of the implicit function theorem: $MF(u^*,\ \alpha) = \mathbf{0},\ MF_w(u^*,\ \alpha) = MR_\alpha = R_\alpha$. So we can get a unique continuously differentiable map $w = (w_1,\ w_2)^T$ in two open neighborhoods δ of 0 in R^2, ϵ of α^0 in R respectively, which satisfies

$$\forall (z,\ \alpha) \in \delta \times \epsilon,\ w(0,\ \alpha) = 0.$$

Substituting $u = u^* + zq_{i_2} + w(z,\ \alpha)$ into the equation (3.16), we have

$$\forall (z,\ \alpha) \in \delta \times \epsilon,\ MF\Big(u^* + zq_{i_2} + w(z,\ \alpha),\ \alpha\Big) = 0. \tag{3.23}$$

Next we apply the implicit function theorem for (3.23) to formulate $w(z,\alpha)$. The step is similar to Subsection (3.1), here we are not describe in detail.

In the same way, for all $\zeta = (\zeta_1, \zeta_2)^T,\ \xi = (\xi_1, \xi_2)^T,\ \varepsilon = (\varepsilon_1,\ \varepsilon_2)^T \in K$, the form of $d^2F_z(\zeta,\xi)$ and $d^3F_z(\zeta,\xi,\varepsilon)$ are respectively

$$\begin{pmatrix} A_1\zeta_1\xi_1 + A_2(\Delta)(\zeta_2\xi_1 + \zeta_1\xi_2) \\ A_3(\Delta)\zeta_1\xi_1 + A_4(\Delta)(\zeta_2\xi_1 + \zeta_1\xi_2) + A_5\zeta_2\xi_2 \end{pmatrix}, \quad \begin{pmatrix} B_1\zeta_1\xi_1\varepsilon_1 + B_2(\zeta_1\xi_2\varepsilon_1 + \zeta_2\xi_1\varepsilon_1 + \zeta_1\xi_1\varepsilon_2) \\ B_3(\Delta)\zeta_1\xi_1\varepsilon_1 + B_4(\Delta)(\zeta_1\xi_2\varepsilon_1 + \zeta_2\xi_1\varepsilon_1 + \zeta_1\xi_1\varepsilon_2) \end{pmatrix}.$$

$$A_1 = 2\Big(\frac{cmu_2^*}{(1+mu_1^*)^3} - 1\Big),\ A_2(\Delta) = \alpha\Delta - \frac{c}{(1+mu_1^*)^2},\ A_3(\Delta) = 2\Big(\frac{\beta^2u_2^*}{(1+\beta u_1^*)^3}\Delta - \frac{mdu_2^*}{(1+mu_1^*)^3}\Big),$$

$$A_4(\Delta) = \frac{d}{(1+mu_1^*)^2} - \frac{\beta\Delta}{(1+\beta u_1^*)^2},\quad A_5 = -2,\quad B_1 = \frac{6cm^2u_2^*}{(1+mu_1^*)^4},\quad B_2 = \frac{2cm}{(1+mu_1^*)^3},$$

$$B_3(\Delta) = 6\Big(\frac{m^2du_2^*}{(1+mu_1^*)^4} - \frac{\beta^3u_2^*}{(1+\beta u_1^*)^3}\Delta\Big),\qquad B_4(\Delta) = 2\Big(\frac{\beta^2\Delta}{(1+\beta u_1^*)^3} - \frac{md}{(1+mu_1^*)^3}\Big).$$

