Nonlinear Analysis 154 (2017) 210–224

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Quasilinear elliptic systems with measure data

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ARTICLE INFO

Article history:
Received 21 March 2016
Accepted 4 April 2016
Communicated by Enzo Mitidieri

Dedicated to Nicola Fusco on his 60th birthday

Keywords:
Elliptic systems
Existence of solutions
Measures

ABSTRACT

We study the existence of solutions of quasilinear elliptic systems involving N equations and a measure on the right hand side, with the form

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{\beta=1}^{N} \sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u) \frac{\partial}{\partial x_j} u^{\beta} \right) = \mu^{\alpha} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha \in \{1, \dots, N\}$ is the equation index, Ω is an open bounded subset of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}^N$ and μ is a finite Randon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^N$. Existence of a solution is proved for two different sets of assumptions on A. Examples are provided that satisfy our conditions, but do not satisfy conditions required on previous works on this matter.

1. Introduction

Let us consider the Dirichlet elliptic problem

$$-\mathrm{div}\,[A(x, u(x), Du(x))] = \mu \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, μ is a measure on $\mathbb{R}^n$ with values into $\mathbb{R}^N$ and A satisfies suitable coercivity and growth conditions. We note that (1.1) is a system of N equations.

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http://dx.doi.org/10.1016/j.na.2016.04.002

First consider the case $N = 1$, i.e. (1.1) is only one single equation. Existence of distributional solutions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ has been deeply studied, starting from [7], see [9,11,8,32,4] and the survey [5]. Uniqueness seems to be a delicate matter, e.g. see [33,3,19] and the introduction of [12]. Regularity results are contained in [28–30,10,22,2] and the survey [31] (see also [6]). Note that existence of solutions is usually obtained by a truncation argument, which shows why the vectorial case $N \geq 2$ is difficult and only few contributions are available in the literature. In fact, for systems $N \geq 2$, the p-Laplacian $A(x, y, \xi) = |\xi|^{p-2}\xi$ is treated in [18,13], and the anisotropic case, in which each component of the gradient $D_i u$ may have a possibly different exponent p_i, is dealt in [23,24]. Let us write (1.1) using components, that is,

$$- \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[A_i^{\alpha}(x, u(x), Du(x)) \right] = \mu^{\alpha} \quad \text{for } \alpha \in \{1, \ldots, N\}. \tag{1.3}$$

We note that systems more general than the p-Laplacian are considered in [14,16], under the assumption

$$0 \leq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^{\alpha}(x, y, \xi)((Id - b \times b)\xi)_i^{\alpha} \tag{1.4}$$

for every $b \in \mathbb{R}^N$ with $|b| \leq 1$. In [34], the author assumes the componentwise sign condition

$$0 \leq \sum_{i=1}^{n} A_i^{\alpha}(x, y, \xi)\xi_i^{\alpha} \tag{1.5}$$

for every $\alpha \in \{1, \ldots, N\}$. When $N = 2$, (1.4) implies (1.5), since it is enough to take first $b = (1, 0)$ and then $b = (0, 1)$. In [25], the authors consider that A is independent of y and satisfies the componentwise coercivity condition

$$\nu|\xi^{\alpha}|^2 - M \leq \sum_{i=1}^{n} A_i^{\alpha}(x, \xi)\xi_i^{\alpha} \tag{1.6}$$

for every $\alpha \in \{1, \ldots, N\}$, for some constants $\nu \in (0, +\infty)$ and $M \in [0, +\infty)$. In [15], they relax (1.4) to some extent

$$-c|\xi|^q - g(x) \leq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^{\alpha}(x, y, \xi)((Id - b \times b)\xi)_i^{\alpha} \tag{1.7}$$

for some $c \in [0, +\infty)$, $g \in L^1(\Omega)$ and $q \in [1, n)$ where $\xi \mapsto A(x, y, \xi)$ is n-coercive.

We note that in [19,17,26] the authors do not truncate u; they modify Du and then adjust via Hodge decomposition; such a procedure requires the dimension n to be the exponent in the coercivity condition for A. The authors use nice estimates for Hodge decomposition, which have been studied in [20] (see also appendix A, in [21]).

In the present paper we consider quasilinear systems, i.e. systems (1.3) with

$$A_i^{\alpha}(x, y, \xi) = \sum_{\beta=1}^{N} \sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x, y)\, \xi_j^{\beta}, \tag{QL}$$

where the coefficients $a_{i,j}^{\alpha,\beta}(x, y)$ are measurable with respect to x and continuous with respect to y. Moreover, we assume ellipticity for the diagonal coefficients $a_{i,j}^{\alpha,\alpha}$, the off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ (with $\alpha \neq \beta$) are sufficiently small, and all the coefficients are bounded. We prove existence of distributional solutions to (1.1)–(1.2) under two sets of hypotheses, with different assumptions on the off-diagonal coefficients.

The first result deals with off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ (with $\alpha \neq \beta$) that have support contained in a "*staircase*" set, along the diagonals of the $y^{\alpha} - y^{\beta}$ plane (see assumption (A_5) in Section 2 and Fig. 1).

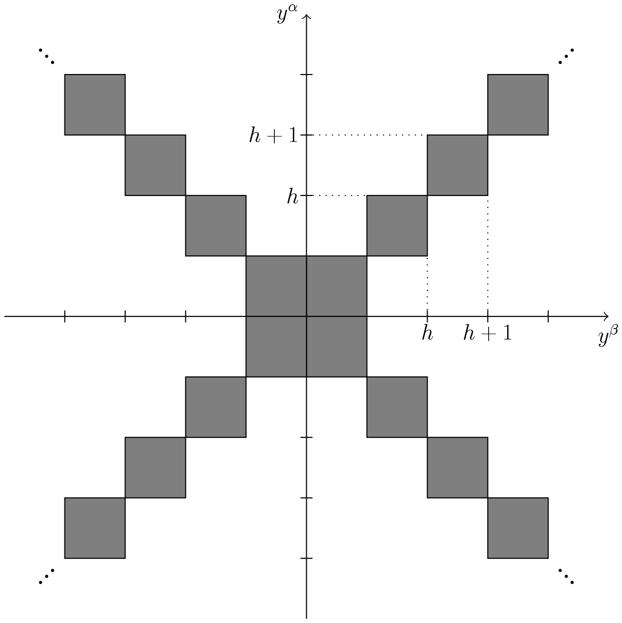

Fig. 1. Support contained in a staircase set (with $h \in \mathbb{N} \cup \{0\}$).

