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Smallness and cancellation in some elliptic systems with measure data

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ABSTRACT

In a bounded open subset $\Omega \subset \mathbb{R}^n$, we study Dirichlet problems with elliptic systems, involving a finite Radon measure μ on $\mathbb{R}^n$ with values into $\mathbb{R}^N$, defined by

$$\begin{cases} -\mathrm{div}\, A(x, u(x), Du(x)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A_i^\alpha(x, y, \xi) = \sum_{\beta=1}^{N} \sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x, y)\, \xi_j^\beta$ with $\alpha \in \{1, \dots, N\}$ the equation index. We prove the existence of a (distributional) solution $u : \Omega \to \mathbb{R}^N$, obtained as the limit of approximations, by assuming: (i) that coefficients $a_{i,j}^{\alpha,\beta}$ are bounded Carathéodory functions; (ii) ellipticity of the diagonal coefficients $a_{i,j}^{\alpha,\alpha}$; and (iii) smallness of the quadratic form associated to the off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ (i.e. $\alpha \neq \beta$) verifying a r-staircase support condition with $r > 0$. Such a smallness condition is satisfied, for instance, in each one of these cases: (a) $a_{i,j}^{\alpha,\beta} = -a_{j,i}^{\beta,\alpha}$ (skew-symmetry); (b) $|a_{i,j}^{\alpha,\beta}|$ is small; (c) $a_{i,j}^{\alpha,\beta}$ may be decomposed into two parts, the first enjoying skew-symmetry and the second being small in absolute value. We give an example that satisfies our hypotheses but does not satisfy assumptions introduced in previous works. A Brezis's type nonexistence result is also given for general (smooth) elliptic-hyperbolic systems.

1. Introduction

Let us consider the Dirichlet elliptic problem

$$-\mathrm{div}\,[A(x, u(x), Du(x))] = \mu \quad \text{in } \Omega, \tag{1.1}$$

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$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, μ is a measure on $\mathbb{R}^n$ with values into $\mathbb{R}^N$ and A satisfies suitable coercivity and growth conditions. We note that (1.1) is a system of N equations.

First consider the case $N = 1$, i.e. (1.1) is only one single equation. Existence of distributional solutions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ has been deeply studied, starting from [11], see [16], [25], [12], [61], [7] and the survey [8]. Uniqueness seems to be a delicate matter, e.g. see [62], [5], [32] and the introduction of [24]. Regularity results are contained in [56], [57], [59], [17], [18], [58], [35], [40], [3], [2], [14] and the survey [60] (see also [9] and [10]). Note that existence of solutions is usually obtained by a truncation argument, which shows why the vectorial case $N \geq 2$ is difficult and only few contributions are available in the literature. In fact, for systems $N \geq 2$, the p-Laplacian $A(x, y, \xi) = |\xi|^{p-2}\xi$ is treated in [31] and [26], and the anisotropic case, in which each component of the gradient $D_i u$ may have a possibly different exponent p_i, is dealt in [50] and [51]. Let us mention [41] for pointwise potential estimates in the framework of vectorial p-Laplacian. Let us write (1.1) using components, that is,

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[A_i^\alpha(x, u(x), Du(x))\right] = \mu^\alpha \quad \text{for } \alpha \in \{1, \dots, N\}. \tag{1.3}$$

We note that systems more general than the p-Laplacian are considered in [27] and [29], under the assumption

$$0 \leq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^\alpha(x, y, \xi)((Id - b \times b)\xi)_i^\alpha \tag{1.4}$$

for every $b \in \mathbb{R}^N$ with $|b| \leq 1$; see also [42], [6], [20] where (1.4) has been used. In [63], the author assumes the componentwise sign condition

$$0 \leq \sum_{i=1}^{n} A_i^\alpha(x, y, \xi)\xi_i^\alpha \tag{1.5}$$

for every $\alpha \in \{1, ..., N\}$. When $N = 2$, (1.4) implies (1.5), since it is enough to take first $b = (1, 0)$ and then $b = (0, 1)$. In [52], the authors consider that A is independent of y and satisfies the componentwise coercivity condition

$$\nu|\xi^\alpha|^2 - M \leq \sum_{i=1}^{n} A_i^\alpha(x, \xi)\xi_i^\alpha \tag{1.6}$$

for every $\alpha \in \{1, ..., N\}$, for some constants $\nu \in (0, +\infty)$ and $M \in [0, +\infty)$. In [28], they relax (1.4) to some extent

$$-c|\xi|^q - g(x) \leq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^\alpha(x, y, \xi)((Id - b \times b)\xi)_i^\alpha \tag{1.7}$$

for some $c \in [0, +\infty)$, $g \in L^1(\Omega)$ and $q \in [1, n)$ where $\xi \mapsto A(x, y, \xi)$ is n-coercive.

We note that in [32], [30] and [53] the authors do not truncate u. They modify Du and then adjust via Hodge decomposition; such a procedure requires the dimension n to be the exponent in the coercivity condition for A. The authors use nice estimates for Hodge decomposition, which have been studied in [33] (see also appendix A, in [34]).

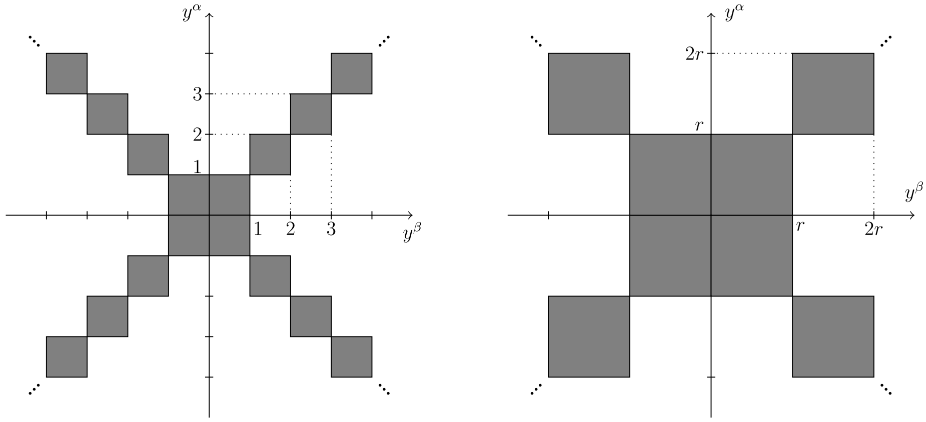

Fig. 1. (left) Support contained in a *staircase* set; (right) Support contained in a *r-staircase* set.

In [54], the authors consider quasilinear systems, i.e. systems (1.3) with

$$A_i^\alpha(x, y, \xi) = \sum_{\beta=1}^{N} \sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x, y)\, \xi_j^\beta, \tag{QL}$$

where the bounded coefficients $a_{i,j}^{\alpha,\beta}(x, y)$ are measurable with respect to x and continuous with respect to y. Moreover, they assume ellipticity for the diagonal coefficients $a_{i,j}^{\alpha,\alpha}$

$$m|\lambda|^2 \le \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha}(x, y)\lambda_i\lambda_j \tag{1.8}$$

for every $\alpha = 1, \cdots, N$, for some constant $m \in (0, +\infty)$; assume that off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ (with $\alpha \neq \beta$) are small

$$\left(\sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha} \sum_{i,j=1}^{n} |a_{i,j}^{\alpha,\beta}(x, y)|^2 \right)^{1/2} \le L < m; \tag{1.9}$$

and require that off-diagonal coefficients have support contained in a *staircase* set, along the diagonals of the $y^\alpha - y^\beta$ plane, defined as

$$\alpha \neq \beta, \quad a_{i,j}^{\alpha,\beta}(x, y) \neq 0 \quad \Longrightarrow \quad y \in \bigcup_{h=0}^{+\infty} \left\{ \begin{array}{l} h \le |y^\alpha| < h+1 \\ h \le |y^\beta| < h+1 \end{array} \right\} \tag{1.10}$$

see Fig. 1 (left). However, smallness condition (1.9) does not take into account cancellations that might occur and condition (1.10) does not allow (general) compact supported off-diagonal coefficients.

