Journal of Functional Analysis 268 (2015) 1153–1166

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Regularizing effect of the interplay between coefficients in some elliptic equations

David Arcoya [a,*,1], Lucio Boccardo [b]

[a] *Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain*
[b] *Dipartimento di Matematica, Università di Roma Sapienza, Piazza A. Moro 2, Roma, Italy*

ARTICLE INFO

Article history:
Received 25 July 2014
Accepted 17 November 2014
Available online 8 December 2014
Communicated by H. Brezis

Keywords:
Linear and nonlinear elliptic equations
Natural growth with respect to the gradient
L^1 coefficients
Regularizing effect

ABSTRACT

We study the regularizing effect of the interaction between the coefficient of the zero order term and the datum in some *nonlinear* Dirichlet problems.
The simplest example is the *linear* problem

$$\begin{cases} \displaystyle\int_\Omega M(x)\nabla u\nabla\varphi + \int_\Omega a(x)u\varphi = \int_\Omega f(x)\varphi, \\ \forall\varphi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega), \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$, M is a bounded elliptic matrix and $0 \le a(x) \in L^1(\Omega)$. Even if $f(x)$ only belongs to $L^1(\Omega)$, the assumption

$$\text{there exists } Q > 0 \text{ such that } |f(x)| \le Qa(x)$$

implies the existence of a weak solution u belonging to $W_0^{1,2}(\Omega)$ and to $L^\infty(\Omega)$.

* Corresponding author.
E-mail addresses: darcoya@ugr.es (D. Arcoya), boccardo@mat.uniroma1.it (L. Boccardo).
[1] The first author is supported by MEC Ministerio de Economía y Competitividad (Spain) MTM-2012-31799 and Junta de Andalucía FQM116, covered in part by EU–FEDER funds.

http://dx.doi.org/10.1016/j.jfa.2014.11.011
0022-1236/

1. Introduction

In this paper we study boundary value problems in a bounded, open subset Ω of $\mathbb{R}^N$, ($N \geq 2$), and datum $f(x) \in L^1(\Omega)$ and related minimization problems. Surprisingly, although (in general) the solution of an elliptic equation having a right-hand side in $L^1(\Omega)$ is not bounded, nor has finite energy, we prove the existence of bounded solutions with finite energy.

Our initial scope was to study quasilinear problems which possess lower order terms having "natural" (i.e. quadratic) growth with respect to the gradient. Nevertheless, the obtained results are new in the particular cases of either linear or semilinear problems. In addition, a simplified proof for these cases can be given. Thus, for the sake of clarity, the first problem considered in this work is the linear one

$$\begin{cases} -\operatorname{div}\big(M(x)\nabla u\big) + a(x)u = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Next we study the semilinear problems

$$\begin{cases} -\operatorname{div}\big(M(x)\nabla u\big) + a(x)g(u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and finally we deal with quasilinear boundary value problem having lower order terms with natural growth

$$\begin{cases} -\operatorname{div}\big(M(x,u)\nabla u\big) + a(x)u = B(x,u,\nabla u) + f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.3}$$

where we assume that $M(x,s)$ (with $M(x,s) = M(x)$ independent of s in the case of (1.1) and (1.2)) is a Carathéodory matrix (that is, measurable with respect to x for every $s \in \mathbb{R}$, and continuous with respect to s for almost every $x \in \Omega$) which satisfies, for some positive constants α, β, a.e. in $x \in \Omega$, for every $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$,

$$\alpha|\xi|^2 \leq M(x,s)\xi\xi \tag{1.4}$$

$$\big|M(x,s)\big| \leq \beta. \tag{1.5}$$

With respect to the coefficient $a(x)$ of the zero order term and to the datum $f(x)$, in addition to imposing that

$$f(x), a(x) \in L^1(\Omega) \tag{1.6}$$

we also assume that there exists $Q > 0$ such that, for a.e. $x \in \Omega$,

$$\big|f(x)\big| \leq Qa(x). \tag{1.7}$$

In the semilinear problem (1.2), the continuous function $g(s)$ satisfies

$$\lim_{s\to-\infty} g(s) = -\infty \quad \text{and} \quad \lim_{s\to\infty} g(s) = \infty; \tag{1.8}$$

while, in the quasilinear problem (1.3), we also suppose that the lower order term $B(x, s, \xi)$ is a Carathéodory function (that is, measurable with respect to x for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s, ξ) for almost every $x \in \Omega$) satisfying for some positive constant γ,

$$\left|B(x, s, \xi)\right| \le \gamma|\xi|^2, \tag{1.9}$$

for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$. Thus the simplest example for quasilinear problems of type (1.3) is

$$\begin{cases} -\Delta u + a(x)u = \gamma|\nabla u|^2 + f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