The second is devoted to systems with $N = 2$ equations, where the off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ (with $\alpha \neq \beta$) are proportional to diagonal coefficients $a_{i,j}^{\beta,\beta}$ (see assumptions (A_8) and (A_9) in Section 2).

Precise assumptions and statements of the results are written in Section 2, where we also give examples of quasilinear systems that fit our conditions but not the ones in the quoted papers. The proofs are based on componentwise truncation arguments. The *a priori* estimates, needed to prove the existence result, are contained in Theorem 1, when A has a staircase support, and in Theorem 2, when off-diagonal coefficients are multiple of diagonal ones. Theorem 3 shows the existence of distributional solutions. Proofs are given in Section 3.

2. Assumptions and results

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ and $\overline{B}(x, r)$ the open and the closed ball with center x and radius r, respectively. For convenience, we define $B_1 = B(0, 1)$ and $\overline{B}_1 = \overline{B}(0, 1)$. Assume Ω is an open bounded subset of $\mathbb{R}^n$, and n, N are integers greater than or equal to two.

We consider the following set of assumptions on A.

$(\mathcal{A})$: *For all $i, j \in \{1, \ldots, n\}$ and all $\alpha, \beta \in \{1, \ldots, N\}$, we consider that $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfy the following conditions:*

$(\mathcal{A}_0)$ *$x \mapsto a_{i,j}^{\alpha,\beta}(x, y)$ is measurable and $y \mapsto a_{i,j}^{\alpha,\beta}(x, y)$ is continuous;*

$(\mathcal{A}_1)$ *(boundedness of all the coefficients) for some positive constant $c > 0$, we have*

$$\left| a_{i,j}^{\alpha,\beta}(x, y) \right| \leq c$$

for almost all $x \in \Omega$ and for all $y \in \mathbb{R}^N$;

$(\mathcal{A}_2)$ *(ellipticity of the diagonal coefficients) for some positive constant $m > 0$, we have*

$$m|\lambda|^2 \leq \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha}(x, y)\, \lambda_i \lambda_j$$

for almost all $x \in \Omega$, for all $y \in \mathbb{R}^N$, and for all $\lambda \in \mathbb{R}^n$;

$(\mathcal{A}_3)$ (*off-diagonal coefficients are small*) *for some positive constant L with $0 < L < m$, we have*

$$\left[\sum_{\alpha=1}^{N} \sum_{\beta\in\{1,\dots,N\}\setminus\{\alpha\}} \sum_{i,j=1}^{n} \left|a_{i,j}^{\alpha,\beta}(x,y)\right|^2\right]^{\frac{1}{2}} \le L;$$

$(\mathcal{A}_4)$ (*staircase support*) *when $\alpha \neq \beta$, we have*

$$a_{i,j}^{\alpha,\beta}(x,y) \neq 0 \Rightarrow y \in \bigcup_{h=0}^{+\infty} \left\{h < |y^\alpha| < h+1, h < \left|y^\beta\right| < h+1\right\}.$$

For any $f \in L^1(\Omega, \mathbb{R}^N)$, we say that a function $u : \Omega \to \mathbb{R}^N$ is a (distributional) solution with respect to f, if $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ and

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u(x)) D_j u^\alpha(x)\, D_i\varphi^\alpha(x)dx = \sum_{\alpha=1}^{N} f^\alpha(x)\varphi^\alpha(x)dx, \tag{2.1}$$

for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$. Our first result is the following.

Theorem 1. *Suppose u is a solution with respect to $f \in L^1(\Omega, \mathbb{R}^N)$. Under the assumptions $(\mathcal{A})$, the following estimate*

$$\sum_{\alpha=1}^{N} \int_{\{k<|u^\alpha|<k+1\}} |Du^\alpha|^2 \le \frac{1}{m-L} \sum_{\alpha=1}^{N} \|f^\alpha\|_{L^1(\Omega)} \tag{E_1}$$

holds for every integer $k \geq 0$.

Remark 1. Take

$$A_i^\alpha(x,y,\xi) = \sum_{\beta=1}^{N} \sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x,y)\, \xi_j^\beta. \tag{2.2}$$

Then assumptions $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ give

$$\sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,y)\, \xi_j^\beta \xi_i^\alpha \ge (m-L)\,|\xi|^2, \tag{2.3}$$

hence

$$\sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^\alpha(x,y,\xi)\, \xi_i^\alpha \ge (m-L)\,|\xi|^2. \tag{2.4}$$

Since $0 < L < m$ and by (2.4), we conclude that

$$\xi \mapsto A(x,y,\xi) \text{ is 2-coercive.}$$

Moreover, assumption $(\mathcal{A}_1)$ implies

$$\sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^\alpha(x,y,\xi)\, \xi_i^\alpha \le cn^2N^2\,|\xi|^2. \tag{2.5}$$

When $n > 2$, (2.5) says that $\xi \mapsto A(x,y,\xi)$ is not n-coercive, therefore we are outside the assumptions of [15,19,17,26].

The next example concerns Theorem 1 and gives $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ with $i, j \in \{1, \dots, n\}$ and $\alpha, \beta \in \{1, \dots, N\}$, which satisfy $(\mathcal{A})$. However $A(x, y, \xi)$, defined by (2.2), does not verify neither (1.4) nor (1.5). Moreover, (1.6) is false. Therefore, the example is outside of the assumptions of the quoted works that use them.

