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An elliptic problem with two singularities

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Abstract. We study a Dirichlet problem for an elliptic equation defined by a degenerate coercive operator and a singular right-hand side. We will show that the right-hand side has some regularizing effects on the solutions, even if it is singular.

Keywords: degenerate elliptic problem, boundary value problem, weak solutions, singular lower order term

1. Introduction

In this paper we study existence of solutions to the following elliptic problem:

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x)\nabla u}{(1+|u|)^p}\right) = \dfrac{f}{|u|^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N, N \geqslant 3$, p and γ are positive reals, f is an $L^m(\Omega)$ non-negative function and $a : \Omega \to \mathbb{R}$ is a measurable function such that $0 < \alpha \leqslant a(x) \leqslant \beta$, for two positive constants α and β. We are able to prove the existence of distributional solutions u in the sense that for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $u \geqslant c_\omega > 0$ in ω and

$$\int_\Omega a(x)\frac{\nabla u \cdot \nabla\varphi}{(1+u)^p} = \int_\Omega \frac{f}{u^\gamma}\varphi \quad \forall\varphi \in C_0^1(\Omega). \tag{1.2}$$

We point out that the operator $v \to -\operatorname{div}(\frac{a(x)\nabla v}{(1+|v|)^p})$ is not coercive on $H_0^1(\Omega)$, when v is large (see [13]). Moreover, the right-hand side is singular in the variable u. We will overcome these two difficulties by approximation, truncating the degenerate coercivity of the operator term and the singularity of the right-hand side (see problems (2.1)). We will prove by Schauder's theorem that these problems admit a bounded finite energy solution u_n with the property that for every subset $\omega \subset\subset \Omega$ there exists a positive constant $c_\omega > 0$ such that $u_n \geqslant c_\omega$ almost everywhere in ω for every $n \in \mathbb{N}$. This condition, combined with some a priori estimates on u_n obtained in Section 3 will allow us to pass to the limit in the approximating problems and get a solution to problem (1.1) in the sense of (1.2).

We remark that the lack of coercivity of the operator term can negatively affect the existence and regularity of solutions to

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x)\nabla u}{(1+|u|)^p}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

This was first pointed out in [7], in the case $p < 1$. In the case where $p > 1$, the authors of [1] proved that no solution exists even for some constant data f.

A natural question is then to search for lower order terms which regularize the solutions to problem (1.3). This leads to the study of existence and regularity of solutions to problems of the form

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x)\nabla u}{(1+|u|)^p}\right) + g(u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

under various hypotheses on $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$. In [6] the case $g = g(s) = s$ was examined. That work takes advantage of the fact that, just as for semilinear elliptic coercive problems (cf. [9]), the summability of the solutions is at least that of the source f. In [10] we analysed two different lower order terms $g = g(s)$. The first, a generalization of the lower order term studied in [6], is $g(s) = |s|^{r-1}s$, $r > 0$. The second is a continuous positive increasing function such that $g(s) \to +\infty$, as $s \to s_0^-$, for some $s_0 > 0$ (see [3]). The lower order term of [4] is, roughly speaking, $b(|s|)|\xi|^2$ where b is continuous and increasing with respect to $|s|$. In [11] we showed the regularizing effects of the lower order term $g(s, \xi) = \frac{|\xi|^2}{s^q}$, $q > 0$, which grows as a negative power with respect to s and has a quadratic dependence on the gradient variable (see [2] and [5] for elliptic coercive problems with the same lower order term).

In all of the above papers a regularizing effect on the solutions to (1.3) was shown under a sign condition on g: either $g(s, \xi) \geqslant 0$ for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and the source is assumed to be positive or $g(s, \xi)s \geqslant 0$ for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

In this paper we study the effects on the solutions of a different term, $\frac{f}{u^\gamma}$, on the right-hand side. Our inspiration is taken from [8] where the authors considered the same right-hand side, for elliptic semilinear problems whose model is $-\Delta u = \frac{f}{u^\gamma}$, with zero Dirichlet condition on the boundary. Our main result shows that, despite the singularity in u, this term has some regularizing effects on the solutions to (1.3). The regularity depends on the different values of $\gamma - p$: we distinguish the cases $p - 1 \leqslant \gamma < p + 1$, $\gamma = p + 1$, and $\gamma > p + 1$. These statements are made more precise in the following theorem.