From (3.23), we have

$$w(z,\ \alpha) = \frac{1}{2}w_2z^2 + \frac{1}{6}w_3z^3 + \cdots,\quad w_2 = -(R_{\alpha^0})^{-1}d^2F_z(q_{i_2},\ q_{i_2}),$$

where

$$(R_{\alpha^0})^{-1} = \frac{1}{\det R_{\alpha^0}}\begin{bmatrix} c_{22}\Delta + a_{22} & -c_{12}\Delta - a_{12} \\ -c_{21}\Delta - a_{21} & c_{11}\Delta + a_{11} \end{bmatrix} = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix},$$

$$-d^2F_z(q_{i_2},\ q_{i_2}) = \begin{pmatrix} A_1(\nu_{i_2}^1)^2 + 2A_2(-\lambda_{i_2})\nu_{i_2}^1\nu_{i_2}^2 \\ A_3(-\lambda_{i_2})(\nu_{i_2}^1)^2 + 2A_4(-\lambda_{i_2})\nu_{i_2}^1\nu_{i_2}^2 + A_5(\nu_{i_2}^2)^2 \end{pmatrix}\phi_{i_2}^2.$$

Substitute $u = u^* + zq_{i_2} + w(z,\ \alpha)$ into (3.20), which can be rewritten as

$$(I - M)F\Big(u^* + zq_{i_2} + w(z,\ \alpha),\ \alpha\Big) = 0. \tag{3.24}$$

To simplify the computations, we denote

$$W_1 = E_1\Big[A_1(v_{i_2}^1)^2 + 2A_2 v_{i_2}^1 v_{i_2}^2\Big] + E_2\Big[A_3(v_{i_2}^1)^2 + 2A_4 v_{i_2}^1 v_{i_2}^2 + A_5(v_{i_2}^2)^2\Big],$$

$$W_2 = E_3\Big[A_1(v_{i_2}^1)^2 + 2A_2 v_{i_2}^1 v_{i_2}^2\Big] + E_4\Big[A_3(v_{i_2}^1)^2 + 2A_4 v_{i_2}^1 v_{i_2}^2 + A_5(v_{i_2}^2)^2\Big].$$

Calculating the inner product of (3.24) with q_{i_2}, derive

$$G^*(z,\ \alpha) = \Big\langle\ q_{i_2},\ F(u^* + zq_{i_2} + w(z,\ \alpha),\ \alpha)\Big\rangle = z\Big((a_{11} - c_{11}\lambda_{i_2})(v_{i_2}^1)^2 + J_{11}z + \frac{1}{6}b_{120}z^2 + \cdots\Big) = 0,$$

where

$$J_{11} = \left[\frac{A_1}{2}(v_{i_2}^1)^3 + \Big(A_2(-\lambda_{i_2}) + \frac{A_3(-\lambda_{i_2})}{2}\Big)(v_{i_2}^1)^2 v_{i_2}^2 + A_4(-\lambda_{i_2})v_{i_2}^1(v_{i_2}^2)^2 + A_5(v_{i_2}^2)^3\right]\int_\Omega \phi_{i_2}^3(x),$$

$$\begin{aligned} b_{120} = \Big[&B_1(v_{i_2}^1)^4 + \Big(3B_2 + B_3(-\lambda_{i_2})\Big)(v_{i_2}^1)^3 v_{i_2}^2 + 3B_4(-\lambda_{i_2})(v_{i_2}^1)^2(v_{i_2}^2)^2 + \\ &3\Big(A_1(v_{i_2}^1)^2 + (A_2 + A_3)v_{i_2}^1 v_{i_2}^2 + A_4(v_{i_2}^2)^2\Big)W_1 + 3\Big(A_2(v_{i_2}^1)^2 + A_4 v_{i_2}^1 v_{i_2}^2 + A_5(v_{i_2}^2)^2\Big)W_2\Big]\int_\Omega \phi_{i_2}^4(x)dx. \end{aligned}$$

If $J_{11} \neq 0$, from the implicit function theorem, we can obtain a unique continuously differentiable map $\alpha \to z_\alpha$ in two open neighborhoods δ of 0 in R, ϵ of α^0 in R, which satisfies $G^*(z,\ \alpha) = 0$ and

$$z_\alpha = \frac{(a_{11} - c_{11}\lambda_{i_2})(v_{i_2}^1)^2}{J_{11}} + o\Big(|\alpha - \alpha^0|\Big).$$

Due to the corresponding relationship between solutions of $G^*(z,\ \alpha) = 0$ and (1.1), we know that if $G^*(z,\ \alpha) = 0$ has a non-trivial solutions z_α, then there is a positive constant δ and a continuously differentiable maps $\alpha \to z_\alpha$ from U to X^2 such that system (1.1) with $\alpha \in U$ has a non-constant steady-states u_α, with $u_\alpha = u^* + z_\alpha\phi_j + w(z_\alpha,\ \alpha)$.