Example 1. Let $N = 2$ and $\delta_{i,j}$ be the usual Kronecker delta function

$$\delta_{i,j} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

For α, $\beta \in \{1, 2\}$ and for $i, j \in \{1, \dots, n\}$ we define $a_{i,j}^{\alpha,\beta}(x, y)$ as follows

$$\begin{aligned} &a_{i,j}^{1,1}(x, y) = a_{i,j}^{2,2}(x, y) = \delta_{i,j}, \\ &a_{i,j}^{2,1}(x, y) = 0, \\ &a_{i,j}^{1,2}(x, y) = b_i(y)\,\delta_{i,j}, \end{aligned}$$

where $b_i(y) = 0$ for all $i \geq 2$ and $b_1(y) \equiv b_1\left(y^1, y^2\right)$ is a bounded continuous function that will be defined later (see (2.7)).

Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(t) = \begin{cases} 0 & \text{if} \quad -\infty < t \leq \dfrac{1}{8}, \\ 8t - 1 & \text{if} \quad \dfrac{1}{8} \leq t \leq \dfrac{1}{4}, \\ 1 & \text{if} \quad \dfrac{1}{4} \leq t \leq \dfrac{3}{4}, \\ -8t + 7 & \text{if} \quad \dfrac{3}{4} \leq t \leq \dfrac{7}{8}, \\ 0 & \text{if} \quad \dfrac{7}{8} \leq t < +\infty. \end{cases} \tag{2.6}$$

Then g is continuous and $\mathrm{supp}(g) = \left[\frac{1}{8}, \frac{7}{8}\right]$, so $0 \leq g(t) \leq 1$ for $t \in \mathbb{R}$. Let

$$\begin{aligned} &g_h(s) = g(s - h) && \text{for } s \in \mathbb{R}, \\ &p_h(y) = g_h\left(y^1\right) g_h\left(y^2\right) && \text{for } y = \left(y^1, y^2\right) \in \mathbb{R}^2, \end{aligned}$$

and

$$b_1(y) = \frac{1}{2} \sum_{h=0}^{\infty} p_h(y) \quad \text{for } y \in \mathbb{R}^2. \tag{2.7}$$

Then $y \mapsto b_1(y)$ is a continuous map on $\mathbb{R}^2$,

$$\begin{aligned} &b_1(y) \in \left[0, \frac{1}{2}\right], \\ &b_1(y) \neq 0 \Rightarrow y \in \bigcup_{h=0}^{\infty} \left\{\left(y^1, y^2\right) \in \mathbb{R}^2 : h < y^1 < h + 1,\ h < y^2 < h + 1\right\}, \end{aligned}$$

and

$$b_1\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}. \tag{2.8}$$

Note that $(\mathcal{A}_1)$ is satisfied with $c = 1$ and $(\mathcal{A}_2)$ is satisfied for $m = 1$. Concerning the off-diagonal coefficients, we have

$$\sum_{\alpha=1}^{2} \sum_{\beta\in\{1,2\}\setminus\{\alpha\}} \sum_{i,j=1}^{n} \left|a_{i,j}^{\alpha,\beta}(x,y)\right|^2 = \sum_{i,j=1}^{n} \left|a_{i,j}^{1,2}(x,y)\right|^2 + \sum_{i,j=1}^{n} \left|a_{i,j}^{2,1}(x,y)\right|^2$$
$$= |b_1(y)|^2,$$

hence, $(\mathcal{A}_3)$ is satisfied for

$$L = \sup_{y\in\mathbb{R}^2} |b_1(y)| = \frac{1}{2} < 1 = m.$$

Now, defining $\mathbf{A}$ to satisfy

$$\sum_{i=1}^{n} A_i^1(x,y,\xi)\xi_i^1 = \left|\xi^1\right|^2 + b_1(y)\,\xi_1^2\xi_1^1 = \mathbf{A},$$

taking $\xi_1^1 = 1$, $\xi_1^2 = t \in \mathbb{R}$ and $\xi_i^1 = \xi_i^2 = 0$ for $i \geq 2$, we get

$$\mathbf{A} = 1 + b_1(y)\,t.$$

If we take $y = \left(\frac{1}{2}, \frac{1}{2}\right)$, then

$$\mathbf{A} = 1 + b_1(y)\,t = 1 + \frac{t}{2} \to -\infty \quad \text{as } t \to -\infty,$$

showing that (1.5) is not satisfied. Moreover, since $N = 2$, and in this case, (1.4) implies (1.5), we get that (1.4) is also not satisfied.

To see that (1.6) is false, remark that, if there exist $\nu > 0$ and $M \geq 0$ such that

$$\nu\left|\xi^1\right|^2 - M \leq \sum_{i=1}^{n} A_i^1(x,y,\xi)\,\xi_i^1$$

then, with our choices, the inequality

$$\nu - M \leq 1 + \frac{t}{2}$$

is false for t sufficiently negative (i.e. when $t \to -\infty$). Therefore, (1.6) is false.

In particular, taking $n = 3$, we obtain

$$a^{1,1}(x,y) = a^{2,2}(x,y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$a^{1,2}(x,y) = \begin{pmatrix} b_1(y) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$a^{2,1}(x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which corresponds to the elliptic problem

$$\begin{cases} -\Delta u^1 - \dfrac{\partial}{\partial x_1}\left[b_1(u^1,u^2)\dfrac{\partial u^2}{\partial x_1}\right] = \mu^1 & \text{in } \Omega, \\ -\Delta u^2 = \mu^2 & \text{in } \Omega, \\ u^1 = u^2 = 0 & \text{on } \partial\Omega, \end{cases}$$

with Ω an open bounded subset of $\mathbb{R}^3$ and μ a finite Radon measure on $\mathbb{R}^3$ with values in $\mathbb{R}^2$.

We now introduce a second set of assumptions $(\mathcal{A}^*)$ confined to a system with two equations (that is, $N = 2$).

$(\mathcal{A}^*)$: *For all* $i, j \in \{1, \ldots, n\}$ *and all* $\alpha, \beta \in \{1, 2\}$, *we consider that* $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ *satisfy the conditions* $(\mathcal{A}_0)$–$(\mathcal{A}_3)$ *but* $(\mathcal{A}_4)$ *is replaced by:*
$(\mathcal{A}_4^*)$ *there exist* $r_1, r_2 \in \mathbb{R}$ *such that*

$$a_{i,j}^{1,2}(x,y) = r_1 a_{i,j}^{2,2}(x,y),$$
$$a_{i,j}^{2,1}(x,y) = r_2 a_{i,j}^{1,1}(x,y),$$

with $r_1 r_2 < 1$.