In the present paper, we also deal with quasilinear systems (QL), assuming that coefficients $a_{i,j}^{\alpha,\beta}$ are bounded Carathéodory functions and satisfy the ellipticity condition (1.8). Nevertheless, instead of (1.9), we require

$$\sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, y)\xi_i^\alpha \xi_j^\beta \ge -\theta \sum_{\alpha=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha}(x, y)\xi_i^\alpha \xi_j^\alpha \tag{1.11}$$

for some constant $\theta \in (0,1)$. This allows to deal with skew-symmetric off-diagonal coefficients $a_{i,j}^{\alpha,\beta} = -a_{j,i}^{\beta,\alpha}$, independently of their magnitude. Indeed, in such situation, the left hand side of (1.11) is zero and the right hand side is negative because of (1.8). Further, condition (1.11) covers condition (1.9), where it is imposed that off-diagonal coefficients are small, by choosing $\theta = L/m$ and using (1.8). The linearity of the condition (1.11) with respect to $a_{i,j}^{\alpha,\beta}$ implies that it also holds when off-diagonal coefficients split as $a_{i,j}^{\alpha,\beta} = \tilde{a}_{i,j}^{\alpha,\beta} + \tilde{\tilde{a}}_{i,j}^{\alpha,\beta}$, with a skew-symmetric part $\tilde{a}_{i,j}^{\alpha,\beta} = -\tilde{a}_{j,i}^{\beta,\alpha}$ and a part $\tilde{\tilde{a}}_{i,j}^{\alpha,\beta}$ that verifies (1.9). Regarding condition (1.10), the *staircase* with step size 1, we let the step size to be any fixed $r > 0$ by define the *r-staircase* condition

$$\alpha \neq \beta, \quad a_{i,j}^{\alpha,\beta}(x,y) \neq 0 \quad \Longrightarrow \quad y \in \bigcup_{h=0}^{+\infty} \left\{ \begin{array}{l} hr \leq |y^\alpha| < (h+1)r \\ hr \leq |y^\beta| < (h+1)r \end{array} \right\}, \tag{1.12}$$

see Fig. 1 (right). Now coefficients with any compact support are allowed.

We prove existence of distributional solutions to (1.1)–(1.2), where precise assumptions and statements of the results are written in Section 2, giving an example of a quasilinear system that fits our conditions but none of the conditions in the above cited works. The proof is based on componentwise truncation arguments in strips and approximation techniques. The *a priori* estimate required to prove existence is contained in Lemma 1. Theorem 1 is the main result of this work, which shows the existence of a distributional solution $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$ for any $1 \leq q < \frac{n}{n-1}$ and finite Radon measure μ on $\mathbb{R}^n$ with values in $\mathbb{R}^N$, under suitable conditions on the coefficients of the operator. In Corollary 1, we give a nonexistence result of a (very) weak solution $u \in L^t(\Omega, \mathbb{R}^N) \cap W_0^{1,q}(\Omega, \mathbb{R}^N)$, with $n > 2$, $t > \frac{n}{n-2}$, $q > \frac{nt'}{n-t'}$, for general (smooth) elliptic-hyperbolic systems when it is involved a Dirac measure $\mu^\alpha = \delta_y$, for some $\alpha \in \{1, \ldots, N\}$, where t' is the Hölder conjugate of t. Proofs are given in Section 3. Let us end this introduction by mentioning regularity results for systems with some special measures μ: [19], [46]. Parabolic equations and systems have been studied in [47], [36], [37], [38], [39], [44], [48]. Special higher order equations are contained in [22]. Singular equations have been considered in [15], [23].

2. Assumptions and main result

Assume Ω is an open bounded subset of $\mathbb{R}^n$, with $n \geq 3$; N is an integer greater or equal than two. For $x \in \mathbb{R}^n$ and $\rho > 0$, we denote by $B(x,\rho)$ and $\overline{B}(x,\rho)$ the open and the closed ball with center x and radius ρ, respectively. For convenience, we define $B_1 = B(0,1)$, $\overline{B}_1 = \overline{B}(0,1)$ and $[N]$ denotes the set $\{1, .., N\}$.

Consider the elliptic Dirichlet problem with measure data

$$\begin{cases} -\sum\limits_{i \in [n]} \frac{\partial}{\partial x_i} \left(\sum\limits_{\beta \in [N]} \sum\limits_{j \in [n]} a_{i,j}^{\alpha,\beta}(x,u) \frac{\partial}{\partial x_j} u^\beta \right) = \mu^\alpha & \text{in } \Omega, \\ u^\alpha = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

for each $\alpha \in [N]$. For any μ that is a finite Radon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^N$, we say that a function $u : \Omega \to \mathbb{R}^N$ is a solution of the problem (2.1), if $u \in W_0^{1,1}(\Omega, \mathbb{R}^N)$ and

$$\sum_{\alpha,\beta \in [N]} \sum_{i,j \in [n]} \int_\Omega a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x)\, D_i \varphi^\alpha(x) dx = \sum_{\alpha \in [N]} \int_\Omega \varphi^\alpha(x) d\mu^\alpha, \tag{2.2}$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$.

We consider the following set of general assumptions on the functions $a^{\alpha,\beta}_{i,j}$.

($\mathcal{A}$): For all $i, j \in [n]$ and $\alpha, \beta \in [N]$, we require that $a^{\alpha,\beta}_{i,j} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfy the following conditions:

($\mathcal{A}_0$) $x \mapsto a^{\alpha,\beta}_{i,j}(x, y)$ is measurable and $y \mapsto a^{\alpha,\beta}_{i,j}(x, y)$ is continuous;

($\mathcal{A}_1$) *(boundedness of all the coefficients)* for some positive constant $c > 0$, we have

$$|a^{\alpha,\beta}_{i,j}(x, y)| \leq c$$

for almost all $x \in \Omega$ and for all $y \in \mathbb{R}^N$;

($\mathcal{A}_2$) *(ellipticity of the diagonal coefficients)* for some positive constant $m > 0$, we have

$$\sum_{i,j\in[n]} a^{\alpha,\alpha}_{i,j}(x, y)\, \lambda_i \lambda_j \geq m|\lambda|^2$$

for almost all $x \in \Omega$, for all $y \in \mathbb{R}^N$, and for all $\lambda \in \mathbb{R}^n$;

($\mathcal{A}_3$) *(r-staircase support of the off-diagonal)* there exists $r \in (0, +\infty)$ such that when $\alpha \neq \beta$,

$$a^{\alpha,\beta}_{i,j}(x, y) \neq 0 \Rightarrow y \in \bigcup_{h=0}^{+\infty} \left\{hr \leq |y^\alpha| < (h + 1)r,\ hr \leq \left|y^\beta\right| < (h + 1)r\right\},$$

see Fig. 1 (right);

($\mathcal{A}_4$) *(smallness of the quadratic form associated to off-diagonal)* for some constant $\theta \in (0, 1)$, we have

$$\sum_{\alpha\in[N]} \sum_{\beta\in[N]\setminus\{\alpha\}} \sum_{i,j\in[n]} a^{\alpha,\beta}_{i,j}(x, y)\xi^\alpha_i \xi^\beta_j \geq -\theta \sum_{\alpha\in[N]} \sum_{i,j\in[n]} a^{\alpha,\alpha}_{i,j}(x, y)\xi^\alpha_i \xi^\alpha_j$$

for almost every $x \in \Omega$, for all $y \in \mathbb{R}^N$ and any $\xi \in \mathbb{R}^{N\times n}$.