If $J_{11} = 0$, then the number of the non-trivial solutions of $G^*(z,\ \alpha) = 0$ is determined by the sign of term $b_{120}(a_{11} - c_{11}\lambda_{i_2})$. When it is negative, $G^*(z,\ \alpha) = 0$ has two non-trivial solutions. Otherwise, $G^*(z,\ \alpha) = 0$ has no solution. So the current work is to analyze the sign of $b_{120}(a_{11} - c_{11}\lambda_{i_2})$.

Recall that $a_{11} < 0$, $c_{11} = 1 + \alpha u_2^* = 0$, $\alpha^* = -\frac{1}{u_2^*}$, $a_{11} - c_{11}\lambda_{i_2} = 0$ and $\lambda_{i_2}^* = \frac{a_{11}}{c_{11}}$. If $\alpha > \alpha^*$, then $a_{11} - c_{11}\lambda_{i_2} < 0$; Provided that $\alpha < \alpha^*$. If $\lambda_{i_2} > \lambda_{i_2}^*$, then $a_{11} - c_{11}\lambda_{i_2} < 0$. If $\lambda_{i_2} < \lambda_{i_2}^*$, then $a_{11} - c_{11}\lambda_{i_2} > 0$.

From the assumption $\alpha < \alpha_T$, we need to check if the parameter threshold value $\alpha^* < \alpha_T$,

$$\alpha_T = -\frac{(1 + \beta u_1^*) + \mu(1 + \beta u_1^*)^2}{u_2^*\Big[\beta u_1^* + (1 + \beta u_1^*) + \mu(1 + \beta u_1^*)^2\Big]} > -\frac{1}{u_2^*}.$$

That is, $\alpha^* < \alpha_T$.

Due to the corresponding relationship between solutions of $G^*(z,\ \alpha) = 0$ and (1.1), we know that if $G^*(z,\ \alpha) = 0$ has two non-trivial solutions $z_\alpha^\pm$, then there is a positive constant δ and two continuously differentiable maps $\alpha \to z_\alpha^\pm$ from $U = \{\alpha \in R \mid |\alpha - \alpha^0| < \delta\}$ to X^2 such that system (1.1) has two non-constant steady-states $u_\alpha^\pm$, with $u_\alpha^\pm = u^* + z_\alpha^\pm\phi_{i_2} + w(z_\alpha^\pm,\ \alpha)$. And $\lim_{\alpha\to\alpha^0} u_\alpha^\pm = u^*$. Moreover, the two non-boundary solutions coalesce into u^* as α goes to α^0, which means that a bifurcation occurs near u^*. Therefore, the following lemma holds.

Lemma 3.2. *Provided that $\alpha^* < \alpha < \alpha_T$. If $b_{120} > 0$ holds, then $G^*(z,\ \alpha) = 0$ has two non-trivial solutions; If $b_{120} < 0$, then $G^*(z,\ \alpha) = 0$ has no non-trivial solutions. Equivalently, If $b_{120} > 0$ holds, then $F(z,\ \alpha) = 0$ has two non-constant steady-state solutions; If $b_{120} < 0$, then $F(z,\ \alpha) = 0$ has no non-constant steady-state solutions.*

Lemma 3.3. *Assume that $\alpha < \alpha^*$, there is a constant $\lambda_{i_2}^*$. If $(\lambda_{i_2}^* - \lambda_{i_2})b_{120} > 0$ holds, then $G^*(z,\ \alpha) = 0$ has two non-trivial solutions, that is, $F(z,\ \alpha) = 0$ has two non-constant steady-state solutions; If $(\lambda_{i_2}^* - \lambda_{i_2})b_{120} < 0$ holds, then $G^*(z,\ \alpha) = 0$ has no non-trivial solutions, which is equivalently to that $F(z,\ \alpha) = 0$ has no non-constant steady-state solutions.*

Furthermore, we can derive the following theorem.