We are in conditions to show the second estimate.

Theorem 2. *Suppose* u *is a solution with respect to* $f \in L^1(\Omega, \mathbb{R}^N)$, *see* (2.1). *Under the assumptions* $(\mathcal{A}^*)$, *the following estimate*

$$\sum_{\alpha=1}^{2} \int_{\{k<|u^\alpha|<k+1\}} |Du^\alpha|^2 \le \frac{1 + |r_1| + |r_2|}{m(1 - r_1 r_2)} \sum_{\alpha=1}^{N} \|f^\alpha\|_{L^1(\Omega)} \tag{E_2}$$

holds for every integer $k \ge 0$.

The following example concerns Theorem 2 and it gives $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ with $i, j \in \{1, \ldots, n\}$ and $\alpha, \beta \in \{1, 2\}$, which satisfy $(\mathcal{A}^*)$. However $A(x, y, \xi)$, defined by (2.2), does not verify neither (1.4) nor (1.5). Moreover, (1.6) is false.

Example 2. Let $N = 2$. For α, $\beta \in \{1, 2\}$ and for $i, j \in \{1, \ldots, n\}$ we define $a_{i,j}^{\alpha,\beta}(x,y)$ as follows

$$a_{i,j}^{1,1}(x,y) = a_{i,j}^{2,2}(x,y) = \delta_{i,j},$$
$$a_{i,j}^{1,2}(x,y) = r_1 a_{i,j}^{2,2}(x,y) = r_1 \delta_{i,j},$$
$$a_{i,j}^{2,1}(x,y) = r_2 a_{i,j}^{1,1}(x,y) = r_2 \delta_{i,j},$$

where the real numbers r_1 and r_2 are such that

$$r_1 > 0, \; r_2 > 0, \quad r_1 r_2 < 1 \quad \text{and} \quad r_1^2 + r_2^2 < \frac{1}{n}. \tag{2.9}$$

Assumption $(\mathcal{A}_1)$ is satisfied with $c = 1$ and $(\mathcal{A}_2)$ is satisfied for $m = 1$. Concerning the off-diagonal coefficients, we have that assumption $(\mathcal{A}_3)$ is satisfied for

$$L := \left[n\left(r_1^2 + r_2^2\right)\right]^{\frac{1}{2}},$$

since the last inequality in (2.9) implies that $L < m$.

To show that (1.5) is not satisfied, remark that for $\alpha = 1$, defining $\mathbf{B}$ to satisfy

$$\sum_{i=1}^{n} A_i^1(x,y,\xi)\xi_i^1 = \sum_{i=1}^{n} \left(\xi_i^1\right)^2 + \sum_{i=1}^{n} r_1 \xi_i^2 \xi_i^1 = \mathbf{B},$$

and choosing

$$\xi_1^1 = 1, \qquad \xi_1^2 = t, \quad \text{and} \quad \xi_i^1 = \xi_i^2 = 0 \quad \text{for } i \ge 2, \tag{2.10}$$

we have

$$\mathbf{B} = 1 + r_1 t \to -\infty \quad \text{as } t \to -\infty \text{ (remember } r_1 > 0).$$

Hence, (1.5) is not satisfied. Moreover, since $N = 2$ and in this case, (1.4) implies (1.5), we get that (1.4) is not satisfied.

To see that (1.6) is false, remark that, from (2.10), we have

$$\sum_{i=1}^{n} A_i^1(x,y,\xi)\,\xi_i^1 = 1 + r_1 t, \qquad \left|\xi^1\right|^2 = \sum_{i=1}^{n}\left(\xi_i^1\right)^2 = 1.$$

If there exist $\nu > 0$ and $M \geq 0$ such that

$$\nu\,|\xi^\alpha|^2 - M \leq \sum_{i=1}^{n} A_i^\alpha(x,y,\xi)\,\xi_i^\alpha$$

then

$$\nu - M = \nu\left|\xi^1\right|^2 - M \leq \sum_{i=1}^{n} A_i^1(\xi)\,\xi_i^1 = 1 + r_1 t$$

and since $r_1 > 0$, this is false for $t \to -\infty$. Therefore (1.6) is false.

Taking $n = 3$ and $r_1 = r_2 = \frac{1}{4}$, we verify (2.9) and the example have the simple form

$$a^{1,1}(x,y) = a^{2,2}(x,y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$a^{1,2}(x,y) = a^{2,1}(x,y) = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix},$$

which corresponds to the elliptic problem

$$\begin{cases} -\Delta u^1 - \frac{1}{4}\Delta u^2 = \mu^1 & \text{in } \Omega, \\ -\frac{1}{4}\Delta u^1 - \Delta u^2 = \mu^2 & \text{in } \Omega, \\ u^1 = u^2 = 0 & \text{on } \partial\Omega, \end{cases}$$

with Ω an open bounded subset of $\mathbb{R}^3$ and μ a finite Radon measure on $\mathbb{R}^3$ with values in $\mathbb{R}^2$.

Under the assumptions of either Theorem 1 or Theorem 2, we prove the main existence result.

Theorem 3. *Assume that either $(\mathcal{A})$ or $(\mathcal{A}^*)$ holds, and μ is a finite Radon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^N$. Then there exists $u \in \bigcap_{1<q<\frac{n}{n-1}} W_0^{1,q}\left(\Omega, \mathbb{R}^N\right)$ such that*

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, u(x))\, D_j u^\beta(x)\, D_i\varphi^\alpha(x)\, dx = \sum_{\alpha=1}^{N} \int_\Omega \varphi^\alpha(x)\, d\mu^\alpha(x)$$

for every $\varphi \in C_0^\infty\left(\Omega, \mathbb{R}^N\right)$.