Theorem 1.1. *Let* $\gamma \geqslant p - 1$.

(1) *Let* $\gamma < p + 1$.

  (a) *If* $f \in L^m(\Omega)$, *with* $m \geqslant \frac{2^*}{2^*-p-1+\gamma}$, *there exists a solution* $u \in H_0^1(\Omega)$ *to* (1.1) *in the sense of* (1.2). *If* $\frac{2^*}{2^*-p-1+\gamma} \leqslant m < \frac{N}{2}$, *then* u *belongs to* $L^{m^{**}(\gamma+1-p)}(\Omega)$.

  (b) *If* $f \in L^m(\Omega)$, *with* $\max\{1, \frac{1^*}{2\cdot 1^*-p-1+\gamma}\} < m < \frac{2^*}{2^*-p-1+\gamma}$, *there exists a solution* $u \in W_0^{1,\sigma}(\Omega)$, $\sigma = \frac{Nm(\gamma+1-p)}{N-m(p+1-\gamma)}$, *to* (1.1) *in the sense of* (1.2).

(2) *Let* $\gamma = p + 1$ *and assume that* $f \in L^1(\Omega)$. *Then there exists a solution* $u \in H_0^1(\Omega)$ *to* (1.1) *in the sense of* (1.2).

(3) *Let* $\gamma > p + 1$ *and assume that* $f \in L^1(\Omega)$. *Then there exists a solution* $u \in H^1_{\mathrm{loc}}(\Omega)$ *to* (1.1) *in the sense of* (1.2), *such that* $u^{\frac{\gamma+1-p}{2}} \in H_0^1(\Omega)$.

(4) *Let* $f \in L^m(\Omega)$, *with* $m > \frac{N}{2}$. *Then the solution found above is bounded.*

Let us point out the regularizing effects of the right-hand side. It is useful to recall the results obtained in [7] for problem (1.3). Let $p < 1$ and $q = \frac{Nm(1-p)}{N-m(1+p)}$.

(a) If $1 < m \leqslant \frac{2N}{N+2-p(N-2)}$, then there exists $u \in W_0^{1,q}(\Omega)$ or $|\nabla u|^s \in L^1(\Omega)$, $\forall s < q$.
(b) If $\frac{2N}{N+2-p(N-2)} \leqslant m < \frac{N}{2}$, then there exists $u \in H_0^1(\Omega) \cap L^{m^{**}(1-p)}(\Omega)$.
(c) If $m > \frac{N}{2}$, then there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

We now compare the summabilities obtained in Theorem 1.1 to the previous ones. First of all, we have a solution for every $p > 0$, if $\gamma \geqslant p - 1$. This is not the case for problem (1.3), as proved in [1]. Under the same conditions on f, the summability of the solutions to (1.1) is better than or equal to that of the solutions to (1.3), since $\sigma > q$ and $m^{**}(\gamma + 1 - p) > m^{**}(1 - p)$. Moreover, we get $H_0^1(\Omega)$ solutions for less regular sources than in [7]. Indeed, if $p - 1 \leqslant \gamma < p + 1$, one has $\frac{2^*}{2^*-p-1+\gamma} < \frac{2N}{N(1-p)+2(p+1)}$; if $\gamma = p + 1$ we get a finite energy solution for every $L^1(\Omega)$ source.

