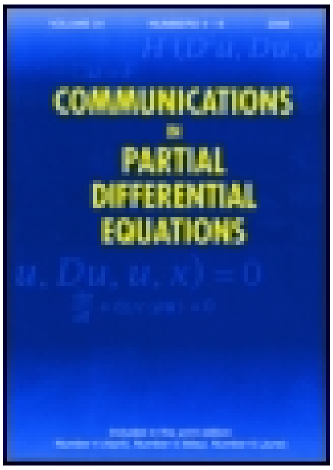

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J. -M. Rakotoson [a]

[a] Université de Poitiers , 40, Avenue du Recteur Pineau, 86022, France
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QUASILINEAR EQUATIONS AND SPACES OF CAMPANATO-MORREY TYPE

J.-M. Rakotoson

Université de Poitiers

40, Avenue du Recteur Pineau

86022 - POITIERS Cedex — FRANCE

0. Introduction

In recent papers [R], [R-Z], we studied some optimal conditions relating the solution u of $Pu = Au + F(u,\nabla u) = T$ and the choice of the right hand side T.

Namely in [R-Z], for a class of quasilinear operator P, for $T = \mu$ a non negative Radon measure of $W^{-1,q}(\Omega)$, we have shown the following equivalence $u \in C^{0,\alpha}_{loc}(\Omega)$ if and only if $\forall \Omega' \Subset \Omega$, $\exists c(\Omega') > 0$, $\exists \varepsilon > 0$ such that :

$$\mu\big(B(x,r) \le c.r^{N-p+\varepsilon} \ , \quad \frac{1}{p} + \frac{1}{q} = 1$$

for any ball of radius $r > 0$: $B(x,2r) \subset \Omega' \subset \mathbb{R}^N$.

A simple and direct consequence of this equivalence is : that we get easily the $W^{1,s}_{loc}(\Omega)$-regularity for variational inequality with irregular obstacle.

Notice that the first result on the proof of this equivalence was obtained by [L-S] in the case where $Pu = \Delta u$.

In [R], we extend such result by enlarging the class of operator P and also the class of right hand side T. We were lead to introduce new spaces in $W^{-1,q}(\Omega)$ that we call Morrey space and denote $M^{-1,q}_{\lambda}(\Omega)$, $\lambda \geq 0$, which is roughly speaking the set of all distributions T in $W^{-1,q}(\Omega)$ such that the restriction on any ball of its norm $\|T\|_{W^{-1,q}(B_r)}$ grows like $r^{\lambda/q}$.

Such space contains not only the class of measure quoted above but also any distribution T which has a decomposition $T = f_0 - \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$ where f_i belong to Campanato-Morrey space $\mathcal{L}^{q,\lambda}(\Omega)$. Such distributions have been also used by Campanato, we will denote this class : $C^{-1,q}_{\lambda}(\Omega)$. One of the new results of this paper is that $C^{-1,q}_{\lambda}(\Omega) = M^{-1,q}_{\lambda}(\Omega)$, $N\text{-}p<\lambda<N$ and $C^{-1,2}_{\lambda}(\Omega) = M^{-1,2}_{\lambda}(\Omega)$, $N<\lambda\leq N+2$. This makes complete the results of [R]. We have shown that for a large class of strongly quasilinear equations $Au + F(u,\nabla u) = T$, if the solution u is locally bounded, then we have the following optimal condition

$$T \in M^{-1,q}_{\lambda,loc}(\Omega) \quad \text{for} \quad \lambda > N\text{-}p, \quad \frac{1}{p} + \frac{1}{q} = 1$$

if and only if

$\forall \Omega' \Subset \Omega \subset \mathbb{R}^N, \quad \exists c(\Omega') > 0, \quad \varepsilon > 0$ such that

$\int_{B(x,r)} |\nabla u|^p dy \leq c.r^{N-p+\varepsilon}$ for all $x \in \Omega'$, $r>0$ such that $B(x,2r) \subset \Omega'$.

Such result extend Campanato's results for linear operator (see [C]).

In particular, this result implies that $u \in C^{0,\alpha}_{loc}(\Omega)$ (as a consequence of the Dirichlet growth of Morrey).

Since we invoked here the fact that the solution has to be locally bounded, we will show that for $T \in M^{-1,q}_{\lambda,loc}(\Omega)$, $\lambda > N-p$ and a class of quasilinear equation P, the solution of $Pu = T$ is locally bounded. A lot of methods has been developped for getting L^∞-estimate (see for instance [S], [H-S], [S-E], [R], [R-T], [D-T], [R-Z] and reference therein). The method, we introduce here is a combination of an idea used in [R] and a method used by [L-U] (see also [D-T] and [R-Z]).

Thanks to the decompositions of T, we can associate a measure m to T, which will allow us to use Adams' inequality.

We will end this paragraph by a remark completing the results of Boccardo-Giachetti [B-G] for L^∞-estimate.

The last paragraph will be devoted to $C^{1,\alpha}_{loc}$-results for quasilinear equations. In similar way, we will study the optimality of the choice of the right hand side T in order to have the solution $u \in C^{1,\alpha}_{loc}$.

The organization of our paper will be

I — Preliminary results

II — A decomposition of an element $T \in M^{-1,q}_{\lambda}(\Omega)$, $N>\lambda>N-p$

III — Local L^∞-estimate for the solution u of $Pu = T$, where $T \in M^{-1,q}_{\lambda,\mathrm{loc}}(\Omega)$, $N-p<\lambda$

IV — On $C^{1,\alpha}_{\mathrm{loc}}$-regularity.

I. Preliminary results — Notation

Ω will denote a subset of $\mathbb{R}^N$, $\Omega_\rho = B(x,\rho) \cap \Omega$, $B_\rho = B(x,\rho) = B(\rho)$ ball of center x and radius $\rho > 0$, $|\Omega_\rho|$ is the measure of Ω_ρ.

I.1.— Classical Campanato-Morrey in $L^q(\Omega)$, $1 \le q \le +\infty$.

DEFINITION 1.— Morrey-spaces $L^{q,\lambda}(\Omega)$, $\lambda \ge 0$.

$$L^{q,\lambda}(\Omega) = \left\{ u \in L^q(\Omega),\ \sup_{\substack{\rho>0 \\ x\in\Omega_\rho}} \rho^{-\lambda/q} \|u\|_{L^q(\Omega_\rho)} \ \textit{is finite} \right\}.$$

DEFINITION 2.— Campanato-Spaces $\mathcal{L}^{q,\lambda}(\Omega)$.

Let us denote the average :

$$u_{x,\rho} = \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} u(y)dy = ⨍_{\Omega_\rho} u(y)dy$$

$$\mathcal{L}^{q,\lambda}(\Omega) = \left\{ u \in L^q(\Omega),\ \sup_{\substack{x\in\Omega \\ \rho>0}} \rho^{-\lambda/q} \|u - u_{x,\rho}\|_{L^q(\Omega_\rho)} \ \textit{is finite} \right\}.$$

For more details on those spaces see [Mo], [C], [Da P]. In particular, we will use the :

PROPOSITION 1.— *We have, for smooth bounded domain* Ω :

$\mathcal{L}^{q,\lambda}(\Omega)$ *is isomorphic to the Hölder-space.* $C^{0,\frac{\lambda-N}{q}}(\bar{\Omega})$, $N<\lambda\leq N+q$.