Remark 2. Theorem 3 deals with existence of a distributional solution to the Dirichlet problem (1.1), (1.2). A glance at our proof in Section 3 shows that we are dealing with a SOLA, that is, a Solution Obtained as Limit of Approximations, see [11] and Section 2.2 in [22]. Entropy solutions have been considered in [3]; see the introduction of [12] for renormalized solutions.

3. Proofs

Proof of Theorem 1. For a fixed integer $k \geq 0$, we consider $T_k : \mathbb{R} \to \mathbb{R}$ defined as follows

$$T_k(s) = \begin{cases} -1 & \text{if} \quad s < -k-1, \\ s+k & \text{if} \quad -k-1 \leq s \leq -k, \\ 0 & \text{if} \quad -k < s < k, \\ s-k & \text{if} \quad k \leq s \leq k+1, \\ 1 & \text{if} \quad k+1 < s. \end{cases} \tag{3.1}$$

Note that

$$-1 \leq T_k(s) \leq 1 \quad \text{for every } s \in \mathbb{R}. \tag{3.2}$$

We choose $\varphi = \left(\varphi^1, \ldots, \varphi^N\right)$ in (2.1) given by

$$\varphi^\alpha = T_k(u^\alpha) \tag{3.3}$$

so that

$$D_i \varphi^\alpha = 1_{\{k<|u^\alpha|<k+1\}} D_i u^\alpha, \tag{3.4}$$

where $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$. Note that inequality (3.2) gives

$$\int_\Omega \sum_{\alpha=1}^N f^\alpha \varphi^\alpha \leq \sum_{\alpha=1}^N \int_\Omega |f^\alpha|. \tag{3.5}$$

Moreover,

$$\begin{aligned} \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta 1_{\{k<|u^\alpha|<k+1\}} D_i u^\alpha &= \int_\Omega \sum_{\alpha=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\alpha} D_j u^\alpha 1_{\{k<|u^\alpha|<k+1\}} D_i u^\alpha \\ &+ \int_\Omega \sum_{\alpha=1}^N \sum_{\beta\in\{1,2,\ldots,N\}\setminus\{\alpha\}} \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta 1_{\{k<|u^\alpha|<k+1\}} D_i u^\alpha \end{aligned}$$

where $a_{i,j}^{\alpha,\beta} \equiv a_{i,j}^{\alpha,\beta}(x, u(x))$. Applying (3.3) in Eq. (2.1), we may define the terms **C**, **D**, **E** as

$$\begin{aligned} \mathbf{C} &= \int_\Omega \sum_{\alpha=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\alpha} D_j u^\alpha 1_{\{k<|u^\alpha|<k+1\}} D_i u^\alpha \\ &= -\int_\Omega \sum_{\alpha=1}^N \sum_{\beta\in\{1,2,\ldots,N\}\setminus\{\alpha\}} \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta 1_{\{k<|u^\alpha|<k+1\}} D_i u^\alpha + \int_\Omega \sum_{\alpha=1}^N f^\alpha \varphi^\alpha = \mathbf{D} + \mathbf{E}. \end{aligned} \tag{3.6}$$

The ellipticity condition for diagonal coefficients $a_{i,j}^{\alpha,\alpha}$ (see $(\mathcal{A}_2)$) implies that

$$\mathbf{C} \geq m \sum_{\alpha=1}^N \int_{\{k<|u^\alpha|<k+1\}} |Du^\alpha|^2. \tag{3.7}$$

Now we use the assumption $(\mathcal{A}_4)$, about the support for off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$, obtaining

$$1_{\{k<|u^\alpha|<k+1\}} a_{i,j}^{\alpha,\beta} = 1_{\{k<|u^\alpha|<k+1\}} 1_{\{k<|u^\beta|<k+1\}} a_{i,j}^{\alpha,\beta}. \tag{3.8}$$

Then

$$\mathbf{D} = -\int_\Omega \sum_{\alpha=1}^N \sum_{\beta\in\{1,2,\ldots,N\}\setminus\{\alpha\}} \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} 1_{\{k<|u^\alpha|<k+1\}} 1_{\{k<|u^\beta|<k+1\}} D_j u^\beta D_i u^\alpha dx \tag{3.9}$$

and using the Cauchy–Schwarz inequality, we get

$$\mathbf{D} \le \int_\Omega \left(\sum_{\alpha=1}^N \sum_{\beta \in \{1,2,\dots,N\} \setminus \{\alpha\}} \sum_{i,j=1}^n \left| a_{i,j}^{\alpha,\beta} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{\alpha=1}^N |Du^\alpha|^2 1_{\{k<|u^\alpha|<k+1\}} \right)^{\frac{1}{2}} \left(\sum_{\beta=1}^N \left|Du^\beta\right|^2 1_{\{k<|u^\beta|<k+1\}} \right)^{\frac{1}{2}}$$

$$\le L \left(\sum_{\gamma=1}^N \int_\Omega |Du^\gamma|^2 1_{\{k<|u^\gamma|<k+1\}} \right) \tag{3.10}$$

where we used assumption $(\mathcal{A}_3)$.

Equality (3.6) and inequalities (3.10), (3.5) and (3.7) give

$$m \sum_{\alpha=1}^N \int_{\{k<|u^\alpha|<k+1\}} |Du^\alpha|^2 \le L \sum_{\alpha=1}^N \int_{\{k<|u^\alpha|<k+1\}} |Du^\alpha|^2 + \sum_{\alpha=1}^N \int_\Omega |f^\alpha|,$$

hence

$$(m-L) \sum_{\alpha=1}^N \int_{\{k<|u^\alpha|<k+1\}} |Du^\alpha|^2 \le \sum_{\alpha=1}^N \|f^\alpha\|_{L^1(\Omega)}.$$

Since $0 < L < m$, we get estimate (E_1). This ends the proof of Theorem 1. □

Proof of Theorem 2. As in the proof of Theorem 1, for a fixed integer $k \ge 0$, we consider $T_k : \mathbb{R} \to \mathbb{R}$ defined by (3.1). We take a new test function $\varphi = \left(\varphi^1, \varphi^2\right)$ in (2.1) as

$$\varphi^\alpha = \sum_{\gamma=1}^2 C_\alpha^\gamma T_k(u^\gamma) \tag{3.11}$$

where C_α^γ are real constants to be chosen later. Then

$$D_i \varphi^\alpha = \sum_{\gamma=1}^2 C_\alpha^\gamma 1_{\{k<|u^\gamma|<k+1\}} D_i u^\gamma. \tag{3.12}$$