Theorem 3.3. *Under the assumption $(H_{i_2}^*)$,*

(i) *if $J_{11} \neq 0$, then there exists a continuously differentiable map $\alpha \to u_\alpha$ in two open neighborhoods δ of 0 in R, ϵ of α^0 in R, which satisfies system (1.1) near u^* has only one non-constant steady-state solution $u(z_\alpha) = u^* + z_\alpha\phi_{i_2} + w(z_\alpha,\ \alpha)$. Moreover, $\lim_{\alpha\to\alpha^0} u(z_\alpha) = u^*$.*
(ii) *Assume that $J_{11} = 0$. If $\alpha^* < \alpha < \alpha_T, b_{120} > 0$ or $\alpha < \alpha^*$, $(\lambda_{i_2}^* - \lambda_{i_2})b_{120} > 0$ hold, then model (1.1) with $\alpha \in U$ has two non-constant steady-state solutions $u(z_\alpha^\pm)$, then there is a bifurcation near u^*. Otherwise, system (1.1) has no bifurcation.*

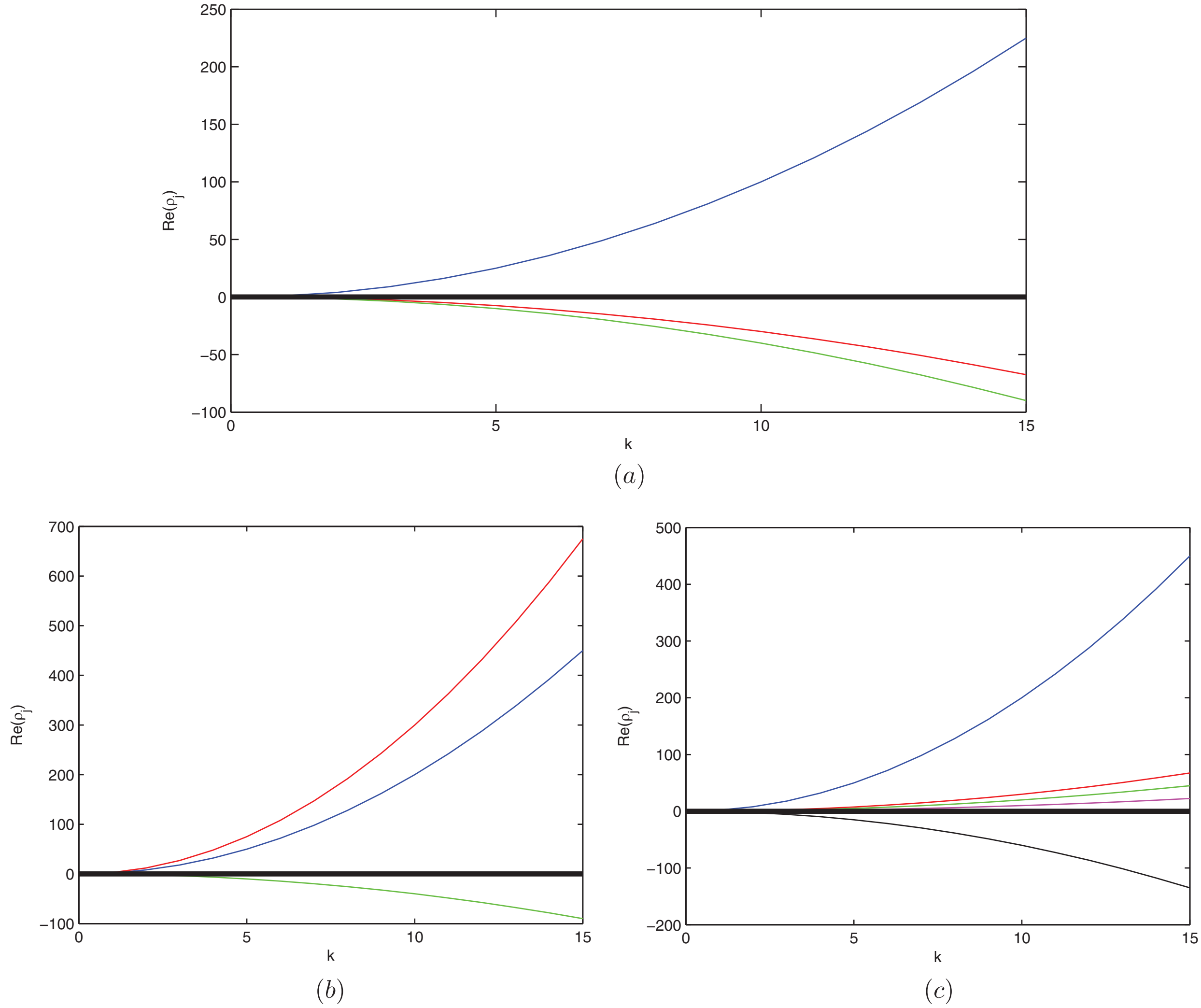

Fig. 1. The stability of u^* changes from the range of α, β expanding in system (1.1).

Remark 3.5. When α close to each α^0 satisfying assumption $(H^*_{i_2})$ with $i_2 \neq 0$, $u(z_\alpha)$ established by Theorem 3.3 is spatially nonhomogeneous and positive on Ω. Otherwise, $u(z_\alpha)$ is spatially homogeneous.

Remark 3.6. From Lemma 3.1, there are two assumptions base on which to investigate the existence of non-constant steady-state solutions near u^*. One is $(H^*_{i_2})$ and another is the following hypothesis (H^*_{jk}):