Remark 1. Assumption ($\mathcal{A}_3$) may be replaced by a stronger condition, which states that off-diagonal elements $y \mapsto a^{\alpha,\beta}_{i,j}(x, y)$ have bounded support uniformly in x, i.e.

$$\rho := \sup \left\{|y| \in \mathbb{R} \ :\ a^{\alpha,\beta}_{i,j}(x, y) \neq 0 \ \text{ for some } x \in \Omega\right\} < +\infty.$$

Hence, ($\mathcal{A}_3$) is satisfied by choosing $r = \rho + 1$ because

$$a^{\alpha,\beta}_{i,j} \neq 0 \Rightarrow |y| \leq \rho < \rho + 1 = r \Rightarrow |y^\alpha| < r \text{ and } |y^\beta| < r.$$

Remark 2. When $\alpha \neq \beta$, assume the skew-symmetry condition

$$a^{\alpha,\beta}_{i,j} = -a^{\beta,\alpha}_{j,i} \tag{2.3}$$

then it results

$$\sum_{\alpha\in[N]} \sum_{\beta\in[N]\setminus\{\alpha\}} \sum_{i,j\in[n]} a^{\alpha,\beta}_{i,j}(x, y)\xi^\beta_j \xi^\alpha_i = 0 \tag{2.4}$$

for every $\xi \in \mathbb{R}^{N\times n}$. Hence ($\mathcal{A}_4$) is satisfied for any $\theta \geq 0$, provided the ellipticity condition ($\mathcal{A}_2$) is in force.

Remark 3. The smallness condition (1.9) guarantees that

$$\begin{aligned}\left|\sum_{\alpha\in[N]}\sum_{\beta\neq\alpha}\sum_{i,j\in[n]} a_{i,j}^{\alpha,\beta}(x,y)\xi_j^\beta\xi_i^\alpha\right| &\le L|\xi|^2 = \frac{L}{m}m|\xi|^2 \\ &\le \frac{L}{m}\sum_{\alpha\in[N]}\sum_{i,j\in[n]} a_{i,j}^{\alpha,\alpha}(x,y)\xi_j^\alpha\xi_i^\alpha, \end{aligned} \tag{2.5}$$

so $(\mathcal{A}_4)$ is satisfied with $\theta = \frac{L}{m}$, provided the ellipticity condition $(\mathcal{A}_2)$ holds.

Remark 4. Assume that off-diagonal coefficients split as

$$a_{i,j}^{\alpha,\beta} = \tilde{a}_{i,j}^{\alpha,\beta} + \tilde{\tilde{a}}_{i,j}^{\alpha,\beta} \tag{2.6}$$

where $\tilde{a}_{i,j}^{\alpha,\beta}$ verify the skew-symmetry condition (2.3) and $\tilde{\tilde{a}}_{i,j}^{\alpha,\beta}$ satisfy the smallness condition (1.9). Using (2.4) for $\tilde{a}_{i,j}^{\alpha,\beta}$ and (2.5) for $\tilde{\tilde{a}}_{i,j}^{\alpha,\beta}$, we get $(\mathcal{A}_4)$ with $\theta = \frac{L}{m}$, provided again that the ellipticity condition $(\mathcal{A}_2)$ holds.

Remark 5. The smallness quadratic form condition $(\mathcal{A}_4)$ implies that

$$\sum_{\alpha\in[N]}\sum_{\beta\in[N]}\sum_{i,j\in[n]} a_{i,j}^{\alpha,\beta}(x,y)\xi_j^\beta\xi_i^\alpha \ge (1-\theta)\sum_{\alpha\in[N]}\sum_{i,j\in[n]} a_{i,j}^{\alpha,\alpha}(x,y)\xi_j^\alpha\xi_i^\alpha \tag{2.7}$$

so, $(\mathcal{A}_2)$ and $(\mathcal{A}_4)$ give

$$\sum_{\alpha\in[N]}\sum_{\beta\in[N]}\sum_{i,j\in[n]} a_{i,j}^{\alpha,\beta}(x,y)\xi_j^\beta\xi_i^\alpha \ge (1-\theta)m|\xi|^2 \tag{2.8}$$

for every $\xi \in \mathbb{R}^{N\times n}$. Moreover, assumption $(\mathcal{A}_1)$ implies

$$\sum_{\alpha=1}^{N}\sum_{i=1}^{n} A_i^\alpha(x,y,\xi)\,\xi_i^\alpha \le cn^2N^2|\xi|^2 \tag{2.9}$$

for every $\xi \in \mathbb{R}^{N\times n}$. When $n \ge 3$, (2.9) says that $\xi \mapsto A(x,y,\xi)$, defined in (QL), is not n-coercive. Therefore, assumptions $(\mathcal{A})$ are not contained in the assumptions of [28], [32], [30] and [53].

A key point in our scheme of proof is that there exists a constant $C_{m,r,\theta} > 0$ such that for every $f \in L^2(\Omega;\mathbb{R}^N)$ the corresponding solution $u \in W_0^{1,2}(\Omega,\mathbb{R}^N)$, of problem (2.1), satisfies

$$\sum_{\alpha\in[N]}\int_{\Gamma_{h,r}^\alpha} |Du^\alpha|^2\,dx \le C_{m,r,\theta}\sum_{\alpha\in[N]}\|f^\alpha\|_{L^1(\Omega)} \tag{E_1}$$

for every integer $h \in \mathbb{N}_0$, where

$$\Gamma_{h,r}^\alpha := \{x\in\Omega\,:\,hr\le|u^\alpha(x)|<(h+1)r\}$$

and $r > 0$ is given in $(\mathcal{A}_3)$. We point out that the right-hand side of (E_1) contains the L^1 norm of f, even if f is in L^2.

Lemma 1. *The estimate* (E_1) *is valid under conditions* ($\mathcal{A}$) *with* $C_{m,r,\theta} = r/[(1-\theta)m]$.