$\mathcal{L}^{q,\lambda}(\Omega)$ *is isomorphic to* $L^{q,\lambda}(\Omega)$, $0\leq\lambda<N$.

$L^{q,N}(\Omega)$ *is isomorphic to* $L^{\infty}(\Omega)$.

I.2.— Morrey-spaces in $W^{-1,q}(\Omega)$.

In [R], we define :

DEFINITION 3.— *We define in* $W^{-1,q}(\Omega)$ $\Big($*dual space of* $W_0^{1,p}(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1\Big)$ *the space* $M_\lambda^{-1,q}(\Omega)$ *by*

$$M_\lambda^{-1,q}(\Omega) = \Big\{T \in W^{-1,q}(\Omega),\ \sup_{\substack{\rho>0\\ x\in\Omega}} \rho^{-\lambda/q}\, \|T\|_{W^{-1,q}(\Omega_\rho)} \ \textit{is finite}\Big\}.$$

Properties of this space can be found in [R], in particular, we have :

PROPOSITION 2.— *If we set :*

$$C_\lambda^{-1,q}(\Omega) = \Big\{T \in W^{-1,q}(\Omega) : T = f_0 - \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i},\quad f_i \in \mathcal{L}^{q,\lambda}(\Omega)\Big\}$$

then $\qquad C_\lambda^{-1,q}(\Omega) \subset M_\lambda^{-1,q}(\Omega)$.

It is not known whether $C_\lambda^{-1,q}(\Omega) = M_\lambda^{-1,q}(\Omega)$ for all $\lambda\geq 0$. One of our results will show : $C_\lambda^{-1,q}(\Omega) = M_\lambda^{-1,q}(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$, Ω bounded for $N-p<\lambda<N$ and $C_\lambda^{-1,2}(\Omega) = M_\lambda^{-1,2}(\Omega)$, $\forall\lambda>N-2$.

I.3.— Notations

From now on, we deal with a solution $u \in W^{1,p}_{loc}(\Omega)$, $1<p<+\infty$ of the equation :

$$Au + F(u,\nabla u) = T \tag{1.1}$$

The Leray-Lions operator A can be written :

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,u,\nabla u)$$

where,

$$\sum_{i=1}^{N} a_i(x,u,z)z_i \geq \nu_0(|u|)|z|^p - \nu_1(x) \tag{1.2}$$

for almost all $x \in \Omega$, all z in $\mathbb{R}^N$, all $u \in \mathbb{R}$, ν_0 is a continuous function such that $\nu_0 > 0$ and :

$$\Big(\sum_{i=1}^{N} |a_i(x,u,z)|^2\Big)^{1/2} \leq a(|u|)\Big(|z|^{p-1} + a_0(x)\Big) \tag{1.3}$$

where : a is an increasing from $\mathbb{R}_+$ to $\mathbb{R}_+$, $\frac{1}{p} + \frac{1}{q} = 1$

$$|F(u,z)| = |F(x,u,z)| \leq a(|u|)\Big(g_0(x) + |z|^s\Big) \tag{1.4}$$

$$T \in W^{-1,q}_{loc}(\Omega).$$

We will precise the class of function where ν_1 , a_0 , g_0 will be and also the value of s.

DEFINITION 4.— *We will say that the operator* A *is strongly monotonic if the coefficients* a_i *satisfy, one of the following inequalities : there exist constants* $c > 0$, $\eta \geq 0$ *such that, for almost all* $x \in \Omega$, *all* $u \in \mathbb{R}$, all z in $\mathbb{R}^N$, $z' \in \mathbb{R}^N$:

(E_1) $1 \le p \le 2$

$$\sum_{i=1}^{N} [a_i(x,u,z)-a_i(x,u,z')][z_i - z'_i] \ge c \frac{|z - z'|^2}{\left(\eta+|z|+|z'|\right)^{2-p}}$$

or

(E_2) $p \ge 2$

$$\sum_{i=1}^{N} [a_i(x,u,z)-a_i(x,u,z')][z_i - z'_i] \ge c\, |z - z'|^p$$

We have shown (see [R], [R-Z]).

THEOREM 0.— *Assume that* $u \in L^{\infty}_{loc}(\Omega) \cap W^{1,p}_{loc}(\Omega)$ *is solution of* **(1.1)** *with the structure* **(1.2)** *to* **(1.4)** *with* $s = p-\varepsilon$ $(\forall \varepsilon>0)$, ν_1 *and* g_0 *on* $L^{1,\lambda}_{loc}(\Omega)$, a_0 *is in* $L^{q,\lambda}_{loc}(\Omega)$, $\lambda>N-p$. *If* A *is strongly monotonic in the sense stated above then :*

$$T \in M^{-1,q}_{\lambda,loc}(\Omega), \quad \lambda>N-p$$

if and only if for every relatively compact open set Ω' *in* Ω, *there exist* $c(\Omega') > 0$, $\varepsilon>0$ *such that*

$$\int_{B(x,r)} |\nabla u|^p dy \le c.r^{N-p+\varepsilon}$$

all x, r *such that* $B(x,2r) \subset \Omega'$.

The following theorem is a direct consequence of a result due to Adams D. For more details, see [M, p.54] or [Z, p.213].

THEOREM 0.1 *(Adams' inequality* [A]*)*.— *Let* m *be a positive Radon measure supported in* Ω *such that there exists a constant* $M > 0$, *for all* $x \in \mathbb{R}^N$, $0<r<+\infty$, *we have*

$$m\left(B(x,r)\right) \leq M\ r^{a}, \quad a = s\left(\frac{N}{p} - 1\right)$$

$1<p<s<+\infty$, $p<N$. *If* $u \in W_0^{1,p}(\Omega)$, *then* :

$$\left[\int_{\Omega} |u|^{s} dm\right]^{1/s} \leq c\ M^{1/s}\ \|\nabla u\|_{L^{p}(\Omega)}$$

$$C = C(p,s,N).$$

(This version is the one given in [Z])

II. A decomposition of an element $T \in M_{\lambda}^{-1,q}(\Omega)$, $\lambda>N-p$

In this paragraph, we want to show :

THEOREM 1.— *Let* $T \in M^{-1,q}_{\lambda,loc}(\Omega)$ *for* $N>\lambda>N-p$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then there exist* $f_i \in L^{q,\lambda}_{loc}(\Omega)$, $N>\lambda>N-p$ *such that*

$$T = -\sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}\ .$$

Proof.— Consider T in $M^{-1,q}_{\lambda,loc}(\Omega)$ and Ω' a relatively compact open set of Ω. Define :

$$\begin{cases} -\mathrm{div}\left(\left(1+|\nabla u|^{2}\right)^{\frac{p}{2}-1} \nabla u\right) = T \quad \text{in} \quad \Omega' \\ u \in W_0^{1,p}(\Omega') \end{cases} \tag{2.1}$$

and v such that :

$$\begin{cases} -\mathrm{div}\left(\left(1+|\nabla v|^{2}\right)^{\frac{p}{2}-1} \nabla v\right) = 0 \quad \text{in} \quad B(x,3r) \\ u - v \in W_0^{1,p}\left(B(x,3r)\right) \end{cases} \tag{2.2}$$

where $B(x,3r) \subset \Omega'$. We will denote by $c>0$ different constants independant of x and r. The following properties are easy to check.