As it has been previously mentioned, our first result is for the linear problem (1.1) under the hypotheses (1.4), (1.5), (1.6) and (1.7). Indeed, thanks to our main assumption (1.7), we show in Theorem 2.1 the existence (and uniqueness) of a bounded weak solution in $W_0^{1,2}(\Omega)$ of (1.1). Furthermore, adding condition (1.8), we also generalize in Theorem 2.4 the proof to cover the existence of a bounded weak solution in $W_0^{1,2}(\Omega)$ of (1.2).

We have earlier observed that the existence of a bounded solution $u \in W_0^{1,2}(\Omega)$ is surprising if the right hand side only belongs to $L^1(\Omega)$. Indeed, the existence result is not true for instance if $a(x) = 0$ (see [8,19]). Therefore, we put in evidence that in spite of the fact that the datum f only belongs to $L^1(\Omega)$, the interplay given by (1.7) between it and the coefficient $a(x)$ of the zero order term provides a regularizing effect on the problems (1.1) and (1.2).

For semilinear problems a strong regularizing effect (of the lower order term) is proved in [5]. We have to mention that the study of the semilinear case with nonregular datum f was initiated in [16]; problems with nonlinear zero order term with "L^1 coefficients" like $a(x)u$ are studied in [14]. In addition, we point out that the existence of solution in a Marcinkiewicz space is studied in [3] (more contributions can be found in [2,12,13,17]) when the datum $f \in L^1(\Omega)$.

In contrast, in Theorems 2.1 and 2.4 we obtain solutions in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. The keystone of the proofs of them is the deduction, by condition (1.7), of an L^∞-estimate of the approximated solutions which implies the corresponding L^∞-estimate of the solution (see (2.2) below).

We recall that for linear Dirichlet problems with lower order terms of order zero like (1.1), with $a(x) \ge a_0 > 0$, the classical estimate

$$\|u\|_{L^p(\Omega)} \le \frac{\|f(x)\|_{L^p(\Omega)}}{a_0}, \quad 1 \le p \le \infty,$$

plays an important role. Thus our L^∞-estimate (2.2) below can be seen as an improvement of the previous one.

Now, following the outline of [11], we can deduce as before by condition (1.7) an L^∞-estimate of the solution and we see in Theorem 3.1 that (1.7) also gives a regularizing effect on the quasilinear problem (1.3) which implies the existence of bounded weak solutions in $W_0^{1,2}(\Omega)$ of (1.3).

For quasilinear problems with lower order term $B(x,s,\xi)$ having natural growth with respect to the gradient, assumptions concerning the sign of $B(x,s,\xi)$ are imposed in order to have a regularizing effect (see [10,15]). We remark explicitly that, in contrast, in this paper, we only assume (1.9) on the function $B(x,s,\xi)$.

Notice that the condition (1.7) implies that $a(x) \ge 0$ a.e. $x \in \Omega$, with $f(x) = 0$ a.e. on the set of these $x \in \Omega$ for which $a(x)$ vanishes. Hence, we point out that (1.7) allows us to consider coefficients $a(x)$ which are zero on a subset of positive measure.

Recall that, even with datum f with stronger summability, $f \in L^{\frac{N}{2}}(\Omega)$, the size of f and the strict positivity of $a(x)$ play an important role (see [6]). More recently, if $f \in L^q(\Omega)$ with $q > \frac{N}{2}$, sufficient conditions for the existence of solution of (1.3) are given in [1] provided that the coefficient satisfies only that $a(x) \ge 0$ (being able to vanish in a set in Ω of positive measure).

Similarly, in a straightforward way, we study in Theorem 4.1 the minimization in $W_0^{1,2}(\Omega)$ of functionals of the type

$$J(v) = \int_\Omega j(x,v,\nabla v) + \int_\Omega \left[\frac{1}{2}a(x)|v|^2 - fv\right], \quad v \in W_0^{1,2}(\Omega),$$

where again a, f satisfy (1.6) and (1.7) and j is a Carathéodory function in $\Omega \times \mathbb{R} \times \mathbb{R}^N$ for which there exist $\beta \ge \alpha > 0$ such that

$$j(x,s,\xi) \text{ is convex on } \xi, \tag{1.10}$$

and

$$\alpha|\xi|^2 \le j(x,s,\xi) \le \beta|\xi|^2, \tag{1.11}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

Observe that (1.11) implies that the first term in the definition of J is finite for all $v \in W_0^{1,2}(\Omega)$, while the second one is bounded from below by (1.7) (see (4.2) below). In particular, $J(v)$ is either finite, or ∞ for every $v \in W_0^{1,2}(\Omega)$. Notice also that when the function j is differentiable, the Euler–Lagrange equation associated to every minimizer in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the integral functional J is of the type (1.3).