Proof. For fixed integers $h \in \mathbb{N}_0$ and $r \in (0, +\infty)$, we consider the truncation function $T_{r,h} : \mathbb{R} \to \mathbb{R}$ defined as

$$T_{r,h}(s) := \begin{cases} -1 & \text{if} \quad s < -(h+1)r, \\ \frac{1}{r}s + h & \text{if} \quad -(h+1)r \le s \le -hr, \\ 0 & \text{if} \quad -hr < s < hr, \\ \frac{1}{r}s - h & \text{if} \quad hr \le s \le (h+1)r, \\ 1 & \text{if} \quad s > (h+1)r, \end{cases} \tag{2.10}$$

which verifies

$$-1 \le T_{r,h}(s) \le 1 \text{ for every } s \in \mathbb{R}. \tag{2.11}$$

Suppose $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ is a solution for $\mu \equiv f \in L^2(\Omega, \mathbb{R}^N)$. Choose $\varphi = \left(\varphi^1, ..., \varphi^N\right)$ in (2.2) as

$$\varphi^\alpha(x) := T_{r,h}(u^\alpha(x)), \tag{2.12}$$

so that

$$D_i\varphi^\alpha(x) = 1_{\Gamma^\alpha_{h,r}}(x)\, r^{-1} D_i u^\alpha(x), \tag{2.13}$$

where $1_E(x) := 1$ if $x \in E$ and $1_E(x) := 0$ if $x \notin E$. The inequality (2.11) gives

$$\sum_{\alpha\in[N]} \int_\Omega f^\alpha \varphi^\alpha dx \le \sum_{\alpha\in[N]} \int_\Omega |f^\alpha|\, dx = \sum_{\alpha\in[N]} \|f^\alpha\|_{L^1(\Omega)}. \tag{2.14}$$

For convenience, we define $\mathbf{C}$ and $\mathbf{D}$ as

$$\begin{aligned} &\sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \int_\Omega a^{\alpha,\beta}_{i,j} D_j u^\beta 1_{\Gamma^\alpha_{h,r}} D_i u^\alpha dx \\ &= \sum_{\alpha\in[N]} \sum_{i,j\in[n]} \int_\Omega a^{\alpha,\alpha}_{i,j} D_j u^\alpha 1_{\Gamma^\alpha_{h,r}} D_i u^\alpha dx \\ &+ \sum_{\alpha\in[N]} \sum_{\beta\in[N]\setminus\{\alpha\}} \sum_{i,j\in[n]} \int_\Omega a^{\alpha,\beta}_{i,j} D_j u^\beta 1_{\Gamma^\alpha_{h,r}} D_i u^\alpha dx =: \mathbf{C} + \mathbf{D} \end{aligned}$$

where $a^{\alpha,\beta}_{i,j} \equiv a^{\alpha,\beta}_{i,j}(x, u(x))$. Applying (2.12) in equation (2.2), we obtain

$$\mathbf{C} + \mathbf{D} = r \sum_{\alpha\in[N]} \int_\Omega f^\alpha \varphi^\alpha dx. \tag{2.15}$$

The ellipticity condition ($\mathcal{A}_2$) for diagonal coefficients $a^{\alpha,\alpha}_{i,j}$ implies that

$$\mathbf{C} \ge m \sum_{\alpha\in[N]} \int_{\Gamma^\alpha_{h,r}} |Du^\alpha|^2\, dx. \tag{2.16}$$

Now, we use the assumption $(\mathcal{A}_3)$, about the support for off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$, obtaining

$$1_{\Gamma_{h,r}^\alpha} a_{i,j}^{\alpha,\beta} = 1_{\Gamma_{h,r}^\alpha} 1_{\Gamma_{h,r}^\beta} a_{i,j}^{\alpha,\beta}. \tag{2.17}$$

Then, from (2.17), we have

$$\begin{aligned}\mathbf{D} &= \int_\Omega \sum_{\alpha\in[N]} \sum_{\beta\in[N]\setminus\{\alpha\}} \sum_{i,j\in[n]} a_{i,j}^{\alpha,\beta} D_j u^\beta 1_{\Gamma_{h,r}^\alpha} D_i u^\alpha dx \\ &= \int_\Omega \sum_{\alpha\in[N]} \sum_{\beta\in[N]\setminus\{\alpha\}} \sum_{i,j\in[n]} a_{i,j}^{\alpha,\beta} \left(1_{\Gamma_{h,r}^\beta} D_j u^\beta\right) \left(1_{\Gamma_{h,r}^\alpha} D_i u^\alpha\right) dx.\end{aligned}$$

Using $(\mathcal{A}_4)$ and (2.7) with $\xi_i^\alpha = 1_{\Gamma_{h,r}^\alpha} D_i u^\alpha$, we get

$$\mathbf{C} + \mathbf{D} \geq (1-\theta)\mathbf{C}. \tag{2.18}$$

Note that (2.18) is due to both $(\mathcal{A}_3)$ and $(\mathcal{A}_4)$, namely, (2.7) can be used because of the r-*staircase* support that implies (2.17). Using $1-\theta > 0$, equality (2.15), inequalities (2.14) and (2.16), we obtain the estimate

$$\sum_{\alpha\in[N]} \int_{\Gamma_{h,r}^\alpha} |Du^\alpha|^2 \, dx \leq \frac{r}{(1-\theta)m} \sum_{\alpha\in[N]} \|f^\alpha\|_{L^1(\Omega)}. \tag{2.19}$$

Hence, the claim is proved. □

Lemma 2. *Assume $f \in L^2(\Omega, \mathbb{R}^N)$ and $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ verify the estimate (E_1) with $\|f\|_{L^1} \leq \tilde{C}$. Then, for $1 \leq q < \frac{n}{n-1}$, we have*

$$\|u\|_{W_0^{1,q}} \leq C \tag{2.20}$$

where C depends only on $\tilde{C}$, $C_{m,r,\theta}$, r, q, n, N and Ω.

Lemma 2 is a version of Lemma 1 in Boccardo–Gallouët [11] adapted to our situation, where we decompose Ω into strips of size $r \in (0, +\infty)$ instead of size one (i.e. $r = 1$). The proof goes along the same line and it is included in the appendix for the convenience of the reader and proof completeness.

Our main result is the following.

Theorem 1. *Suppose that conditions $(\mathcal{A})$ hold. For every finite Radon measure μ on $\mathbb{R}^n$ with values in $\mathbb{R}^N$, there exists $u \in \bigcap_{1\leq q<\frac{n}{n-1}} W_0^{1,q}\left(\Omega, \mathbb{R}^N\right)$ such that*

$$\sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \int_\Omega a_{i,j}^{\alpha,\beta}(x, u(x))\, D_j u^\beta(x)\, D_i\varphi^\alpha(x)\, dx = \sum_{\alpha\in[N]} \int_\Omega \varphi^\alpha(x)\, d\mu^\alpha(x)$$

for every $\varphi \in C_0^\infty\left(\Omega, \mathbb{R}^N\right)$.

In what follows, we give an example which satisfies the hypotheses of the Theorem 1.

Example 1. Let $\delta_{i,j}$ denote the usual Kronecker delta function

$$\delta_{i,j} := \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

For $N = 2$, any $n \geq 3$, and continuous bounded functions $b_i : \mathbb{R}^N \to \mathbb{R}^N$ having the same r-*staircase* support for some $r \in (0, +\infty)$, with $b_1(0) \geq 1$, set

$$a_{i,j}^{\alpha,\beta}(x,y) := \begin{cases} -\delta_{i,j} b_i(y) & \text{if} \quad \beta < \alpha, \\ \delta_{i,j} & \text{if} \quad \beta = \alpha, \\ \delta_{i,j} b_i(y) & \text{if} \quad \beta > \alpha. \end{cases}$$

For $n = 3$, the matrices are the following

$$a^{1,1} = a^{2,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad a^{1,2} = -a^{2,1} = \begin{pmatrix} b_1(y) & 0 & 0 \\ 0 & b_2(y) & 0 \\ 0 & 0 & b_3(y) \end{pmatrix}.$$

The assumptions $(\mathcal{A})$ are verified for $m = 1$ and any $\theta \in (0, 1)$. The associated problem (2.1) may be written as

$$\begin{cases} -\Delta u^1 - \sum\limits_{i \in [n]} \frac{\partial}{\partial x_i} \left(b_i\left(u^1, u^2\right) \frac{\partial}{\partial x_i} u^2 \right) = \mu^1 & \text{in } \Omega, \\ -\Delta u^2 + \sum\limits_{i \in [n]} \frac{\partial}{\partial x_i} \left(b_i\left(u^1, u^2\right) \frac{\partial}{\partial x_i} u^1 \right) = \mu^2 & \text{in } \Omega, \\ u^1 = u^2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.21}$$