PROPOSITION 2.— *There exist constant* $c>0$ *such that*

a) $$⨍_{B(x,3r)} |\nabla v|^p \, dy \le c + c.⨍_{B(x,3r)} |\nabla u|^p \, dy \qquad (1)$$

b) *There exists a constant* $q>p$ *depending only* N *and* p *such that :*

$$\left[⨍_{B(x,r)} |\nabla v|^q \, dy\right]^{1/q} \le c.\left[⨍_{B(x,2r)} |\nabla v|^p \, dy\right]^{1/p} + c.$$

Proof of proposition 2.

a) It suffices to multiply equation **(2.2)** by $v-u$.

b) While for the statement **b)** [(1)] is a classical result due to Meyer (see Giaquinta [G]). The proof relies on using as test function

[(1)] $\int$ denotes an average.

$\varphi^p(v - v_{x,\rho})$ where $0<2\rho\le 5/2\ r$, $\varphi \in C_0^\infty(B(x,2\rho))$, $\varphi = 1$ or $B(x,\rho)$, $0\le\varphi\le 1$, $v_{x,\rho}$ is the average over the ball of radius ρ. ∎

The following computations and results are due to Lewis (see [L]) and easy to derive :

$$\operatorname{div}\left((1+|\nabla v|^2)^{\frac{p}{2}-1} \nabla v\right) = a_{ij}\, v_{x_i x_j} = 0 \qquad (2.3)$$

where repeated indices denote summation from 1 to N and

$$a_{ij} = (1+|\nabla v|^2)^{\frac{p}{2}-1} \left[(p-2)v_{x_i} v_{x_j}(1+|\nabla v|^2)^{-1} + \delta_{ij}\right]$$
$$= (1+|\nabla v|^2)^{\frac{p}{2}-1} b_{ij}\,, \qquad |b_{ij}| \le p-1.$$

If we set $\nu = (1+|\nabla v|^2)^{p/2}$, then ν satisfies

$$L\nu = (b_{ij}\, \nu_{x_j})_{x_i} = pa_{ij}\, v_{x_k x_j}\, v_{x_k x_i} = g \ge 0 \qquad (2.4)$$

and L is a strongly linear operator, indeed :

$$\lambda_1|z|^2 \le b_{ij}\, z_i\, z_j \le \lambda_2|z|^2 \quad \text{where} \tag{2.5}$$
$$\lambda_1 = \min(p-1,1), \quad \lambda_2 = \max(p-1,1).$$

Relation **(2.4)** implies that ν is a supersolution for the operator L. Hence we can apply the following result on linear equation with bounded coefficients (that can be found in [G-T]).

LEMMA 1.— *For all* $\gamma>1$, *there exists a constant* $c(\gamma) > 0$ *independant of* r *and* x *such that :*

$$\operatorname*{Sup}_{y\in B_r(x)} \nu(y) \le c \left[⨍_{B_r(x)} \nu(y)^\gamma \, dy \right]^{1/\gamma} .$$

Here, $\nu = (1+|\nabla v|^2)^{p/2}$ *is a supersolution of* L.

A simple combination of Proposition **2** and lemma **1**, leads to :

PROPOSITION 3.— *There are constants* $c>0$:

$$\operatorname*{Max}_{y\in B_r(x)} |\nabla v(y)| \le c + c \left(⨍_{B(x,3r)} |\nabla u|^p \, dy \right)^{1/p} .$$

Proof.— $\displaystyle\operatorname*{Max}_{B_r(x)} |\nabla v(y)|^p \le \operatorname{Max} \nu \le c \left(⨍_{B(3r)} |\nabla v|^{p\gamma} \, dy \right)^{1/\gamma} + c.$

Choosing γ such that $q = p\gamma$ (where q is define in proposition **2**) we then have :

$$\operatorname*{Max}_{B_r} |\nabla v| \le c \left(⨍_{B(3r)} |\nabla v|^q \, dy \right)^{1/q} + c \le c \left(⨍_{B(3r)} |\nabla v|^p \right)^{1/p} + c$$

$$\operatorname*{Max}_{B_r} |\nabla v| \le c \left(⨍_{B(x,3r)} |\nabla u|^p \, dy \right)^{1/p} + c. \qquad ■$$

LEMMA 2.— *There exist* $c>0$, $\varepsilon>0$, *such that for all* x, r : $B(x,3r) \subset \Omega'$ *and all* $0<\rho<r$:

$$\int_{B(x,3\rho)} |\nabla u|^p \, dy \le c\left(\frac{\rho}{r}\right)^N \int_{B(x,3r)} |\nabla u|^p \, dy + c.r^{N-p+\varepsilon} .$$

Proof.— <u>Case $p\ge 2$</u>

Let us substract equation **(2.2)** by **(2.1)**, then :

$$- \operatorname{div}\Big((1+|\nabla u|^2)^{\frac{p}{2}-1} \nabla u\Big) + \operatorname{div}\Big((1+|\nabla v|^2)^{\frac{p}{2}-1} \nabla v\Big)=T \qquad \textbf{(2.6)}$$

Multipliying by u-v, equation(2.6) gives :

$$\int_{B(x,3r)} \Big((1+|\nabla u|^2)^{\frac{p}{2}-1} \nabla u-(1+|\nabla v|^2)^{\frac{p}{2}-1} \nabla v,\nabla(u-v)\Big)dx = \langle T,u-v\rangle \qquad \textbf{(2.7)}.$$

One can check that there exist $c>0$:

$$\int_{B(x,3r)} \Big((1+|\nabla u|^2)^{\frac{p}{2}-1} \nabla u-(1+|\nabla v|^2)^{\frac{p}{2}-1} \nabla v,\nabla(u-v)\Big)dy\ge c \int_{B(x,3r)} |\nabla(u-v)|^p dy$$

and

$$|\langle T,u-v\rangle| \le \|T\|_{W^{-1,q}(B(x,3r))} . \|\nabla(u-v)\|_{L^p(B(x,3r))} .$$