Under these hypotheses, we prove in Theorem 4.1 the existence of a minimizer $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ among all functions in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Remark 1.1. We note that we set all our problems in $W_0^{1,2}(\Omega)$. However the results still hold, with the same proofs, for differential operators with growth of order $p > 1$ in the $W_0^{1,p}(\Omega)$ framework.

2. Linear and semilinear Dirichlet problems

In the first part of this section we study the problem (1.1). For a solution of it we understand

$$\begin{cases} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega): \\ \int_\Omega M(x)\nabla u \nabla\varphi + \int_\Omega a(x)u\varphi = \int_\Omega f(x)\varphi, \\ \quad \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{2.1}$$

Theorem 2.1. *Assume (1.4), (1.5), (1.6), (1.7) and that the matrix $M(x,s) = M(x)$ does not depend on s. Then there exists a unique solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the problem (1.1). Moreover,*

$$\|u\|_{L^\infty(\Omega)} \le Q. \tag{2.2}$$

Proof. We define

$$f_n(x) = \frac{f(x)}{1 + \frac{1}{n}|f(x)|}, \qquad a_n(x) = \frac{a(x)}{1 + \frac{Q}{n}|a(x)|}, \tag{2.3}$$

and, by the Schauder Theorem, we consider the solution u_n of the approximated problems

$$u_n \in W_0^{1,2}(\Omega): \quad -\operatorname{div}\big(M(x)\nabla u_n\big) + a_n(x)u_n = f_n(x), \tag{2.4}$$

i.e., satisfying

$$\int_\Omega M(x)\nabla u_n \nabla\varphi + \int_\Omega a_n(x)u_n\varphi = \int_\Omega f_n(x)\varphi, \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

Since $\psi(s) = s(1 + \frac{s}{n})^{-1}$ is increasing, we deduce by (1.7) that

$$\big|f_n(x)\big| = \frac{|f(x)|}{1 + \frac{1}{n}|f(x)|} \le \frac{Qa(x)}{1 + \frac{Q}{n}a(x)} = Qa_n(x). \tag{2.5}$$

We will use the following function defined for $s \in \mathbb{R}$ by

$$G_k(s) = \begin{cases} 0, & \text{if } |s| \le k, \\ s-k, & \text{if } s > k, \\ s+k, & \text{if } s < -k. \end{cases}$$

The use of $G_Q(u_n)$ as a test function in (2.4) gives, thanks to (1.4) and (2.5),

$$\alpha \int_\Omega \left|\nabla G_Q(u_n)\right|^2 + \int_\Omega a_n(x)\big[|u_n| - Q\big]\left|G_Q(u_n)\right| \le 0,$$

which implies

$$|u_n| \le Q. \tag{2.6}$$

Then, again by (1.4), the use of u_n as a test function in (2.4) gives, dropping a positive term,

$$\alpha \int_\Omega |\nabla u_n|^2 \le Q \int_\Omega |f|.$$

Thus the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and then there exist a function u in $W_0^{1,2}(\Omega)$ and a subsequence, still denoted $\{u_n\}$, such that u_n converges to u weakly in $W_0^{1,2}(\Omega)$ and a.e. to u and $|u| \le Q$. Moreover, taking $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ as a test function in (2.4) we have

$$\int_\Omega M(x)\nabla u_n \nabla\varphi + \int_\Omega a_n(x) u_n \varphi = \int_\Omega f_n \varphi$$

which, by taking into account the $L^1(\Omega)$-convergence of $a_n(x)u_n\varphi$ and $f_n(x)\varphi$ to $a(x)u\varphi$ and $f(x)\varphi$ and passing to the limit, implies that u is a solution of (1.1) in the sense of (2.1) which satisfies (2.2).

To conclude the proof we just have to observe the uniqueness of solution of the linear problem (1.1) which is a clear consequence of the uniqueness for the homogeneous problem ($f = 0$). □

Remark 2.2. As we show in Section 4, if the matrix $M(x)$ is symmetric, an alternative proof can be obtained by finding a bounded minimizer in $W_0^{1,2}(\Omega)$ of the (bounded from below) functional J defined by

$$J(v) = \int_\Omega M(x)\nabla v \nabla v + \int_\Omega \left[\frac{1}{2}a(x)|v|^2 - fv\right], \quad v \in W_0^{1,2}(\Omega).$$

Observe that the boundedness of the minimizer implies that it is a solution of (1.1).