Then, Theorem 1 ensures the existence of a solution of the problem (2.21). Our example does not satisfy the assumptions of previous works, cited in the introduction. Indeed, when $y = 0$, we have

$$\sum_{\alpha \in [N]} \sum_{\beta \in [N] \setminus \{\alpha\}} \sum_{i,j \in [n]} \left| a_{i,j}^{\alpha,\beta}(x, 0) \right|^2 = 2|b_1(0)|^2 + 2 \sum_{i=2}^{n} |b_i(0)|^2 \geq 2,$$

so, there is no constant L with $0 < L < m = 1$ satisfying the condition (1.9) that was required in the set of assumptions $(\mathcal{A})$ of [54]. Note that the function b_1 verifies the r-staircase support condition so $b(y) = 0$ outside the support, e.g. for $y = (0, 3r/2)$. On the other hand, we have $b_1(0) \geq 1$, thus b_1 is not constant everywhere. Hence, it is not possible to find a constant $r_1 \in \mathbb{R}$ satisfying

$$a_{i,j}^{1,2}(x,y) = \delta_{ij} b_i(y) = r_1 \delta_{ij} = r_1 a_{i,j}^{2,2}(x,y)$$

then the set of assumptions $(\mathcal{A}^*)$ of [54] is also not satisfied by the present example. Moreover, for $\xi \in \mathbb{R}^{N \times n}$ and $\alpha = 1$, we have

$$\begin{aligned} \sum_{i \in [n]} A_i^\alpha(x, 0, \xi) \xi_i^\alpha &= \sum_{i \in [n]} \sum_{\beta \in [N]} \sum_{j \in [n]} a_{i,j}^{\alpha,\beta} \xi_j^\beta \xi_i^\alpha \\ &= \sum_{i,j \in [n]} a_{i,j}^{1,1} \xi_j^1 \xi_i^1 + \sum_{i,j \in [n]} a_{i,j}^{1,2} \xi_j^2 \xi_i^1 \\ &= \sum_{i,j \in [n]} \delta_{i,j} \xi_j^1 \xi_i^1 + \sum_{i,j \in [n]} \delta_{i,j} b_i(0) \xi_j^2 \xi_i^1 \end{aligned}$$

$$= \sum_{i \in [n]} (\xi_i^1)^2 + \sum_{i \in [n]} b_i(0) \xi_i^2 \xi_i^1 =: \mathbf{A}(\xi).$$

Choosing

$$\xi_1^1 = 1 \text{ and } \xi_i^1 = \xi_i^2 = 0 \text{ for } i \geq 2,$$

we have that

$$\mathbf{A}(\xi) = 1 + b_1(0)\xi_1^2 \to -\infty \text{ as } \xi_1^2 \to -\infty.$$

Therefore, condition (1.5) is not valid, implying that condition (1.4) is also not valid. The example is concluded.

The existence result of Theorem 1 applies for any finite Radon measure, when $1 \leq q < \frac{n}{n-1}$, since the set of Radon measures are compactly embedded in $W^{-1,q'}(\Omega, \mathbb{R}^N)$ and the unique set with zero harmonic capacity (i.e. $\mathrm{cap}_{W^{2,q'}}$) is the empty set. However, in Brezis's seminar [13] there is a nonexistence result involving the delta measure. In fact, when the integrability of u and Du is high enough, we may apply the same idea to the system (2.1). For that, we will consider the notion of a very weak solution and a new set of assumptions, where $a_{i,j}^{\alpha,\beta}$ are more regular.

($\mathcal{A}^*$): For all $i, j, s \in [n]$ and $\alpha, \beta, \gamma \in [N]$, we assume that $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfy the following conditions:
($\mathcal{A}_0^*$) $a_{i,j}^{\alpha,\beta} \in C^1(\Omega \times \mathbb{R}^N)$;
($\mathcal{A}_1^*$) for some positive constant $c > 0$, we have

$$|a_{i,j}^{\alpha,\beta}(x, y)| \leq c, \ \ |b_{i,j,s}^{\alpha,\beta}(x, y)| \leq c \ \text{ and } \ |c_{i,j}^{\alpha,\beta,\gamma}(x, y)| \leq c$$

for all $x \in \Omega$ and for all $y \in \mathbb{R}^N$, where $b_{i,j,s}^{\alpha,\beta}(x, y) := \frac{\partial}{\partial x_s} a_{i,j}^{\alpha,\beta}(x, y)$ and $c_{i,j}^{\alpha,\beta,\gamma}(x, y) := \frac{\partial}{\partial y_\gamma} a_{i,j}^{\alpha,\beta}(x, y)$.

Let μ be a finite Radon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^N$. We say that $u \in L^1(\Omega, \mathbb{R}^N)$ is a very weak solution of problem (2.1) if it holds

$$\begin{aligned} &- \sum_{\alpha,\beta \in [N]} \sum_{i,j \in [n]} \int_\Omega a_{i,j}^{\alpha,\beta} u^\beta D_j D_i \varphi^\alpha dx \\ &- \sum_{\alpha,\beta \in [N]} \sum_{i,j \in [n]} \int_\Omega u^\beta D_j a_{i,j}^{\alpha,\beta} D_i \varphi^\alpha dx = \sum_{\alpha \in [N]} \int_\Omega \varphi^\alpha d\mu^\alpha \end{aligned} \tag{2.22}$$

for every $\varphi \in C_0^\infty\left(\Omega, \mathbb{R}^N\right)$, where $a_{i,j}^{\alpha,\beta} \equiv a_{i,j}^{\alpha,\beta}(x, u(x))$. Under assumptions ($\mathcal{A}^*$), the nonexistence of very weak solutions imply the nonexistence of weak solutions.

Lemma 3. *If $n > 2$, $t > \frac{n}{n-2}$, $q > \frac{nt'}{n-t'}$ and $u \in L^t(\Omega, \mathbb{R}^N) \cap W_0^{1,q}(\Omega, \mathbb{R}^N)$ is a (very) weak solution, then*

$$\mu(\{y\}) = 0 \qquad \forall y \in \Omega. \tag{2.23}$$

The previous Lemma allows us to get the following nonexistence result.

Corollary 1. *If $n > 2$, $t > \frac{n}{n-2}$, $q > \frac{nt'}{n-t'}$ and $\mu^\alpha = \delta_y$ for some $\alpha \in [N]$ and $y \in \Omega$, then there is no (very) weak solution $u \in L^t(\Omega, \mathbb{R}^N) \cap W_0^{1,q}(\Omega, \mathbb{R}^N)$ of (2.22).*

In the previous nonexistence result, no ellipticity condition was required so it may be applied to elliptic-hyperbolic systems. See also [49], [21], [45], [43] for very weak solutions of elliptic systems.

As a final remark, we point out that Theorem 1 guarantees existence of solutions $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$ with any $q < \frac{n}{n-1}$; Sobolev embedding gives $u \in L^t(\Omega, \mathbb{R}^N)$ for every $t < \left(\frac{n}{n-1}\right)^* = \frac{n}{n-2}$: the nonexistence result in Corollary 1 requires $t > \frac{n}{n-2}$; see also [4].