2. Approximating problems

As explained in the Introduction, we will work on the following approximating problems:

$$\begin{cases} -\mathrm{div}\Big(\dfrac{a(x)\nabla u_n}{(1+|T_n(u_n)|)^p}\Big) = \dfrac{T_n(f)}{(|u_n|+\frac{1}{n})^\gamma} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}$ and

$$T_n(s) = \begin{cases} -n, & s \leqslant -n, \\ s, & -n \leqslant s \leqslant n, \\ n, & s \geqslant n. \end{cases} \tag{2.2}$$

Observe that we have "truncated" the degenerate coercivity of the operator term and the singularity of the right-hand side.

Proposition 2.1. *Problems* (2.1) *are well posed, that is, there exists a non-negative solution* $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *for every fixed* $n \in \mathbb{N}$.

Proof. In this proof we will use the same technique as in [8]. Let $S : L^2(\Omega) \to L^2(\Omega)$ be the map which associates to every $v \in L^2(\Omega)$ the solution $w_n \in H_0^1(\Omega)$ to

$$\begin{cases} -\mathrm{div}\Big(\dfrac{a(x)\nabla w_n}{(1+|T_n(w_n)|)^p}\Big) = \dfrac{T_n(f)}{(|v|+\frac{1}{n})^\gamma} & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Observe that S is well-defined by the results of [12] and w_n is bounded by the results of [14]. Let us choose w_n as a test function. Then,

$$\alpha \int_\Omega \frac{|\nabla w_n|^2}{(1+n)^p} \leqslant \int_\Omega \frac{a(x)|\nabla w_n|^2}{(1+|T_n(w_n)|)^p} = \int_\Omega \frac{T_n(f) w_n}{(|v|+\frac{1}{n})^\gamma} \leqslant n^{\gamma+1} \int_\Omega |w_n| \leqslant |\Omega|^{1/2} n^{\gamma+1} \|w_n\|_{L^2(\Omega)}$$

by the hypotheses on a and Hölder's inequality on the right-hand side. Poincaré's inequality on the left-hand side implies

$$\alpha\mathcal{P}\|w_n\|^2_{L^2(\Omega)} \leqslant |\Omega|^{1/2}(1+n)^p n^{\gamma+1}\|w_n\|_{L^2(\Omega)}.$$

Thus there exists an invariant ball for S. Moreover it is easily seen that S is continuous and compact by the $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ embedding. By Schauder's theorem, S has a fixed point. Therefore there exists a solution $u_n \in H^1_0(\Omega)$ to problems (2.1). Observe that u_n is bounded by the results of [14]; by the maximum principle, u_n is non-negative since f is non-negative. □

Remark 2.2. We remark that the existence of solutions to (2.1) could not be inferred by the results established in [7].

Proposition 2.3. *Let u_n be the solution to problem* (2.1). *Then $u_n \leqslant u_{n+1}$ a.e. in Ω. Moreover for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $u_n \geqslant c_\omega$ a.e. in ω for every $n \in \mathbb{N}$.*

Proof. We will use the same technique as in [13] to prove uniqueness of the solutions to problem (1.3). In the proof C will denote a positive constant independent of n (depending on α, β, p and the constant $\mathcal{P}$ of Poincaré's inequality). The solution u_n to problem (2.1), being non-negative, satisfies

$$-\mathrm{div}\left(\frac{a(x)\nabla u_n}{(1+T_n(u_n))^p}\right) = \frac{T_n(f)}{(u_n+\frac{1}{n})^\gamma} \leqslant \frac{T_{n+1}(f)}{(u_n+\frac{1}{n+1})^\gamma}.$$

Therefore

$$-\mathrm{div}\left(\frac{a(x)\nabla u_n}{(1+T_n(u_n))^p} - \frac{a(x)\nabla u_{n+1}}{(1+T_{n+1}(u_{n+1}))^p}\right) = T_{n+1}(f)\left[\frac{1}{(u_n+\frac{1}{n+1})^\gamma} - \frac{1}{(u_{n+1}+\frac{1}{n+1})^\gamma}\right] \leqslant 0.$$