Assuming that $T \in M^{-1,q}_{\lambda,loc}(\Omega)$ for $\lambda>N-p$ we derive the existence of $c>0$, $\varepsilon>0$:

$$\int_{B(x,3r)} |\nabla(u-v)|^p \, dy \le c.r^{N-p+\varepsilon}.$$

Thus if $0\le\rho\le r/3$ (The case $\rho>r/3$ is obviuos) :

$$\int_{B(3\rho)} |\nabla u|^p \, dy \le c \int_{B(x,3r)} |\nabla(u-v)|^p \, dy + c \int_{B(x,3\rho)} |\nabla v|^p \, dy$$

$$\le c.r^{N-p+\varepsilon} + c\, \rho^N \operatorname*{Max}_{B_r} |\nabla v|^p$$

$$\le c.r^{N-p+\varepsilon} + c\left(\frac{\rho}{r}\right)^N \int_{B(x,3r)} |\nabla u|^p \, dy + c\, \rho^N$$

$$\int_{B(x,3\rho)} |\nabla u|^p \, dy \le c\left(\frac{\rho}{r}\right)^N \int_{B(x,3r)} |\nabla u|^p \, dy + c.r^{N-p+\varepsilon} \qquad \textbf{(2.8)}$$

■

<u>End of the proof of theorem **1** for $p \geq 2$</u>

We use need the following whose proof can be found in [G] see also [C],

LEMMA 3.— *Let* ϕ *be a non negative and non decreasing function. Suppose that there are constants* $A>0$, $B>0$, $0<\beta<\alpha$, $\varepsilon>0$

$$\phi(\rho) \leq A\left(\frac{\rho}{r}\right)^{\alpha} \left[\phi(r)+\varepsilon\right] + Br^{\beta}$$

for all $\rho \leq r < \mathrm{diam}(\Omega)$. *Then, there exist* $c>0$ *and* ε_0 *depending on* (A,α,β) *if* $0<\varepsilon\leq\varepsilon_0$, *we have :*

$$\phi(\rho) \leq c\left(\frac{\rho}{r}\right)^{\beta} \left[\phi(r)+Br^{\beta}\right].$$

Applying this lemma with lemma 2 and setting :

$$\phi(\rho) = \int_{B(x,3\rho)} |\nabla u|^{p}\, dy, \quad \beta = N-p+\varepsilon, \quad \alpha = N.$$

We conclude that $(1+|\nabla u|^2)^{\frac{p}{2}-1} \nabla u$ is in $\left(L^{q,\lambda}_{loc}(\Omega)\right)^N$ for $\lambda>N-p$, $\frac{1}{p}+\frac{1}{q} = 1$ thus if we set $f_i = (1+|\nabla u|^2)^{\frac{p}{2}-1} \frac{\partial u}{\partial x_i}$, we have

$$T = -\sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \qquad f_i \in L^{q,\lambda}_{loc}(\Omega). \qquad \blacksquare$$

If <u>$1\leq p<2$,</u> the operator $Au = -\mathrm{div}\left((1+|\nabla u|^2)^{\frac{p}{2}-1} \nabla u\right)$ is still strongly monotonic and we have instead of **(2.7)** the following inequality, whose proof is simple :

$$\int_{B(x,3r)} \left((1+|\nabla u|^2)^{\frac{p}{2}-1} \nabla u-(1+|\nabla v|^2)^{\frac{p}{2}-1} \nabla v,\nabla(u-v)\right)dy \geq$$

$$c\int_{B(x,3r)} \frac{|\nabla(u-v)|^2}{(1+|\nabla u|+|\nabla v|)^{2-p}}\, dy$$

Using the reverse Hölder inequality as in [R] we deduce :

$$\left[\int_{B(3r)} |\nabla(u-v)|^p \, dy\right]^{\frac{2}{p}} \left[\int_{B(3r)} \big(|\nabla u|+|\nabla v|+1\big)^p \, dy\right]^{\frac{p-2}{p}} \leq |<T,w>|.$$

Thus, using young inequality and relation **a)** of proposition **2** :

$$\int_{B(3r)} |\nabla(u-v)|^p \, dy \leq |<T,w>|^{\frac{p}{2}} . \left(\int_{B(3r)} (|\nabla u|+|\nabla v|+1)^p \, dy\right)^{\frac{2-p}{2}} .$$

For all $\eta > 0$

$$\int_{B(3r)} |\nabla(u-v)|^p \, dy \leq r^{-\eta} |<T,w>| + r^{\frac{p}{2-p}\eta} \left(\int_{B(3r)} (|\nabla u|+|\nabla v|+1)|^p \, dy\right).$$

Since $T \in M^{-1,q}_{\lambda,loc}(\Omega)$, $\lambda>N-p$, we deduce there exist $c>0$, $\varepsilon>0$, $\eta>0$

$$\int_{B(3r)} |\nabla(u-v)|^p \, dy \leq r^{\eta} \int_{B(3r)} |\nabla u|^p \, dy + c.r^{N-p+\varepsilon} \tag{2.9}$$

As before for any $0\leq\rho\leq r$:

$$\int_{B(\rho)} |\nabla u|^p \, dy \leq c \int_{B(3r)} |\nabla(u-v)|^p \, dy + c \int_{B(\rho)} |\nabla v|^p \, dy \tag{2.10}$$

From **(2.9)**, **(2.10)** and proposition **3**, we get easily

$$\int_{B(3\rho)} |\nabla u|^p \, dy \leq c \left[\left(\frac{\rho}{r}\right)^N + r^{\eta}\right] \int_{B(3r)} |\nabla u|^p \, dy + c.r^{N-p+\varepsilon}.$$

Applying lemma 3, we get that $(1+|\nabla u|^2)^{\frac{p}{2}-1} \nabla u$ is in $\left(L^{q,\lambda}_{loc}(\Omega)\right)^N$ for $\lambda>N-p$. Setting again $f_i = (1+|\nabla u|^2)^{\frac{p}{2}-1} \frac{\partial u}{\partial x_i}$, we have $T = -\sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$, $f_i \in L^{q,\lambda}_{loc}(\Omega)$. ■

III. An L^∞-local estimate for the solution

$$Au + F(u,\nabla u) = T \in M^{-1,q}_{\lambda,loc}(\Omega), \quad N-p < \lambda .$$

In this paragraph, we consider $u \in W^{1,p}_{loc}(\Omega)$ solution of $Au+F(u,\nabla u) = T$, where A, F satisfies the structure **(1.2)** to **(1.4)** with :

$\mathbf{e}_0$: $\nu_0(|u|) =$ constant ν_0 , $\nu_1 \in L^{1,\lambda}_{loc}(\Omega)$, $\lambda > N-p$

$\mathbf{e}_1$: $a(|u|) =$ constant a , $a_0 \in L^{q,\lambda}_{loc}(\Omega)$, $\lambda > N-p$

$\mathbf{e}_2$: F has the following decomposition

$$F(x,u,z) = F_1(x,u,z) + F_2(x,u,z)$$

$$\begin{cases} uF_1(x,u,z) \geq 0 \\ |F_2(x,u,z)| \leq c\left(g_0(x)+|z|^{p-1}\right) \end{cases}$$

with $g_0 \in L^{q,\lambda}_{loc}(\Omega)$, $\lambda > N-p$

$\mathbf{e}_3$: $T \in M^{-1,q}_{\lambda,loc}(\Omega)$, $N-p<\lambda\leq N$.