Remark 2.3. The argument based in (1.7) and used to deduce the estimate (2.6) can be also applied to conclude that every solution u of (1.1) (in the sense of (2.1)) has to satisfy $\|u\|_\infty \le Q$ (instead of $G_Q(u_n)$, take $G_Q(u)$ as a test function). Observe that essentially we are using that $(u-Q)^+$ is subharmonic (Kato's inequality [18]).

In the following result we show that the technique developed in the previous theorem also handles semilinear Dirichlet problem (1.2). In this case, for a solution of (1.2) we understand that

$$\begin{cases} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), a(x)g(u) \in L^1(\Omega), \\ \displaystyle\int_\Omega M(x)\nabla u \nabla \varphi + \int_\Omega a(x)g(u)\varphi = \int_\Omega f(x)\varphi, \\ \quad \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases}$$

Theorem 2.4. *Assume (1.4), (1.5), (1.6), (1.7) and (1.8) with the matrix $M(x,s) = M(x)$ independent on s. Then there exists a solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the problem (1.2).*

Proof. Following the argument of Theorem 2.1, we use the Schauder Theorem to take a solution u_n of the approximated problem

$$u_n \in W_0^{1,2}(\Omega): \quad -\mathrm{div}\big(M(x)\nabla u_n\big) + a_n(x)g(u_n) = f_n(x),$$

where recall that $a_n(x)$ and $f_n(x)$ are given by (2.3). By (1.8), we can choose $k_0 > 0$ such that

$$g(s)s \ge 0, \tag{2.7}$$

and

$$\big|g(s)\big| \ge Q \tag{2.8}$$

for every $s \ge k_0$. The use of $G_{k_0}(u_n)$ as a test function in the approximated problem gives, thanks to (1.4) and (2.5) (which is deduced by (1.7)),

$$\alpha \int_\Omega \big|\nabla G_{k_0}(u_n)\big|^2 + \int_\Omega a_n(x)g(u_n)G_{k_0}(u_n) \le \int_\Omega f_n\big|G_{k_0}(u_n)\big| \le Q\int_\Omega a_n(x)\big|G_{k_0}(u_n)\big|.$$

By (2.7), this means

$$\alpha \int_\Omega \big|\nabla G_{k_0}(u_n)\big|^2 + \int_\Omega a_n(x)\big[\big|g(u_n)\big| - Q\big]\big|G_{k_0}(u_n)\big| \le 0,$$

which, by (2.8) implies that $\|u_n\|_{L^\infty(\Omega)} \leq k_0$ and the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$.

As a consequence, we can use u_n as a test function to deduce again by (1.4) that

$$\alpha \int_\Omega |\nabla u_n|^2 - \max_{|s|\leq k_0} |g(s)s| \int_\Omega a_n(x) \leq k_0 \int_\Omega |f_n|,$$

i.e.,

$$\alpha \int_\Omega |\nabla u_n|^2 \leq k_0 \int_\Omega |f| + \max_{|s|\leq k_0} |g(s)s| \int_\Omega a(x),$$

and we obtain that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, too. Thus there exist a function u in $W_0^{1,2}(\Omega)$ and a subsequence, still denoted $\{u_n\}$, which converges weakly in $W_0^{1,2}(\Omega)$ and a.e. to u with $\|u\|_{L^\infty(\Omega)} \leq k_0$.

Moreover, using that $|a_n(x)g(u_n)| \leq a(x) \max_{|s|\leq k_0} |g(s)|$, we obtain by the dominated convergence theorem the $L^1(\Omega)$ convergence of the sequence $\{a_n(x)g(u_n)\}$ to $a(x)g(u)$, which together with the $L^1(\Omega)$ convergence of $f_n(x)$ to $f(x)$ allows to pass to the limit in the approximated problems to conclude that u satisfies (1.2). $\square$

3. Quasilinear Dirichlet problems having natural growth terms

This section is devoted to prove the existence of solution of (1.3). Specifically, for a solution of it we understand that

$$\begin{cases} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega): \\ \int_\Omega M(x,u)\nabla u \nabla \varphi + \int_\Omega a(x)u\varphi = \int_\Omega B(x,u,\nabla u)\varphi + \int_\Omega f\varphi, \\ \quad \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{3.1}$$

and we show the following theorem.