- The condition (iii) of Proposition 2.1 holds;
- There exists $\alpha^0 \in R$ such that

$$\det Q(\lambda_j,\ \alpha^0) = \det Q(\lambda_k,\ \alpha^0) = 0, \det Q(\lambda_s,\ \alpha^0) \neq 0,\ \ \forall\ s \in N^0 \setminus \{j, k\}.$$

By combining the step of the research under assumption (H_2) and the results of $(H^*_{i_2})$, we can obtain the existence of non-constant steady-state solutions near u^* with hypothesis (H^*_{jk}). Here, we are not repeat.

4. Numerical examples

In this section, we give some numerical simulations in one-dimensional spatial domain $\Omega = (0,\ \pi)$ to verify and complement the analysis results. The method of parameter selection is based on reference [12]. Thus, in the system (1.1) without diffusion, we take the values of

$$a = 1,\ \ b = 0.7,\ \ c = 0.4,\ \ d = 0.6,\ \ m = 0.2,\ \ \mu = 1.$$

Obviously, system (1.1) has a trivial steady-state solution $u_0(0, 0)$, two boundary equilibria $u_{01}(1, 0), u_{02}(0, 0.7)$ and a interior equilibrium point $u^*(0.1429, 0.7836)$. Noticing that, under these parameter values, condition (H1) is satisfied and the steady-state solution $u^*(0.1429, 0.7836)$ is asymptotically stable (see Fig. 1).