THEOREM 2.— *Let* $u \in W^{1,p}_{loc}(\Omega)$, $1<p\leq N$, be a weak solution of :

$$Au + F(u,\nabla u) = T \tag{2.11}$$

with the structure **(1.2)** *to* **(1.4)**, (e_0) *to* (e_3). *Then,* u *is locally bounded. Moreover, there is a non negative Radon measure* m *such that : for all* Ω' *relatively compact open set in* Ω, *there is* $c>0$, $\varepsilon>0$ *:*

$$m\big(B(x,r)\big) \leq c.r^{N-p+\varepsilon}, \quad B(x,2r) \subset \Omega'$$

and :

$$\sup_{B(x,\sigma r)} |u| \leq \frac{c}{(1-\sigma)^{1/\alpha} r^{1/\alpha}} \left[\int_{B(x,r)} |u|^p dm\right]^{1/p} + c$$

for all $0<\sigma<1$ *and* c, α, β *are absolute constants.*

Proof.— Let $\Omega' \Subset \Omega$. For a value of k that will determined later, we set :

$$k_i = k(1 - 2^{-i}), \quad i=0,1,2,\ldots$$

and

$$r_i = \sigma r + 2^{-i} r(1-\sigma), \quad \bar{r}_i = \frac{1}{2}(r_i + r_{i+1})$$
$$\bar{\bar{r}}_i = \sigma r + \frac{3}{4} 2^{-i} r(1-\sigma), \quad i=0,1,2,\dots$$

We consider the corresponding balls

$$B_i = B(r_i), \quad \tilde{B}_i = B(x, \bar{r}_i) = B(\bar{r}_i).$$

We denote by φ_i the cut-off function whose support is contained in $\tilde{B}_i$ such that :

$$\varphi_i = 1 \quad \text{on} \quad B_{i+1}, \quad 0 \le \varphi_i \le 1$$

and

$$|\nabla\varphi_i| \le \frac{2^{i+2}}{r(1-\sigma)}$$

$\Big($Observe that : $(r_i - r_{i+1})^{-1} = \dfrac{2^{i+1}}{r(1-\sigma)}$ and $(r_i - \bar{r}_i)^{-1} = \dfrac{2^{i+2}}{r(1-\sigma)}\Big)$

Let $\psi = \varphi_i^p(|u| - k_{i+1})_+ \text{sign } u$. Multiplying **(2.11)** by ψ, we get :

$$\int_\Omega a_j(y,u,\nabla u)\frac{\partial\psi}{\partial x_j}\,dy + \int_\Omega F(u,\nabla u)\psi = \langle T,\psi\rangle \tag{2.12}$$

By theorem **1**, we have the existence of $f_i \in L^{q,\lambda}_{loc}(\Omega)$, $\lambda \in\,]N-p,N]$ such that : $T = -\sum_{i=1}^{N} \dfrac{\partial f_i}{\partial x_i}$, we set $f = \Big(\sum_{i=1}^{N} f_i^2\Big)^{1/2}$,

$A_i = B(r_i) \cap \{(|u|-k_{i+1})_+ > 0\}$. Using the structure of A, F and the decomposition of T, we get the following inequalities :

for all $\eta > 0$:

$$* \int_{B_i} a_j(y,u,\nabla u)\frac{\partial\psi}{\partial x_j}\,dy \ge \nu_0 \int_{\tilde{B}_i} |\nabla(|u|-k_{i+1})_+|^p\,\varphi_i^p\,dy - \int_{A_i} [\nu_1(y)+a_0^q(y)]dy$$
$$- \eta \int_{\tilde{B}_i} |\nabla(|u|-k_{i+1})_+|^p\,\varphi_i^p\,dy - c\,\frac{2^{ip}}{(1-\sigma)^p r^p}\int_{\tilde{B}_i} (|u|-k_{i+1})_+^p\,dy$$

$$* \int_{\tilde{B}_i} F(u,\nabla u)\psi \, dy \geq \int_{\tilde{B}_i} F_2(u,\nabla u)\psi \, dy$$

$$* \int_{\tilde{B}_i} |F_2(u,\nabla u)\psi| \, dy \leq c \frac{2^{ip}}{(1-\sigma)^p r^p} \int_{\tilde{B}_i} (|u|-k_{i+1})_+^p \, dy +$$

$$+ \eta \int_{\tilde{B}_i} |\nabla(|u|-k_{i+1})_+|^p \varphi_i^p \, dy + \int_{A_i} g_0(y)^q \, dy + c\, k^p |A_i|$$

with $|A_i|$ = Lebesgue measure of A_i

$$* \ |\langle T,\psi\rangle| \leq c \frac{2^{ip}}{(1-\sigma)^p r^p} \int_{\tilde{B}_i} (|u|-k_{i+1})_+^p \, dy +$$

$$+ \int_{A_i} f(y)^q \, dy + \eta \int_{\tilde{B}_i} |\nabla(|u|-k_{i+1})_+|^p \varphi_i^p \, dy$$

We define the following measure m by setting :

$$dm = [1 + \nu_1(y) + a_0^q(y) + g_0^q(y) + f^q(y)]dy.$$

A simple choice of η and a combination of these last inequalities lead to the fundamental energy estimate :

$$\int_{B_{i+1}} |\nabla(|u|-k_{i+1})_+|^p dy \leq \int_{\tilde{B}_i} |\nabla(|u|-k_{i+1})|^p \varphi_i^p dy \leq$$

$$\leq c \frac{2^{ip}}{(1-\sigma)^p r^p} \int_{\tilde{B}_i} |(|u|-k_{i+1})_+|^p dy + c(k^p+1) m(A_i) \tag{2.13}$$

Using assumption on ν_1 , a_0 , g_0 and T, one can check that for any ball, $B(x,2r) \subset \Omega'$:

$$m\big(B(x,r)\big) \leq c.r^{N-p+\varepsilon} \quad \text{for some } c>0, \ \ \varepsilon>0.$$

So if we define s by $\frac{s}{p}(N-p) = N-p+\varepsilon$, then the previous result of

Adams (see theorem **0.1**), $\psi \in L^s(dm)$. Now :

$$\int_{B_{i+1}} (|u|-k_{i+1})_+^p \, dm \le \int_{\tilde{B}_i} |(|u|-k_{i+1})_+ \varphi_i|^p \, dm$$

$$\le \left[\int_{\tilde{B}_i} |(|u|-k_{i+1})_+ \varphi_i|^s \, dm\right]^{p/s} . m(A_i)^{1-\frac{p}{s}}.$$