Theorem 3.1. *Under the assumptions (1.4), (1.5), (1.6), (1.7) and (1.9), there exists a solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the problem (1.3). Moreover, u satisfies (2.2).*

Proof. We define the functions a_n and f_n given by (2.3) and

$$B_n(x,s,\xi) = \frac{B(x,s,\xi)}{1 + \frac{1}{n}|B(x,s,\xi)|}.$$

We now consider the approximated Dirichlet problems

$$\begin{cases} u_n \in W_0^{1,2}(\Omega): \\ -\operatorname{div}\big(M(x,u_n)\nabla u_n\big) + a_n(x)u_n = B_n(x,u_n,\nabla u_n) + f_n(x). \end{cases} \tag{3.2}$$

By the Schauder Theorem, for every $n \in \mathbb{N}$, there exists a solution $u_n \in W_0^{1,2}(\Omega)$ of (3.2). Moreover every $u_n \in L^\infty(\Omega)$ (see [19]). Following [11], this allows us to choose for fixed $\lambda > \frac{\gamma}{2\alpha}$, $v_n = (e^{2\lambda|G_Q(u_n)|} - 1)\,\mathrm{sgn}(G_Q(u_n))$ as a test function in (3.2) to obtain

$$2\lambda \int_\Omega M(x,u_n)\nabla u_n \cdot \nabla G_Q(u_n) e^{2\lambda|G_Q(u_n)|} + \int_\Omega a_n(x)|u_n|\big(e^{2\lambda|G_Q(u_n)|} - 1\big)$$
$$\leq \int_\Omega \big|B_n(x,u_n,\nabla u_n)\big|\big(e^{2\lambda|G_Q(u_n)|} - 1\big) + \int_\Omega |f_n|\big(e^{2\lambda|G_Q(u_n)|} - 1\big).$$

Using the ellipticity condition (1.4) and hypothesis (1.9) we deduce that

$$(2\alpha\lambda - \gamma)\int_\Omega \big|\nabla G_Q(u_n)\big|^2 e^{2\lambda|G_Q(u_n)|} + \int_\Omega a_n(x)|u_n|\big(e^{2\lambda|G_Q(u_n)|} - 1\big)$$
$$\leq \int_\Omega |f_n|\big(e^{2\lambda|G_Q(u_n)|} - 1\big).$$

Moreover, assumption (1.7) again gives (2.5) which implies that

$$(2\alpha\lambda - \gamma)\int_\Omega \big|\nabla G_Q(u_n)\big|^2 + \int_\Omega a_n(x)\big(e^{2\lambda|G_Q(u_n)|} - 1\big)\big[|u_n| - Q\big] \leq 0.$$

Observing that the integrand in the second integral is zero if $|u_n(x)| \leq Q$, we deduce that both integrals in the above inequality are positive (recall that $\lambda > \frac{\gamma}{2\alpha}$), and consequently we have

$$\int_\Omega \big|\nabla G_Q(u_n)\big|^2 = 0,$$

which implies again (2.6).

For $\lambda > \frac{\gamma}{2\alpha}$ let now $v = (e^{2\lambda|u_n|} - 1)\,\mathrm{sgn}(u_n)$ as a test function in (3.2) to obtain

$$2\lambda \int_\Omega M(x,u_n)\nabla u_n \cdot \nabla u_n e^{2\lambda|u_n|} + \int_\Omega a_n(x)|u_n|\big(e^{2\lambda|u_n|} - 1\big)$$
$$\leq \int_\Omega \big|B_n(x,u_n,\nabla u_n)\big|\big(e^{2\lambda|u_n|} - 1\big) + \int_\Omega |f_n|\big(e^{2\lambda|u_n|} - 1\big).$$

Using again the ellipticity condition (1.4) and hypothesis (1.9) we deduce that

$$(2\alpha\lambda - \gamma)\int_\Omega |\nabla u_n|^2 e^{2\lambda|u_n|} + \int_\Omega a_n(x)|u_n|\big(e^{2\lambda|u_n|} - 1\big) \leq \int_\Omega |f|\big(e^{2\lambda|u_n|} - 1\big),$$

from which, by (2.6) and dropping the second positive term

$$(2\alpha\lambda - \gamma) \int_\Omega |\nabla u_n|^2 e^{2\lambda|u_n|} \le \left(e^{2\lambda|Q|} - 1\right) \int_\Omega |f|.$$

In particular, since $\lambda > \frac{\gamma}{2\alpha}$, the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and thus there exist a function u in $W_0^{1,2}(\Omega)$ and a subsequence, still denoted $\{u_n\}$, such that u_n converges weakly in $W_0^{1,2}(\Omega)$ and a.e. to u. Moreover $|u| \le Q$.