Note that inequality (3.2) gives

$$\int_\Omega \left(\sum_{\alpha=1}^2 f^\alpha \varphi^\alpha \right) dx \le \sum_{\alpha=1}^2 \sum_{\gamma=1}^2 |C_\alpha^\gamma| \int_\Omega |f^\alpha|\, dx. \tag{3.13}$$

Moreover,

$$\begin{aligned}
\int_\Omega \sum_{\alpha,\beta=1}^2 \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta D_i \varphi^\alpha &= \int_\Omega \sum_{\alpha,\beta=1}^2 \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta \left(\sum_{\gamma=1}^2 C_\alpha^\gamma 1_{\{k<|u^\gamma|<k+1\}} D_i u^\gamma \right) \\
&= \int_\Omega \sum_{i,j=1}^n \sum_{\gamma=1}^2 a_{i,j}^{1,1} D_j u^1 C_1^\gamma 1_{\{k<|u^\gamma|<k+1\}} D_i u^\gamma \\
&\quad + \int_\Omega \sum_{i,j=1}^n \sum_{\gamma=1}^2 a_{i,j}^{1,2} D_j u^2 C_1^\gamma 1_{\{k<|u^\gamma|<k+1\}} D_i u^\gamma \\
&\quad + \int_\Omega \sum_{i,j=1}^n \sum_{\gamma=1}^2 a_{i,j}^{2,1} D_j u^1 C_2^\gamma 1_{\{k<|u^\gamma|<k+1\}} D_i u^\gamma \\
&\quad + \int_\Omega \sum_{i,j=1}^n \sum_{\gamma=1}^2 a_{i,j}^{2,2} D_j u^2 C_2^\gamma 1_{\{k<|u^\gamma|<k+1\}} D_i u^\gamma
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,1} D_j u^1 C_1^1 1_{\{k<|u^1|<k+1\}} D_i u^1 \\
&\quad + \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,1} D_j u^1 C_1^2 1_{\{k<|u^2|<k+1\}} D_i u^2 \\
&\quad + \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,2} D_j u^2 C_1^1 1_{\{k<|u^1|<k+1\}} D_i u^1 \\
&\quad + \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,2} D_j u^2 C_1^2 1_{\{k<|u^2|<k+1\}} D_i u^2 \\
&\quad + \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,1} D_j u^1 C_2^1 1_{\{k<|u^1|<k+1\}} D_i u^1 \\
&\quad + \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,1} D_j u^1 C_2^2 1_{\{k<|u^2|<k+1\}} D_i u^2 \\
&\quad + \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,2} D_j u^2 C_2^1 1_{\{k<|u^1|<k+1\}} D_i u^1 \\
&\quad + \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,2} D_j u^2 C_2^2 1_{\{k<|u^2|<k+1\}} D_i u^2 \\
&= \mathbf{F}
\end{aligned}
$$

where $a_{i,j}^{\alpha,\beta} \equiv a_{i,j}^{\alpha,\beta}(x, u(x))$. Using assumptions $(\mathcal{A}_4^*)$, we get

$$
\begin{aligned}
\mathbf{F} &= C_1^1 \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,1} D_j u^1 1_{\{k<|u^1|<k+1\}} D_i u^1 + C_1^2 \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,1} D_j u^1 1_{\{k<|u^2|<k+1\}} D_i u^2 \\
&\quad + r_1 C_1^1 \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,2} D_j u^2 1_{\{k<|u^1|<k+1\}} D_i u^1 + r_1 C_1^2 \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,2} D_j u^2 1_{\{k<|u^2|<k+1\}} D_i u^2 \\
&\quad + r_2 C_2^1 \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,1} D_j u^1 1_{\{k<|u^1|<k+1\}} D_i u^1 + r_2 C_2^2 \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,1} D_j u^1 1_{\{k<|u^2|<k+1\}} D_i u^2 \\
&\quad + C_2^1 \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,2} D_j u^2 1_{\{k<|u^1|<k+1\}} D_i u^1 + C_2^2 \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,2} D_j u^2 1_{\{k<|u^2|<k+1\}} D_i u^2 \\
&= \left(C_1^1 + r_2 C_2^1\right) \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,1} D_j u^1 1_{\{k<|u^1|<k+1\}} D_i u^1 + \left(C_1^2 + r_2 C_2^2\right) \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,1} D_j u^1 1_{\{k<|u^2|<k+1\}} D_i u^2 \\
&\quad + \left(r_1 C_1^1 + C_2^1\right) \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,2} D_j u^2 1_{\{k<|u^1|<k+1\}} D_i u^1 + \left(r_1 C_1^2 + C_2^2\right) \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,2} D_j u^2 1_{\{k<|u^2|<k+1\}} D_i u^2 \\
&= \mathbf{G}.
\end{aligned}
$$

Now, choosing

$$
C_1^2 = -r_2 C_2^2 \quad \text{and} \quad C_2^1 = -r_1 C_1^1,
$$

hence

$$
\begin{aligned}
&C_1^1 + r_2 C_2^1 = C_1^1 - r_1 r_2 C_1^1 = (1 - r_1 r_2)\, C_1^1, \\
&C_1^2 + r_2 C_2^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&r_1 C_1^1 + C_2^1 = 0, \\
&r_1 C_1^2 + C_2^2 = -r_1 r_2 C_2^2 + C_2^2 = (1 - r_1 r_2)\, C_2^2,
\end{aligned}
$$

we get

$$
\mathbf{G} = (1 - r_1 r_2)\, C_1^1 \int_\Omega \sum_{i,j=1}^n a_{i,j}^{1,1} D_j u^1 D_i u^1 1_{\{k<|u^1|<k+1\}} + (1 - r_1 r_2)\, C_2^2 \int_\Omega \sum_{i,j=1}^n a_{i,j}^{2,2} D_j u^2 D_i u^2 1_{\{k<|u^2|<k+1\}}.
$$