By choosing $T_k((u_n - u_{n+1})^+)$ as a test function we get

$$\alpha\int_\Omega \frac{|\nabla T_k((u_n-u_{n+1})^+)|^2}{(1+T_n(u_n))^p}$$
$$\leqslant \beta\int_\Omega \nabla u_{n+1}\cdot\nabla T_k\big((u_n-u_{n+1})^+\big)\left[\frac{1}{(1+T_{n+1}(u_{n+1}))^p} - \frac{1}{(1+T_n(u_n))^p}\right]$$

by the hypotheses on a. In $\{0 \leqslant u_n - u_{n+1} \leqslant k\}$ one has

$$\left|\frac{1}{(1+T_{n+1}(u_{n+1}))^p} - \frac{1}{(1+T_n(u_n))^p}\right| \leqslant k\left[\frac{1}{(1+T_{n+1}(u_{n+1}))^p} + \frac{1}{(1+T_n(u_n))^p}\right].$$

Therefore

$$\int_\Omega \frac{|\nabla T_k((u_n-u_{n+1})^+)|^2}{(1+T_n(u_n))^p}$$
$$\leqslant Ck\int_{\{0\leqslant u_n-u_{n+1}\leqslant k\}} |\nabla u_{n+1}|\big|\nabla T_k\big((u_n-u_{n+1})^+\big)\big|\left|\frac{1}{(1+T_{n+1}(u_{n+1}))^p} + \frac{1}{(1+T_n(u_n))^p}\right|.$$

For sufficiently small k, one has in $\{0 \leqslant u_n - u_{n+1} \leqslant k\}$

$$\frac{1}{2^p}\frac{1}{(1+T_n(u_n))^p} \leqslant \frac{1}{(1+T_{n+1}(u_{n+1}))^p} \leqslant \frac{2^p}{(1+T_n(u_n))^p}.$$

This implies that

$$\int_\Omega \frac{|\nabla T_k((u_n-u_{n+1})^+)|^2}{(1+T_n(u_n))^p} \leqslant Ck\int_{\{0\leqslant u_n-u_{n+1}\leqslant k\}} \frac{|\nabla u_{n+1}|}{(1+T_{n+1}(u_{n+1}))^{p/2}}\frac{|\nabla T_k((u_n-u_{n+1})^+)|}{(1+T_n(u_n))^{p/2}}.$$

Hölder's inequality on the right-hand side gives

$$\begin{aligned}&\int_\Omega \frac{|\nabla T_k((u_n-u_{n+1})^+)|^2}{(1+T_n(u_n))^p}\\ &\quad\leqslant Ck\biggl[\int_{\{0\leqslant u_n-u_{n+1}\leqslant k\}} \frac{|\nabla u_{n+1}|^2}{(1+T_{n+1}(u_{n+1}))^p}\biggr]^{\frac12}\biggl[\int_{\{0\leqslant u_n-u_{n+1}\leqslant k\}} \frac{|\nabla T_k((u_n-u_{n+1})^+)|^2}{(1+T_n(u_n))^p}\biggr]^{\frac12},\end{aligned}$$

and then

$$\int_\Omega \frac{|\nabla T_k((u_n-u_{n+1})^+)|^2}{(1+T_n(u_n))^p} \leqslant Ck^2\int_{\{0\leqslant u_n-u_{n+1}\leqslant k\}} \frac{|\nabla u_{n+1}|^2}{(1+T_{n+1}(u_{n+1}))^p}. \tag{2.4}$$

On the other hand, by Poincaré's inequality and (2.4)

$$\begin{aligned}k^2\bigl|\{0\leqslant u_n-u_{n+1}\leqslant k\}\bigr| &\leqslant \int_\Omega \frac{|\nabla T_k((u_n-u_{n+1})^+)|^2}{(1+T_n(u_n))^p}\bigl(1+T_n(u_n)\bigr)^p\\ &\leqslant Ck^2\int_{\{0\leqslant u_n-u_{n+1}\leqslant k\}} \frac{|\nabla u_{n+1}|^2(1+n)^p}{(1+T_{n+1}(u_{n+1}))^p},\end{aligned}$$

that is,

$$\bigl|\{0\leqslant u_n-u_{n+1}\leqslant k\}\bigr| \leqslant C\int_{\{0\leqslant u_n-u_{n+1}\leqslant k\}} \frac{|\nabla u_{n+1}|^2(1+n)^p}{(1+T_{n+1}(u_{n+1}))^p}.$$