If $\lambda > \frac{\gamma}{\alpha}$, we take now $v_n = (e^{2\lambda|u_n-u|} - 1)\operatorname{sgn}(u_n - u)$ as a test function in (3.2) to obtain

$$\begin{aligned} 2\lambda \int_\Omega M(x,u_n)\nabla u_n \nabla(u_n - u)e^{2\lambda|u_n-u|} + \int_\Omega a_n(x)u_n v_n \\ = \int_\Omega B(x,u_n,\nabla u_n)v_n + \int_\Omega f_n v_n. \end{aligned}$$

Subtracting $2\lambda \int_\Omega M(x,u_n)\nabla u \nabla(u_n - u)e^{2\lambda|u_n-u|}$, we have

$$\begin{aligned} 2\lambda \int_\Omega M(x,u_n)\nabla(u_n - u)\nabla(u_n - u)e^{2\lambda|u_n-u|} + \int_\Omega a_n(x)u_n v_n \\ = \int_\Omega B(x,u_n,\nabla u_n)v_n + \int_\Omega f_n v_n - 2\lambda \int_\Omega M(x,u_n)\nabla u \nabla(u_n - u)e^{2\lambda|u_n-u|}. \end{aligned}$$

By (1.4) and (1.9), we get

$$\begin{aligned} 2\lambda\alpha \int_\Omega \left|\nabla(u_n - u)\right|^2 e^{2\lambda|u_n-u|} + \int_\Omega a_n(x)u_n v_n \\ \le \gamma \int_\Omega |\nabla u_n|^2|v_n| + \int_\Omega |f||v_n| - 2\lambda \int_\Omega M(x,u_n)\nabla u \nabla(u_n - u)e^{2\lambda|u_n-u|}, \end{aligned}$$

which implies that

$$\begin{aligned} \int_\Omega \left|\nabla(u_n - u)\right|^2[2\lambda\alpha - 2\gamma] + \int_\Omega a_n(x)u_n v_n \\ \le 2\gamma \int_\Omega |\nabla u|^2|v_n| + \int_\Omega |f||v_n| + 2\lambda \int_\Omega M(x,u_n)\nabla u \nabla(u_n - u)e^{2\lambda|u_n-u|}. \end{aligned}$$

Observing that thanks to the Lebesgue Theorem and (2.6), the last four integrals tends to zero as n goes to infinity, we obtain that

$$\|u_n - u\|_{W_0^{1,2}(\Omega)} \to 0.$$

Therefore, it is possible to pass to the limit in the weak formulation of (3.2) to conclude that u satisfies (3.1) with $\|u\|_{L^\infty(\Omega)} \le Q$. $\square$

4. A related minimization problem

In this section, we assume the hypotheses (1.6), (1.7), (1.10) and (1.11) to study the functional J defined in $W_0^{1,2}(\Omega)$ by

$$J(v) = \int_\Omega j(x, v, \nabla v) + \int_\Omega \left[\frac{1}{2}a(x)|v|^2 - fv\right], \quad v \in W_0^{1,2}(\Omega). \tag{4.1}$$

First, note that J is bounded from below. Indeed, by (1.11) the first integral of J is bounded from below; while, thanks to (1.7), we have

$$\int_\Omega \left[\frac{1}{2}a(x)|v|^2 - fv\right] \ge \int_\Omega a(x)\left[\frac{1}{2}|v|^2 - Q|v|\right] \ge -\frac{Q^2}{2}\int_\Omega a, \tag{4.2}$$

and the second integral of J is bounded from below, too.

Our result is the following.

Theorem 4.1. *Let J be given by (4.1) and assume that conditions (1.6), (1.7), (1.10) and (1.11) hold. Then there exists $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that*

$$J(u) \le J(\varphi), \quad \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \tag{4.3}$$

Proof. For every $n \in \mathbb{N}$, let f_n be given by (2.3) and consider the approximated functional J_n defined for each $v \in W_0^{1,2}(\Omega)$ by setting

$$J_n(v) = \begin{cases} \displaystyle\int_\Omega j(x, v, \nabla v) + \frac{1}{2}\int_\Omega a(x)|v|^2 - \int_\Omega f_n v, & \text{if } a(x)|v|^2 \in L^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