3. Proof of the main result

For any $f \in L^1\left(\Omega, \mathbb{R}^N\right)$, we say that a function $u : \Omega \to \mathbb{R}^N$ is a (distributional) solution with respect to f, if $u \in W_0^{1,2}\left(\Omega, \mathbb{R}^N\right)$ and

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x)\, D_i \varphi^\alpha(x) dx = \int_\Omega \sum_{\alpha=1}^N f^\alpha(x) \varphi^\alpha(x) dx, \tag{3.1}$$

for all $\varphi \in W_0^{1,2}\left(\Omega, \mathbb{R}^N\right) \cap L^\infty\left(\Omega, \mathbb{R}^N\right)$.

Theorem 1 deals with existence of a distributional solution to the Dirichlet problem (1.1)–(1.2). The following proof shows that we are dealing with a Solution Obtained as Limit of Approximations (SOLA), see [25] and Section 2.2 in [35]. Entropy solutions have been considered in [5]; see the introduction of [24] for renormalized solutions.

Proof of Theorem 1. Let $\mu : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}^N$ be a finite Radon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^N$ and let $|\mu|$ denote its total variation. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers, that is $\rho_\varepsilon(x) = \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right)$, where $\rho \in C_c^\infty(\mathbb{R}^n)$ satisfies

$$\rho(x) \geq 0,\ \ \rho(-x) = \rho(x),\ \ \operatorname{supp}(\rho) \subseteq B_1 \quad \text{and} \quad \int_{\mathbb{R}^n} \rho(x)\, dx = 1.$$

Recall that the convolution $\mu * \rho_\varepsilon$ defined by

$$(\mu * \rho_\varepsilon)(x) = \int_{\overline{B}(x,\varepsilon)} \rho_\varepsilon(x-y)\, d\mu(y)$$

is a regular function, that is $\mu * \rho_\varepsilon \in C^\infty\left(\mathbb{R}^n, \mathbb{R}^N\right)$.

Let $(\varepsilon_h)_{h\in\mathbb{N}}$ be such that $0 < \varepsilon_h \leq 1$ and $\varepsilon_h \to 0$ as $h \to \infty$. Then $\mu * \rho_{\varepsilon_h} \in C^\infty\left(\mathbb{R}^n, \mathbb{R}^N\right)$ and

$$\int_\Omega |\mu * \rho_{\varepsilon_h}|(x)\, dx \leq |\mu|(I_{\varepsilon_h}(\Omega)) \leq |\mu|(\mathbb{R}^n) < +\infty, \tag{3.2}$$

where $I_\varepsilon(\Omega)$ denotes the open ε-neighborhood of Ω (see [1], Theorem 2.2, p. 42). We define

$$f_h = \mu * \rho_{\varepsilon_h} \in L^2\left(\Omega, \mathbb{R}^N\right).$$

By assumptions $(\mathcal{A})$ and (2.8), the conditions of Leray–Lions Theorem (see [55]) are satisfied, so for each $h \in \mathbb{N}$ we obtain

$$u_h \in W_0^{1,2}\left(\Omega, \mathbb{R}^N\right)$$

and, for all $\varphi \in W_0^{1,2}\left(\Omega, \mathbb{R}^N\right)$,

$$\sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \int_\Omega a_{i,j}^{\alpha,\beta}(x, u_h(x)) D_j u_h^\beta (x)\, D_i \varphi^\alpha (x) dx$$
$$= \int_\Omega \sum_{\alpha=1}^{N} f_h^\alpha (x) \varphi^\alpha (x) dx = \sum_{\alpha=1}^{N} \int_\Omega \varphi^\alpha d\mu_{\varepsilon_h}^\alpha . \tag{3.3}$$

Applying Lemmas 1 and 2 we have that, for $1 \leq q < \frac{n}{n-1}$, there exists $C > 0$ such that

$$\|u_h^\alpha\|_{W_0^{1,q}(\Omega)} \leq C.$$

So, passing to a subsequence if needed, we may assume that there exists $u_\infty^\alpha \in W_0^{1,q}(\Omega)$ such that

$$u_h^\alpha \overset{w}{\to} u_\infty^\alpha \text{ in } W_0^{1,q}(\Omega) \quad \text{ and } \quad u_h^\alpha \to u_\infty^\alpha \text{ in } L^q(\Omega), \tag{3.4}$$

where $\overset{w}{\to}$ denotes weak convergence. Again passing to a subsequence if needed, we may assume that

$$u_h^\alpha (x) \to u_\infty^\alpha (x) \quad \text{ for almost all } x \in \Omega,$$

and, by the continuity of $y \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ (see $(\mathcal{A}_0)$), it follows that

$$a_{i,j}^{\alpha,\beta}(x, u_h(x)) \to a_{i,j}^{\alpha,\beta}(x, u_\infty(x)) \quad \text{ for almost all } x \in \Omega.$$

For presentation convenience, we set $a_{i,j}^{\alpha,\beta}(u_h) \equiv a_{i,j}^{\alpha,\beta}(x, u_h(x))$ and $a_{i,j}^{\alpha,\beta}(u_\infty) \equiv a_{i,j}^{\alpha,\beta}(x, u_\infty(x))$. For every $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$, we have

$$\begin{aligned}
&\left| \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(u_h) D_j u_h^\beta D_i \varphi^\alpha dx \right.\\
&\left. - \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(u_\infty) D_j u_\infty^\beta D_i \varphi^\alpha dx \right|\\
&= \left| \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \left[a_{i,j}^{\alpha,\beta}(u_h) - a_{i,j}^{\alpha,\beta}(u_\infty)\right] D_j u_h^\beta D_i \varphi^\alpha dx \right.\\
&+ \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(u_\infty) D_j u_h^\beta D_i \varphi^\alpha dx\\
&\left. - \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(u_\infty) D_j u_\infty^\beta D_i \varphi^\alpha dx \right|\\
&\leq \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \left[\int_\Omega \left| a_{i,j}^{\alpha,\beta}(u_h) - a_{i,j}^{\alpha,\beta}(u_\infty)\right|^{q'} |D_i \varphi^\alpha|^{q'} \right]^{\frac{1}{q'}} \left(\int_\Omega \left| D_i u_h^\beta \right|^q \right)^{\frac{1}{q}} dx\\
&+ \left| \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \left[\int_\Omega a_{i,j}^{\alpha,\beta}(u_\infty) D_i \varphi^\alpha D_j u_h^\beta - \int_\Omega a_{i,j}^{\alpha,\beta}(u_\infty) D_i \varphi^\alpha D_j u_\infty^\beta \right] \right| .
\end{aligned} \tag{3.5}$$

Using the dominated convergence theorem, we obtain

$$\int_\Omega \left| a_{i,j}^{\alpha,\beta}(u_h) - a_{i,j}^{\alpha,\beta}(u_\infty)\right|^{q'} |D_i \varphi^\alpha|^{q'} \to 0 \tag{3.6}$$

and, by the weak convergence in (3.4), it follows that

$$\int_\Omega a_{i,j}^{\alpha,\beta}(u_\infty) D_i\varphi^\alpha D_j u_h^\beta \to \int_\Omega a_{i,j}^{\alpha,\beta}(u_\infty) D_i\varphi^\alpha D_j u_\infty^\beta. \tag{3.7}$$

From (3.5), (3.6) and (3.7), it follows that

$$\begin{aligned} &\textstyle\int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(x, u_h(x)) D_j u_h^\beta(x) D_i\varphi^\alpha(x)\, dx \\ &\to \textstyle\int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(x, u_\infty(x)) D_j u_\infty^\beta(x) D_i\varphi^\alpha(x)\, dx. \end{aligned} \tag{3.8}$$