It is enough to choose $C_1^1 = C_2^2 = 1$, use the fact that $r_1 r_2 < 1$, and the ellipticity condition $(\mathcal{A}_2)$ to obtain

$$
\mathbf{G} \geq m\,(1 - r_1 r_2) \int_{\{k<|u^1|<k+1\}} \left|Du^1\right|^2 + m\,(1 - r_1 r_2) \int_{\{k<|u^2|<k+1\}} \left|Du^2\right|^2,
$$

and

$$
\begin{aligned}
\int_\Omega \sum_{\alpha,\beta=1}^2 \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta D_i \varphi^\alpha \geq{}& m\,(1 - r_1 r_2) \int_{\{k<|u^1|<k+1\}} \left|Du^1\right|^2 \\
&+ m\,(1 - r_1 r_2) \int_{\{k<|u^2|<k+1\}} \left|Du^2\right|^2.
\end{aligned} \tag{3.14}
$$

We use Eq. (2.1), estimates (3.13) and (3.14) to obtain

$$
\begin{aligned}
&m\,(1 - r_1 r_2) \int_{\{k<|u^1|<k+1\}} \left|Du^1\right|^2 + m\,(1 - r_1 r_2) \int_{\{k<|u^2|<k+1\}} \left|Du^2\right|^2 \\
&\leq \sum_{\alpha=1}^2 \sum_{\gamma=1}^2 |C_\alpha^\gamma| \int_\Omega |f^\alpha| \\
&= (1 + |r_2|) \int_\Omega \left|f^1\right| dx + (1 + |r_1|) \int_\Omega \left|f^2\right| dx \\
&\leq (1 + |r_1| + |r_2|) \left(\int_\Omega \left|f^1\right| dx + \int_\Omega \left|f^2\right| dx \right).
\end{aligned}
$$

This ends the proof. □

Proof of Theorem 3. Let $\mu : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}^N$ be a finite Radon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^N$ and let $|\mu|$ denote its total variation. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers, that is $\rho_\varepsilon(x) = \varepsilon^{-n} \rho\left(\frac{x}{\varepsilon}\right)$, where $\rho \in C_c^\infty(\mathbb{R}^n)$ satisfies

$$
\rho(x) \geq 0, \qquad \rho(-x) = \rho(x), \qquad \operatorname{supp}(\rho) \subseteq B_1 \quad \text{and} \quad \int_{\mathbb{R}^n} \rho(x)\, dx = 1.
$$

Recall that the convolution $\mu * \rho_\varepsilon$ defined by

$$
(\mu * \rho_\varepsilon)(x) = \int_{\overline{B}(x,\varepsilon)} \rho_\varepsilon(x - y)\, d\mu(y)
$$

is a regular function, that is $\mu * \rho_\varepsilon \in C^\infty\left(\mathbb{R}^n, \mathbb{R}^N\right)$.

Let $(\varepsilon_h)_{h\in\mathbb{N}}$ be such that $0 < \varepsilon_h \leq 1$ and $\varepsilon_h \to 0$ as $h \to \infty$. Then $\mu * \rho_{\varepsilon_h} \in C^\infty\left(\mathbb{R}^n, \mathbb{R}^N\right)$ and

$$
\int_\Omega |\mu * \rho_{\varepsilon_h}|(x)\, dx \leq |\mu|\left(I_{\varepsilon_h}(\Omega)\right) \leq |\mu|(\mathbb{R}^n) < +\infty, \tag{3.15}
$$

where $I_\varepsilon(\Omega)$ denotes the open ε-neighborhood of Ω (see [1], Theorem 2.2, p. 42). We define

$$
f_h = \mu * \rho_{\varepsilon_h} \in L^2\left(\Omega, \mathbb{R}^N\right).
$$

By assumptions $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$, we have (2.3). Then the conditions of Leray–Lions Theorem (see [27]) are satisfied, so for each $h \in \mathbb{N}$ we obtain

$$u_h \in W_0^{1,2}\left(\Omega, \mathbb{R}^N\right)$$

and, for all $\varphi \in W_0^{1,2}\left(\Omega, \mathbb{R}^N\right)$,

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, u_h(x)) D_j u_h^\alpha(x)\, D_i \varphi^\alpha(x) dx = \int_\Omega \sum_{\alpha=1}^{N} f_h^\alpha(x) \varphi^\alpha(x) dx = \sum_{\alpha=1}^{N} \int_\Omega \varphi^\alpha d\mu_{\varepsilon_h}^\alpha. \tag{3.16}$$

Now, we use the estimate (E_1), if the assumptions of Theorem 1 are satisfied, and estimate (E_2), if the assumptions of Theorem 2 are satisfied. We obtain that there exists $\widetilde{c} > 0$ such that

$$\sum_{\alpha=1}^{N} \int_{\left\{k<\left|u_h^\alpha\right|<k+1\right\}} \left|Du_h^\alpha\right|^2 \le \widetilde{c} \sum_{\alpha=1}^{N} \|f_h^\alpha\|_{L^1(\Omega)} \quad \text{for all } k \ge 0. \tag{3.17}$$

The definition of f_h^α and inequalities (3.15) and (3.17) allow us to use Lemma 1 of Boccardo–Gallouet [7], i.e. for $\alpha \in \{1, \dots, N\}$ and $1 < q < \frac{n}{n-1}$ there exists $C_2 > 0$ such that

$$\|u_h^\alpha\|_{W_0^{1,q}(\Omega)} \le C_2.$$

Hence, passing to a subsequence if needed, we may assume that there exists $u_\infty^\alpha \in W_0^{1,q}(\Omega)$ such that

$$u_h^\alpha \xrightarrow{w} u_\infty^\alpha \quad \text{in } W_0^{1,q}(\Omega) \quad \text{and} \quad u_h^\alpha \to u_\infty^\alpha \quad \text{in } L^q(\Omega), \tag{3.18}$$

where $\xrightarrow{w}$ denotes weak convergence. Again passing to a subsequence if needed, we may assume that

$$u_h^\alpha(x) \to u_\infty^\alpha(x) \quad \text{for almost all } x \in \Omega,$$

and, by the continuity of $y \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ (see $(\mathcal{A}_0)$), it follows that

$$a_{i,j}^{\alpha,\beta}(x, u_h(x)) \to a_{i,j}^{\alpha,\beta}(x, u_\infty(x)) \quad \text{for almost all } x \in \Omega.$$