By (1.10), (1.11) and the Fatou Lemma, J_n is weakly lower semicontinuous and coercive (from (4.2)). Thus, there exists $u_n \in W_0^{1,2}(\Omega)$ minimizing J_n. In particular,

$$\int_\Omega j(x, u_n, \nabla u_n) + \frac{1}{2}\int_\Omega a(x)|u_n|^2 - \int_\Omega f_n u_n \le J_n(0) = 0.$$

By (1.11), we deduce that

$$\alpha \int_\Omega |\nabla u_n|^2 + \frac{1}{2} \int_\Omega a(x)|u_n|^2 \le \int_\Omega |f||u_n|.$$

On the other hand, we also have

$$\int_\Omega |f||u_n| \le \frac{1}{4} \int_{\{4|f| \le a(x)|u_n|\}} a(x)|u_n|^2 + \int_{\{4|f| > a(x)|u_n|\}} |f||u_n|.$$

Hence, by (1.7), we obtain

$$\alpha \int_\Omega |\nabla u_n|^2 + \frac{1}{4} \int_\Omega a(x)|u_n|^2 \le 4Q \int_\Omega |f|,$$

which implies the existence of $R > 0$ such that

$$\int_\Omega |\nabla u_n|^2 \le R, \tag{4.4}$$

and

$$\int_\Omega a(x)|u_n|^2 \le R.$$

In consequence, thanks to (4.4), there exist u in $W_0^{1,2}(\Omega)$ and a subsequence (not relabeled) $\{u_n\}$ weakly converging in $W_0^{1,2}(\Omega)$ and almost everywhere converging to u.

Note that $j(x, s, 0) = 0$ by (1.11). Thus, if $T_Q(s) := \min\{Q, \max\{-Q, s\}\}$, the minimality inequality $J_n(u_n) \le J_n(T_Q(u_n))$, i.e.,

$$\begin{aligned} &\int_\Omega j(x, u_n, \nabla u_n) + \frac{1}{2} \int_\Omega a(x)|u_n|^2 - \int_\Omega f_n u_n \\ &\quad \le \int_{\{|u_n|<Q\}} j(x, u_n, \nabla u_n) + \frac{1}{2} \int_\Omega a(x)\big|T_Q(u_n)\big|^2 - \int_\Omega f_n T_Q(u_n) \end{aligned}$$

implies, dropping a positive term, that

$$\frac{1}{2} \int_\Omega a(x)\big[|u_n|^2 - \big|T_Q(u_n)\big|^2\big] \le \int_\Omega f_n G_Q(u_n) \le \int_\Omega |f|\big|G_Q(u_n)\big|,$$

and by (1.7) then

$$\frac{1}{2} \int_\Omega a(x)\big[\big|G_Q(u_n)\big|^2 + 2Q\big|G_Q(u_n)\big|\big] \le \int_\Omega Q a(x)\big|G_Q(u_n)\big|.$$

Thus we have

$$\int_\Omega a(x)\left|G_Q(u_n)\right|^2 \le 0,$$

that is, $|u_n| \le Q$.

To conclude the proof, we use that u_n minimizes J_n, i.e.,

$$\int_\Omega j(x,u_n,\nabla u_n) + \frac{1}{2}\int_\Omega a(x)|u_n|^2 - \int_\Omega f_n u_n \le J_n(\varphi),$$

for every $\varphi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Observe that we can pass to the limit in the first term (by weak lower semicontinuity in $W_0^{1,2}(\Omega)$), in the second term (by Fatou Lemma) and in the third term (by Lebesgue Theorem, since $|f_n u_n| \le Q|f|$). Thus we prove that

$$u \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega): \quad J(u) \le J(\varphi), \quad \forall \varphi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega). \qquad \square$$

Remark 4.2. Even if it is only assumed that f belongs to $L^1(\Omega)$, we do not need the T-minima framework of [4] for the study of the minimization problem (4.3).

Remark 4.3. Observe that every minimizer in $W_0^{1,2}(\Omega)$ of J has to belong to $L^\infty(\Omega)$. Indeed, similarly to the argument used in the proof for a minimizer in $W_0^{1,2}(\Omega)$ of J_n, if for every minimizer $u \in W_0^{1,2}(\Omega)$ of the functional J given by (4.1), we use that $J(u) \le J(T_Q(u))$, it is easy to deduce that the minimizer $u \in L^\infty(\Omega)$.