The right-hand side of the above inequality tends to 0, as $k\to 0$. Therefore $|\{0\leqslant u_n-u_{n+1}\leqslant k\}|\to 0$ as $k\to 0$. This implies that $u_n\leqslant u_{n+1}$ a.e. in Ω.

We remark that u_1 is bounded, that is, $|u_1|\leqslant c$, for some positive constant c. Setting $h(s)=\int_0^s \frac{\mathrm{d}t}{(1+T_1(t))^p}$, we have

$$-\mathrm{div}\bigl(a(x)\nabla\bigl(h(u_1)\bigr)\bigr) = -\,\mathrm{div}\biggl(a(x)\frac{\nabla u_1}{(1+T_1(u_1))^p}\biggr) \geqslant \frac{T_1(f)}{(c+1)^\gamma}.$$

Let z be the $H_0^1(\Omega)$ solution to $-\mathrm{div}(a(x)\nabla z)=\frac{T_1(f)}{(c+1)^\gamma}$. By the strong maximum principle, for every $\omega\subset\subset\Omega$ there exists a positive constant c_ω such that $z\geqslant c_\omega$ a.e. in ω. By the comparison principle, we have $h(u_1) \geqslant z$ a.e. in Ω. The strict monotonicity of h implies the existence of a constant $c_\omega > 0$, for every $\omega \subset\subset \Omega$, such that $u_1 \geqslant c_\omega$ a.e. in ω. Since u_n is an increasing sequence, as proved above, $u_n \geqslant c_\omega$ a.e. in ω for every $n \in \mathbb{N}$. $\square$

3. Existence results

We are going to prove in the following three lemmata some a priori estimates on the solutions u_n to problems (2.1). They will allow us to prove Theorem 1.1. In the proofs C will denote a positive constant independent of n.

Lemma 3.1. *Assume that* $p-1 \leqslant \gamma < p+1$.

(1) *Let* $f \in L^m(\Omega)$, *with* $m \geqslant \frac{2^*}{2^*-p-1+\gamma}$. *Then the solutions* u_n *to* (2.1) *are uniformly bounded in* $H_0^1(\Omega)$. *If* $\frac{2^*}{2^*-p-1+\gamma} \leqslant m < \frac{N}{2}$ *then the solutions* u_n *are uniformly bounded in* $L^{m^{**}(\gamma+1-p)}(\Omega)$.

(2) *Let* $f \in L^m(\Omega)$, *with* $\max\{1, \frac{1^*}{2\cdot 1^*-p-1+\gamma}\} < m < \frac{2^*}{2^*-p-1+\gamma}$. *Then the solutions* u_n *to* (2.1) *are uniformly bounded in* $W_0^{1,\sigma}(\Omega)$, $\sigma = \frac{Nm(\gamma+1-p)}{N+\gamma m-m(p+1)}$.

Proof. In case (1) let us choose $(1+u_n)^{p+1}-1$ as a test function; the hypotheses on a imply that

$$\alpha(p+1)\int_\Omega \frac{|\nabla u_n|^2}{(1+T_n(u_n))^p}(1+u_n)^p \leqslant C\int_\Omega |f|u_n^{p+1-\gamma}.$$