On the other hand,

$$\int_\Omega f_h^\alpha(x)\varphi^\alpha(x)\, dx = \int_\Omega \varphi^\alpha d\mu_{\varepsilon_h}^\alpha \to \int_\Omega \varphi^\alpha d\mu^\alpha.$$

Therefore, for each $q \in \left[1, \frac{n}{n-1}\right)$ there exists $u_{\infty,q} \in W_0^{1,q}\left(\Omega, \mathbb{R}^N\right)$ and a subsequence, again denoted by $(u_h)_h$, such that

$$u_h \overset{w}{\to} u_{\infty,q} \quad \text{in } W_0^{1,q}\left(\Omega, \mathbb{R}^N\right) \tag{3.9}$$

and

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{i,j}^{\alpha,\beta}(x, u_{\infty,q}(x)) D_j u_{\infty,q}^\beta(x) D_i\varphi^\alpha(x)\, dx \\ &= \int_\Omega \sum_{\alpha=1}^{N} \varphi^\alpha(x)\, d\mu^\alpha(x), \end{aligned}$$

i.e. $u_{\infty,q}$ is a (distributional) solution of our problem.

We claim that $u_{\infty,q}$ does not depend on a fixed q. Indeed, if $q < \widetilde{q} < \frac{n}{n-1}$ then

$$\|u_h^\alpha\|_{W_0^{1,\widetilde{q}}(\Omega)} \le C_6(\widetilde{q})$$

and, passing to a subsequence of $(u_h)_h$ satisfying (3.9), we can assume that

$$u_h \overset{w}{\to} u_{\infty,\widetilde{q}} \quad \text{in } W_0^{1,\widetilde{q}}\left(\Omega, \mathbb{R}^N\right). \tag{3.10}$$

By (3.9), (3.10) and the fact that $q < \widetilde{q}$, we obtain

$$u_{\infty,q} = u_{\infty,\widetilde{q}}.$$

Therefore,

$$u_{\infty,q} \in W_0^{1,\widetilde{q}}\left(\Omega, \mathbb{R}^N\right) \qquad \text{for all } \widetilde{q} \in \left[1, \frac{n}{n-1}\right),$$

and the proof is concluded. $\square$

4. Proofs of the lemmas

Proof of Lemma 2. Suppose $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ satisfies the estimate (E_1) for all $h \in \mathbb{N}_0$. Applying the Hölder's inequality when $1 \leq q < 2$, we have

$$\int_\Omega |Du^\alpha|^q\,dx = \sum_{h=1}^{+\infty} \int_{\Gamma^\alpha_{h,r}} |Du^\alpha|^q\,dx$$
$$\leq \sum_{h=1}^{+\infty} \left(\int_{\Gamma^\alpha_{h,r}} |Du^\alpha|^2\,dx \right)^{\frac{q}{2}} \left(\operatorname{meas}(\Gamma^\alpha_{h,r})\right)^{\frac{2-q}{2}}.$$

Since $\operatorname{meas}(\Gamma^\alpha_{h,r}) \leq (hr)^{-q^*} \int_{\Gamma^\alpha_{h,r}} |u^\alpha|^{q^*}\,dx$, we may continue by applying the estimate (E_1) and the discrete Hölder's inequality so

$$\int_\Omega |Du^\alpha|^q\,dx \leq C_0 \sum_{h=1}^{+\infty} h^{-\frac{q^*(2-q)}{2}} \left(\int_{\Gamma^\alpha_{h,r}} |Du^\alpha|^2\,dx \right)^{\frac{q}{2}} \left(\int_{\Gamma^\alpha_{h,r}} |u^\alpha|^{q^*}\,dx \right)^{\frac{2-q}{2}}$$
$$\leq C_1 \left(\sum_{h=1}^{+\infty} h^{-\frac{q^*(2-q)}{q}} \right)^{\frac{q}{2}} \left(\sum_{h=1}^{+\infty} \int_{\Gamma^\alpha_{h,r}} |u^\alpha|^{q^*}\,dx \right)^{\frac{2-q}{2}}$$
$$\leq C_2 \left(\int_\Omega |u^\alpha|^{q^*}\,dx \right)^{\frac{2-q}{2}}. \tag{4.1}$$

For $1 \leq q < \frac{n}{n-1}$, Sobolev embedding gives

$$\|u^\alpha\|_{q^*}^{q^*} \leq C_3 \left(\|u^\alpha\|_{q^*}^{q^*} \right)^{\frac{(2-q)q^*}{2q}} \quad \Rightarrow \quad \|u^\alpha\|_{q^*} \leq C_4.$$

Apply the above result in inequality (4.1), obtaining

$$\|Du\|_q \leq C_5. \quad \square$$

Proof of Lemma 3. Suppose $u \in L^1(\Omega, \mathbb{R}^N)$ is a very weak solution. Let $\varphi \in C_0^\infty\left(B_1(0), \mathbb{R}^N\right)$, $\varphi_k(x) := \varphi(k(x-y))$ for all $k \in \mathbb{N}$, where $y \in \Omega$. We claim that, when $k \to +\infty$,

$$\sum_{\alpha \in [N]} \int_\Omega \varphi_k^\alpha(x)\,d\mu^\alpha(x) \to 0. \tag{4.2}$$

Indeed, we start noting that $B_{1/k}(y) \subset\subset \Omega$ when $k > [\operatorname{dist}(y, \partial\Omega)]^{-1}$. Moreover $\varphi(k(x-y)) = 0$ when $x \notin B_{1/k}(y)$. Then $\varphi_k \in C_0^\infty(\Omega, \mathbb{R}^N)$ when $k > [\operatorname{dist}(y, \partial\Omega)]^{-1}$. We use φ_k as a test function in (2.17). Define, for presentation clarification, the following

$$\Phi := \sum_{\alpha,\beta \in [N]} \sum_{i,j \in [n]} \int_\Omega a^{\alpha,\beta}_{i,j} u^\beta D_j D_i \varphi^\alpha_k dx + \int_\Omega u^\beta D_j a^{\alpha,\beta}_{i,j} D_i \varphi^\alpha_k dx$$

$$= \sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \int_\Omega a_{i,j}^{\alpha,\beta} u^\beta D_j D_i \varphi_k^\alpha dx + \int_\Omega u^\beta b_{i,j,j}^{\alpha,\beta} D_i \varphi_k^\alpha dx$$
$$+ \sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \sum_{\gamma\in[N]} \int_\Omega u^\beta c_{i,j}^{\alpha,\beta,\gamma} D_j u^\gamma D_i \varphi_k^\alpha dx.$$

Then using $(\mathcal{A}^*)$ and the support of φ_k, there exists a constant $C_0 > 0$ such that

$$C_0^{-1}|\Phi| \le \sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \int_{B_{1/k}(y)} \left|u^\beta D_j D_i \varphi_k^\alpha\right| dx + \sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \int_{B_{1/k}(y)} \left|u^\beta D_i \varphi_k^\alpha\right| dx$$
$$+ \sum_{\alpha,\beta,\gamma\in[N]} \sum_{i,j\in[n]} \int_{B_{1/k}(y)} \left|u^\beta D_j u^\gamma D_i \varphi_k^\alpha\right| dx$$
$$=: \Phi_1 + \Phi_2 + \Phi_3.$$