From Adams' inequality, we have :

$$\int_{B_{i+1}} (|u|-k_{i+1})_+^p \, dm$$

$$\le c \left[\int_{\tilde{B}_i} |\nabla(|u|-k_{i+1})_+|^p \varphi_i^p \, dx + \int_{\tilde{B}_i} (|u|-k_{i+1})_+^p |\nabla\varphi_i|^p \, dx\right] \times m(A_i)^{1-\frac{p}{s}}.$$

using **(2.13)**, we get :

$$\int_{B_{i+1}} (|u|-k_{i+1})_+^p dm \le \left[c \frac{2^{ip}}{(1-\sigma)^p r^p} \int_{\tilde{B}_i} (|u|-k_{i+1})_+^p dy + c(k^p+1) m(A_i)\right] \times$$

$$\times \, m(A_i)^{1-\frac{p}{s}}.$$

Defining, $Y_i = k^{-p} \int_{B_i} (|u|-k_i)_+^p dm$, we have :

$$m(A_i) \le 2^{(i+1)p} \, k^{-p} \int_{B_i} (|u|-k_i)_+^p dy = 2^{(i+1)p} \, Y_i$$

and then observing, $k_{i+1} \ge k_i$, we deduce :

$$k^{-p} \int_{B_{i+1}} (|u|-k_{i+1})_+^p dm \le c \frac{2^{ip}}{(1-\sigma)^p r^p} Y_i \, (2^{ip} \, Y_i)^{1-\frac{p}{s}} +$$

$$+ \, c \frac{(k^p+1)}{k^p} (2^{ip} \, Y_i)(2^{ip} \, Y_i)^{1-\frac{p}{s}}.$$

Choosing, $k \ge 1$, we deduce

$$Y_{i+1} \le c \, \frac{2^{ip(2-\frac{p}{s})}}{(1-\sigma)^p r^p} \, Y_i^{2-\frac{p}{s}} \qquad \textbf{(2.14)}$$

We set $b = 2^{p(2-\frac{p}{s})} > 1$, $\alpha = 1-\frac{p}{s} > 0$. Then **(2.14)** yields

$Y_{i+1} \le \dfrac{c\,.\,b^i}{(1-\sigma)^p r^p} \, Y_i^{1+\alpha}$. Choosing k such that

$$Y_0 = k^{-p} \int_{B(x,r)} |u|^p dm \le c^{\frac{-1}{\alpha}} (1-\sigma)^{\frac{p}{\alpha}} r^{\frac{p}{\alpha}} b^{\frac{-1}{\alpha^2}} , \quad k \ge 1.$$

Lemma **(4.7)** of [L-U], yields :

$$Y_i \longrightarrow 0 \quad \text{as} \quad i \longrightarrow +\infty. \qquad ■$$

Thus we get the desired result.

COROLLARY 1.— *Under the same assumptions as in theorem 2, if furthermore,* A *is strongly monotonic and* $s = p-\varepsilon$ *($\forall\varepsilon>0$) for the growth of* F. *Then* u *satisfies the Dirichlet growth in* Ω *and in particular* u *is in* $C^{0,\alpha}_{loc}(\Omega)$.

Proof.— Combine theorem **0** with theorem **2**. ■

COROLLARY 2.— *Under the same assumptions as in theorem 2. If* μ *is a signed Radon measure, such that the total variation* $|\mu|$ *satisfies the following growth*

$$|\mu|\big(B(x,r)\big) \le c.r^{N-p+\varepsilon} \quad \textit{on any ball.}$$

$$B(x,2r) \subset \Omega' \Subset \Omega,$$

then, u *is* $C^{0,\alpha}_{loc}(\Omega)$.

This results is more general than one we get in [R-Z] since the monotoni- city of A is weaker and μ is a signed measure.

Remark on L^∞-global estimate

In their paper [B-G], Boccardo-Giachetti studied the following Dirichlet problem :

$$\begin{cases} Au + F(u,\nabla u) = T \text{ in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

where $Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,u,\nabla u) - \operatorname{div}\big(\phi(u)\big)$, with the structure **(1.2)**, ν_0 = constant, $\nu_1 = 0$, $a(|u|)$ = constant, a_0 is in $L^q(\Omega)$.

$uF(u,\nabla u) \geq 0$ and has a p-growth with respect to the gradient.

ϕ is continuous function such that : $|\phi(u)| \leq c.|u|^\gamma$ for some $\gamma \geq 0$.

They showed similar results as we did in [R 2], [R-T], [R 1]. One of the novelties in their paper is the additionnal term $\operatorname{div}\big(\phi(u)\big)$. In particular they showed if $T \in W^{-1,r}(\Omega)$, $r > N/(p-1)$. Then u is bounded.

In fact, their results can be extended to degenerate operators A as we did in [R 1], [R 2], [R-T]. We just have to observe the following lemma :

LEMMA 4.— *Let* $S_{\theta,h}$ *be the following lipchitz function :* $S_{\theta,h}(t) = 0$ *if* $|t| \leq 0$, $h>0$. $S_{\theta,h}(t) = \text{sign } t$ *of* $|t| \geq \theta+h$ *and affine elsewhere. Let* $u \in W_0^{1,p}(\Omega)$, *such that* $\phi(u) \in L^q(\Omega)$. *Then*

$$\langle \operatorname{div}\big(\phi(u)\big),\ S_{\theta,h}(u) \rangle = 0.$$

Proof of lemma 1.— For $\varepsilon>0$, define $S^\varepsilon_{\theta,h}$ by $S^\varepsilon_{\theta,h}(t) = \varepsilon t + S_{\theta,h}(t)$.

Then one can check that $S^{\varepsilon}_{\theta,h}$ is invertible from $\mathbb{R}$ to $\mathbb{R}$, $S^{\varepsilon}_{\theta,h}$ remains in a bounded set of $W^{1,\infty}(\mathbb{R})$ and for any $u \in W^{1,p}_0(\Omega)$, $S^{\varepsilon}_{\theta,h}(u)$ converge to $S_{\theta,h}(u)$ in $W^{1,p}_0(\Omega)$ strong when ε goes to zero.

One has, for $u \in C^1_0(\bar{\Omega})$:

$$<\operatorname{div} \phi(u),\ S^{\varepsilon}_{\theta,h}(u)> = - \int_{\Omega} \phi(u) \operatorname{div}\big(S^{\varepsilon}_{\theta,h}(u)\big)dx$$

if we define

$$F_{\varepsilon}(x) = \int_0^{S^{\varepsilon}_{\theta,h}(u)(x)} \phi\big((S^{\varepsilon}_{\theta,h})^{-1}(t)\big)dt, \quad F_{\varepsilon} \in W^{1,1}_0(\Omega).$$

Then, one has from Green formula :

$$\int_{\Omega} \phi(u) \operatorname{div}\big(S^{\varepsilon}_{\theta,h}(u)\big)dx = \int_{\Omega} \operatorname{div}\big(F_{\varepsilon}(x)\big)dx = 0.$$

Then : $<\operatorname{div} \phi(u), S^{\varepsilon}_{\theta,h}(u)> = 0$. Letting ε go to zero and then using density, we get the desired result. ∎

IV. On $C^{1,\alpha}_{loc}$-regularity

The aim of this paragraph is to derive results similar to those obtained for the $C^{0,\alpha}$-regularity. Again, our results will extend those obtained by Lewy-Stampacchia [L-S], Campanato [C] (see also [G]). The arguments that we use here is analogous to those that Campanato introduced. In this paragraph, α will denote different holder exponents.