By Sobolev's inequality on the left-hand side and Hölder's inequality with exponent $\overline{m} = \frac{2^*}{2^*-p-1+\gamma} = \frac{2N}{N(\gamma+1-p)+2(p+1-\gamma)}$ (>1) on the right one, we have

$$\mathcal{S}\alpha(p+1)\|u_n\|^2_{L^{2^*}(\Omega)} \leqslant \alpha(p+1)\|\nabla u_n\|^2_{L^2(\Omega)} \leqslant \|f\|_{L^{\overline{m}}(\Omega)}\left[\int_\Omega |u_n|^{\overline{m}'(p+1-\gamma)}\right]^{\frac{1}{\overline{m}'}}.$$

We remark that $2^* = \overline{m}'(p+1-\gamma)$. Moreover, $\frac{2}{2^*} \geqslant \frac{1}{\overline{m}'}$ as $\gamma \leqslant p+1$. Then the above estimate implies that the sequence u_n is bounded in $L^{2^*}(\Omega)$ and in $H_0^1(\Omega)$.

We are now going to prove that u_n is bounded in $L^{m^{**}(\gamma+1-p)}(\Omega)$, if $m < \frac{N}{2}$. Let us choose $(1+u_n)^\delta - 1$ as a test function: by the hypotheses on a, one has

$$\begin{aligned}
&\frac{4\alpha\delta}{(-p+\delta+1)^2}\int_\Omega \left|\nabla\left[(1+u_n)^{\frac{-p+\delta+1}{2}}-1\right]\right|^2 \\
&\quad = \alpha\delta\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{p-\delta+1}} \leqslant \alpha\delta\int_\Omega \frac{|\nabla u_n|^2}{(1+T_n(u_n))^p}(1+u_n)^{\delta-1} \\
&\quad \leqslant \int_\Omega \frac{T_n(f)}{(u_n+\frac{1}{n})^\gamma}\left[(u_n+1)^\delta - 1\right] \leqslant C + C\int_\Omega \frac{|f|}{(u_n+1)^{-\delta+\gamma}}.
\end{aligned}$$

By Sobolev's inequality on the left-hand side and Hölder's inequality on the right-hand one we have

$$\left[\int_\Omega \left|\left[(1+u_n)^{\frac{-p+\delta+1}{2}}-1\right]\right|^{2^*}\right]^{\frac{2}{2^*}} \leqslant C\|f\|_{L^m(\Omega)}\left[\int_\Omega |u_n+1|^{m'(\delta-\gamma)}\right]^{\frac{1}{m'}}.$$

Let δ be such that $\frac{(1+\delta-p)N}{N-2} = \frac{(\delta-\gamma)m}{m-1}$ and $\frac{2}{2^*} \geqslant \frac{1}{m'}$, that is,

$$\delta = \frac{(1-p)N(m-1) + \gamma m(N-2)}{N-2m}$$

and $m \leqslant \frac{N}{2}$. We observe that $(-p+\delta+1)\frac{2^*}{2} = m^{**}(\gamma+1-p) > 1$. This implies that u_n is bounded in $L^{m^{**}(\gamma+1-p)}(\Omega)$.

In case (2), let us choose $(1+u_n)^\theta - 1$, $\theta = \frac{(p-1)N(m-1)-\gamma m(N-2)}{2m-N}$, as a test function. With the same arguments as before, we have

$$\left[\int_\Omega \left|\left[(1+u_n)^{\frac{-p+\theta+1}{2}} - 1\right]\right|^{2^*}\right]^{\frac{2}{2^*}}$$
$$\leqslant C\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{p-\theta+1}} \leqslant C\|f\|_{L^m(\Omega)}\left[\int_\Omega |u_n+1|^{m'(\theta-\gamma)}\right]^{\frac{1}{m'}}.$$