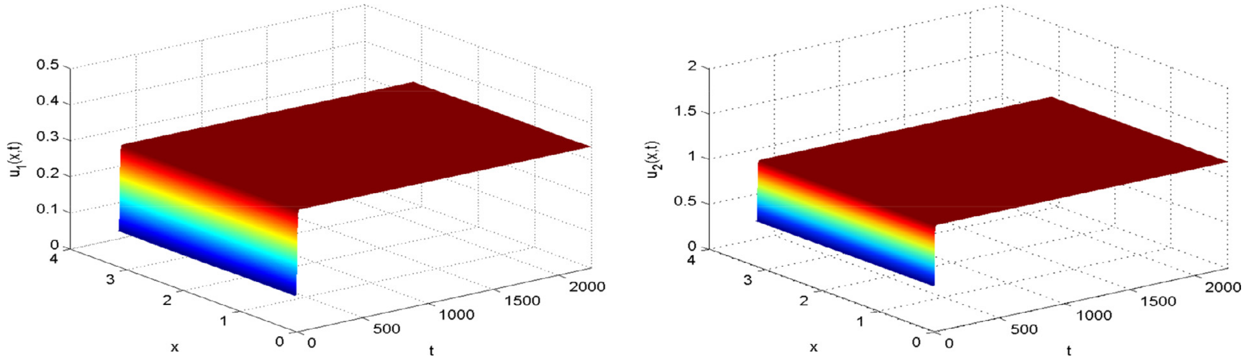

Fig. 2. Solutions of models (1.1) with $\alpha^* < \alpha < \alpha_T$ tend to a positive steady state.

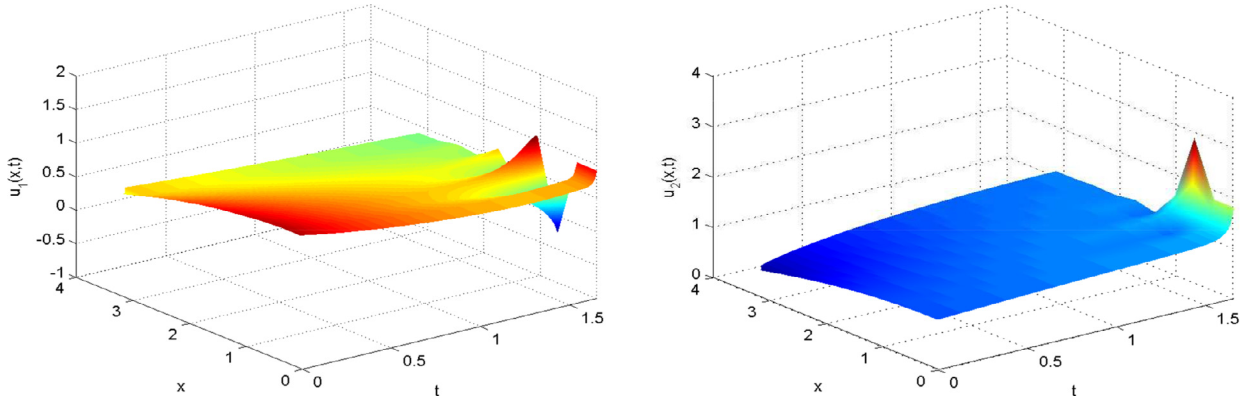

Fig. 3. Model (1.1) with $\alpha < \alpha^*$, $(\lambda^*_{i_2} - \lambda_{i_2})b_{120} > 0$ has the non-constant steady-state near u^*.

Example 4.1. We simulated three tendencies of from different parameter values can be displayed in Fig. 2. The curves of (a) of Fig. 2 depict the circumstance of $\beta = 0$, that is, system (1.1) only have the cross-diffusion term $\Delta\alpha u_1 u_2$, $\alpha_T = -2.3529$. We can see that the stability of u^* changes as α. The blue and green curves represent that the real part of largest eigenvalue ρ_j of u^* are negative for both $\alpha = 0 > \alpha_T$ and $\alpha = 3.1000 > \alpha_T$. That is, u^* is uniformly asymptotically stable. While $\alpha = -6.3000 < \alpha_T$, the red curve represents that $Re(\rho_j)$ of u^* become positive and $\Delta\alpha u_1 u_2$ causes Turing instability of u^*.