We consider the following operators :

$$Au = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left[a_{ij}(x,u) \frac{\partial u}{\partial x_j}\right] \tag{4.1}$$

$$a_{ij} \in C^{0,\beta}_{loc}(\Omega \times \mathbb{R}) \tag{4.2}$$

$\forall z \in \mathbb{R}^N$, $\forall u \in \mathbb{R}$, for almost all $x \in \Omega$:

$$\sum_{i,j=1}^{N} a_{ij}(x,u)z_i z_j \geq \nu_0(|u|)|z|^2, \quad \nu_0 \in C(\mathbb{R}), \quad \nu_0 > 0 \tag{4.3}$$

$$|F(u,z)| = |F(x,u,z)| \leq a(|u|)(1 + |z|^2) \tag{4.4}$$

with : $a : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ increasing.

Let $u \in H^1_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$, solution of :

$$Au + F(u,\nabla u) = T \in H^{-1}_{loc}(\Omega).$$

We have the following optimal condition.

THEOREM 3.

a) *If* $u \in C^{1,\alpha}_{loc}(\Omega)$ *then* $T \in M^{-1,2}_{\lambda,loc}$ *for some* $\lambda \in]N,N+2]$.

b) *Conversely, if* $T \in M^{-1,2}_{\lambda,loc}$, $N<\lambda\leq N+2$ *then* $u \in C^{1,\alpha}_{loc}(\Omega)$.

Proof.— a) Let Ω' be a relatively compact open set, let $\rho>0$, $\overline{B}(x_0,\rho) \subset \Omega'$, $\varphi \in H^1_0\big(B(x_0,\rho)\big)$, define $f_i(x) = \sum_{j=1}^{N} a_{ij}(x,u).\frac{\partial u}{\partial x_j} \in C^{0,\alpha}(\Omega')$. Using the structure of F, g, we have :

$$|\langle T,\varphi\rangle| \leq \left|\int_{B(x_0,\rho)} |f_i - (f_i)_{x_0,\rho}| \, \left|\frac{\partial \varphi}{\partial x_i}\right| dx\right|$$
$$+ c\int_{B(x_0,\rho)} |\nabla u|^2 |\varphi| \, dx + c\int_{B(x_0,\rho)} |\varphi| \, dx.$$

where $(f_i)_{x_0,\rho}$ is the average over the ball $B(\rho)$ using Schwartz's inequality, and Poincaré's inequality :

$$\|T\|_{H^{-1}(B(\rho))} \leq \left[c.\rho^\alpha.\rho^{N/2} + c.\rho^{N/2}.\rho\right] \times |\nabla\varphi|_{L^2(B(\rho))}.$$

Thus, there exist $c>0$ and $\lambda>N$ such that for all ball $B(x_0,\rho)$

$$\rho^{-\lambda/2}\|T\|_{H^{-1}(B(\rho))} \le c. \Longrightarrow T \in M^{-1,2}_{\lambda,loc}(\Omega) \qquad \blacksquare$$

b) Conversely, we want to show that u is $C^{1,\alpha}_{loc}(\Omega)$ if $T \in M^{-1,2}_{loc}(\Omega)$. For this, we need the following decomposition.

LEMMA 5.— *Let* $T \in M^{-1,2}_{\lambda,loc}(\Omega)$, $N<\lambda\le N+2$. *Then there exist* $f_i \in C^{0,\alpha}_{loc}(\Omega)$ *such that :*

$$T = -\sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$$

and conversely,

Proof of lemma 5.— Consider, $\Omega' \Subset \Omega$ and define $u \in H^1_0(\Omega)$ solution of $-\Delta u = T$ in Ω. For $r>0$ such that : $\overline{B}(x_0,r) \subset \Omega'$, denote by $v \in H^1(B_r)$ such that :

$$\Delta v = 0 \qquad B(r)$$

$$v = u \quad \text{on} \quad \partial B_r .$$

We want to show that $u \in C^{1,\alpha}_{loc}$. For this, we divide the proof into two steps.

<u>*1^st step*</u>. $|\nabla u| \in L^\infty_{loc}(\Omega)$, we have :

$$-\Delta(u-v) = T \quad \text{in} \quad B(r).$$

Multiplying by u-v, we derive : (using the fact $T \in M^{-1,2}_{\lambda,loc}(\Omega)$)

$$\int_{B(x_0,r)} |\nabla(u-v)|^2 dx \le c.\|T\|^2_{H^{-1}(B(r))} \le c.r^\lambda, \quad \lambda>N.$$

Let $0\le\rho\le r$, from Caccioppoli-estimate (see [G]), we derive :

$$\int_{B(x_0,\rho)} |\nabla v|^2 dx \le c.\Big(\frac{\rho}{r}\Big)^N \int_{B(x_0,r)} |\nabla u|^2 dx.$$

Thus :

$$\int_{B(x_0,\rho)} |\nabla u|^2 dx \le c \int_{B(x_0,r)} |\nabla(u-v)|^2 dx + c \int_{B(x_0,\rho)} |\nabla v|^2 dx$$

$$\le c.r^\lambda + c\Big(\frac{\rho}{r}\Big)^N \int_{B(x_0,r)} |\nabla u|^2 dx, \quad \lambda>N$$

applying lemma 3, we get $|\nabla u| \in L^{2,N}_{loc}(\Omega) \simeq L^\infty_{loc}(\Omega)$.

Step 2. $u \in C^{1,\alpha}_{loc}(\Omega)$.

Using Campanato's result ([C], corollary 7.11), we have

$$\int_{B(\rho)} |\nabla v-(\nabla v)_{x_0,\rho}|^2 dx \le c.\Big(\frac{\rho}{r}\Big)^{N+2} \int_{B(r)} |\nabla v-(\nabla v)_{x_0,r}|^2 dx.$$

On the other hand, a simple decomposition leads to :

$$\int_{B(\rho)} |\nabla u-(\nabla u)_{x_0,\rho}|^2 dx \le c \int_{B(r)} |\nabla(u-v)|^2 dx + c \int_{B(\rho)} |\nabla v-(\nabla v)_{x_0,\rho}|^2 dx$$

$$\le c\, r^\lambda + c\Big(\frac{\rho}{r}\Big)^{N+2} \int_{B(r)} |\nabla v-(\nabla v)_{x_0,r}|^2 dx.$$

Since $(\nabla v)_{x_0,r} = (\nabla u)_{x_0,r}$ (by Green formula) we deduce

$$\int_{B(\rho)} |\nabla u-(\nabla u)_{x_0,\rho}|^2 dx \le c\Big(\frac{\rho}{r}\Big)^{N+2} \int_{B(r)} |\nabla u-(\nabla u)_{x_0,r}|^2 dx + c.r^\lambda.$$

Using Campanato's lemma (lemma 6.11 [C]), we derive that $\nabla u \in \mathcal{L}^{2,\lambda}_{loc}(\Omega)$, for some $\lambda>N$, then, $u \in C^{1,\alpha}_{loc}(\Omega)$. ∎

Proof of theorem 3 (converse).— Let u be in $L^\infty_{loc}(\Omega) \cap H^1_{loc}(\Omega)$ solution of $-\frac{\partial}{\partial x_i}\Big[a_{ij}(x,u)\frac{\partial u}{\partial x_j}\Big] + F(u,\nabla u) = T$ with the structure

(4.1) to **(4.4)**, if $T \in M^{-1,2}_{\lambda,loc}(\Omega)$ from the decomposition (lemma **5**), we can write : $T = -\sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$, $f_i \in C^{0,\alpha}_{loc}(\Omega)$, let v be solution of :

$$\begin{cases} -\frac{\partial}{\partial x_i}\left[a_{ij}(x,u).\frac{\partial v}{\partial x_j}\right] = 0 & \text{on } B(x_0,r) \\ u-v = 0 & \text{on } \partial B(r) \end{cases}$$

with $\overline{B}(x_0,r) \subset \Omega' \Subset \Omega$.