As above, we infer that u_n is bounded in $L^{\frac{N(1+\theta-p)}{N-2}}(\Omega)$. We observe that $p-\theta+1 > 0$ and $1 < \sigma = \frac{Nm(\gamma+1-p)}{N-m(p+1-\gamma)} < 2$, by the assumptions on m. Writing

$$\int_\Omega |\nabla u_n|^\sigma = \int_\Omega \frac{|\nabla u_n|^\sigma}{(1+u_n)^{\sigma\frac{p-\theta+1}{2}}}(1+u_n)^{\sigma\frac{p-\theta+1}{2}}$$

and using Hölder's inequality with exponent $\frac{2}{\sigma}$, we obtain

$$\int_\Omega |\nabla u_n|^\sigma \leqslant \left[\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{p-\theta+1}}\right]^{\frac{\sigma}{2}}\left[\int_\Omega (1+u_n)^{\sigma\frac{p-\theta+1}{2-\sigma}}\right]^{\frac{2-\sigma}{2}}.$$

The above estimates imply that the sequence u_n is bounded in $W_0^{1,\sigma}(\Omega)$ if $\sigma\frac{p-\theta+1}{2-\sigma} = \frac{N(1+\theta-p)}{N-2}$, that is, $\sigma = \frac{Nm(\gamma+1-p)}{N-m(p+1-\gamma)}$. □

Lemma 3.2. *Assume that $\gamma = p+1$ and $f \in L^1(\Omega)$. Then the solutions u_n to* (2.1) *are uniformly bounded in $H_0^1(\Omega)$.*

Proof. Let us choose $(1+u_n)^{p+1} - 1$ as a test function. Using that $a(x) \geqslant \alpha$ a.e. in Ω, we have

$$\alpha(p+1)\int_\Omega \frac{|\nabla u_n|^2}{(1+T_n(u_n))^p}(1+u_n)^p \leqslant C\int_\Omega |f|.$$

The previous estimate implies that the sequence u_n is bounded in $H_0^1(\Omega)$. □

Lemma 3.3. *Assume that $\gamma > p+1$ and $f \in L^1(\Omega)$. Then the solutions u_n to* (2.1) *are such that $u_n^{\frac{\gamma+1-p}{2}}$ is uniformly bounded in $H_0^1(\Omega)$, u_n is uniformly bounded in $L^{\frac{\gamma+1-p}{2}2^*}(\Omega)$ and in $H_{\mathrm{loc}}^1(\Omega)$.*

Proof. If we choose u_n^γ as a test function and use the hypotheses on a we get

$$\frac{4\alpha\gamma}{(\gamma+1-p)^2}\int_\Omega \left|\nabla\left(u_n^{\frac{\gamma+1-p}{2}}\right)\right|^2 = \alpha\gamma\int_\Omega |\nabla u_n|^2 u_n^{\gamma-1-p} \leqslant \int_\Omega |f|.$$

This proves that the sequence $u_n^{\frac{\gamma+1-p}{2}}$ is bounded in $H_0^1(\Omega)$. Sobolev's inequality on the left-hand side applied to $u_n^{\frac{\gamma+1-p}{2}}$ gives

$$\int_\Omega u_n^{\frac{\gamma+1-p}{2}2^*} \leqslant C. \tag{3.1}$$

Let us prove that u_n is bounded in $H^1_{\mathrm{loc}}(\Omega)$. For $\varphi \in C_0^1(\Omega)$ we choose $[(u_n+1)^{p+1}-1]\varphi^2$ as a test function in (2.1). Then, if $\omega = \{\varphi \neq 0\}$, one has by the hypotheses on a and Lemma 2.3

$$\begin{aligned}\alpha(p+1)\int_\Omega |\nabla u_n|^2\varphi^2 + 2\alpha\int_\Omega \nabla u_n\cdot\nabla\varphi\varphi u_n &\leqslant \int_\Omega \frac{|f|}{(u_n+\frac{1}{n})^\gamma}\left[(u_n+1)^{p+1}-1\right]\varphi^2\\ &\leqslant \int_\Omega C_\omega |f|\varphi^2 \leqslant C_\omega\|\varphi\|^2_{L^\infty(\Omega)}\int_\Omega |f|,\end{aligned}$$

where C_ω is a positive constant depending only on $\omega \subset\subset \Omega$ and p. Then

$$(p+1)\int_\Omega |\nabla u_n|^2\varphi^2 \leqslant -2\int_\Omega \nabla u_n\cdot\nabla\varphi\varphi u_n + \frac{C_\omega}{\alpha}\|\varphi\|^2_{L^\infty(\Omega)}\int_\Omega |f|. \tag{3.2}$$