From the curves of (b) and (c) in Fig. 2, we find the following phenomena. If $\alpha = -6.3000$ is fixed and keep others same except β. $\beta = \frac{9}{10u_1^*} > 0$, u^* is still unstable, but $\beta = -\frac{9}{10u_1^*}$, u^* is stable. They may express the information: when $\beta = \beta(u_1^*)$, $\Delta\left(\dfrac{u_2}{1+\beta u_1^*}\right)$ is not related to u_1^* and can impact on the appearance of Turing instability. Moreover, (c) may survey the regularity of stability shift velocity when $\dfrac{1}{1+\beta u_1^*}$ close to $-\frac{1}{u_1^*}$. From the right: $\beta = -\frac{9}{100u_1^*}$ is far away from $-\frac{1}{u_1^*}$, the stability not change; when $\beta = -\frac{9}{10u_1^*}$ is close to $-\frac{1}{u_1^*}$, u^* become stable from unstable. From the left, $\beta = -\frac{109}{100u_1^*}$ close to $-\frac{1}{u_1^*}$, u^* become unstable; $\beta = -\frac{1009}{1000u_1^*}$ more closer to $-\frac{1}{u_1^*}$, u^* become unstable faster. In conclusion, $\dfrac{1}{1+\beta u_1^*}$ more closer to $-\frac{1}{u_1^*}$, u^* become unstable more faster.

In next example, we present a special case of Theorem 3.3 where the space dimension is 1, the situation of u_0 is similar to obtain.

Example 4.2. In this special case, $-\Delta$ subject to the homogeneous Neumann boundary condition on $\partial\Omega$ has eigenvalues $\lambda_n = n^2$, $n \in N^0$, ϕ_n is the eigenfunction associated with the eigenvalue λ_n, and $\phi_0(x) = \frac{1}{\sqrt{\pi}}$, $\phi_n(x) = \sqrt{\frac{2}{\pi}}\cos nx$, $n \in N$.

With the virtue of Matlab, it is easy to see that if we choose $\alpha^* < \alpha = -3.3500 < \alpha_T$, then $\lambda^*_{i_2} = -0.5386$, $b_{120} < 0$, from Theorem 2.1, the steady-state solution $u^*(0.1429, 0.7836)$ is asymptotically stable, and (1.1) has no non-constant steady-state solution (see Fig. 2). When we choose $\alpha = -3.9000 < \alpha^* = -2.7319$, $\lambda^*_{i_2} = 0.5318$, $\lambda_{i_2} = 2.1433$, $b_{120} < 0$, from (ii) of Theorem 3.3, (1.1) has two non-constant steady-state solution bifurcates from u^* (see Fig. 3).

5. Discussion and conclusions

In this paper, we study a nonlinear reaction cross-diffusion predator-prey system under Neumann boundary condition. System (1.1) is very general, From a biology viewpoint, negative diffusion coefficients with local accumulation effect of prey are introduced. We compute analytically Turing instability and bifurcation of spatially non-homogeneous steady-state solutions. In particular, the classical diffusion-driven instability induced by self-diffusion does not occur and the important role of negative diffusion coefficient in Turing instability is discussed in detail, i.e., the model generates spatial patterns only in the presence of $\alpha < 0$ (or $\beta < 0$). Furthermore, we present the existence and multiplicity of spatially nonhomogeneous steady-state solutions by Lyapunov–Schmidt reduction.

We split two aspects to analyze our work. On the one hand, the model is different from other similar system [1,19]. The main reason is that the prey cross-diffusion parameter range change, expanding non-negative values to all real numbers. The expansion of the parameters led to the study of model more complicated, but improve the biological reality and significance of the nonlinear cross-diffusion coefficients in this system. In addition, the boundary conditions is homogeneous Neumann boundary conditions, which is also different from the model used in [1,19]. On the other, about the method, when research the existence, stability and Turing instability of equilibria of the model, we do not use new method. But in Section 3, we apply the Lyapunov–Schmidt reduction method to the study of the bifurcation induced by nonlinear cross-diffusion term $\Delta\alpha u_1 u_2(\alpha \in R)$ and $\Delta\left(\frac{u_2}{1+\beta u_1}\right)(\beta \in R)$. Our results showed that Lyapunov–Schmidt reduction method is suitable and can establish some sufficient conditions of the existence of non-constant steady-state solutions near trivial and non-trivial solutions.

There are two problems that keep open. One is the existence of the spatially time-periodic nonhomogeneous steady-state solutions. The other is the effect of two nonlinear cross-diffusion term with delay. To our knowledge, many scholars [20–22] to think about the influence of reaction diffusion with delay, but few papers report about this work. These could be the building blocks of later research. We leave them for future work.

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