By the preceeding result on $C^{0,\alpha}$-regularity, we have that $u \in C^{0,\alpha}_{loc}$ and thus $a_{ij}\big(x,u(x)\big)$ is in $C^{0,\alpha}_{loc}(\Omega)$.

From here, the proof is similar to the proof of lemma **5**. We divide it into two steps.

1st step. $|\nabla u| \in L^{\infty}_{loc}(\Omega)$

Multiply by u-v the following equation :

$$-\frac{\partial}{\partial x_i}\left[a_{ij}(x,u)\,\frac{\partial}{\partial x_j}(u\text{-}v)\right] + F(u,\nabla u) = T.$$

Then :

$$\int_{B(r)} |\nabla(u-v)|^2 dx \le \|T\|_{H^{-1}(B(r))} \|u-v\|_{H^1_0(B(r))}$$

$$+ c\int_{B(r)} |u-v|\,|\nabla u|^2 dx + c\int_{B(r)} |u-v|\,dx \tag{4.5}$$

A simple application of maximum's principle leads :

$$\inf_{B_r} u \le v \le \sup_{B_r} u \Longrightarrow |u-v| \le \sup_{B_r} u - \inf_{B_r} u \le c.r^{\alpha}.$$

Equation **(4.5)** becomes :

$$\int_{B(r)} |\nabla(u-v)|^2 dx \le c\ \|T\|^2_{H^{-1}(B(r))} + c\ r^{\alpha}\int_{B(r)} |\nabla u|^2 dx + c.r^{N+\alpha}$$

$$\le c.r^{N+\varepsilon} + c\ r^{\alpha}\int_{B(r)} |\nabla u|^2 dx \tag{4.6}$$

Using lemma 8.1 of Campanato's [C] and using the fact that $a_{ij} \in C^{0,\alpha}$, we dirive : $0<\rho\le r$

$$\int_{B(\rho)} |\nabla v|^2 dx \le c\left[\left(\frac{\rho}{r}\right)^N + r^\alpha\right] \int_{B(r)} |\nabla v|^2 dx$$

$$\le c.\left[\left(\frac{\rho}{r}\right)^N + r^\alpha\right] \int_{B(r)} |\nabla u|^2 dx. \tag{4.7}$$

We deduce

$$\int_{B(\rho)} |\nabla u|^2 dx \le c \int_{B(r)} |\nabla(u-v)|^2 dx + c \int_{B(\rho)} |\nabla v|^2 dx \tag{4.8}$$

So from **(4.5)** to **(4.8)**, we derive :

$$\int_{B(\rho)} |\nabla u|^2 dx \le c \left[\left(\frac{\rho}{r}\right)^N + r^\alpha\right] \int_{B(r)} |\nabla u|^2 dx + c.r^{N+\epsilon}.$$

From lemma 3, we deduce that

$$|\nabla u| \in L^{2,N}_{loc}(\Omega) \simeq L^\infty_{loc}(\Omega). \qquad \blacksquare$$

Step 2. $u \in C^{1,\alpha}_{loc}(\Omega)$

Using Campanato's lemma, (lemma 8.II, [C]) we derive for $0<\rho<r$:

$$\int_{B(\rho)} |\nabla(v)-(\nabla v)_{x_0,\rho}|^2 dx \le c \left(\frac{\rho}{r}\right)^{N+2} \int_{B(r)} |\nabla v - (\nabla v)_{x_0,r}|^2 dx$$

$$+ c.r^\alpha \int_{B(r)} |\nabla v|^2 dx$$

As in relation **(4.5)**, on has easily

$$\int_{B(r)} |\nabla(u-v)|^2 dy \le c.\sum_{i=1}^{N} \int_{B(r)} |f_i - (f_i)_{x_0,r}|^2 dy$$

$$+ c.r^\alpha \int_{B(r)} |\nabla u|^2\, dy + c.r^{N+\alpha} \tag{4.9}$$

where $\underset{\sim}{T} = - \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$, $f_i \in C^{0,\alpha}_{loc}$. A simple decomposition of :

$$\int_{B(\rho)} |\nabla u-(\nabla u)_{x_0,\rho}|^2 dx$$

analogous to step two of lemma 5 lead to : (using **(4.9)**, $f_i \in C^{0,\alpha}_{loc}$:

$$\int_{B(\rho)} |\nabla u-(\nabla u)_{x_0,\rho}|^2 dx \le c \left(\frac{\rho}{r}\right)^{N+2} \int_{B(r)} |\nabla u - (\nabla u)_{x_0,r}|^2 dx$$
$$+ c.r^\alpha \int_{B(r)} |\nabla u|^2 dx + c.r^{N+\alpha}.$$

Since $|\nabla u| \in L^\infty_{Loc}(\Omega)$

$$\int_{B(\rho)} |\nabla u-(\nabla u)_{x_0,\rho}|^2 dx \le c \left(\frac{\rho}{r}\right)^{N+2} \int_{B(r)} |\nabla u - (\nabla u)_{x_0,r}|^2 dx + c.r^{N+\alpha}.$$

From lemma 6.II of Campanato's [C], we deduce that $|\nabla u| \in \mathcal{L}^{2,\lambda}_{loc}(\Omega)$.
Thus, $u \in C^{1,\alpha}_{loc}(\Omega)$. ■

COROLLARY OF THEOREM 3.— *Let* μ *be a signed Radon measure whose total variation grows as follows :*

$$|\mu|(B(x,r)) \le c.r^{N-1+\varepsilon} \tag{4.10}$$

for all $x, r : B(x,2r) \subset \Omega' \Subset \Omega$, $c>0$, $\varepsilon>0$.

Then, $\mu \in M^{-1,2}_{\lambda,loc}$ *for* $N<\lambda\le N+2$.

In particular, under the same assumptions as in theorem 3, the solution u of $Au + F(u,\nabla u) = \mu$, $u \in L^\infty_{loc}(\Omega) \cap H^1_{loc}(\Omega)$ is in $C^{1,\alpha}_{loc}(\Omega)$.

Proof.— Applying the result of Lewy-Stampacchia [L-S], we know that the solution v of $\begin{cases}\Delta v = \mu \\ v = 0\end{cases}$, in in $C^{1,\alpha}_{loc}$. Thus, $\mu \in M^{-1,\alpha}_{\lambda,loc}(\Omega)$.
Applying theorem 3, we get that $u \in C^{1,\alpha}_{loc}(\Omega)$.

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