Young's inequality implies that

$$2\left|\int_\Omega \nabla u_n\cdot\nabla\varphi\varphi u_n\right| \leqslant \int_\Omega |\nabla u_n|^2\varphi^2 + \int_\Omega |\nabla\varphi|^2 u_n^2.$$

From (3.2) we infer that

$$p\int_\Omega |\nabla u_n|^2\varphi^2 \leqslant \int_\Omega |\nabla\varphi|^2 u_n^2 + \frac{C_\omega}{\alpha}\|\varphi\|^2_{L^\infty(\Omega)}\int_\Omega |f|.$$

Since u_n is uniformly bounded in $L^2(\Omega)$ by (3.1), this proves our result. □

We are now able to prove Theorem 1.1.

Proof. We will prove point (1); the second and the third point can be proved in a similar way. Lemma 3.1 gives the existence of a function $u \in H_0^1(\Omega)$ such that $u_n \to u$ weakly in $H_0^1(\Omega)$ and a.e. in Ω, up to a subsequence. We will prove that u is a solution to (1.1) passing to the limit in (2.1). For every $\varphi \in C_0^1(\Omega)$,

$$\int_\Omega \frac{a(x)\nabla u_n\cdot\nabla\varphi}{(1+T_n(u_n))^p} \quad\longrightarrow\quad \int_\Omega \frac{a(x)\nabla u\cdot\nabla\varphi}{(1+u)^p},$$

since $\frac{1}{(1+T_n(u_n))^p} \to \frac{1}{(1+u)^p}$ in $L^r(\Omega)$, for every $r \geqslant 1$. For the limit of the right-hand side of (2.1), let $\omega = \{\varphi \neq 0\}$. One can use Lebesgue's theorem, since

$$\left|\frac{T_n(f)\varphi}{(u_n + \frac{1}{n})^\gamma}\right| \leqslant \frac{|\varphi||f|}{c_\omega^\gamma},$$

where c_ω is the constant given by Lemma 2.3. In the case where $\frac{2^*}{2^*-p-1+\gamma} \leqslant m < \frac{N}{2}$, since $m^{**}(\gamma + 1 - p) > 1$, Lemma 3.1 implies that u_n converges weakly in $L^{m^{**}(\gamma+1-p)}(\Omega)$ to some function, which is u by identification.

To prove that u is bounded for $\gamma \geqslant p - 1$, let us choose $[(u_n + 1)^{\gamma+1} - (k+1)^{\gamma+1}]_+$ as a test function in (2.1):

$$\alpha(\gamma+1)\int_{A_k} \frac{|\nabla u_n|^2}{(1+u_n)^{p-\gamma}} \leqslant \int_{A_k} |f| \frac{(u_n+1)^{\gamma+1} - (k+1)^{\gamma+1}}{(u_n + \frac{1}{n})^\gamma} \leqslant c(\gamma) \int_{A_k} |f|(u_n - k), \tag{3.3}$$

where $A_k = \{u_n \geqslant k\}$ and $c(\gamma)$ denotes a positive constant depending only on γ.

For $p - \gamma \leqslant 0$, (3.3) is the starting point of the proof of Theorem 4.1 in [14]. For $0 < p - \gamma \leqslant 1$, (3.3) is the starting point of Lemma 2.2 in [7]. In both cases u_n is uniformly bounded in $L^\infty(\Omega)$ and therefore (at the a.e. limit) the solutions u found in the previous results are bounded. $\square$

Remark 3.4. We observe that we have the boundedness of the solution to problem (1.1) for any value of $\gamma \geqslant p - 1$.

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