For presentation convenience, we set $a_{i,j}^{\alpha,\beta}(u_h) \equiv a_{i,j}^{\alpha,\beta}(x, u_h(x))$ and $a_{i,j}^{\alpha,\beta}(u_\infty) \equiv a_{i,j}^{\alpha,\beta}(x, u_\infty(x))$. For every $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$, we have

$$\begin{aligned}
&\left| \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(u_h) D_j u_h^\beta D_i \varphi^\alpha dx - \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(u_\infty) D_j u_\infty^\beta D_i \varphi^\alpha dx \right| \\
&= \left| \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \left[a_{i,j}^{\alpha,\beta}(u_h) - a_{i,j}^{\alpha,\beta}(u_\infty)\right] D_j u_h^\beta D_i \varphi^\alpha dx + \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(u_\infty) D_j u_h^\beta D_i \varphi^\alpha dx \right. \\
&\quad \left. - \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(u_\infty) D_j u_\infty^\beta D_i \varphi^\alpha dx \right| \\
&\le \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \left[\int_\Omega \left| a_{i,j}^{\alpha,\beta}(u_h) - a_{i,j}^{\alpha,\beta}(u_\infty) \right|^{q'} \left|D_i \varphi^\alpha\right|^{q'} \right]^{\frac{1}{q'}} \left(\int_\Omega \left| D_i u_h^\beta \right|^q \right)^{\frac{1}{q}} dx \\
&\quad + \left| \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \left[\int_\Omega a_{i,j}^{\alpha,\beta}(u_\infty) D_i \varphi^\alpha D_j u_h^\beta - \int_\Omega a_{i,j}^{\alpha,\beta}(u_\infty) D_i \varphi^\alpha D_j u_\infty^\beta \right] \right|.
\end{aligned} \tag{3.19}$$

Using the dominated convergence theorem, we obtain

$$\int_\Omega \left| a_{i,j}^{\alpha,\beta}(u_h) - a_{i,j}^{\alpha,\beta}(u_\infty) \right|^{q'} \left|D_i \varphi^\alpha\right|^{q'} \to 0 \tag{3.20}$$

and, by the weak convergence in (3.18), it follows that

$$\int_{\Omega} a_{i,j}^{\alpha,\beta}(u_\infty) D_i\varphi^\alpha D_j u_h^\beta \to \int_{\Omega} a_{i,j}^{\alpha,\beta}(u_\infty) D_i\varphi^\alpha D_j u_\infty^\beta. \tag{3.21}$$

From (3.19)–(3.21), it follows that

$$\int_{\Omega} \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(x, u_h(x)) D_j u_h^\beta(x) D_i\varphi^\alpha(x)\, dx$$
$$\to \int_{\Omega} \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(x, u_\infty(x)) D_j u_\infty^\beta(x) D_i\varphi^\alpha(x)\, dx. \tag{3.22}$$

On the other hand,

$$\int_{\Omega} f_h^\alpha(x)\varphi^\alpha(x)\, dx = \int_{\Omega} \varphi^\alpha d\mu_{\varepsilon_h}^\alpha \to \int_{\Omega} \varphi^\alpha d\mu^\alpha.$$

Therefore, for each $q \in \left(1, \frac{n}{n-1}\right)$ there exist $u_{\infty,q} \in W_0^{1,q}(\Omega, \mathbb{R}^N)$ and a subsequence, again denoted by $(u_h)_h$, such that

$$u_h \xrightarrow{w} u_{\infty,q} \quad \text{in } W_0^{1,q}(\Omega, \mathbb{R}^N) \tag{3.23}$$

and

$$\int_{\Omega} \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(x, u_{\infty,q}(x)) D_j u_{\infty,q}^\beta(x) D_i\varphi^\alpha(x)\, dx = \int_{\Omega} \sum_{\alpha=1}^{N} \varphi^\alpha(x)\, d\mu^\alpha(x),$$

i.e. $u_{\infty,q}$ is a (distributional) solution of our problem.

We claim that $u_{\infty,q}$ does not depend on a fixed q. Indeed, if $q < \widetilde{q} < \frac{n}{n-1}$ then

$$\|u_h^\alpha\|_{W_0^{1,\widetilde{q}}(\Omega)} \le C_2(\widetilde{q})$$

and, passing to a subsequence of $(u_h)_h$ satisfying (3.23), we can assume that

$$u_h \xrightarrow{w} u_{\infty,\widetilde{q}} \quad \text{in } W_0^{1,\widetilde{q}}(\Omega, \mathbb{R}^N). \tag{3.24}$$

By (3.23), (3.24) and the fact that $q < \widetilde{q}$, we obtain

$$u_{\infty,q} = u_{\infty,\widetilde{q}}.$$

Therefore,

$$u_{\infty,q} \in W_0^{1,\widetilde{q}}(\Omega, \mathbb{R}^N) \quad \text{for all } \widetilde{q} \in \left(1, \frac{n}{n-1}\right),$$

and the proof is concluded. □

Acknowledgments

This joint work was started when the second and third authors were visiting the Department of Information Engineering, Computer Science and Mathematics (DISIM) of the University of L'Aquila (Italy), where the third author spent part of his sabbatical leave. Partial support by Portuguese funds through CIDMA—Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (FCT), within the project PEst-OE/MAT/UI4106/2014, and, for the third author, by the Sabbatical Fellowship SFRH/BSAB/113647/2015, as well as the hospitality and partial support of DISIM is gratefully acknowledged. The first author gladly acknowledges the support of MIUR, UNIVAQ, GNAMPA, INdAM.

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