By Hölder's inequality with $t \in (1, +\infty)$ and $\sigma \in (1, +\infty)$, the chain rule that gives $D_i\varphi_k(x) = kD_i\varphi(k(x-y))$ and $D_jD_i\varphi_k(x) = k^2 D_j D_i \varphi(k(x-y))$, the change of variables $z = k(x-y)$, there exist positive constants A_Φ, B_Φ, C_Φ such that

$$\Phi_1 = \sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \int_{B_{1/k}(y)} \left|u^\beta D_j D_i \varphi_k^\alpha\right| dx$$
$$\le \sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \left(\int_{B_{1/k}(y)} \left|u^\beta\right|^t dx \right)^{\frac{1}{t}} \left(\int_{B_{1/k}(y)} \left|D_j D_i \varphi_k^\alpha\right|^{t'} dx \right)^{\frac{1}{t'}}$$
$$= k^2 \sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \left(\int_{B_{1/k}(y)} \left|u^\beta\right|^t dx \right)^{\frac{1}{t}} \left(\int_{B_{1/k}(y)} \left|D_j D_i \varphi^\alpha\left(k(x-y)\right)\right|^{t'} dx \right)^{\frac{1}{t'}}$$
$$= k^{2-\frac{n}{t'}} \sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \left(\int_{B_{1/k}(y)} \left|u^\beta\right|^t dx \right)^{\frac{1}{t}} \left(\int_{B_1(0)} \left|D_j D_i \varphi^\alpha\left(z\right)\right|^{t'} dz \right)^{\frac{1}{t'}}$$
$$\le k^{2-\frac{n}{t'}} \sum_{\beta\in[N]} \|u^\beta\|_t \sum_{\alpha\in[N]} \sum_{i,j\in[n]} \left(\int_{B_1(0)} \left|D_j D_i \varphi^\alpha\right|^{t'} \right)^{\frac{1}{t'}}$$
$$\le A_\Phi k^{2-\frac{n}{t'}} \sum_{\beta\in[N]} \|u^\beta\|_t$$

$$\Phi_2 = \sum_{\alpha,\beta\in[N]} \sum_{i,j\in[n]} \int_{B_{1/k}(y)} \left|u^\beta D_i \varphi_k^\alpha\right| dx$$
$$\le \sum_{\beta\in[N]} \left(\int_{B_{1/k}(y)} \left|u^\beta\right|^t dx \right)^{\frac{1}{t}} \sum_{\alpha\in[N]} \sum_{i,j\in[n]} \left(\int_{B_{1/k}(y)} \left|D_i \varphi_k^\alpha\right|^{t'} dx \right)^{\frac{1}{t'}}$$

$$
\begin{aligned}
&= k \sum_{\beta\in[N]} \left(\int_{B_{1/k}(y)} \left|u^\beta\right|^t dx \right)^{\frac{1}{t}} \sum_{\alpha\in[N]} \sum_{i,j\in[n]} \left(\int_{B_{1/k}(y)} \left|D_i\varphi^\alpha\left(k(x-y)\right)\right|^{t'} dx \right)^{\frac{1}{t'}} \\
&= k^{1-\frac{n}{t'}} \sum_{\beta\in[N]} \left(\int_{B_{1/k}(y)} \left|u^\beta\right|^t dx \right)^{\frac{1}{t}} \sum_{\alpha\in[N]} \sum_{i,j\in[n]} \left(\int_{B_1(0)} \left|D_i\varphi^\alpha\left(z\right)\right|^{t'} dz \right)^{\frac{1}{t'}} \\
&\le B_\Phi k^{1-\frac{n}{t'}} \sum_{\beta\in[N]} \|u^\beta\|_t
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3 &= \sum_{\alpha,\beta,\gamma\in[N]} \sum_{i,j\in[n]} \int_{B_{1/k}(y)} \left|u^\beta D_j u^\gamma D_i\varphi_k^\alpha\right| dx \\
&\le \sum_{\alpha,\beta,\gamma\in[N]} \sum_{i,j\in[n]} \left(\int_{B_{1/k}(y)} \left|u^\beta\right|^t dx \right)^{\frac{1}{t}} \left(\int_{B_{1/k}(y)} \left|D_j u^\gamma D_i\varphi_k^\alpha\right|^{t'} dx \right)^{\frac{1}{t'}} \\
&\le \sum_{\alpha,\beta,\gamma\in[N]} \sum_{i,j\in[n]} \|u^\beta\|_t \|D_j u^\gamma\|_{\sigma t'} \left(\int_{B_{1/k}(y)} \left|D_i\varphi_k^\alpha\right|^{\frac{t'\sigma}{\sigma-1}} dx \right)^{\frac{\sigma-1}{\sigma t'}} \\
&= k \sum_{\alpha,\beta,\gamma\in[N]} \sum_{i,j\in[n]} \|u^\beta\|_t \|D_j u^\gamma\|_{\sigma t'} \left(\int_{B_{1/k}(y)} \left|D_i\varphi^\alpha\left(k(x-y)\right)\right|^{\frac{t'\sigma}{\sigma-1}} dx \right)^{\frac{\sigma-1}{\sigma t'}} \\
&= k^{1-\frac{n(\sigma-1)}{\sigma t'}} \sum_{\alpha,\beta,\gamma\in[N]} \sum_{i,j\in[n]} \|u^\beta\|_t \|D_j u^\gamma\|_{\sigma t'} \left(\int_{B_1(0)} \left|D_i\varphi^\alpha\left(z\right)\right|^{\frac{t'\sigma}{\sigma-1}} dz \right)^{\frac{\sigma-1}{\sigma t'}} \\
&\le C_\Phi k^{1-\frac{n(\sigma-1)}{\sigma t'}} \sum_{\beta,\gamma\in[N]} \sum_{j\in[n]} \|u^\beta\|_t \|D_j u^\gamma\|_{\sigma t'}.
\end{aligned}
$$

Suppose that $u \in L^t(\Omega,\mathbb{R}^N) \cap W_0^{1,q}(\Omega,\mathbb{R}^N)$ for $t > \frac{n}{n-2}$ and $q > \frac{nt'}{n-t'}$. Then, by choosing $\sigma = q/t'$, we have $\sigma > \frac{n}{n-t'}$ and there exists a constant $\tilde{C} > 0$ such that

$$
|\Phi| \le \tilde{C} \left(k^{2-\frac{n}{t'}} + k^{1-\frac{n}{t'}} + k^{1-\frac{n(\sigma-1)}{\sigma t'}} \right).
$$

Note that $t > \frac{n}{n-2}$ implies $2 - \frac{n}{t'} < 0$ and $1 - \frac{n}{t'} < 0$. Moreover $\sigma > \frac{n}{n-t'}$ implies $1 - \frac{n(\sigma-1)}{\sigma t'} < 0$. Hence, $\Phi \to 0$ as $k \to +\infty$, which proves (4.2).

Now we fix $\alpha \in [N]$ and we take $\varphi^\alpha \in C_0^\infty(B_1(0))$ such that $0 \le \varphi^\alpha \le 1$ and $\varphi^\alpha(0) = 1$. Then, for every $x \in \Omega$, we have

$$
\varphi^\alpha(k(x-y)) \to 1_{\{y\}}(x) \tag{4.3}
$$

when $k \to +\infty$. We can use Lebesgue convergence theorem in order to get

$$
\int_\Omega \varphi^\alpha(k(x-y)) d\mu^\alpha(x) \to \int_\Omega 1_{\{y\}}(x) d\mu^\alpha(x) = \mu^\alpha(\{y\}) \tag{4.4}
$$

Putting together (4.4) and (4.2) gives (2.23). $\square$

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