Remark 4.4. The approach used in the previous problems can be used to prove the existence of a bounded solution of some unilateral problem. Indeed, let

$$\mathcal{K} = \left\{v \in W_0^{1,2}(\Omega):\ v(x) \ge 0 \text{ in } \Omega\right\}$$

and consider the unilateral problems

$$\begin{cases} 0 \le u_n \in W_0^{1,2}(\Omega): \\ \displaystyle\int_\Omega M(x)\nabla u_n \nabla(u_n - v) + \int_\Omega a(x)u_n(u_n - v) \le \int_\Omega f_n(x)(u_n - v), \\ \quad \forall v \in W_0^{1,2}(\Omega), \end{cases}$$

under the assumption of Theorem 2.1. Again we can prove the estimate

$$|u_n| \le Q,$$

which, following the proof of Theorem 2.1, has as a consequence the existence of solution u of the unilateral problems

$$\begin{cases} 0 \le u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega): \\ \displaystyle\int_\Omega M(x)\nabla u \nabla(u-\varphi) + \int_\Omega a(x)u(u-\varphi) \le \int_\Omega f(x)(u-\varphi), \\ \quad \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases}$$

Thus, even if it is only assumed that f belongs to $L^1(\Omega)$, we do not need the framework of papers [7,9] for the study of the above unilateral problem.

Acknowledgment

We thank to the referee for his valuable comments (in particular, Remarks 2.2, 2.3 and 4.3) and for bringing [17] to our attention.

References

[1] D. Arcoya, C. de Coster, L. Jeanjean, K. Tanaka, Continuum of solutions for an elliptic problem with critical growth in the gradient, arXiv:1304.3066, submitted for publication.
[2] Ph. Bénilan, H. Brezis, Nonlinear problems related to the Thomas–Fermi equation, in: Dedicated to Philippe Bénilan, J. Evol. Equ. 3 (2003) 673–770.
[3] Ph. Bénilan, H. Brezis, M.G. Crandall, A semilinear equation in $L^1(\mathbb{R}^N)$, Ann. Sc. Norm. Super. Pisa Cl. Sci. 2 (1975) 523–555.
[4] L. Boccardo, T-minima: an approach to minimization problems in L^1, in: Contributi dedicati alla memoria di Ennio De Giorgi, Ric. Mat. 49 (2000) 135–154.
[5] L. Boccardo, On the regularizing effect of strongly increasing lower order terms, J. Evol. Equ. 3 (2003) 225–236.
[6] L. Boccardo, Hardy potential and quasi-linear elliptic problems having natural growth terms, in: Proceedings of the Conference Held in Gaeta on the Occasion of the 60th Birthday of Haim Brezis, in: Progr. Nonlinear Differential Equations Appl., vol. 63, Birkhäuser, Basel, 2005, pp. 67–82.
[7] L. Boccardo, R. Cirmi, Existence and uniqueness of solutions of unilateral problems with L^1 data, J. Convex Anal. 6 (1999) 195–206.
[8] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, J. Funct. Anal. 87 (1989) 149–169.
[9] L. Boccardo, T. Gallouët, Problèmes unilatéraux dans L^1, C. R. Acad. Sci. Paris 311 (1990) 617–619.
[10] L. Boccardo, T. Gallouët, Strongly nonlinear elliptic equations having natural growth terms and L^1 data, Nonlinear Anal. 19 (1992) 573–579.
[11] L. Boccardo, F. Murat, J.-P. Puel, L^∞-estimate for nonlinear elliptic partial differential equations and application to an existence result, SIAM J. Math. Anal. 23 (1992) 326–333.
[12] H. Brezis, Problèmes elliptiques et paraboliques non linéaires avec données mesures, in: Goulaouic–Meyer–Schwartz Seminar 1981/1982, Exp. No. XX, Ecole Polytech., Palaiseau, 1982, 13 pp.
[13] H. Brezis, Semilinear equations in $\mathbb{R}^N$ without condition at infinity, Appl. Math. Optim. 12 (1984) 271–282.
[14] H. Brezis, F.H. Browder, Strongly nonlinear elliptic boundary problems, Ann. Sc. Norm. Super. Pisa Cl. Sci. 5 (1978) 587–603.
[15] H. Brezis, L. Nirenberg, Removable singularities for nonlinear elliptic equations, Topol. Methods Nonlinear Anal. 9 (1997) 201–219.
[16] H. Brezis, W.A. Strauss, Semi-linear second-order elliptic equations in L^1, J. Math. Soc. Japan 25 (1973) 565–590.
[17] T. Gallouët, J.-M. Morel, Resolution of a semilinear equation in L^1, Proc. Roy. Soc. Edinburgh Sect. A 96 (1984) 275–288.
[18] T. Kato, Schrödinger operators with singular potentials, Israel J. Math. 13 (1972) 135–148.
[19] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble) 15 (1965) 189–258.